POLS 6481. Research Design and Quantitative Methods II

Lecture 7. Standard Errors of and Hypothesis Tests for Regression Coefficients

Readings: Wooldridge, *Introductory Econometrics 5e*, 2.5 + 3.3 + 3.4 + 3.5 + 4.2 + 4.3

Outline:

1. Classical Linear Regression Model (CLRM) assumptions
2. Regression models and their sampling variances
3. Regression coefficients and their sampling distributions
4. The $t$ distribution
5. Confidence intervals for a regression coefficient
6. Null hypothesis retention region for a regression coefficient

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## 0. Motivation: *Fun with Simulations*

Simulate $N$ = 1,000 observations from a known data process

$$y = \beta_0 + \beta_1 x + \beta_2 w + \beta_3 v + u$$

Assume $x$ and $w$ are modestly positively correlated

$v$ is Normally distributed and uncorrelated to $x$ or $w$ ; $v$ can be a disturbance term if we omit $v$

$u$ is uniform distributed and uncorrelated to $x$ or $w$ ; $u$ can be a disturbance term if we omit $u$

For concreteness, let $\beta_0 = 0$

$\beta_1 = 1$

$\beta_2 = 1$

$\beta_3 = 1$

Generate $x$, $w$, $v$, & $u$ by simulation, then generate $y$ using the population regression function

Next, draw samples, and regress $y$ on $x$ (and possibly on $w$ and/or $v$) to estimate parameters

Then, resample repeatedly and examine the distributions of parameter estimates

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## 1. Classical Linear Regression Model assumptions

| | | |
|---|---|---|
| MLR.1 | Linear in Parameters | $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$ |
| MLR.2 | Random Sampling | $\{(x_{i1}, x_{i2}, \ldots, x_{ik}, y_i) : \; i = 1, \ldots n\}$<br>$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + u_i$ |
| MLR.3 | No Perfect Collinearity | $R_j^2 < 1 \qquad j = 0, 1, \ldots, k$ |
| MLR.4 | Zero Conditional Mean | $E(u_i \mid x_{i1}, x_{i2}, \ldots, x_{ik}) = 0$ |
| MLR.5 | Homoskedasticity | $Var(u_i \mid x_{i1}, x_{i2}, \ldots, x_{ik}) = \sigma^2$ |

**MLR.1 – MLR.5 are the Gauss-Markov assumptions**

THM 3.4 = Properties of OLS

$$Var(\hat{\beta}_j) \leq Var(\tilde{\beta}_j) \qquad j = 0, 1, \ldots, k$$

for all $\tilde{\beta}_j = \sum_{i=1}^{n} w_{ij} y_i$ for which $E(\tilde{\beta}_j) = \beta_j, j = 0, \ldots, k$

**THM 3.4 is the Gauss-Markov Theorem**; find "best" and "linear" and "unbiased"…

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1. Classical Linear Regression Model assumptions

MLR.6 Normality of disturbances $u_i \sim N(0, \sigma^2)$ independently of $x_{i1}, x_{i2}, \ldots, x_{ik}$

Therefore it follows that: $y|\mathbf{x} \sim N(\beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k, \sigma^2)$

**MLR.1 – MLR.6 are the Classical Linear Regression Model (CLRM) Assumptions**

## 2. Regression models and their sampling variances

THM 3.3 Unbiased estimation of error variance*

$$MLR.1 - MLR.5 \quad \Rightarrow \quad E(\hat{\sigma}^2) = \sigma^2 \qquad\qquad \hat{\sigma}^2 = \left(\sum_{i=1}^{n} \hat{u}_i^2\right) / \,[n-k-1]$$

* THM 5.2 adds that under MLR.1 – MLR.5, $\widehat{\sigma^2}$ is a consistent estimator of $\sigma^2$ = *var*(*u*) (i.e., $plim\ \hat{\sigma}^2 = \sigma^2$ )

## 3. Regression coefficients and their sampling distributions

THM 3.1 Unbiased estimation of regression coefficients

$$MLR.1{-}MLR.4 \quad \Rightarrow \quad E(\hat{\beta}_j) = \beta_j, \quad j = 0, 1, \ldots, k$$

