POLS 6481. Research Design and Quantitative Methods II
Lecture 22. Polynomial Functional Forms II: Standard Errors, Confidence Intervals, *t*
Readings: Wooldridge, *Introductory Econometrics 5e*, 6.2b

Outline

1. Standard Error of the Marginal Effect
2. Confidence Intervals for Marginal Effects
3. Example: Parliamentary government duration and parliamentary support

Lecture 21 = *focus on the 'numerator'* (*coefficients, marginal effects*)

Lecture 22 = *focus on the 'denominator'* (*standard error*) *and 'ratio'* (*t statistic*)

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0. Recap: Coefficients vs Marginal Effects in Polynomial models

	Linear	Quadratic
equation	$y_i = \beta_0 + \beta_1 x_i + u_i$	$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i$
graph	line	parabola (convex/concave)
effect of *x*	$\frac{\Delta y}{\Delta x} = \beta_1$ (constant)	$\frac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x_i$ (conditional on *x*)

Note: be careful of multicollinearity, as x^2 and x will be correlated!

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Brambor, Clark & Golder 2005 *Political Analysis*, "Understanding Interaction Models"

1. Standard Error of Marginal Effects in Polynomial models

	Linear	Quadratic
equation	$y_i = \beta_0 + \beta_1 x_i + u_i$	$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i$
graph	line	parabola (convex/concave)
effect of *x*	$\frac{\Delta y}{\Delta x} = \beta_1$ (constant)	$\frac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x_i$ (conditional on *x*)
std.err.	$\sqrt{var(\widehat{\beta_1})}$ (constant)	$\sqrt{var(\widehat{\beta_1}) + 4x^2 \cdot var(\widehat{\beta_2}) + 4x \cdot cov(\widehat{\beta_1}, \widehat{\beta_2})}$ (conditional on *x*)

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1. Standard Error of Marginal Effects in Polynomial models

Case	Equation	Marginal Effect	Variance
1	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2$	$\frac{\partial Y}{\partial X} = \beta_1 + 2\beta_2 X$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial X}} = var(\hat{\beta}_1) + 4X^2 var(\hat{\beta}_2) + 4X cov(\hat{\beta}_1 \hat{\beta}_2)$
2	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 Z$	$\frac{\partial Y}{\partial X} = \beta_1 + 2\beta_2 X$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial X}} = var(\hat{\beta}_1) + 4X^2 var(\hat{\beta}_2) + 4X cov(\hat{\beta}_1 \hat{\beta}_2)$
3a	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 Z + \beta_4 XZ$	$\frac{\partial Y}{\partial X} = \beta_1 + 2\beta_2 X + \beta_4 Z$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial X}} = var(\hat{\beta}_1) + 4X^2 var(\hat{\beta}_2) + Z^2 var(\hat{\beta}_4)$ $+4X cov(\hat{\beta}_1 \hat{\beta}_2) + 2Z cov(\hat{\beta}_1 \hat{\beta}_4)$ $+4XZ cov(\hat{\beta}_2 \hat{\beta}_4)$
3b	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 Z + \beta_4 XZ$	$\frac{\partial Y}{\partial Z} = \beta_3 + \beta_4 X$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial Z}} = var(\hat{\beta}_3) + X^2 var(\hat{\beta}_4) + 2X cov(\hat{\beta}_3 \hat{\beta}_4)$
4a	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 Z + \beta_4 XZ$ $+\beta_5 X^2 Z$	$\frac{\partial Y}{\partial X} = \beta_1 + 2\beta_2 X + \beta_4 Z$ $+2\beta_5 XZ$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial X}} = var(\hat{\beta}_1) + 4X^2 var(\hat{\beta}_2) + Z^2 var(\hat{\beta}_4)$ $+4X^2 Z^2 var(\hat{\beta}_5)$ $+4X cov(\hat{\beta}_1 \hat{\beta}_2) + 2Z cov(\hat{\beta}_1 \hat{\beta}_4)$ $+4XZ cov(\hat{\beta}_2 \hat{\beta}_4) + 4XZ cov(\hat{\beta}_1 \hat{\beta}_5)$ $+8X^2 Z cov(\hat{\beta}_2 \hat{\beta}_5) + 4XZ^2 cov(\hat{\beta}_4 \hat{\beta}_5)$
4b	$\hat{Y} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 Z + \beta_4 XZ$ $+\beta_5 X^2 Z$	$\frac{\partial Y}{\partial Z} = \beta_3 + \beta_4 X + \beta_5 X^2$	$\hat{\sigma}^2_{\frac{\partial Y}{\partial Z}} = var(\hat{\beta}_3) + X^2 var(\hat{\beta}_4) + X^4 var(\hat{\beta}_5)$ $+2X cov(\hat{\beta}_3 \hat{\beta}_4) + 2X^2 cov(\hat{\beta}_3 \hat{\beta}_5)$ $+2X^3 cov(\hat{\beta}_4 \hat{\beta}_5)$

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2. Confidence Intervals for Marginal Effects in Polynomial models

