Numerical Methods in Finance II

Lecture 6- Characteristic Function Pricing, Stochastic Volatility & The Fast Fourier Transform

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Introduction

- In this module, using the characteristic function of the log terminal stock price, we derive a general methodology for the pricing of a European call (or put) option.
- As an example, the Black-Scholes price of a call option is recovered when the characteristic function for geometric Brownian motion is used.
- We then introduce the Heston model and the "little trap" formulation for the corresponding characteristic function.
- By pricing and then finding the implied volatility (using the Black-Scholes equation), this allows us to show how the Heston model produces various smiles and skews under changes of parameters.
- Using a Milstein discretisation of the Heston model, a Monte Carlo simulation of a call option is compared with the characteristic pricing method.
- Finally, we will describe an efficient algorithm for discrete computation known as the Fast Fourier Transform.

The Fourier Transform

Definition 6.1 (Fourier Transform) Let $f : \mathbb{R} \to \mathbb{C}$ be a function that is *integrable* in the sense that $\int_{\mathbb{R}} |f(x)|\,dx < \infty$, the *Fourier transform* of f is defined as

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx.$$

- Recall that the complex exponential at the heart of this transform may be expanded using *Euler's formula*: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
- The set of complex exponentials is complete and orthogonal, leading to the following fundamental result.

Theorem 6.1 (Fourier Inversion Formula) *Suppose that f and $\widehat{f}$ are integrable and that f is a continuous function, then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\widehat{f}(\omega)\,d\omega.$$

- Note that in the literature, you may encounter a number of other sign and scaling conventions (eg. $\omega = 2\pi\nu$). Here we follow the convention used by most probabilists.

Characteristic Functions

Definition 6.2 (Characteristic Function) If X is a random variable, then its *characteristic function* $\phi_X : \mathbb{R} \to \mathbb{C}$ is defined by

$$\phi_X(u) = \mathbb{E}\left[e^{iuX}\right].$$

If X has a probability density function $f_X : \mathbb{R} \to \mathbb{R}$ then the characteristic function may be written as

$$\phi_X(u) = \int_{-\infty}^{\infty} e^{iux} f_X(x)\,dx.$$

- The second part of this definition implies that the characteristic function is essentially a Fourier transform of the density function, which means that if we have a characteristic function then the density can be reconstructed using the Fourier inversion formula.

Theorem 6.2 (Uniqueness Theorem) *If two random variables X and Y have the same characteristic functions,*

$$\phi_X(u) = \phi_Y(u)$$

then they have the same cumulative distribution functions.

Examples

1. The standard normal distribution
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \Rightarrow \quad \phi_X(u) = e^{-u^2/2}.$$
2. The uniform distribution
$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad \phi_X(u) = \frac{e^{iu} - 1}{iu}.$$
3. The double exponential (Laplace) distribution
$$f_X(x) = \frac{1}{2} e^{-|x|} \quad \Rightarrow \quad \phi_X(u) = \frac{1}{1+u^2}.$$

Exercise: By direct application of the definition of a characteristic function show that the above relationships hold. Hint: for the normal distribution, perform a completion of squares for the exponent and make the substitution $y = x - iu$ (for justification of this, see p. 187 of Grimmett and Stirzaker).

Proposition 6.1 (Characteristic Function Properties) *Suppose that X and Y are independent random variables, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. Then*
$$\begin{aligned} \phi_{\mu+X}(u) &= e^{i\mu u}\phi_X(u), \\ \phi_{\sigma X}(u) &= \phi_X(\sigma u), \quad \textit{and} \\ \phi_{X+Y}(u) &= \phi_X(u)\phi_Y(u). \end{aligned}$$

Proof. Exercise.

European Call Price

- The price of a call option may be written as
$$\begin{aligned} C(K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\mathbb{I}_{S_T>K}\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbb{I}_{S_T>K}\right] - Ke^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{S_T>K}\right] \\ &= S_0\mathbb{E}^{\mathbb{Q}}\left[\frac{S_T/S_0}{\beta_T/\beta_0}\mathbb{I}_{S_T>K}\right] - Ke^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{S_T>K}\right] \qquad (1) \\ &= S_0\mathbb{E}^{\mathbb{Q}_\mathsf{s}}\left[\mathbb{I}_{S_T>K}\right] - Ke^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{S_T>K}\right], \qquad (2) \end{aligned}$$
where $\beta_t = e^{rt}$ is the numeraire asset (bank account). Here $\mathbb{Q}$ is the risk-neutral measure and $\mathbb{Q}_\mathsf{s}$ is the measure defined by
$$\frac{d\mathbb{Q}_\mathsf{s}}{d\mathbb{Q}} = \frac{S_T/S_0}{\beta_T/\beta_0}. \qquad (3)$$
- The expectations in (2) may be thought of as probabilities and we have
$$\begin{aligned} C(K) &= S_0\mathbb{Q}_\mathsf{s}\{S_T > K\} - Ke^{-rT}\mathbb{Q}\{S_T > K\} \\ &= S_0\mathbb{Q}_\mathsf{s}\{\ln(S_T) > \ln(K)\} - Ke^{-rT}\mathbb{Q}\{\ln(S_T) > \ln(K)\} \\ &= S_0P_1 - Ke^{-rT}P_2. \end{aligned}$$
- Of course, P_1 and P_2 are recognisable in the Black-Scholes model as $\Phi(d_1)$ and $\Phi(d_2)$. We have, however, said nothing about the dynamics of S in the analysis above.

- We now introduce a theorem that will enable the computation of these probabilities given a suitable characteristic function.

