Numerical Methods in Finance II

Lecture 3 - American Options

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Introduction

- In this module, we investigate two techniques for computing American options.
- To start with, we introduce some of the theory relating to American options with an initial emphasis on formulating the problem as an optimal stopping problem.
- This naturally leads to a linear complementarity formulation of the problem which is amenable to solution using finite-differences. It is easy to obtain solutions using an explicit scheme, with solutions for the implicit or Crank-Nicholson schemes provided using a modified SOR algorithm.
- An alternative formulation for American options is given by the dynamic programming approach.
- This is used to motivate the least-squares Monte Carlo. Implementation entails generating stochastic realisations of the price process and then performing a least squares optimisation to solve the dynamic programming problem in a backward recursive manner.

Problem Formulation

- As usual, let S_t represent our stock price, assumed Markov, on a finite time horizon $t \in [0, T]$ and let $H_t(\cdot)$ represent the payoff of the claim. We allow H to be a time varying payoff.
- The general formulation of our problem is to find the price

$$V_0 = \sup_{\bar{\tau} \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[e^{-r\bar{\tau}} H_{\bar{\tau}}(S_{\bar{\tau}})] \tag{1}$$

  where $\mathcal{T}$ is the set of admissible stopping times on $[0, T]$.
- This includes the case of the American put option

$$V_0 = \sup_{\bar{\tau} \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[e^{-r\bar{\tau}} (K - S_{\bar{\tau}})^+],$$

  where K is the strike price.
- Alternatively, this may be viewed (pathwise) as finding an optimal stopping time τ^* with the form

$$\tau^* = \inf\{t \geq 0 : S_t \leq b(t)\}$$

  for some exercise boundary $b(t)$. See the Figure 1 below.
- We use this formulation to form a linear complementarity specification of the problem. Later we reformulate the problem as a dynamic programming problem.

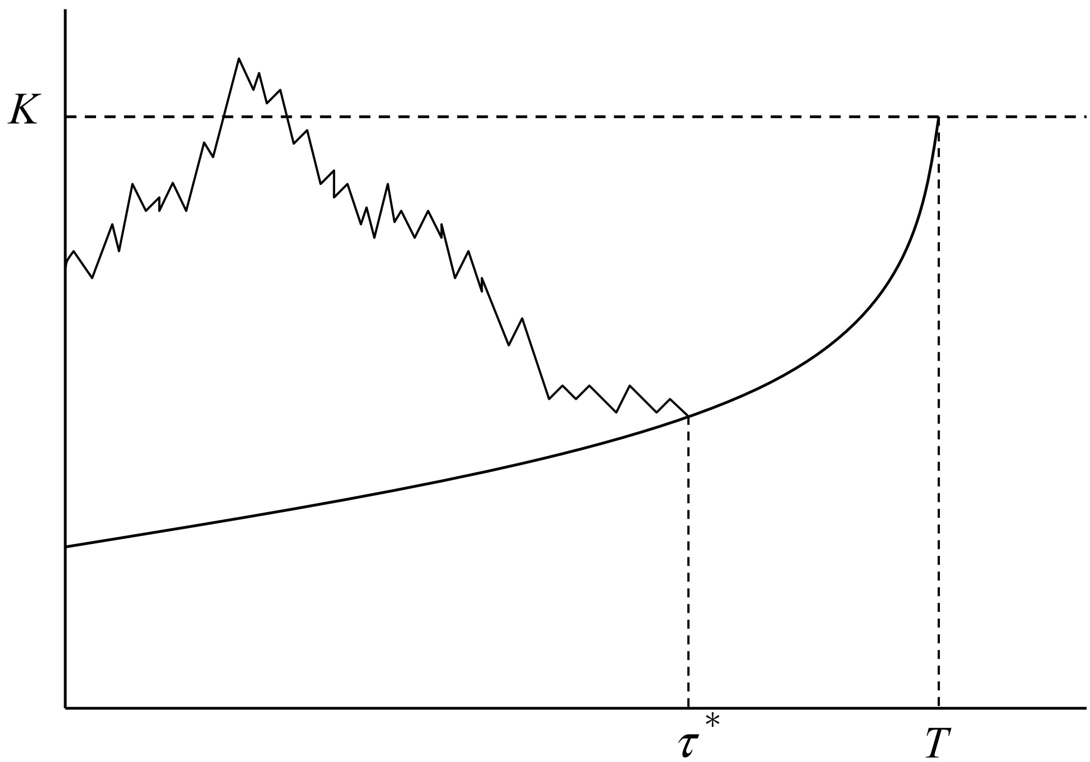

Figure: Early exercise boundary for an American put option.