Marginal effect $$\frac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x$$

Standard error $$\widehat{s.e.}\left({}^{\Delta y}/_{\Delta x}\right) = \sqrt{var\left(\widehat{\beta_1}\right) + 4x^2 \cdot var\left(\widehat{\beta_2}\right) + 4x \cdot cov\left(\widehat{\beta_1}, \widehat{\beta_2}\right)}$$

t statistic $$\frac{{}^{\Delta y}/_{\Delta x}}{\widehat{s.e.}\left({}^{\Delta y}/_{\Delta x}\right)} = \frac{\beta_1 + 2\beta_2 x}{\sqrt{var\left(\widehat{\beta_1}\right) + 4x^2 \cdot var\left(\widehat{\beta_2}\right) + 4x \cdot cov\left(\widehat{\beta_1}, \widehat{\beta_2}\right)}}$$

95% Confidence interval $$\frac{\Delta y}{\Delta x} \pm t^* \times \widehat{s.e.}\left({}^{\Delta y}/_{\Delta x}\right)$$

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2. Confidence Intervals for Marginal Effects in Polynomial models

Recall wages and experience example: $\underset{}{\widehat{wage}} = \underset{(.35)}{3.73} + \underset{(.041)}{.298}\, exper - \underset{(.0009)}{.0061}\, exper^2$

Marginal effect $\dfrac{\partial wage}{\partial exper} = .298 - 2(.0061)exper = .298 - .012\ exper$

Standard error $\widehat{s.e.}\left({}^{\Delta wage}/_{\Delta exper}\right) = \sqrt{var(\widehat{\beta_1}) + 4 \cdot exper^2 \cdot var(\widehat{\beta_2}) + 4 \cdot exper \cdot cov(\widehat{\beta_1}, \widehat{\beta_2})}$

$= \sqrt{(.0016782) + exper^2(.0000033) - exper(.0001421)}$

95% conf. interval $(.298 - .012\ exper) \pm 1.96 \times \widehat{s.e.}\left({}^{\Delta wage}/_{\Delta exper}\right)$

a) choose a few values of *exper*, then calculate marginal effects, standard errors, t statistics

b) or go to Lecture 15 wages.R; graph marginal effects and 95% confidence interval

```
> round(vcov(quad), digits=6)
            (Intercept)  exper       expersq
(Intercept) 0.119674    -0.011737    0.000219
exper      -0.011737     0.001678   -0.000036
expersq     0.000219    -0.000036    0.000001
```

3. Example: Parliamentary Government Duration and Parliamentary Support

Quadratic model $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i \quad = 95.2 - 2.734 \cdot PS + .0257 \cdot PS^2$

Marginal effect $\dfrac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x_i \quad = -2.734 + .0514 \cdot PS$

Vertex $x^* = -\dfrac{\beta_1}{2\beta_2} \quad = -\dfrac{-2.734}{2(.0257)} = 53.2$

TABLE 7. OLS Regression Results, *Government Duration:* Quadratic-Term Model

	Coefficient (standard error) *p*-Value
Parliamentary Support (PS)	−2.734
	(2.061)
	0.200
Parliamentary Support, squared (PS^2)	0.0257
	(0.017)
	0.142
Intercept	95.20
	(62.44)
	0.144
N (*df*)	22 (19)
Adjusted R^2	0.158
$P > F$	0.075

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided *p*-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

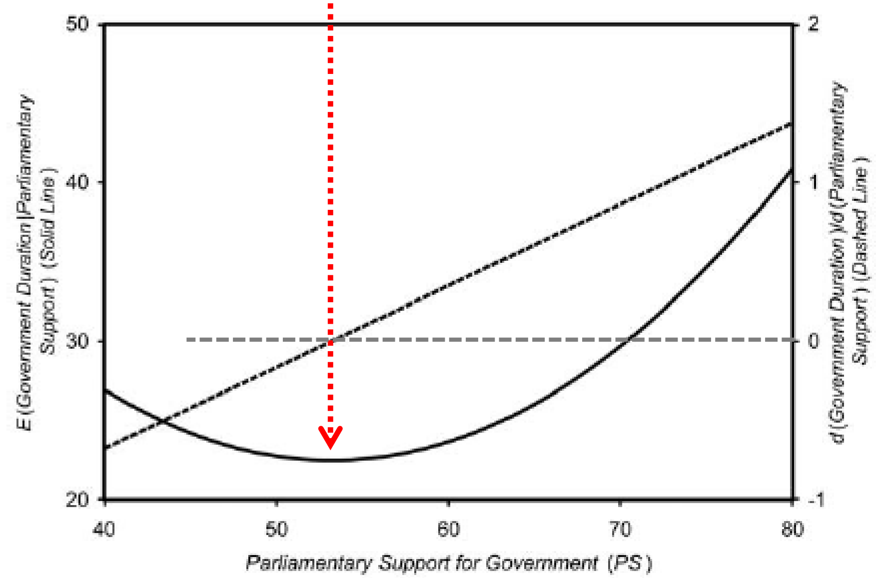

Fig. 2. Predicted *Government Duration* by *Parliamentary Support for Government*, quadratic model

3. Example: Parliamentary Government Duration and Parliamentary Support

Marginal effect $\frac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x_i$ $= -\,2.734 + .0514 \cdot PS$

