A short remark on Feller's square root condition.

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In this short notice we present a proof of the popular Feller's square root condition which provides the existence of the positive solution of the Cox-Ingersoll-Ross (CIR , 1985) model of the short interest rate. CIR model deals with the short interest rate dynamics, which follows a scalar stochastic differential equation (SDE)

$$d r (t) = k [\theta - r (t)] d t + \sigma \sqrt{r(t)} \, d w (t) \tag{1}$$

where $w (t) , t > 0$ is a scalar Wiener process on a complete probability space.

Theorem. Let k , θ , σ^2 be positive constants and $r (0) > 0$. Then, if the Feller's condition

$$2 k \theta > \sigma^2 \tag{2}$$

is satisfied then there exists an unique positive solution of the equation (1) on each finite time interval $t \in [0 , + \infty)$.

Proof. Let $\varepsilon > 0$ be a small number and denote $\tau_\varepsilon = \min \{ t : r (t) \le \varepsilon \}$ and put $\tau_\varepsilon \wedge t = \min \{ \tau_\varepsilon , t \}$. Then there exists a unique solution of the equation (1) on the interval $[0 , \tau_\varepsilon \wedge t]$. Let us show that $P \{ \tau_\varepsilon \wedge t < t \} \to 0$ when $\varepsilon \to 0$. Define a positive constant

$$m = \frac{2k\theta - \sigma^2}{\sigma^2} \tag{3}$$

Applying Ito formula for the function $f(x) = x^{-m}$ we note that

$$r^{-m}(\tau_\varepsilon \wedge t) = r^{-m}(0) - \int_0^{\tau_\varepsilon \wedge t} mk[\theta - r(s)]r^{-(m+1)}(s)ds -$$

$$- \int_0^{\tau_\varepsilon \wedge t} m\sigma\sqrt{r(s)}\, r^{-(m+1)}(s)dw(s) + \frac{1}{2}\int_0^{\tau_\varepsilon \wedge t} m(m+1)\sigma^2 r(s)r^{-(m+2)}(s)ds =$$

$$= r^{-m}(0) + mk\int_0^t r^{-m}(\tau_\varepsilon \wedge s)ds - \int_0^t m\sigma r^{-(m+0.5)}(\tau_\varepsilon \wedge s)dw(s) +$$

$$+ \int_0^t \left[\frac{m(m+1)}{2}\sigma^2 - mk\theta\right] r^{-(m+1)}(\tau_\varepsilon \wedge s)ds$$

Bearing in mind (3) and taking expectation in the latter equality, we arrive at the estimate

$$Er^{-m}(\tau_\varepsilon \wedge t) \le r^{-m}(0) + mk\int_0^t Er^{-m}(\tau_\varepsilon \wedge s)ds$$

Applying Gronwall inequality, we get estimate

$$Er^{-m}(\tau_\varepsilon \wedge t) \le r^{-m}(0) \exp mkt$$

Next using Chebyshev inequality we note that

$$P\{\tau_\varepsilon \le t\} = P\{r^m(\tau_\varepsilon) \le \varepsilon^m\} = P\{r^{-m}(\tau_\varepsilon) \ge \varepsilon^{-m}\} \le \varepsilon^m r^{-m}(0)\exp mt$$

where the constant m is defined by (3). It is easy to note that the right hand side tends to 0 when $\varepsilon > 0$ tends to 0 for each $t \in [0, +\infty)$. Theorem is proved.

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