Theorem 6.3 (Gil-Pelaez) *Let ϕ_X be the characteristic function associated with random variable X, then, for $k \in \mathbb{R}$,*

$$\Pr\{X > k\} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\left[\frac{e^{-iuk}\phi_X(u)}{iu}\right] du.$$

Proof. Firstly, note that the signum function, defined as

$$\operatorname{sgn}\alpha = \begin{cases} \alpha/|\alpha| & \alpha \neq 0 \\ 0 & \alpha = 0, \end{cases}$$

has the following integral representation

$$\operatorname{sgn}\alpha = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin x\alpha}{x}\, dx. \tag{4}$$

Now, for f_X, the density associated with ϕ_X, we have

$$\begin{aligned} \Pr\{X > k\} &= \int_k^\infty f_X(x)\, dx \\ &= \frac{1}{2\pi} \int_k^\infty \int_{-\infty}^\infty e^{-iux}\phi_X(u)\, du\, dx \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_k^\infty e^{-iux}\, dx\, \phi_X(u)\, du \\ &= -\frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-iux}}{iu}\phi_X(u)\, du\bigg|_{x=k}^{\infty}. \end{aligned} \tag{5}$$

To evaluate this integral at the upper limit, note that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-iux}}{iu}\phi_X(u)\, du &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-iux}}{iu} \int_{-\infty}^\infty e^{iuy} f_X(y)\, dy\, du \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{iu(y-x)}}{iu}\, du\, f_X(y)\, dy \\ &= \frac{1}{2} \int_{-\infty}^\infty \operatorname{sgn}(y-x) f_X(y)\, dy. \end{aligned} \tag{6}$$

The last step in this equation follows by using Euler's identity

$$\frac{e^{iu(y-x)}}{iu} = \frac{1}{i}\frac{\cos(u(y-x))}{u} + \frac{\sin(u(y-x))}{u}$$

and recognising that the first term is odd (hence evaluates to zero) and that the second term may be evaluated using (4). Now, for some $x \in \mathbb{R}$, we may write

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{sgn}(y-x) f_X(y)\, dy &= -\int_{-\infty}^{x} f_X(y)\, dy + \int_{x}^{\infty} f_X(y)\, dy \\ &= -F_X(x) + 1 - F_X(x) \\ &= 1 - 2F_X(x),\end{aligned} \tag{7}$$

where F_X is the associated distribution function. Thus, (6) evaluated with $x = \infty$ is equal to $-\frac{1}{2}$, which means that (5) may be written as

$$\begin{aligned}\Pr\{X > k\} &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iuk}\phi_X(u)}{iu}\, du \\ &= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \operatorname{Re}\left[\frac{e^{-iuk}\phi_X(u)}{iu}\right] du,\end{aligned}$$

since the real part of the integrand is even and the imaginary part is odd. ■

Exercise: Show the last step of the proof in detail using Euler's identity.

- Let $s_T = \ln(S_T)$ and $k = \ln(K)$. Then, to compute the probability P_2, we have the following characteristic function inversion theorem

$$\mathbb{Q}\{s_T > k\} = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} \operatorname{Re}\left[\frac{e^{-iuk}\phi_{s_T}(u)}{iu}\right] du. \tag{8}$$

- Here the characteristic function of the log terminal stock price (under $\mathbb{Q}$) may be written as

$$\phi_{s_T}(u) = \mathbb{E}^{\mathbb{Q}}\left[e^{ius_T}\right] = \int_{-\infty}^{\infty} e^{ius} f_{s_T}(s)\, ds,$$

  where f_{s_T} is the density of s_T.

- To compute the probability P_1, note that $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$, and the change of measure (3) may be written as

$$\frac{d\mathbb{Q}_{\mathsf{s}}}{d\mathbb{Q}} = \frac{S_T}{\mathbb{E}^{\mathbb{Q}}[S_T]} = \frac{e^{s_T}}{\mathbb{E}^{\mathbb{Q}}\left[e^{s_T}\right]} = \frac{e^{s_T}}{\phi_{s_T}(-i)}.$$

- This allows us to relate the density $f^{\mathsf{s}}_{s_T}$ under $\mathbb{Q}_{\mathsf{s}}$ to the density f_{s_T} under $\mathbb{Q}$ as

$$f^{\mathsf{s}}_{s_T}(s)\, ds = \frac{e^{s}}{\phi_{s_T}(-i)} f_{s_T}(s)\, ds.$$

- Then, the characteristic function of s_T under $\mathbb{Q}_{\mathsf{s}}$ is given by

$$\begin{aligned}
\phi^{\mathsf{s}}_{s_T}(u) &= \int_{-\infty}^{\infty} e^{ius} f^{\mathsf{s}}_{s_T}(s)\, ds \\
&= \frac{1}{\phi_{s_T}(-i)} \int_{-\infty}^{\infty} e^{ius} e^{s} f_{s_T}(s)\, ds \\
&= \frac{1}{\phi_{s_T}(-i)} \int_{-\infty}^{\infty} e^{i(u-i)s} f_{s_T}(s)\, ds \\
&= \frac{\phi_{s_T}(u-i)}{\phi_{s_T}(-i)}.
\end{aligned}$$

- Thus, P_1 may be computed using

$$\begin{aligned}
\mathbb{Q}_{\mathsf{s}}\{s_T > k\} &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{-iuk}\phi^{\mathsf{s}}_{s_T}(u)}{iu}\right] du \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{e^{-iuk}\phi_{s_T}(u-i)}{iu\phi_{s_T}(-i)}\right] du. \qquad (9)
\end{aligned}$$

- We are now in a position to price the call option. We do this by implementing the integrals in (9) and (8) using quadrature.
- Care must be taken to avoid evaluating these expressions at zero since the variable of integration appears in the denominator of the integrand.
- Thus, we propose using simple quadrature, where the quadrature points are at the center of each interval.
- The integrals may then be written as

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{N} \mathrm{Re}\left[\frac{e^{-iu_n k}\phi_{s_T}(u_n - i)}{iu_n\phi_{s_T}(-i)}\right] \Delta u$$

and

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{N} \mathrm{Re}\left[\frac{e^{-iu_n k}\phi_{s_T}(u_n)}{iu_n}\right] \Delta u$$