The Linear Complementarity Formulation

- Assuming that the driving process for our stock is a geometric Brownian motion, recall that the time reversed Black-Scholes PDE is given by

$$\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rU = 0. \tag{2}$$

- Arbitrage arguments ensure that the following set of constraints uniquely value an American option
  - The option value U must always be greater than or equal to the exercise value (at any time).
  - Where the option is not exercised, the value should satisfy the Black-Scholes equation.
  - The option value must be a continuous function of the stock.
  - The option delta should be continuous.
- In terms of the early exercise boundary $b(\tau)$ (recall $\tau = T - t$), we then have the following specification for the American put

$$\begin{aligned} &\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rU = 0 && \text{on } S > b(\tau), \\ &U = (K - S)^+ && \text{on } S \leq b(\tau), \end{aligned}$$

with the initial condition $U_0(S, 0) = V_T(S) = (K - S)^+$ and the boundary conditions

$$U^0(0, \tau) = V^0(0, T - \tau) = K \quad \text{and} \quad U^\infty(S_{\max}, \tau) = V^\infty(S_{\max}, T - \tau) = 0.$$

- We have not worked out a way to determine the early exercise boundary. It is, however, possible to reformulate the problem without referring to it.
- Subject to U and $\partial U / \partial S$ being continuous, the linear complementarity specification of the problem is

$$\left(\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rU\right)\left(U - (K - S)^+\right) = 0,$$

subject to the constraints

$$\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rU \geq 0, \qquad U - (K - S)^+ \geq 0,$$

with the initial condition $U_0(S, 0) = (K - S)^+$ and the boundary conditions

$$U^0(0, \tau) = K \quad \text{and} \quad U^\infty(S_{\max}, \tau) = 0.$$

- Here the inequality in the first constraint results from the fact that when early exercise is appropriate, we have $U = K - S$ for $S < K$. Substitution of this solution into the left hand side of Black-Scholes PDE (2) evaluates to $rK > 0$. Of course, when early exercise is not appropriate, then (2) holds with equality.
- This formulation is amenable to easy computation in the finite-difference schemes we have already explored.

- Consider the explicit finite-difference scheme of Lecture 2. It is relatively easy to modify such a scheme to implement the complementarity formulation.
- Recall that the explicit formulation, without early exercise, is $\mathbf{U}_{m+1} = \mathbf{F}\mathbf{U}_m + \mathbf{b}_m$. The system is solved with early exercise by merely imposing the early exercise condition on each newly computed vector of values $\mathbf{U}_{m+1} = \max(\mathbf{F}\mathbf{U}_m + \mathbf{b}_m, (K - \mathbf{S})^+)$. Here $\mathbf{S}$ is the vector of stock prices applicable for the mesh upon which the solution is based and the $\max$ function is applied element-wise.
- It is clear that the above approach produces a solution U that is continuous. What is not clear is that $\partial U/\partial S$ is continuous. Proof of this is rather lengthy and requires a fair deal of analysis. We can, however, motivate the procedure by analogy to the binomial tree method which, in essence, is a simplified explicit finite-difference scheme.
- Things are slightly more complicated for the implicit, Crank-Nicolson and Theta-method formulations.

- Recall, from the previous module, that the theta method finite-difference scheme for the Black-Scholes PDE had a solution at time step $m+1$, given in terms of the solution at the previous time step, of $\mathbf{U}_{m+1} = \mathbf{A}^{-1}\mathbf{y}$ where

$$\begin{aligned} \mathbf{A} &= \theta\mathbf{G} + (1-\theta)\mathbf{I} \\ \mathbf{y} &= ((1-\theta)\mathbf{F} + \theta\mathbf{I})\mathbf{U}_m + (1-\theta)\mathbf{b}_m + \theta\mathbf{b}_{m+1}. \end{aligned}$$

- Since the constituents of $\mathbf{U}_{m+1}$ are related to each other (through $\mathbf{A}$), it is not correct to solve the system and then impose the early exercise on the solution. We require a way to impose the early exercise condition so that internal consistency of the solution is ensured.
- We do this using a modified SOR algorithm on the system $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{x}$ represents the iterated solution of $\mathbf{U}_{m+1}$.
- Recall the decomposition $\mathbf{A} = \mathbf{D} + \mathbf{U} + \mathbf{L}$. Denoting $\mathbf{x}^{(k)}$ as the kth iteration of a solution for $\mathbf{x}$, an iteration including the early exercise condition is then written as

$$\mathbf{x}^{(k+1)} = \max\left((\mathbf{D} + \omega\mathbf{L})^{-1}(\omega\mathbf{y} + ((1-\omega)\mathbf{D} - \omega\mathbf{U})\mathbf{x}^{(k)}), (K - \mathbf{S})^+\right).$$

- A convenient initial guess $\mathbf{x}^{(0)} = \mathbf{U}_m$ is the vector at the previous time step in the finite difference solution.
- This is known as the *projected SOR algorithm* and is guaranteed to produce a solution with U and $\partial U/\partial S$ continuous. Again, proof of this is beyond the scope of this material.