THM 3.2 Sampling variance of regression slope coefficients

$$MLR.1 - MLR.5 \quad \Rightarrow \quad Var(\hat{\beta}_j) = \frac{\sigma^2}{SST_j(1 - R_j^2)}, \quad j = 1, \ldots, k$$

$$sd(\hat{\beta}_j) = \sqrt{Var(\hat{\beta}_j)} = \sqrt{\sigma^2 / \left[SST_j(1 - R_j^2)\right]}$$

$$MLR.1 - MLR.5 \quad \Rightarrow \quad \widehat{Var}(\hat{\beta}_j) = \frac{\hat{\sigma}^2}{SST_j(1 - R_j^2)}$$

$$se(\hat{\beta}_j) = \sqrt{\widehat{Var}(\hat{\beta}_j)} = \sqrt{\hat{\sigma}^2 / \left[SST_j(1 - R_j^2)\right]}$$

3. Regression coefficients and their sampling distributions

THM 4.1 Normality of sampling distribution: Under MLR.1 – MLR.6 , OLS slope estimators are Normally distributed around the true parameters: $\hat{\beta}_j \sim N(\beta_j, Var(\hat{\beta}_j))$

i.e., with mean: $E(\hat{\beta}_j) = \beta_j$ and variance: $Var(\hat{\beta}_j) = \dfrac{\sigma^2}{SST_j(1-R_j^2)}, \quad j = 1, \ldots, k$

THM 4.1 Therefore, under MLR.1 – MLR.6 , the standardized slope estimators follow a standard Normal distribution: $\dfrac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim N(0,1)$ for each *j*

THM 5.2 Under MLR.1 – MLR.5, OLS estimators are <u>asymptotically</u> Normally distributed, <u>and</u> the standardized slope estimators follow a standard Normal distribution:

$$\frac{(\hat{\beta}_j - \beta_j)}{se(\hat{\beta}_j)} \stackrel{a}{\sim} N(0,1)$$

Readings: Wooldridge, *Introductory Econometrics 5e*, 2.5 + 3.3 + 3.4 + 3.5 + 4.2 + 4.3

## 3. Regression coefficients and their sampling distributions

THM 4.1 Under MLR.1 – MLR.6 , the standardized slope estimators follow a standard Normal distribution: $\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim N(0,1)$ for each $j$

Interpretation:

1. $P(-1.96 < z < 1.96) = .95$
2. $P(-1.96 < \frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} < 1.96) = .95$
3. $P(-1.96 \cdot sd(\hat{\beta}_j) < \hat{\beta}_j - \beta_j < 1.96 \cdot sd(\hat{\beta}_j)) = .95$

4A. $P(\beta_j - 1.96 \cdot sd(\hat{\beta}_j) < \hat{\beta}_j < \beta_j + 1.96 \cdot sd(\hat{\beta}_j)) = .95$

The standard Normal distribution has the property that 95% of $z$ scores are between -1.96 and +1.96

There is a 95% probability that the sampling error will be less than 1.96 std. deviations (in either direction).

There is a 95% probability that $\hat{\beta}_j$ will be within 1.96 $sd(\hat{\beta}_j)$ of $\beta_j$

## 3. Regression coefficients and their sampling distributions

THM 4.1 Under MLR.1 – MLR.6 , the standardized slope estimators follow a standard Normal distribution: $\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim N(0,1)$ for each $j$

Interpretation:

1. $P(-1.96 < z < 1.96) = .95$

2. $P(-1.96 < \frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} < 1.96) = .95$

3. $P(-1.96 \cdot sd(\hat{\beta}_j) < \hat{\beta}_j - \beta_j < 1.96 \cdot sd(\hat{\beta}_j)) = .95$

There is a 95% probability that the sampling error will be less than 1.96 std. deviations (in either direction).

4B. $P(-\hat{\beta}_j - 1.96 \cdot sd(\hat{\beta}_j) < -\beta_j < -\hat{\beta}_j + 1.96 \cdot sd(\hat{\beta}_j)) = .95$

5. $P(\hat{\beta}_j + 1.96 \cdot sd(\hat{\beta}_j) > \beta_j > \hat{\beta}_j - 1.96 \cdot sd(\hat{\beta}_j)) = .95$

6. $P(\hat{\beta}_j - 1.96 \cdot sd(\hat{\beta}_j) < \beta_j < \hat{\beta}_j + 1.96 \cdot sd(\hat{\beta}_j)) = .95$

95% confidence interval for $\beta_j$:
$\hat{\beta}_j \pm 1.96 \cdot sd(\hat{\beta}_j)$

## 4. The *t* distribution

THM 4.2 Under the CLRM assumptions (MLR.1 – MLR.6), $\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1}$ [4.3]

where *n*–*k*–1 is the degrees of freedom (as *k*+1 is the number of unknown parameters in the population model: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$ )

Because the mean of the *t* distribution is zero, the numerator implies unbiasedness of the $\hat{\beta}_j$; the *t* distribution comes from the fact that the constant $\sigma$ in $sd(\hat{\beta}_j)$ has been replaced with the random variable $\hat{\sigma}$ [in $se(\hat{\beta}_j)$]. ... THM 4.2 allows us to test hypotheses involving the $\hat{\beta}_j$.