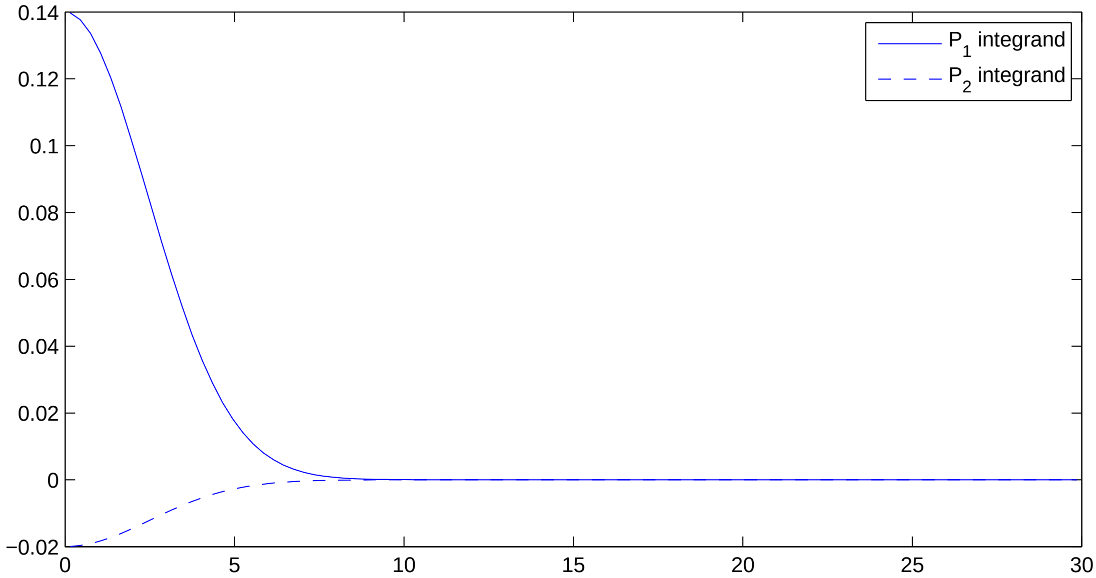
where $u_n = \left(n - \frac{1}{2}\right)\Delta u$. Here $\Delta u = u_{\mathsf{max}}/N$, where the integration limits have been restricted to the interval $[0, u_{\mathsf{max}}]$.

Example: geometric Brownian motion

- **Exercise:** Show that the characteristic function for the log-stock price s_T driven by GBM is

$$\phi_{s_T}(u) = \exp\left(iu\left(\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T\right) - \frac{1}{2}\sigma^2 T u^2\right).$$

- For stock parameters $S_0 = 50$, $\sigma = 40\%$, $T = 1$, $r = 6\%$ and $K = 50$ and for algorithm specific parameters $u_{\text{max}} = 30$ and $N = 100$, the price is the same as the Black-Scholes solution of 9.2363. The graph shows the integrands decreasing to zero over the integration range.



The Heston Model

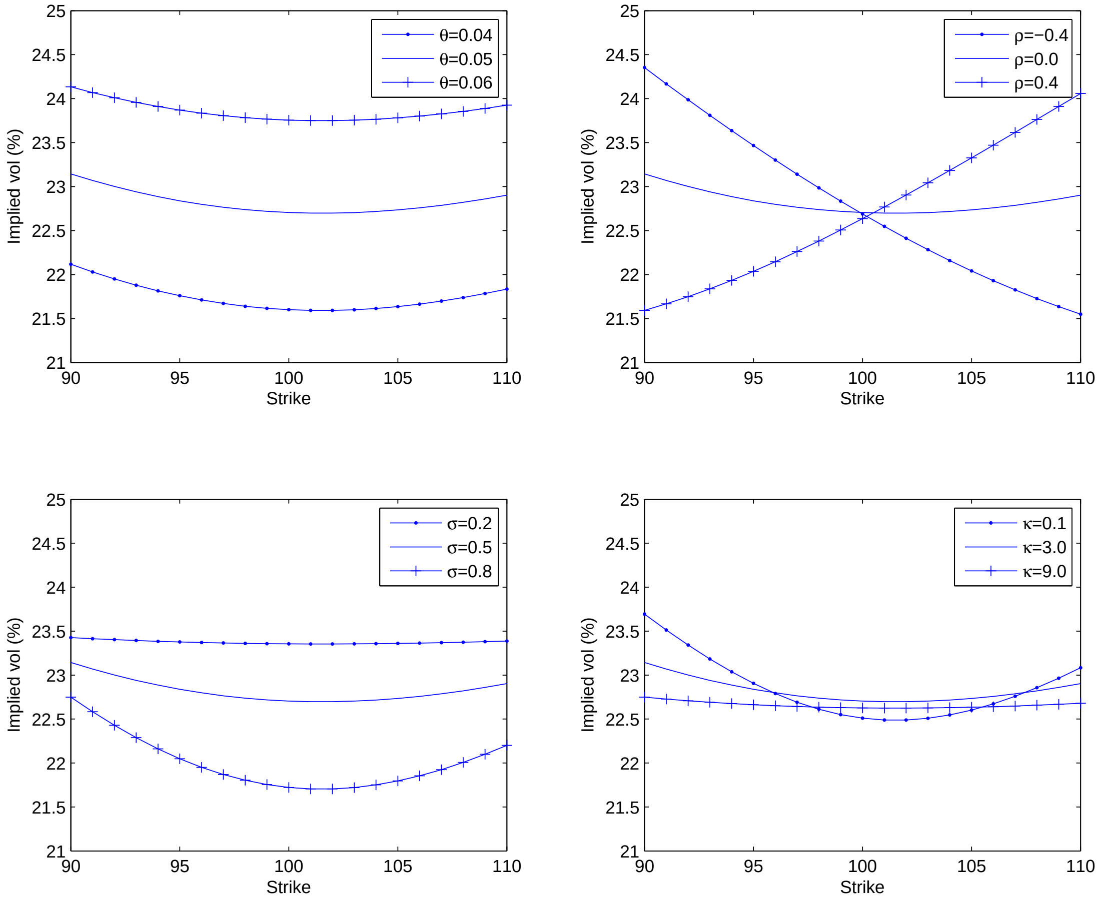
- In order to produce a more realistic model for the dynamics of stock prices Heston (1993) proposed the following model

$$dS_t = \mu S_t\,dt + \sqrt{\nu_t}S_t\,dW_t^{(1)}$$
$$d\nu_t = \kappa(\theta - \nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_t^{(2)},$$

where the Brownian motions are correlated with correlation constant ρ.

- In the above, μ is the drift, which in the risk neutral case is just r, κ is the rate of mean-reversion of the variance to the mean reversion level, given by θ, and σ is the volatility of variance.
- As is usual, the time zero stock price is given by S_0 and we will now also need the time zero value of the variance which we denote by ν_0.
- To price under these dynamics, Heston provided a characteristic function pricing approach, which was later generalised by Bakshi and Madan (2000).
- The following graph shows the kind of smiles that are possible using the Heston model. It shows the Black-Scholes implied volatilities for call option prices. The model parameters not being varied in the graphs are as follows: $S_0 = 100$, $\nu_0 = 0.06$, $\kappa = 3$, $\theta = 0.05$, $\sigma = 0.5$, $T = 0.5$, $r = 3\%$.