Example: Black-Scholes American Put

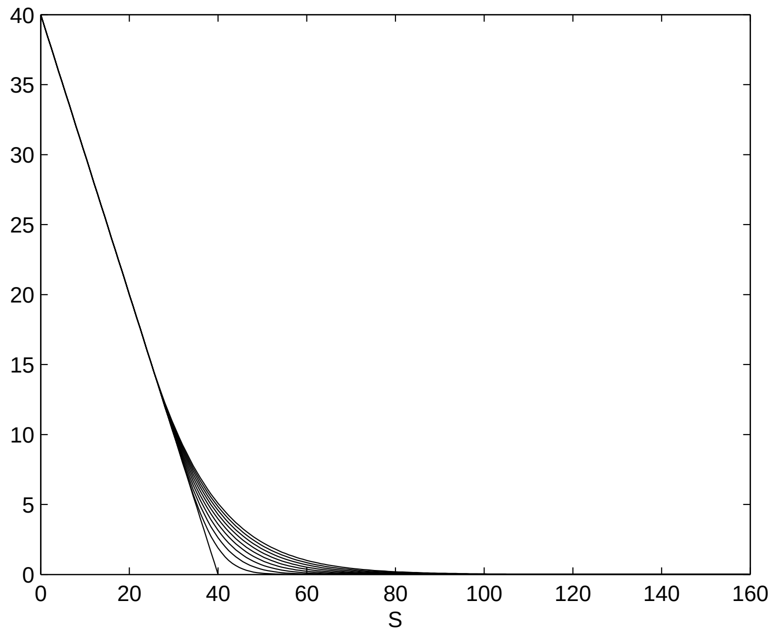

Figure: Output from the direct implicit finite-difference method for an American put, using parameters from the European put example. Different contours correspond to different times, showing that the value of the option never drops below intrinsic value. The price of the option is computed as 5.3007.

Dynamic Programming Formulation

- The original problem formulation (1) is specified for exercise in continuous time. To make things computationally tractable, we shall now consider only a finite set of discrete exercise times $0 < t_1 < t_2 < \cdots < t_N = T$. To simplify notation, we use only the time index when referring to time, i.e. $S_i := S_{t_i}$.
- Because we are allowing only a finite number of exercise times, we are actually valuing a *Bermudan* option — but if the set of times is large enough, the computed value will approach that of an American option.
- The value of the option is determined recursively, backward in time, as follows

$$V_N(s) = H_N(s),$$
$$V_{i-1}(s) = \max\left\{H_{i-1}(s),\, \mathbb{E}[e^{-r\Delta t_i} V_i(S_i)|S_{i-1} = s]\right\}, \tag{3}$$

  for $N \geq i \geq 1$ where $\Delta t_i = t_i - t_{i-1}$.
- The expectation is taken under the risk-neutral measure, or alternatively, the stock evolves with a drift equal to the risk-free rate. For now, we only consider a deterministic risk-free rate.
- The second term in the $\max$ function is known as the continuation value at time t_{i-1}. It is interpreted as the expected value of the option conditioned on the option not having been exercised before t_{i-1} and the share price having the value $S_{i-1} = s$.

The Least Squares Monte Carlo Algorithm

- We now explore a fundamentally different approach to valuing American options based on Monte Carlo simulation.
- This is advantageous from a number of perspectives:
  - it is efficient;
  - more easily enables computation under multiple factors; and
  - is applicable to a wider range of processes (including jump diffusions and non-Markovian processes).
- As we have seen, the difficulty with valuing American options is that either one must estimate an early exercise boundary (this was done implicitly in the finite difference approach) or one must evaluate a series of conditional expectations in order to solve the dynamic programming equation (3).
- The key insight of Carriere (1996), popularized by Longstaff and Schwartz (2001), is that the conditional expectation in (3) can be estimated using least squares on cross-sectional information in a Monte Carlo simulation.
- This is achieved by assuming the conditional expectation can be written as

  $$\mathbb{E}[e^{-r\Delta t_i}V_i(S_i)|S_{i-1}=x] = f(\hat{\beta}_{i-1}, x),$$

  where $f(\hat{\beta}_{i-1}, x)$ is a suitable parametric function. The parameter vector $\hat{\beta}_{i-1}$ is found by regressing realised payoffs (at each exercise time) from continuation values on the function of state variables (prices).
- Using the estimate of conditional expectation, it is then possible to decide (pathwise) whether or not early exercise or continuation should occur.

