Numerical Methods in Finance II

Lecture 4 - Local Volatility

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Introduction

- In this lecture, we consider generalisations of the Black-Scholes equation by relaxing the assumption of constant volatility.
- Instead, we assume that the volatility may be expressed as a deterministic function of time and stock $(\sigma(t, S_t))$. Because the volatility is deterministic, the market remains complete.
- We explore two methods for determining the $\sigma(t, S_t)$ from the market prices and implied volatilities.
- We start by deriving the Kolmogorov forward equation for an arbitrary SDE.
- Then, we derive the Breeden-Litzenberger equations for European options.
- These are used to derive the Dupire equation, which allows estimation of the local volatility function $(\sigma(t, S_t))$ from market prices.
- To illustrate the principle, we apply the Dupire equation to the CEV model.
- To show how prices may be computed using a local volatility model, we extend the theta finite-difference method presented in module two.
- Finally, we show how the local volatility function may be determined using the market implied volatility surface, rather than prices.

The Kolmogorov Forward Equation

- Let $H(x)$ and $f(x)$ be arbitrary C^2-functions, where, in addition, H has compact support.
- Then, using integration by parts, we have

$$\int_{-\infty}^{\infty} H(x)\frac{\partial f(x)}{\partial x}\,dx = H(x)f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial H(x)}{\partial x} f(x)\,dx$$
$$= -\int_{-\infty}^{\infty} \frac{\partial H(x)}{\partial x} f(x)\,dx.$$

- Similarly, applying integration by parts twice, we have

$$\int_{-\infty}^{\infty} H(x)\frac{\partial^2 f(x)}{\partial x^2}\,dx = \int_{-\infty}^{\infty} \frac{\partial^2 H(x)}{\partial x^2} f(x)\,dx.$$

- Consider the general Stochastic process X defined by

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t, \qquad X_0 \text{ a constant},$$

where a, b are C^2-functions.

- Applying Itō's formula to $H(X_t)$ gives

$$dH(X_t) = \left(a(t, X_t)\frac{\partial H(X_t)}{\partial x} + \frac{1}{2}b^2(t, X_t)\frac{\partial^2 H(X_t)}{\partial x^2}\right) dt + b(t, X_t)\frac{\partial H(X_t)}{\partial x}\,dW_t.$$

which means that

$$\mathbb{E}[H(X_t)] = H(X_0) + \mathbb{E}\left[\int_0^t a(s, X_s)\frac{\partial H(X_s)}{\partial x} + \frac{1}{2}b^2(s, X_s)\frac{\partial^2 H(X_s)}{\partial x^2}\,ds\right].$$

- Taking a partial derivative with respect to time of these expectations gives

$$\frac{\partial}{\partial t}\mathbb{E}[H(X_t)] = \mathbb{E}\left[a(t, X_t)\frac{\partial H(X_t)}{\partial x} + \frac{1}{2}b^2(t, X_t)\frac{\partial^2 H(X_t)}{\partial x^2}\right],$$

- Let $\varphi(t, x)$ be the density of X_t, given that X starts from X_0, then the expectations may be written explicitly as

$$\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} H(x)\varphi(t, x)\,dx\right] = \int_{-\infty}^{\infty} H(x)\frac{\partial \varphi(t, x)}{\partial t}\,dx$$
$$= \int_{-\infty}^{\infty} a(t, X_t)\frac{\partial H(x)}{\partial x}\varphi(t, x) + \frac{1}{2}b^2(t, x)\frac{\partial^2 H(x)}{\partial x^2}\varphi(t, x)\,dx$$
$$= \int_{-\infty}^{\infty} -H(x)\frac{\partial}{\partial x}\left[a(t, x)\varphi(t, x)\right] + \frac{1}{2}H(x)\frac{\partial^2}{\partial x^2}\left[b^2(t, x)\varphi(t, x)\right] dx$$

where the last step follows using integration by parts.