Heston Smiles



Milstein Scheme for Heston

- There are a number of discretisation schemes available for the Heston scheme, see for example Rouah (2013) who mentions no less than nine.
- For the purposes of comparison with the pricing method previously outlined, we shall use the Milstein scheme given by

$$\hat{S}_i = \begin{cases} S_0 & \text{if } i = 0, \\ \hat{S}_{i-1} \exp\left(\left(r - \frac{1}{2}\hat{\nu}_{i-1}\right)\Delta t + \sqrt{\hat{\nu}_{i-1}}\sqrt{\Delta t} Z_{s,i}\right) & \text{if } i > 0. \end{cases} \tag{10}$$

where

$$\hat{\nu}_i = \begin{cases} \nu_0, & \text{if } i = 0, \\ \left.\begin{aligned} &\Big(\hat{\nu}_{i-1} + \kappa(\theta - \hat{\nu}_{i-1})\Delta t \\ &\qquad + \sigma\sqrt{\hat{\nu}_{i-1}}\sqrt{\Delta t} Z_{\nu,i} + \frac{1}{4}\sigma^2(Z_{\nu,i}^2 - 1)\Delta t\Big)^+ \end{aligned}\right\} & \text{if } i > 0. \end{cases} \tag{11}$$

Here, the standard normal random variables $Z_{s,i}$ and $Z_{\nu,i}$ are generated with correlation ρ.

- Note that here we have used a truncation scheme to prevent the variance from going negative. We could also have used reflection, in which case the last line would have been specified using $|\cdot|$ instead of $(\cdot)^+$.

The "Little Trap" Heston Characteristic Function

- While it is possible to perform numerics using Heston's original specification of the characteristic function, we shall use a different specification due to Albrecher *et al.* (2006) which is numerically more stable.
- The characteristic function of the log stock price under the risk-neutral measure is given by

$$\phi_{s_T}(u) = \exp(C + D\nu_0 + iu\log(S_0)), \tag{12}$$

where

$$C = rTiu + \theta\kappa\left(Tx_- - \frac{1}{a}\log\left(\frac{1-ge^{-Td}}{1-g}\right)\right)$$
$$D = \frac{1-e^{-Td}}{1-ge^{-Td}}x_-$$

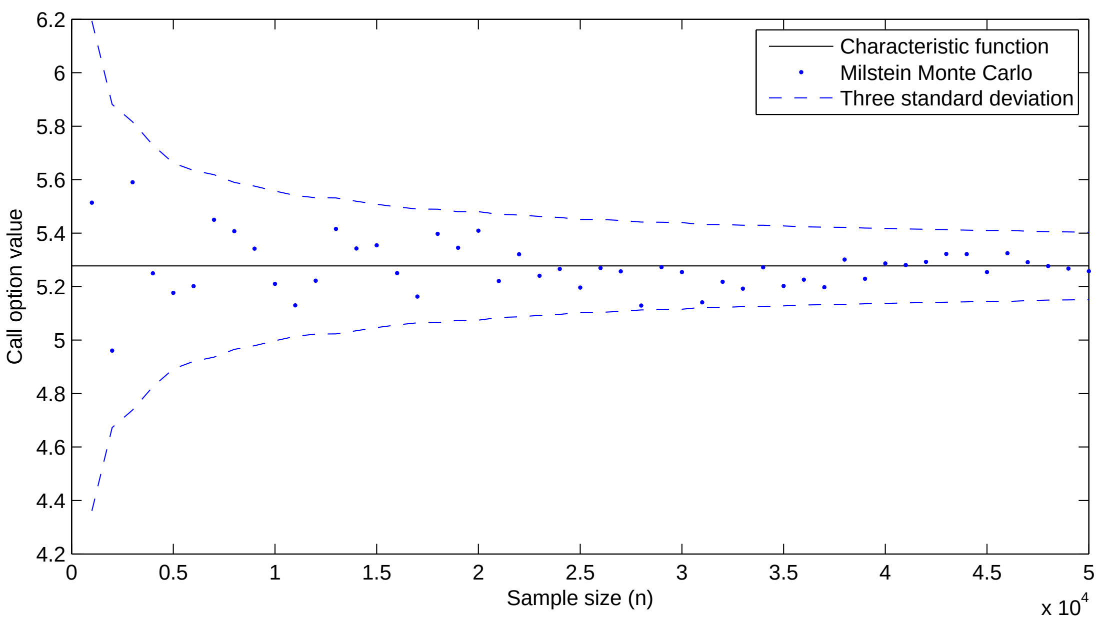
with

$$a = \frac{\sigma^2}{2}, \qquad b = \kappa - \rho\sigma iu, \qquad c = -\frac{u^2+iu}{2}, \qquad d = \sqrt{b^2-4ac},$$

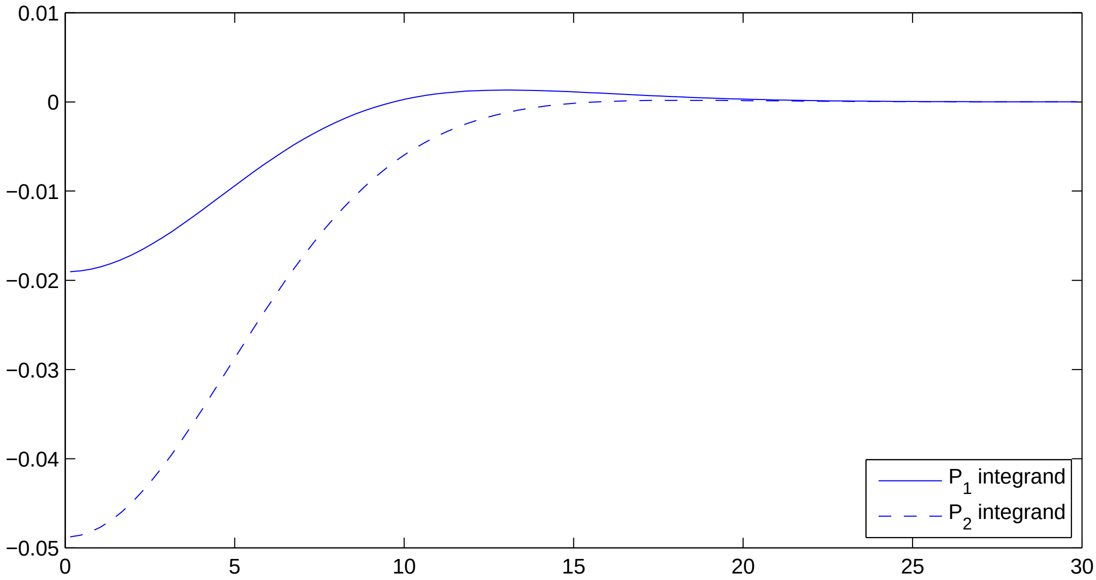
$$x_\pm = \frac{b \pm d}{2a} \quad \text{and} \quad g = \frac{x_-}{x_+}.$$