Summary of Least-Squares Monte Carlo Algorithm

- The following is a slightly modified version of the LSM algorithm as described by Longstaff and Schwartz.

  1. Generate Monte Carlo realisations of stock paths with values at each of the exercise dates.
  2. Compute the vector V_N of terminal payoffs.
  3. For $i = N, N-1, \ldots, 2$ repeat steps 4–7.
  4. Compute the realised continuation values as $V_{i-1} = e^{-r\Delta t_i}V_i$.
  5. Identify the paths for which early exercise is greater than zero.
  6. Let the vector X be the stock prices for these paths (S_{i-1}) and Y be the corresponding realised continuation values. Perform least squares regression on Y and $f(\hat{\beta}, X)$ to produce an estimate of $\hat{\beta}$.
  7. For the paths where early exercise is greater than $f(\hat{\beta}, X)$ set V_{i-1} to the early exercise values.
  8. Compute the value of the option as $V_0 = \mathbb{E}\left[e^{-r\Delta t_1}V_1\right]$.

- Note the last step assumes that early exercise is not applicable at $t = 0$.
- The algorithm presented by Longstaff and Schwartz computes a matrix of stoping rules for each path and then discounts the stopped cash flows to the valuation date. We have chosen to apply the early exercise decisions as they become available, thus discounting the cash flows incrementally — this ensures that we always have the pathwise continuation values at hand, thus simplifying the algorithm.
- Finally, we derive the least squares formulation required in Step 6.

Least Squares

- In performing the least squares optimisation, we shall assume the functional form
$$f(\hat{\beta}, x) = \sum_{r=0}^{R} \beta_r \phi_r(x)$$
in terms of the basis functions $\phi_r(x)$, $0 \leq r \leq R$, and $\hat{\beta} = [\beta_0, \beta_1, \ldots, \beta_R]^T$.
- There are many choices applicable for the basis functions including Hermite, Legendre, Chebyshev, Gegenbauer and Jacobi polynomials. We shall use the Laguerre polynomials
$$\begin{aligned}
\phi_0(x) &= 1, \\
\phi_1(x) &= 1 - x, \\
\phi_2(x) &= 1 - 2x + x^2/2, \\
\phi_r(x) &= \frac{e^x}{r!} \frac{d^r}{dx^r}(x^r e^{-x}).
\end{aligned}$$
- Note that Longstaff and Schwartz choose to use the Laguerre polynomials weighted by the factor $e^{-x/2}$. This is implemented in conjunction with a renormalisation procedure. To keep our implementation as simple as possible we choose not to implement this procedure, but note that it may produce better numerical results when matrix scaling problems occur.

- In terms of the observations $x_j \sim x$ and $y_j \sim y$, with $1 \leq j \leq n$, we wish to minimize the mean square error
$$\mathbb{E}\left[\left(y - \sum_{r=0}^{R} \beta_r \phi_r(x)\right)^2\right]$$
with respect to the coefficients β_r.
- Thus, differentiating this expression with respect to β_s and equating to zero we have
$$\sum_{r=0}^{R} \beta_r \mathbb{E}[\phi_r(x)\phi_s(x)] = \mathbb{E}[y\phi_s(x)],$$
for $0 \leq s \leq R$.
- **Exercise:** Reformulate the solution above to show that
$$\hat{\beta} = (FF^T)^{-1}FY \quad \text{and} \quad f(\hat{\beta}, X) = F^T\hat{\beta}$$
where X and Y are written as column vectors of the observations and F is the matrix
$$F = \begin{bmatrix} \phi_0(x_1) & \phi_0(x_2) & \cdots & \phi_0(x_n) \\ \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_n) \\ \vdots & \vdots & & \vdots \\ \phi_R(x_1) & \phi_R(x_2) & \cdots & \phi_R(x_n) \end{bmatrix}.$$
- While we use ordinary least squares, in some cases it may be advantageous to implement more advanced methods such as weighted least squares, generalised least squares or the generalised method of moments.

