Lab session 6: Propositional logic

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May 3, 2024

1 Propositional logic: Base questions

Question 1. Write the truth table for the propositions $\phi_1 = a \to (b \vee (b \to a))$

Question 2. Which of the following propositions are tautologies? Unsatisfiable?

$$\begin{aligned}
\phi_1 &= a \to (b \vee (b \to a)) \\
\phi_2 &= (a \to b) \vee \neg(b \to a) \\
\phi_3 &= ((b \to c) \leftrightarrow (a \wedge \neg c)) \to (b \vee \neg c) \\
\phi_4 &= \neg c \wedge ((a \leftrightarrow b) \to c) \wedge (a \leftrightarrow b)
\end{aligned}$$

2 Substitutions in tautologies

Remark: The theorem presented in this exercise is important and is considered to be part of what you must know[1].

The substitution of a propositional symbol x by a propositional formula ψ in a propositional formula ϕ is the propositional formula, denoted by $\phi[x \setminus \psi]$, similar to ϕ, but with all instances of x replaced by ψ.

Question 3. For $\phi = ((a \vee b) \to (b \wedge c))$ and $\psi = (a \leftrightarrow c)$, what is $\phi[a \setminus \psi]$?

We will now prove the following result:

Theorem 1. If ϕ is a tautology, then any substitution $\phi[x \setminus \psi]$ (for any propositional symbol x and any proposition ψ) is a tautology.

Question 4. Let $\mathcal{I}$ be an interpretation. What is $(\phi[x \setminus \psi])^{\mathcal{I}}$? Prove the theorem.

We will now use this result to prove that, for all propositional sentences α, $\neg(\neg\alpha) \equiv \alpha$.

Question 5. Using the deduction theorem, show that $\neg(\neg\alpha) \equiv \alpha$ if and only if $\neg(\neg\alpha) \to \alpha$ and $\alpha \to \neg(\neg\alpha)$ are tautologies.

Question 6. Show that, for a propositional symbol x, we have $\neg(\neg(x)) \equiv x$. Using the deduction and substitution theorems, as well as the previous answer, show the result for any proposition α.

[1] In other words, don't be surprised if you have a question about it in the exam.

3 Enumeration method

Question 7. Given the knowledge base $KB = (a \vee c) \wedge (b \vee \neg c)$ and the propositional formula $\alpha = a \vee b$, show that $KB \models \alpha$ using the enumeration method.

The complexity of a propositional formula is defined recursively as follows. An atomic formula (i.e. a propositional symbol) has complexity 0. A formula of complexity $n + 1$ is a compound proposition of 1 or 2 propositions of complexity at most n (the composition can be of 1 or 2 variables).

Question 8. What is time complexity of a naive implementation of the enumeration method, with a KB of complexity c_1, a proposition of complexity c_2 and a language of n atomic variables.