Example: Call option price for the Heston Model

- The following graph shows the price of a call option with the option related parameters $S_0 = 100$, $\nu_0 = 0.06$, $\kappa = 9$, $\theta = 0.06$, $\sigma = 0.5$, $\rho = -0.4$, $T = 0.5$, $r = 3\%$ and $K = 105$.
- For the Mistein approach, $N_{\text{m}} = 10$ updates (intervals) were used over the time interval $[0, T]$.
- For the characteristic pricing method, the number of quadrature steps was $N = 100$ with upper integration limit being $u_{\max} = 30$.



- The following graph shows the behaviour of the integrands over the range of integration.



Discrete Fourier Transform

- As mentioned previously, a number of different sign and scaling conventions may be found in the literature for Fourier transforms.
- Now, consider the following standard form of the discrete Fourier transform (DFT) of the sequence $x_0, x_1, \ldots, x_{N-1}$

$$\widehat{x}_m = \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}nm} x_n, \tag{13}$$

  for $m = 0, 1, \ldots, N-1$, where N is usually a power of 2.
- The inverse discrete Fourier transform (IDFT) is then

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}nm} \widehat{x}_m.$$

- The link between the discrete Fourier transform and the previous definition of the Fourier transform is that, modulo scaling factors, the DFT is essentially a discrete approximation of the Fourier inversion formula and the IDFT is a discretisation of the Fourier Transform.

Fast Fourier Transform

- The sum in (13) may be split into two sums over the even and odd sequences

$$
\begin{aligned}
\widehat{x}_m &= \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}nm} x_n \\
&= \sum_{n=0}^{N/2-1} e^{-i\frac{2\pi}{N}(2n)m} x_{2n} + \sum_{n=0}^{N/2-1} e^{-i\frac{2\pi}{N}(2n+1)m} x_{2n+1} \\
&= \sum_{n=0}^{N/2-1} e^{-i\frac{2\pi}{N/2}nm} x_{2n} + e^{-i\frac{2\pi}{N}m} \sum_{n=0}^{N/2-1} e^{-i\frac{2\pi}{N/2}nm} x_{2n+1} \\
&= \widehat{e}_m + \widehat{o}_m e^{-i\frac{2\pi}{N}m},
\end{aligned}
$$

  for $m = 0, 1, \ldots, N-1$.

- Now, due to the fact that $e^{-i\frac{2\pi}{N}(m+N/2)} = -e^{-i\frac{2\pi}{N}m}$ and that $\widehat{o}$ and $\widehat{e}$ are periodic, i.e., $\widehat{e}_{m+N/2} = \widehat{e}_m$ and $\widehat{o}_{m+N/2} = \widehat{o}_m$, we may express $\widehat{x}$ as

$$
\widehat{x}_m = \widehat{e}_m + \widehat{o}_m e^{-i\frac{2\pi}{N}m} \quad \text{and} \quad \widehat{x}_{m+N/2} = \widehat{e}_m - \widehat{o}_m e^{-i\frac{2\pi}{N}m},
$$

  for $m = 0, 1, \ldots, \frac{N}{2} - 1$.

FFT Algorithm

- This suggests the following recursive algorithm for computation of the Fourier transform:

  `Function` $\widehat{x}$ =`FFT`(x)

  1. `Set` N =`length`(x);
  2. `If` $N = 1$ `then return` $\widehat{x} = x$; `otherwise`
  3. `Set` $\widehat{e}$ =`FFT`$([x_0, x_2, \ldots, x_{N-2}])$;
  4. `Set` $\widehat{o}$ =`FFT`$([x_1, x_3, \ldots, x_{N-1}])$;
  5. `Set` $m = [0, 1, \ldots, N/2 - 1]$;
  6. `Return` $\widehat{x}$ `formed by concatenating` $\widehat{e} + \widehat{o} \cdot e^{-i\frac{2\pi}{N}m}$ `and` $\widehat{e} - \widehat{o} \cdot e^{-i\frac{2\pi}{N}m}$ `(where` $\cdot$ `represents elementwise multiplication).`

- This is known as the fast Fourier transform (FFT) algorithm and is due to Cooley and Tukey (1965).
- Analysis of computational efficiency shows that the algorithm is $O(N \log N)$.

Numerical Methods in Finance II

Lecture 7 - Pricing Using Characteristic Functions II: The Fast Fourier Transform Method and the Cosine Method

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2020

Introduction

- In this module we apply the fast Fourier transform to price options across many strikes.
- Using a damping function to ensure integrability, the method of Carr and Madan (1999) for pricing options is introduced.
- We then look at the Cosine Method of Fang and Oosterlee (2008), which has rapidly been adopted by industry for pricing options using characteristic functions.
- We start by introducing the Fourier-cosine expansion, which allows a representation of a real-valued function in terms of sums of cosine functions.
- The Cosine Method provides a convenient partition of the pricing problem into a stock price process component, specified in terms of the characteristic function, and a payoff specific component which is independent of the price process.
- The fact that these two components are separate means that computing the Greeks for any option is relatively simple, relying on a modification of the price process component.
- While we look only at pricing of vanilla call options under geometric Brownian motion, these techniques are applicable for a very wide range of price processes, many of which may not easily be computed under some of the other techniques we have explored.