"The proof that [replacing $\sigma$ with $\hat{\sigma}$] leads to a *t* distribution with *n*–*k*–1 degrees of freedom is <u>difficult and not especially instructive</u>. Essentially, the proof shows that [4.3] can be written as the ratio of the standard normal random variable $(\hat{\beta}_j - \beta_j)/sd(\hat{\beta}_j)$ over the square root of $(\widehat{\sigma^2}/\sigma^2)$. These random variables can be shown to be independent, and $(n\text{–}k\text{–}1)\cdot(\widehat{\sigma^2}/\sigma^2) \sim \chi^2_{n-k-1}$. The result follows from the definition of a *t* random variable."

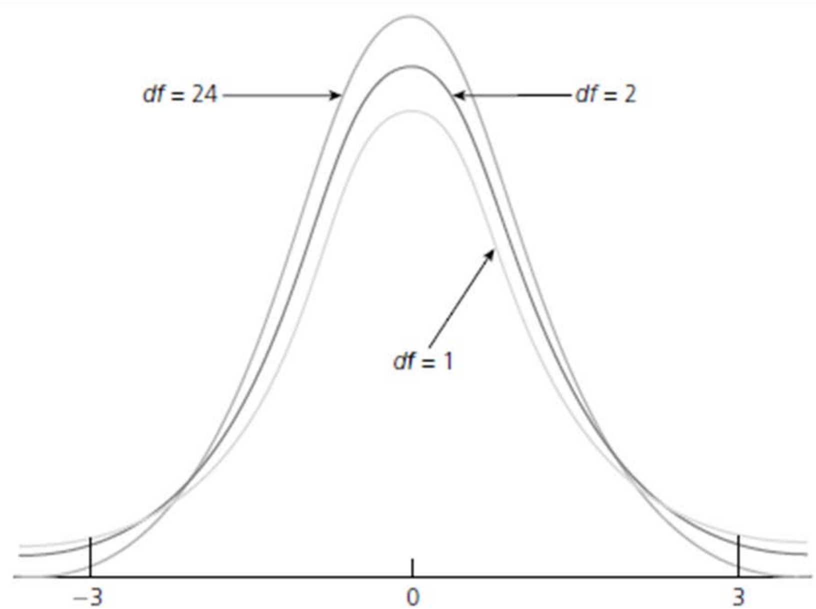
4. The *t* distribution

The pdf of the *t* distribution has a shape similar to that of the standard normal distribution, except that it is more spread out and therefore has more area in the tails.

The expected value of a *t* distributed random variable is 0, and the variance is $^{n}/_{n-2}$ for $n > 2$.

As the degrees of freedom gets large, the *t* distribution approaches the standard normal distribution (which has expected value 0 and variance 1).

FIGURE B.10
The *t* distribution with various degrees of freedom.



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| | | Significance Level | | | | |
|---|---|---|---|---|---|---|
| 1-Tailed: | | .10 | .05 | .025 | .01 | .005 |
| 2-Tailed: | | .20 | .10 | .05 | .02 | .01 |
| Degrees of Freedom | 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| | 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| | 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| | 4 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 |
| | 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| | 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| | 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| | 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| | 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| | 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| | 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| | 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| | 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| | 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 |
| | 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |
| | 16 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 |
| | 17 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 |
| | 18 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 |
| | 19 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 |
| | 20 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 |
| | 21 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 |
| | 22 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 |
| | 23 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 |
| | 24 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |
| | 25 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 |
| | 26 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 |
| | 27 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 |
| | 28 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 |
| | 29 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 |
| | 30 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 |
| | 40 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 |
| | 60 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 |
| | 90 | 1.291 | 1.662 | 1.987 | 2.368 | 2.632 |
| | 120 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 |
| | ∞ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |

## 5. Confidence intervals for a regression coefficient

Begin with a significance level (conventionally 5%)
The confidence level is 1 minus significance (i.e., 95% confidence)
The $t$ distribution gives a critical value for a given *confidence level* and *degrees of freedom*

