from $n-1$ to 0,

$$M\widetilde{v}_i = N v_{i+1} + \mathrm{h} w_i, \qquad v_i = \max(\psi, \widetilde{v}_i). \tag{8.33}$$

By the results of Barles et al. [23], reviewed in Sect. 13.2.3, this scheme converges to the unique viscosity solution v of (8.32).

8.5 Finite Differences for Bidimensional Vanilla Options

Alternate Direction Implicit (ADI) schemes are the industry finite difference standard to cope with multivariate pricing problems. We now describe such a scheme in a bivariate Black–Scholes setting, where two underlying stock prices satisfy the following stochastic differential equations:

$$\begin{cases} dS_t^1 = S_t^1\big(\kappa_1\,dt + \sigma_{11}\,dW_t^1 + \sigma_{12}\,dW_t^2\big) \\ dS_t^2 = S_t^2\big(\kappa_2\,dt + \sigma_{21}\,dW_t^1 + \sigma_{22}\,dW_t^2\big), \end{cases}$$

for independent Brownian motions W^1 and W^2, with

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} r-q_1 \\ r-q_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$$

(see Sect. 6.7.2). In order to apply the ADI method, it is better to work directly with the independent Brownian motions W^1 and W^2. Let us introduce

$$\varphi(t,x,y) = \phi\big(t, S_0^1 e^{b_1 t + \sigma_1 x}, S_0^2 e^{b_2 t + \sigma_2(\rho x + \sqrt{1-\rho^2} y)}\big),$$

with $(b_1, b_2) = (\kappa_1 - \frac{1}{2}\sigma_1^2, \kappa_2 - \frac{1}{2}\sigma_2^2)$. The payoff process $\phi(S_t^1, S_t^2)$ is thus rewritten as $\varphi(t, W_t^1, W_t^2)$.

The time-t price of a European option with payoff $\phi(S_T^1, S_T^1)$ at T is then given by

$$\mathbb{E}_t e^{-r(T-t)}\phi\big(S_T^1, S_T^2\big) = w\big(t, S_t^1, S_t^2\big),$$

or equally well

$$\mathbb{E}\, e^{-r(T-t)}\varphi\big(T, W_T^1, W_T^2\big) = u\big(t, W_t^1, W_t^2\big),$$

where w satisfies a bivariate Black–Scholes equation in the S-variables and $u = u(t,x,y)$ solves the following bivariate heat equation:

$$\begin{cases} u(T,x,y) = \varphi(T,x,y) \quad \text{on } \mathbb{R}^2 \\ \partial_t u(t,x,y) + \dfrac{1}{2}\partial_{x^2}^2 u(t,x,y) + \dfrac{1}{2}\partial_{y^2}^2 u(t,x,y) - ru(t,x,y) = 0 \quad \text{on } [0,T)\times\mathbb{R}^2. \end{cases} \tag{8.34}$$

For the numerical solution of (8.34) by finite differences:

- ***localize*** the domain in space to a set $\mathcal{O} = (-\ell, \ell)^2$, introducing a suitable condition at the spatial boundary of $[0, T] \times \mathcal{O}$;
- introduce a time-space mesh $(t, x, y) = (i\mathrm{h}, j_1\mathrm{k}_1, j_2\mathrm{k}_2)$ on $[0, T] \times \mathcal{O}$, with mesh steps h, k_1, k_2, and ***discretize*** the localized problem on the mesh by a suitable finite difference scheme, such as the one described in the next subsection.

Regarding the deltas, note that we have $w(t, S_1, S_2) = u(t, x, y)$, where

$$\begin{pmatrix} \ln S_1 \\ \ln S_2 \end{pmatrix} = \begin{pmatrix} \ln S_0^1 + b_1 T \\ \ln S_0^2 + b_2 T \end{pmatrix} + \Sigma \begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore

$$\begin{aligned}\begin{pmatrix} S_1 \partial_{S_1} \\ S_2 \partial_{S_2} \end{pmatrix} w(t, S_1, S_2) &= \Sigma^{-1} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u(t, x, y) \\ &= \frac{1}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2 \sqrt{1-\rho^2} & 0 \\ -\sigma_2 \rho & \sigma_1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u(t, x, y).\end{aligned}$$

The time-0 deltas $\Delta_0^1 = \partial_{S_1} w(0, S_0^1, S_0^2)$ and $\Delta_0^2 = \partial_{S_2} w(0, S_0^1, S_0^2)$ are then given in terms of u as

$$\Delta_0^1 = \frac{e^{-x}}{\sigma_1} (\partial_x u)(0, 0, 0), \qquad \Delta_0^2 = \frac{e^{-y}}{\sqrt{1-\rho^2}} \left(\frac{-\rho \partial_x u}{\sigma_1} + \frac{\partial_y u}{\sigma_2} \right)(0, 0, 0). \tag{8.35}$$