Fourier Transform pricing: Carr and Madan (1998)

- Consider a stock price S_T at time T which has initial price S_0. A call option, with strike K, has terminal payoff $(S_T - K)\mathbb{I}_{\{S_T \geq K\}}$.
- Let $s_T = \ln(S_T)$ and $k = \ln(K)$. The risk-neutral density of the log price s_T is denoted $q_{s_T}(s)$. Then, we may write the price of the option (as a function of k) as the usual discounted expected payoff

$$\begin{aligned} C(k) &= e^{-rT} \int_{-\infty}^{\infty} (e^s - e^k)\mathbb{I}_{\{s \geq k\}} q_{s_T}(s)\, ds \\ &= e^{-rT} \int_{k}^{\infty} (e^s - e^k) q_{s_T}(s)\, ds. \end{aligned}$$

- Note that as $k \to -\infty$ (i.e. $K \to 0$) $C(k) \to S_0$. Thus C is not (square-)integrable — this is a problem because we wish to take a Fourier transform of this function.
- Thus, we apply a damping factor of $e^{\alpha k}$ and consider the function

$$c(k) = e^{\alpha k} C(k),$$

for $\alpha > 0$. This function is integrable.
- Now, consider the Fourier transform of $c(k)$

$$\widehat{c}(v) = \int_{-\infty}^{\infty} e^{ivk} c(k)\, dk.$$

- To obtain the price of the call in terms of $\widehat{c}(v)$ we apply the inverse Fourier transform and divide through by the damping factor

$$C(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \widehat{c}(v)\, dv = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-ivk} \widehat{c}(v)\, dv, \tag{1}$$

where the second equality holds because $C(k)$ is real, which means that $\widehat{c}(v)$ is odd in its imaginary part and even in its real part.
- Now we express $\widehat{c}(v)$ in terms of the characteristic function $\phi_{s_T}(\cdot)$ for s_T

$$\begin{aligned} \widehat{c}(v) &= \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} \int_{k}^{\infty} e^{-rT}(e^s - e^k) q_{s_T}(s)\, ds\, dk \\ &= e^{-rT} \int_{-\infty}^{\infty} q_{s_T}(s) \int_{-\infty}^{s} e^{s+(\alpha+iv)k} - e^{(1+\alpha+iv)k}\, dk\, ds \\ &= e^{-rT} \int_{-\infty}^{\infty} q_{s_T}(s) \left(\frac{e^{(\alpha+1+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha + 1 + iv} \right) ds \\ &= \frac{e^{-rT} \int_{-\infty}^{\infty} e^{i(v-(\alpha+1)i)s} q_{s_T}(s)\, ds}{(\alpha + iv)(\alpha + 1 + iv)} \\ &= \frac{e^{-rT} \phi_{s_T}(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \end{aligned} \tag{2}$$

- Thus, we can compute prices by evaluating the closed form expression (2) in terms of the characteristic function for our stock price and performing the inverse Fourier transform (1) in terms of these values.

Using the Fast Fourier Transform

- We now discretise the problem and specify it in terms of the fast Fourier transform.
- We can approximate the integral (1) using quadrature (not trapezoidal!), with step length δ_v by

$$C(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{n=0}^{N-1} e^{-iv_n k} \widehat{c}(v_n)\delta_v, \tag{3}$$

  where $v_n = n\delta_v$. In doing so, we have imposed a finite upper limit of integration of $a = N\delta_v$.
- We now discretise the log-strike prices, in the range $k \in [-b, b]$ using a step length of $\delta_k = 2b/(N-1)$ so that the k values are given by

$$k_m = -b + m\delta_k \quad \text{for } m = 0, 1, \ldots, N-1.$$

- In terms of the discretised log-strikes, (3) becomes

$$\begin{aligned} C(k_m) &\approx \frac{e^{-\alpha k_m}}{\pi} \sum_{n=0}^{N-1} e^{-iv_n(-b+m\delta_k)} \widehat{c}(v_n)\delta_v \\ &= \frac{e^{-\alpha k_m}}{\pi} \sum_{n=0}^{N-1} e^{-i\delta_v \delta_k nm} e^{ibv_n} \widehat{c}(v_n)\delta_v, \end{aligned}$$

  for $m = 0, 1, \ldots, N-1$.

- Thus, by choosing $\delta_k\delta_v = 2\pi/N$ (known as the Nyquist relation) and defining

$$x_n := e^{ibv_n}\widehat{c}(v_n)\delta_v, \tag{4}$$

  the system may be written as

$$C(k_m) \approx \frac{e^{-\alpha k_m}}{\pi} \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}nm} x_n.$$

- Since

$$\widehat{x}_m = \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}nm} x_n$$

  is the FFT of x_n, the prices may thus be computed using

$$C(k_m) \approx \frac{e^{-\alpha k_m}}{\pi} \mathrm{Re}\left\{\widehat{x}_m\right\} \quad \text{for } m = 0, 1, \ldots, N-1, \tag{5}$$

  where we have restricted the solution to have real values.
- To improve the accuracy of the integration, Carr and Madan suggest using composite Simpson's rule weighting factors, in which case (4) may be written as

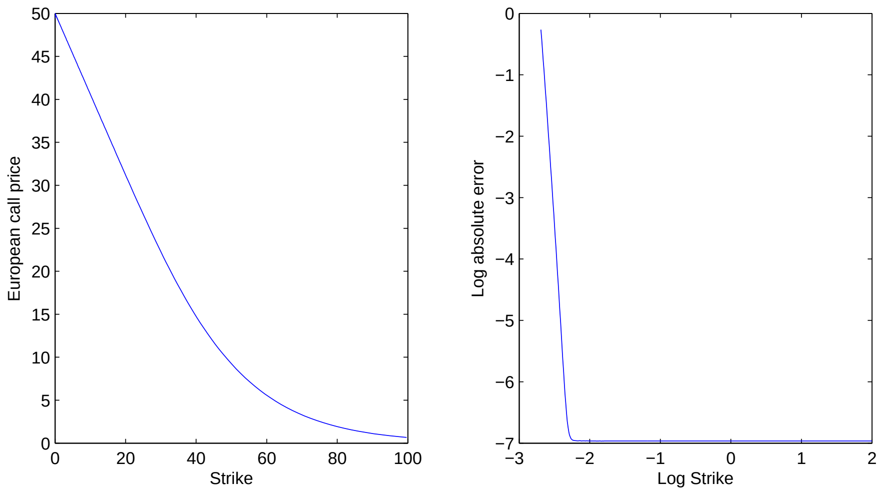
$$x_n = e^{ibv_n}\widehat{c}(v_n)\frac{\delta_v}{3}\left(3 + (-1)^{n+1} - \mathbb{I}_{\{n=0\}}\right). \tag{6}$$

Example: European Call using GBM

- Recall that the characteristic function for the log-stock price s_T driven by GBM is

$$\phi_{s_T}(u) = \exp\left(iu\left(\ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T\right) - \frac{1}{2}\sigma^2 T u^2\right).$$

- The left graph below shows Fourier prices as a function of strike using the FFT pricing equation (5) in terms of (6) and the stock parameters $S_0 = 50$, $\sigma = 0.4$, $T = 1$ and $r = 6\%$. The algorithm specific parameters used were $N = 2^{10}$, $\delta_v = 0.25$ and $\alpha = 1.5$.
- The right graph shows the $\log_{10}$ absolute difference between these prices and the corresponding Black-Scholes prices, on a log scale for strike.