- This means that

$$\int_{-\infty}^{\infty} H(x)\left(\frac{\partial \varphi(t,x)}{\partial t} + \frac{\partial}{\partial x}\left[a(t,x)\varphi(t,x)\right] - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b^2(t,x)\varphi(t,x)\right]\right) dx = 0$$

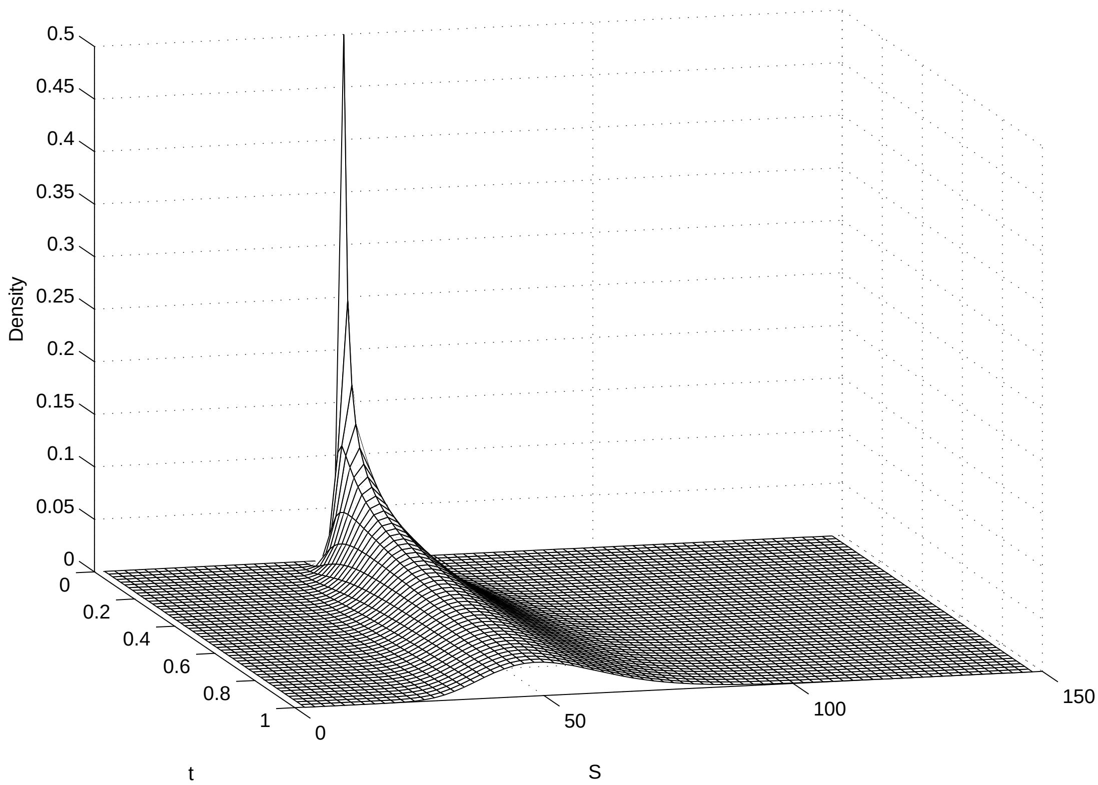
and since $H(\cdot)$ was arbitrary we have

$$\frac{\partial \varphi(t,x)}{\partial t} = -\frac{\partial}{\partial x}\left[a(t,x)\varphi(t,x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b^2(t,x)\varphi(t,x)\right]. \qquad (1)$$

- This is known as the Kolmogorov forward equation, or alternatively the Fokker-Planck equation. It describes the dynamics of the density function.
- As an example, consider geometric Brownian motion. Then $a(t,x) = \mu x$ and $b(t,x) = \sigma x$, which means that the Kolmogorov forward equation may be written as

$$\frac{\partial \varphi(t,x)}{\partial t} + (\mu x - 2\sigma^2 x)\frac{\partial \varphi(t,x)}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \varphi(t,x)}{\partial x^2} + (\mu - \sigma^2)\varphi(t,x) = 0.$$

- The figure below shows a finite difference solution of this equation with the initial condition given by $\varphi(0,x) = \delta(x - S_0)$ and boundary conditions equal to zero, where $\delta(\cdot)$ is the Dirac delta function. Parameters used were $S_0 = 50$, $\mu = 0.06$ $\sigma = 0.25$ and $T = 1$.
- The density in the final time step, modulo approximation error, is the same as the familiar log-normal density.



