SF1811 HOME ASSIGNMENT 3
MARKOWITZ PORTFOLIO PROBLEM

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1. Markowitz model

In this assignment you formulate Markowitz model for portfolio optimization as a quadratic optimization problem, and solve it using Matlab. Markowitz work[1] is highly cited and worthwhile to study. It describes the motivation and analysis of the problem in elementary terms. Markowitz received the 1990 Alfred Nobel Memorial Prize in Economic Sciences.

The model can briefly be described as follows. A fraction x_j of the total capital is invested in asset j, $j = 1, \ldots, n$. The annual (or daily) return of each asset is modelled as a random variable ξ_j. The random variables can be correlated and it is assumed that the expected values $\mu_j := \mathbb{E}[\xi_j]$ and the covariance matrix $C_{ij} := \mathbb{E}[(\xi_i - \mu_i)(\xi_j - \mu_j)]$ are known, $i, j = 1, \ldots, n$. The annual return of an asset is defined as the relative change of the asset price in a year. For given expected returns $\mu = (\mu_1, \ldots, \mu_n)^\mathsf{T}$, given correlation matrix C and a given expected return of the portfolio, $r = \boldsymbol{\mu}^\mathsf{T}\mathbf{x}$, Markowitz portfolio optimization problem is then to determine $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$ that minimize the variance of the portfolio, i.e.

$$
\begin{array}{rl}
\text{minimize} & \mathbf{x}^\mathsf{T} C\, \mathbf{x} \\
\text{subject to} & \boldsymbol{\mu}^\mathsf{T}\mathbf{x} = r, \\
& \mathbf{e}^\mathsf{T}\mathbf{x} = 1, \quad \text{where } \mathbf{e} = (1, \ldots, 1)^\mathsf{T}, \\
& \mathbf{x} \geq \mathbf{0}.
\end{array}
\tag{1.1}
$$

2. Problems to be solved

Exercise 1.
Use the `Matlab` function `quadprog` to solve the above problem (1.1) for the following 25 different values of the right hand side r:
$r = 3.00,\ 3.25,\ 3.50,\ 3.75, \ldots\ldots,\ 8.50,\ 8.75,\ 9.00.$
Save the obtained 25 values of $\sigma(\mathbf{x}) = \sqrt{\mathbf{x}^\mathsf{T}\mathbf{C}\,\mathbf{x}}$ and $\mu(\mathbf{x}) = \boldsymbol{\mu}^\mathsf{T}\mathbf{x}$ in two vectors. Plot these vectors in a figure showing σ on the horizontal and μ on the vertical axis. Use the following Matlab code for generating your vector $\boldsymbol{\mu}$ and matrix C. Note that the generated data uses input

[1] Harry Markowitz, Portfolio selection, J. of Finance, 7 (1952) 77–91.

of random numbers, so that the values of the matrix C will vary with each run.

```
n=8;
Corr=zeros(n,n);
for i=1:n
    for j=1:n
        Corr(i,j)=(-1)^abs(i-j)/(abs(i-j)+1);
    end
end
sigma=zeros(n,1);
mu=zeros(n,1);
sigma(1)=2;
mu(1)=3;
for i=1:n-1
    sigma(i+1)=sigma(i)+2*rand;
    mu(i+1)=mu(i)+1;
end
D=diag(sigma);
C2=D*Corr*D;
C=0.5*(C2+C2');
```

Exercise 2.
Repeat Exercise 1, including `quadprog` runs, but modified as follows: Assume that it is not necessary to invest the whole capital K, and that the *not invested* fraction $1 - \sum_j^n x_j$ of K can be saved without any return and without any "risk". Motivate that this situation can be modelled by simply changing the constraint $\mathbf{e}^\mathsf{T}\mathbf{x} = 1$ in the problem (1.1) to $\mathbf{e}^\mathsf{T}\mathbf{x} \leq 1$. Then comment on the difference between your obtained new figure and your figure from Exercise 1.

Exercise 3.
Repeat Exercise 1, including `quadprog` runs, but now modified as follows: Assume that the constraint $\boldsymbol{\mu}^\mathsf{T}\mathbf{x} = r$ in problem (1.1) is changed to $\boldsymbol{\mu}^\mathsf{T}\mathbf{x} \geq r$. Give an interpretation of this change in the model, and motivate the difference between your obtained new figure and your figure from Exercise 1.

Exercise 4.
Repeat Exercise 1, including `quadprog` runs, but now modified as follows: Assume that so called "short selling" is possible. This essentially means that it is possible to borrow assets today and immediatly sell them for today's market price. Then, at a future date agreed upon, (e.g. one year from today) the borrowed assets must be bought back, for the market price at that future date, and returned to the lender. Motivate that this situation can be modelled by simply removing the constraints $\mathbf{x} \geq \mathbf{0}$ in the problem (1.1). Then comment briefly on the difference between your obtained new figure and your figure from Exercise 1.

Note: Short selling might be *dangerous* since there is in principles no upper bound on the possible loss (if the market price on the borrowed asset increases dramatically).

Optional exercise 5. The www-page
`https://www.portfoliovisualizer.com/asset-correlations`
provides historic data for assets annual returns and correlations. Choose some assets, download the annual returns and correlation matrix, and solve Markowitz portfolio problem in Matlab. Try also to find a mathematical formulation of the random variables ξ_j that would not be well modelled using Markowitz setting.

Comment on the plotting: It is recommended that you, instead of presenting four separate figures with one pair of vectors in each figure, present the following three figures:
Figure 1: The two pairs of vectors corresponding to Exercises 1 and 2.
Figure 2: The two pairs of vectors corresponding to Exercises 1 and 3.
Figure 3: The two pairs of vectors corresponding to Exercises 1 and 4.

3. Instructions for the report

The aim of the homework assignment is to practice using mathematical concepts and methods and to write a good report. This means that a solution with only formulas is not acceptable. The solution should be similar to the presentation of examples in the course literature. The purpose of the report is to well explain the problem, theoretical background and results for a master student who has taken the course SF1811 but not done this assignment. Write using your own words and include additional explanations for the steps. In the grading, the teacher considers how well the report:

- explains the problem,
- describes the theoretical background,
- presents the results.

For instance, the teacher takes into account

- is the report correct,
- is the report well written,
- are the figures and derivations well chosen,
- will the reader of the report learn something.

The report does not have to be long, probably shorter than 10 pages. Matlab code should be included, e.g in an appendix. The most important is that what is written in the report is correct and that the reader learns something. The form of the report is not important, e.g. it does not matter if the there is table of context or a section "conclusion".