The Fourier-Cosine Series Expansion

Theorem 7.1 (Fourier-Cosine Expansion) *Suppose that $f : [0, \pi] \to \mathbb{R}$ is integrable on $[0, \pi]$, then f may be written as*

$$f(\theta) = {\sum_{n=0}^{\infty}}' A_n \cos(n\theta),$$

where $\sum'$ indicates that the first term in the sum is multiplied by $\frac{1}{2}$ and the Fourier-cosine expansion coefficients A_n, for $n \in \mathbb{N}$, are given by

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)\, d\theta.$$

- In order to support functions on a finite interval $[a, b] \in \mathbb{R}$, the expansion may be modified using a change of variables:

$$\theta = \pi\frac{x-a}{b-a} \quad \Rightarrow \quad x = \frac{b-a}{\pi}\theta + a,$$

in which case

$$f(x) = {\sum_{n=0}^{\infty}}' A_n \cos\left(n\pi\frac{x-a}{b-a}\right) \quad \text{with} \quad A_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(n\pi\frac{x-a}{b-a}\right) dx.$$

The Cosine Method: Fang and Oosterlee (2008)

- Consider a European option with maturity T and strike price K.
- We shall work with the scaled terminal stock price $s_T = \ln(S_T/K)$ with the pay-off of the option is specified as $v(s_T)$.
- In terms of the risk-neutral density function of the scaled stock price, $q_{s_T}(s)$, the price of the option at inception ($t = 0$) is then the usual discounted expectation

$$\begin{aligned} V &= e^{-rT} \int_{-\infty}^{\infty} v(s) q_{s_T}(s)\, ds \\ &\approx e^{-rT} \int_{a}^{b} v(s) q_{s_T}(s)\, ds. \end{aligned} \tag{7}$$

  for suitable choices of $a, b \in \mathbb{R}$.
- We now express $q_{s_T}(s)$ on $[a, b]$ by its cosine expansion

$$q_{s_T}(s) = {\sum_{n=0}^{\infty}}' A_n \cos\left(n\pi \frac{s-a}{b-a} \right),$$

  in terms of Fourier-cosine coefficients A_n, for $n \in \mathbb{N}$.

- The coefficients A_n may be expressed as

$$\begin{aligned} A_n &= \frac{2}{b-a} \int_a^b q_{s_T}(s) \cos\left(n\pi \frac{s-a}{b-a} \right) ds \\ &= \frac{2}{b-a} \int_a^b q_{s_T}(s) \mathrm{Re}\left\{ e^{in\pi \frac{s-a}{b-a}} \right\} ds \\ &= \frac{2}{b-a} \int_a^b q_{s_T}(s) \mathrm{Re}\left\{ e^{i\frac{n\pi}{b-a}s} e^{-in\pi \frac{a}{b-a}} \right\} ds \\ &= \frac{2}{b-a} \mathrm{Re}\left\{ \left(\int_a^b q_{s_T}(s) e^{i\frac{n\pi}{b-a}s}\, ds \right) e^{-in\pi\frac{a}{b-a}} \right\} \\ &\approx \frac{2}{b-a} \mathrm{Re}\left\{ \phi_{s_T}\left(\frac{n\pi}{b-a} \right) e^{-in\pi \frac{a}{b-a}} \right\}. \end{aligned} \tag{8}$$

- We have thus expressed $q_{s_T}(s)$ as an expansion in terms of its characteristic function ϕ_{s_T}.
- The price of the option (7) may then be written as

$$\begin{aligned} V &\approx e^{-rT} \int_a^b v(s) {\sum_{n=0}^{\infty}}' A_n \cos\left(n\pi \frac{s-a}{b-a} \right) ds \\ &= e^{-rT} {\sum_{n=0}^{\infty}}' A_n \int_a^b v(s) \cos\left(n\pi \frac{s-a}{b-a} \right) ds. \end{aligned}$$

- If we define
$$v_n = \frac{2}{b-a}\int_a^b v(s)\cos\left(n\pi\frac{s-a}{b-a}\right)ds \tag{9}$$
for $n \in \mathbb{N}$ and expand A_n using (8) we obtain the COS formula
$$V \approx e^{-rT}\sum_{n=0}^{N-1}{}' \operatorname{Re}\left\{\phi_{s_T}\left(\frac{n\pi}{b-a}\right)e^{-in\pi\frac{a}{b-a}}\right\}v_n, \tag{10}$$
where only the first $N \in \mathbb{N}$ terms in the expansion are summed.
- This representation is convenient in that it compartmentalises the valuation into two separate components:
  - a process specific component (the term in the curly brackets of (10)) which is evaluated in terms of the process characteristic function; and
  - a payoff specific component, v_n.
- We now deduce expressions for the payoff specific coefficients (v_n) for European call and put options.

Proposition 7.2 (Cosine Series Coefficients) *The cosine series coefficients, χ_n, of $g(s) = e^s$ on $[c,d] \subset [a,b]$,*
$$\chi_n(c,d) = \int_c^d e^s \cos\left(n\pi\frac{s-a}{b-a}\right)ds,$$
and the cosine series coefficients, ψ_n, of $g(s) = 1$ on $[c,d] \subset [a,b]$,
$$\psi_n(c,d) = \int_c^d \cos\left(n\pi\frac{s-a}{b-a}\right)ds,$$
are given analytically by
$$\chi_n(c,d) = \left[1+\left(\frac{n\pi}{b-a}\right)^2\right]^{-1}\left\{\cos\left(n\pi\frac{d-a}{b-a}\right)e^d - \cos\left(n\pi\frac{c-a}{b-a}\right)e^c + \frac{n\pi}{b-a}\left[\sin\left(n\pi\frac{d-a}{b-a}\right)e^d - \sin\left(n\pi\frac{c-a}{b-a}\right)e^c\right]\right\}$$
and
$$\psi_n(c,d) = \begin{cases} d-c & n=0,\\ \frac{b-a}{n\pi}\left[\sin\left(n\pi\frac{d-a}{b-a}\right) - \sin\left(n\pi\frac{c-a}{b-a}\right)\right] & n>0.\end{cases}$$

Proof. Exercise.