The Breeden-Litzenberger Equations

- Consider the SDE of a stock under the risk-neutral measure
$$dS_t = rS_t\,dt + \sigma(t,S_t)S_t\,d\tilde{W}_t, \qquad S_0 \text{ constant},$$
where the local volatility function is a deterministic function of t and S_t. We only consider the case of deterministic short rate r and no dividends.
- Consider a put option with strike K and maturity T. Its price is
$$P(T,K) = e^{-rT}\int_0^K (K-S)\varphi(T,S)\,dS, \tag{2}$$
where $\varphi(T,S)$ is the density function for S_T.
- We differentiate (2) with respect to K twice to obtain the B-L equations
$$\begin{aligned}\frac{\partial P(T,K)}{\partial K} &= e^{-rT}\int_0^\infty \frac{\partial}{\partial K}\left[(K-S)\mathbb{I}_{\{S\le K\}}\right]\varphi(T,S)\,dS\\ &= e^{-rT}\int_0^\infty \left[\mathbb{I}_{\{S\le K\}} + (K-S)\delta(K-S)\right]\varphi(T,S)\,dS\\ &= e^{-rT}\int_0^K \varphi(T,S)\,dS = e^{-rT}\Phi(T,K)\\ \frac{\partial^2 P(T,K)}{\partial K^2} &= e^{-rT}\varphi(T,K), \end{aligned}\tag{3}$$
where $\delta(\cdot)$ is the Dirac delta function and $\Phi(\cdot,\cdot)$ is the distribution of S_T.
- Note that (3) holds for calls as well.

The Dupire Equation

- Now, we derive the Dupire Equation. Start by differentiating (2) with respect to T
$$\frac{\partial P(T,K)}{\partial T} = -rP(T,K) + e^{-rT}\int_0^K (K-S)\frac{\partial \varphi(T,S)}{\partial T}\,dS. \tag{4}$$
- To derive the partial derivative in the last term above we use the Breeden-Litzenberger equation (3):
$$\varphi(T,S) = e^{rT}\frac{\partial^2 P(T,S)}{\partial S^2}$$
and substitute this into the Kolmogorov forward equation to get
$$\begin{aligned}\frac{\partial \varphi(T,S)}{\partial T} &= -\frac{\partial}{\partial S}[rS\varphi(T,S)] + \frac{1}{2}\frac{\partial^2}{\partial S^2}[\sigma^2(T,S)S^2\varphi(T,S)]\\ &= -e^{rT}\frac{\partial}{\partial S}\left[rS\frac{\partial^2 P(T,S)}{\partial S^2}\right] + \frac{1}{2}e^{rT}\frac{\partial^2}{\partial S^2}\left[\sigma^2(T,S)S^2\frac{\partial^2 P(T,S)}{\partial S^2}\right].\end{aligned}$$
- Now
$$rS\frac{\partial^2 P(T,S)}{\partial S^2} = \frac{\partial}{\partial S}\left[rS\frac{\partial P(T,S)}{\partial S} - rP(T,S)\right],$$
thus
$$\frac{\partial \varphi(T,S)}{\partial T} = e^{rT}\frac{\partial^2}{\partial S^2}\left[-rS\frac{\partial P(T,S)}{\partial S} + rP(T,S) + \frac{1}{2}\sigma^2(T,S)S^2\frac{\partial^2 P(T,S)}{\partial S^2}\right]. \tag{5}$$

- Substituting (5) into (4) gives

$$
\begin{aligned}
rP(T,K) + \frac{\partial P(T,K)}{\partial T} \\
&= \int_0^K (K-S)\frac{\partial^2}{\partial S^2}\left[-rS\frac{\partial P(T,S)}{\partial S} + rP(T,S) + \frac{1}{2}\sigma^2(T,S)S^2\frac{\partial^2 P(T,S)}{\partial S^2}\right] dS \\
&= \int_0^K \frac{\partial}{\partial S}\left[-rS\frac{\partial P(T,S)}{\partial S} + rP(T,S) + \frac{1}{2}\sigma^2(T,S)S^2\frac{\partial^2 P(T,S)}{\partial S^2}\right] dS \\
&= \left[-rS\frac{\partial P(T,S)}{\partial S} + rP(T,S) + \frac{1}{2}\sigma^2(T,S)S^2\frac{\partial^2 P(T,S)}{\partial S^2}\right]_{S=0}^{K} \\
&= -rK\frac{\partial P(T,K)}{\partial K} + rP(T,K) + \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 P(T,K)}{\partial K^2},
\end{aligned}
$$

  where the second step follows by integration by parts and the third follows by the Fundamental Theorem of Calculus. In the final step we have used the fact that $P(T,0)=0$.