Standard error $\sqrt{var(\widehat{\beta_1}) + 4x^2 \cdot var(\widehat{\beta_2}) + 4x \cdot cov(\widehat{\beta_1}, \widehat{\beta_2})} = \sqrt{4.247 + PS^2 \cdot .001124 - PS \cdot .1372}$

TABLE 7. OLS Regression Results, *Government Duration:* Quadratic-Term Model

	Coefficient (standard error) *p*-Value
Parliamentary Support (PS)	−2.734
	(2.061)
	0.200
Parliamentary Support, squared (PS²)	0.0257
	(0.017)
	0.142
Intercept	95.20
	(62.44)
	0.144
N (*df*)	22 (19)
Adjusted R^2	0.158
$P > F$	0.075

Note: Cell entries are the estimated coefficient, with standard error in parentheses, and two-sided *p*-level (probability $|T| > t$) referring to the null hypothesis that $\beta = 0$ in italics.

$$\widehat{V\left(\frac{\partial \widehat{GD}}{PS}\right)} = \widehat{V(\hat{\beta}_{ps} + 2\hat{\beta}_{ps^2}PS)}$$
$$= \widehat{V(\hat{\beta}_{ps})} + \widehat{V(2\hat{\beta}_{ps^2}PS)} + 2\widehat{C(\hat{\beta}_{ps}, 2\hat{\beta}_{ps^2}PS)}$$
$$= \widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 2PS \times 2\widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})}$$
$$= \widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 4PS \times \widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})} \qquad (27)$$

The relevant portion of the estimated variance-covariance matrix of these coefficient estimates is

$$\widehat{V(\hat{\beta}_{ps})} \approx 4.247 \qquad \widehat{C(\hat{\beta}_{ps^2}, \hat{\beta}_{ps})} \approx -\,0.0343$$
$$\widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})} \approx -0.0343 \qquad \widehat{V(\hat{\beta}_{ps^2})} \approx 0.000281$$

3. Example: Parliamentary Government Duration and Parliamentary Support

Marginal effect $\frac{\Delta y}{\Delta x} = \beta_1 + 2\beta_2 x_i \qquad = -\,2.734 + .0514 \cdot PS$

Standard error $\sqrt{var(\widehat{\beta_1}) + 4x^2 \cdot var(\widehat{\beta_2}) + 4x \cdot cov(\widehat{\beta_1}, \widehat{\beta_2})} = \sqrt{4.247 + PS^2 \cdot .001124 - PS \cdot .1372}$

Graphical presentation of estimates and estimated effects in nonlinear models is especially useful, and including some representation of the certainty of those estimates and estimated effects is equally crucial. Accordingly, figure 10 adds 90 percent confidence intervals to the straight line (the estimated marginal conditional effect line) in figure 2, using the square root of the expression in (27) to calculate the estimated standard error of the estimated marginal conditional effect. (We discuss construction of the confidence interval around the curved line, the predicted values, subsequently.) We take the estimated marginal effect and add (subtract) the product of the *t*-critical value and the estimated standard error to obtain the upper (lower) bound of the confidence interval:

$$(\hat{\beta}_{ps} + 2\hat{\beta}_{ps^2}PS) \pm 1.729 \times \left[\widehat{V(\hat{\beta}_{ps})} + 4PS^2 \times \widehat{V(\hat{\beta}_{ps^2})} + 4PS \times \widehat{C(\hat{\beta}_{ps}, \hat{\beta}_{ps^2})}\right]^{0.5}$$

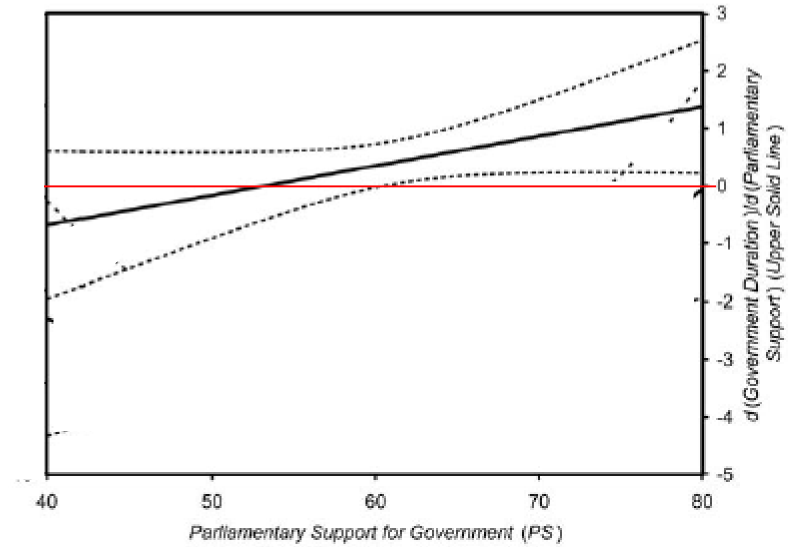

Fig. 10. Marginal effect of *Parliamentary Support* and predicted *Government Duration*, quadratic-term model, with 90 percent confidence intervals