- Given the previous proposition, we are now in a position to work out the coefficients v_n for the specific cases of call and put options.
- Consider a stock price S_T at time T which has initial price S_0. We consider pricing a call option with strike K.
- Let $s_T = \ln(S_T/K)$. The payoff of a call option with strike K is then written as

$$v^{\text{call}}(s) = K(e^s - 1)\mathbb{I}_{\{s \geq 0\}}.$$

- Substituting this expression into (9) and applying Proposition 7.2 the coefficients are

$$\begin{aligned} v_n^{\text{call}} &= \frac{2}{b-a}\int_0^b K(e^s - 1)\cos\left(n\pi\frac{s-a}{b-a}\right) ds \\ &= \frac{2}{b-a}K\left(\chi_n(0,b) - \psi_n(0,b)\right), \end{aligned} \tag{11}$$

  for $n \in \mathbb{N}$.
- Similarly, the coefficients for a put option are

$$v_n^{\text{put}} = \frac{2}{b-a}K\left(-\chi_n(a,0) + \psi_n(a,0)\right),$$

  for $n \in \mathbb{N}$.

- There are three sources of error in the COS method approximation as a result of:
  1. the truncation of the integration range to $[a, b]$ in (7),
  2. the substitution of the characteristic function for the integral over the truncated range in (8), and
  3. the summation of only N terms in the expansion (10)
- Fang and Oosterlee provide an error analysis showing that the COS method exhibits exponential convergence (as a function of N) and has a computational complexity that is linear. We do not reproduce this analysis.
- To minimise the error resulting from the first error source above, the following interval of integration is recommended

$$[a, b] = \left[c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}}\right],$$

  where c_n denotes the nth cumulant of $\ln(S_T/K)$ and $L = 10$.
- For geometric Brownian motion, these cumulants are as follows:

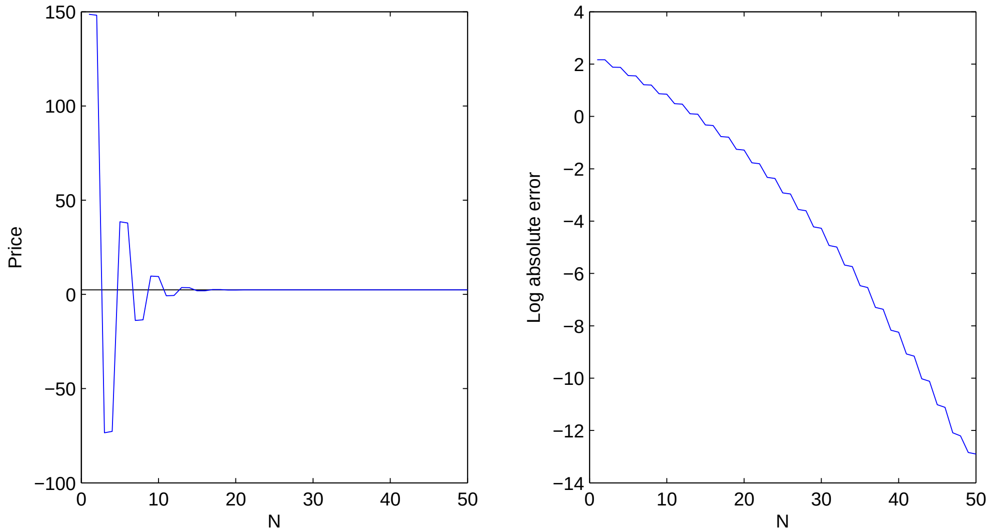
$$\begin{aligned} c_1 &= \mu T, \\ c_2 &= \sigma^2 T \quad \text{and} \\ c_4 &= 0. \end{aligned}$$

Example: European Call using GBM

- **Exercise:** Show that the characteristic function for the scaled log-stock price s_T driven by GBM is

$$\phi_{s_T}^{\text{GBM}}(u) = \exp\left(iu\left(\ln(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T\right) - \frac{1}{2}\sigma^2 T u^2\right).$$

- The left graph below shows the COS method price as a function of N using the parameters $S_0 = 7$, $\sigma = 0.4$, $T = 2$, $K = 6$ and $r = 6\%$.
- The right graph shows the log absolute difference between this price and the corresponding Black-Scholes price.



Calculating Greeks

- Due to the fact that the functions v_n do not directly contain terms dependent on s_T, calculation of the Greeks only requires modification of the process-specific component of the COS formula (10).
- With the exception of Rho, which applies to the discount factor and the characteristic function, all the other Greeks only require one to perform partial derivatives on the characteristic function.
- For example, we may compute the Delta of the option V as

$$\begin{aligned}\Delta &= \frac{\partial V}{\partial S_0} \\ &\approx e^{-rT} \sum_{n=0}^{N-1}{}' \operatorname{Re}\left\{\frac{\partial \phi_{s_T}}{\partial S_0}\left(\frac{n\pi}{b-a}\right) e^{-in\pi\frac{a}{b-a}}\right\} v_n.\end{aligned}$$

- In the case of GBM, we have

$$\frac{\partial \phi_{s_T}^{\text{GBM}}(u)}{\partial S_0} = \phi_{s_T}^{\text{GBM}}(u)\frac{iu}{S_0},$$

in which case

$$\Delta \approx e^{-rT} \sum_{n=0}^{N-1}{}' \operatorname{Re}\left\{\phi_{s_T}^{\text{GBM}}\left(\frac{n\pi}{b-a}\right)\frac{in\pi}{S_0(b-a)} e^{-in\pi\frac{a}{b-a}}\right\} v_n.$$

- The left graph below shows COS method Delta of the call option as a function of N using the same parameters as above.
- The right graph shows the log absolute difference between the Delta and the corresponding Black-Scholes Delta.