Example: Least Squares Continuation Boundary

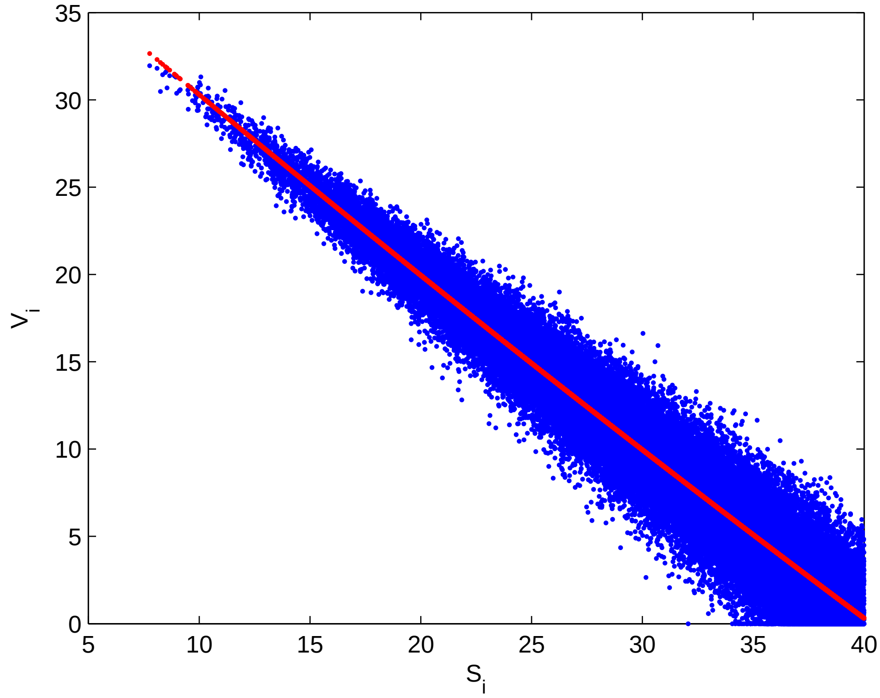

Figure: Using parameters from the European put example and 50 exercise times, the graph illustrates the realised (dark/blue) and estimated expected (light/red) continuation values as a function of stock price one time step before expiry of the American put option. The price of the option is computed as 5.2919.

Early Exercise Boundary

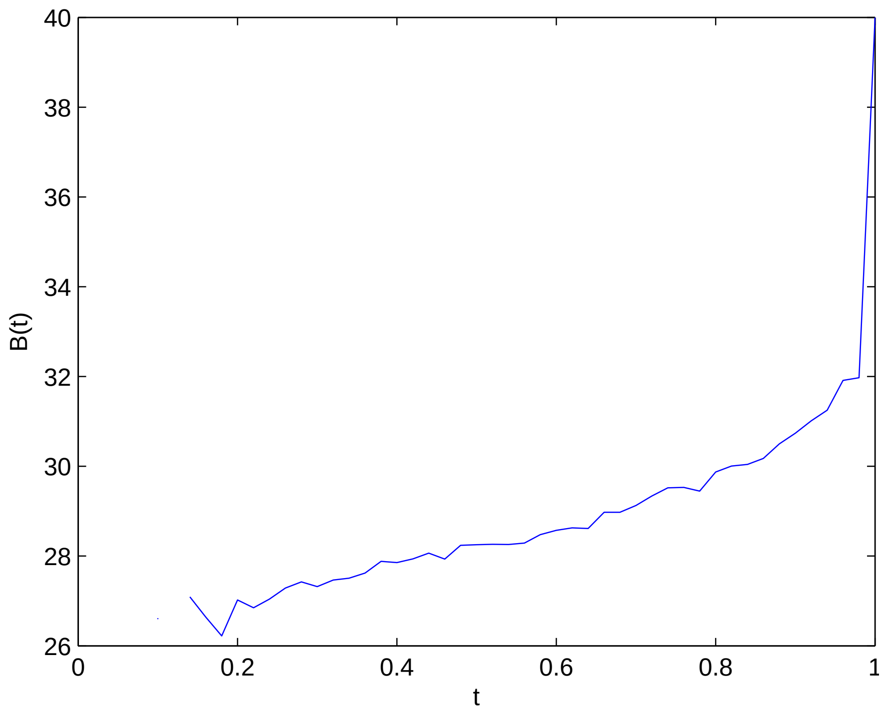

Figure: The graph illustrates the approximate early exercise boundary computed using least squares Monte Carlo. Note that, at early times, no stock price samples are small enough for early exercise to occur and, as a result, no boundary is estimated for these times.