Let $u_i^{j_1, j_2}$ denote the solution to the discretized pricing equation. Approximate deltas are then retrieved from (8.35) by substituting finite differences $\delta_x u$ and $\delta_y u$ for $\partial_x u$ and $\partial_y u$, e.g.

$$(\delta_x u)_i^{j_1, j_2} = \frac{u_i^{j_1+1, j_2} - u_i^{j_1-1, j_2}}{2\mathrm{k}_1}, \qquad (\delta_y u)_i^{j_1, j_2} = \frac{u_i^{j_1, j_2+1} - u_i^{j_1, j_2-1}}{2\mathrm{k}_2}.$$

8.5.1 ADI Scheme

ADI schemes [207, 220] consist in decomposing each time step into two parts, the first implicit in x and the second implicit in y, resulting in the following approximation scheme for (8.34): $u_n = \varphi$ and, for $i = n-1, \ldots, 0$,

$$\begin{cases} \frac{2}{\mathrm{h}}(u_{i+1} - u_{i+\frac{1}{2}}) + \frac{1}{2}\delta_{x^2}^2 u_{i+\frac{1}{2}} + \frac{1}{2}\delta_{y^2}^2 u_{i+1} - \frac{1}{2} r u_{i+\frac{1}{2}} - \frac{1}{2} r u_{i+1} = 0 \\ \frac{2}{\mathrm{h}}(u_{i+\frac{1}{2}} - u_i) + \frac{1}{2}\delta_{x^2}^2 u_{i+\frac{1}{2}} + \frac{1}{2}\delta_{y^2}^2 u_i - \frac{1}{2} r u_{i+\frac{1}{2}} - \frac{1}{2} r u_i = 0 \end{cases}$$

or, equivalently,

$$\begin{cases} \left[\left(1+\frac{\mathrm{h}r}{4}\right)I-\frac{\mathrm{h}}{4}\delta_{x^2}^2\right]u_{i+\frac{1}{2}}=\left[\left(1-\frac{\mathrm{h}r}{4}\right)I+\frac{\mathrm{h}}{4}\delta_{y^2}^2\right]u_{i+1} \\ \left[\left(1+\frac{\mathrm{h}r}{4}\right)I-\frac{\mathrm{h}}{4}\delta_{y^2}^2\right]u_i=\left[\left(1-\frac{\mathrm{h}r}{4}\right)I+\frac{\mathrm{h}}{4}\delta_{x^2}^2\right]u_{i+\frac{1}{2}}, \end{cases} \tag{8.36}$$

in which

$$\left(\delta_{x^2}^2 u\right)_i^{j_1,j_2}=\frac{u_i^{j_1+1,j_2}-2u_i^{j_1,j_2}+u_i^{j_1-1,j_2}}{\mathrm{k}_1^2},$$

$$\left(\delta_{y^2}^2 u\right)_i^{j_1,j_2}=\frac{u_i^{j_1,j_2+1}-2u_i^{j_1,j_2}+u_i^{j_1,j_2-1}}{\mathrm{k}_2^2}.$$

Each time step i takes the form

$$\begin{cases} M^{j_2}u_{i+\frac{1}{2}}^{\cdot,j_2}=N^{j_2}u_{i+1}^{\cdot,j_2}, \quad \text{for every } j_2 \\ P^{j_1}u_i^{j_1,\cdot}=Q^{j_1}u_{i+\frac{1}{2}}^{j_1,\cdot}, \quad \text{for every } j_1, \end{cases} \tag{8.37}$$

for suitable "one-dimensional tridiagonal" matrices M^{j_2}, N^{j_2}, P^{j_1}, Q^{j_1}. So each time step reduces to (m_1+m_2) implicit tridiagonal one-dimensional problems, each solvable by the Thomas algorithm of paragraph "Solution by Gauss Factorization" on p. 224. This is in general a far better alternative than having to solve the $(m_1 m_2)$-dimensional linear system that would arise from a bivariate implicit discretization.[5]

Unless simple transformations as in the above case allow elimination of the correlation from a pricing problem, additional cross-derivatives show up in the equations. These can be dealt with explicitly (i.e. put on the right-hand side in (8.36)) and the ADI scheme is still applicable, but subject to stability conditions which become stringent in multivariate settings. See Hout and Welfert [145] or Duffy [104] for alternative schemes.

8.5.1.1 American Options

The time-t price of an American vanilla option is given as

$$\operatorname*{esssup}_{\tau\in\mathcal{T}_t}\mathbb{E}\,e^{-r(\tau-t)}\varphi\left(\tau,W_\tau^1,W_\tau^2\right)=v\left(t,W_t^1,W_t^2\right),$$

[5]However, the sparseness of the corresponding matrix may be exploited in an iterative solution.