A confidence interval is centered at $\widehat{\beta}_j$, and extends just far enough to cover 95% of the area under the $t$ distribution:

$$P\left(\widehat{\beta}_j - \underbrace{c_{0.05} \cdot se(\widehat{\beta}_j)}_{-\text{ margin of error}} \leq \beta_j \leq \widehat{\beta}_j + \underbrace{c_{0.05} \cdot se(\widehat{\beta}_j)}_{+\text{ margin of error}}\right) = 0.95$$

The margin of error equals the product of:

the $t$ critical value (denoted $c_{0.05}$) $\times$ the standard error of the coefficient (denoted $se(\widehat{\beta}_j)$)

One can carry out a hypothesis test using a confidence interval:

If the interval does not contain 0, reject $H_0 : \beta_j = 0$ in favor of $H_1 : \beta_j \neq 0$

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6. Null hypothesis retention region for a regression coefficient

Null hypothesis: $H_{0j} : \beta_j = 0$

Retention region is an interval centered at the null hypothesized value (0) , and extends just far enough to cover 95% of the area under the $t$ distribution

$$P(-c_{0.05} \cdot se(\widehat{\beta}_j) \leq 0 \leq c_{0.05} \cdot se(\widehat{\beta}_j)) = 0.95$$

Null hypothesis: $H_{0j} : \beta_j = a_j$

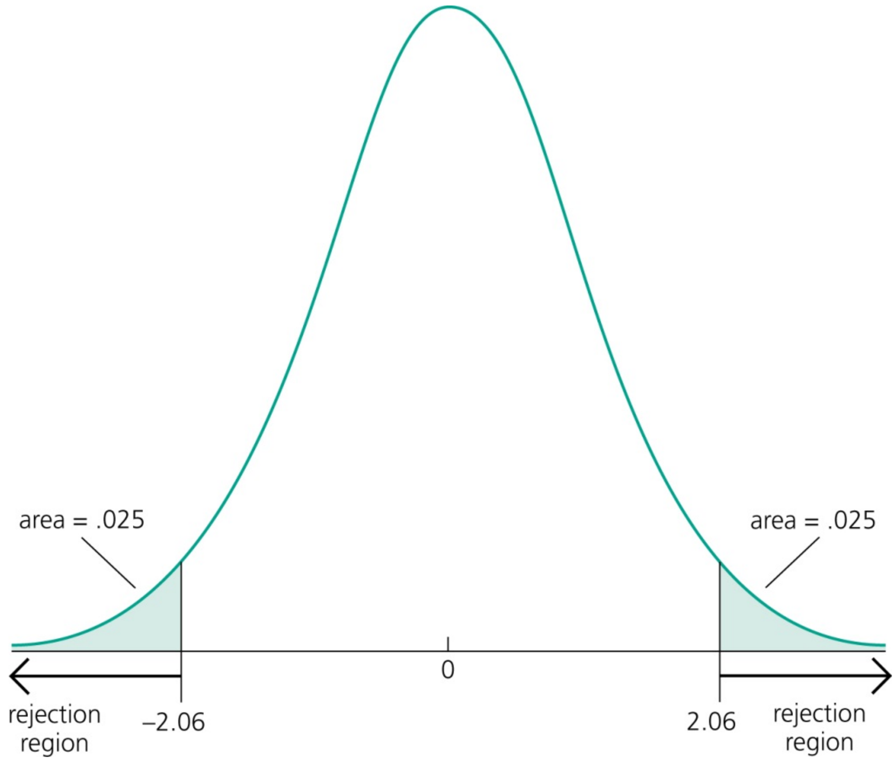
Retention region is an interval centered at the null hypothesized value ($a_j$) , and extends just far enough to cover 95% of the area under the $t$ distribution

$$P(a_j - c_{0.05} \cdot se(\widehat{\beta}_j) \leq a_j \leq a_j + c_{0.05} \cdot se(\widehat{\beta}_j)) = 0.95$$

The $t$ critical value for 95% confidence ($c_{0.05}$) is found using the $t$ table with $n-k-1$ *d.f.*

## 6. Null hypothesis retention region for a regression coefficient

For the following figure, note that the $t$ critical value with 25 *d.f.* equals ±2.06




retain $H_0 : \beta_j = 0$

Note that we never *accept* the alternative hypothesis ($H_{1j} : \beta_j \neq 0$); rather we *reject* the null if the $t$ statistic falls in the rejection region.