- Solving for $\sigma(T,K)$ gives Dupire's equation

$$
\sigma(T,K) = \frac{\sqrt{2}}{K}\sqrt{\frac{\frac{\partial P(T,K)}{\partial T} + rK\frac{\partial P(T,K)}{\partial K}}{\frac{\partial^2 P(T,K)}{\partial K^2}}}. \tag{6}
$$

Using the Dupire Equation

- We have derived the Dupire equation for put prices, but it is also valid, in the same form, for call prices.
- It is difficult to apply this formula directly to market data. Thus, the usual practice is to preprocess price data before applying the formula. This takes the form of either interpolating/extrapolating the data or fitting a parametric model. It is then easier to sample the processed prices and use the Dupire equation to compute the local-vol surface. The local-vols generated may then be used in conjunction with numerical methods for pricing, including finite difference and Monte Carlo techniques.
- Rather than using the prices directly, the same approach may instead be used with Black-Scholes implied volatilities. This requires us to express the local volatility function as a function of the implied volatility surface. We derive this in a later section.
- In both cases, care must be taken to ensure that the surface generated is arbitrage-free. See Gatheral (2006) for more detail.
- We now demonstrate the Dupire formula using the CEV model.

The CEV Model

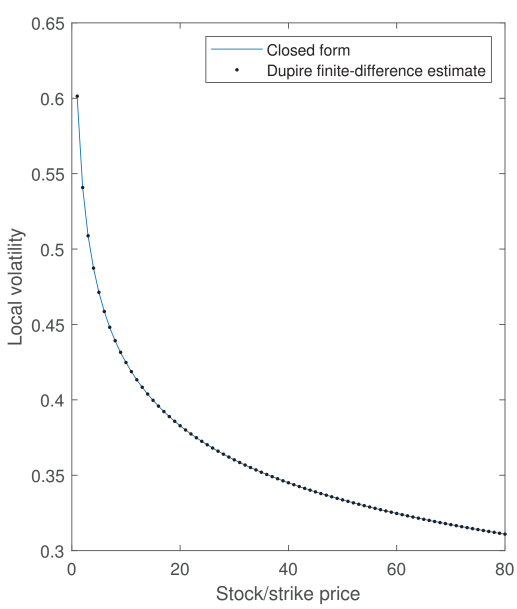
- The constant elasticity of variance (CEV) model is a simple parametric local volatility model, where the stock is modeled using the following SDE
$$dS_t = rS_t\,dt + \sigma S_t^{\alpha}\,dW_t, \qquad S_0 \text{ constant},$$
where $\alpha \in [0, 1]$.
- For values of $\alpha < 1$, this stock model produces behavior consistent with the leverage effect (i.e. as stock prices fall, volatility increases).
- The model derives its name from the fact that, for the variance function $v^2(t, S) = \sigma^2 S^{2\alpha}$, the elasticity of variance
$$\frac{\partial v^2(t,S)}{\partial S} \Big/ \frac{v^2(t,S)}{S} = 2\alpha$$
is constant.
- By inspection, the local volatility function is given as
$$\sigma(t, S_t) = \sigma S_t^{(\alpha-1)}. \tag{7}$$
Notice that the local volatility function has no explicit time dependence.
- For the following values of α we recover less general processes
  - $\alpha = 1$: Geometric Brownian motion.
  - $\alpha = 0.5$: A square root process — similar to CIR process, without mean reversion.
  - $\alpha = 0$: A process similar to the Ornstein-Uhlenbeck process, without mean reversion.

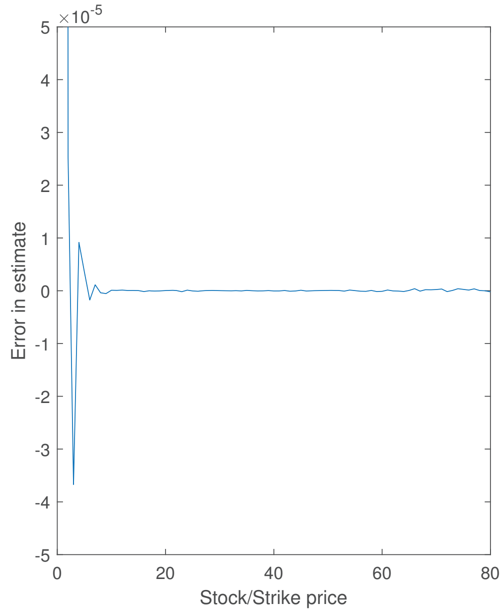
  These are the only cases which have analytical solutions for stock price.

- It is, therefore, surprising that closed form option pricing formulae are available, with the value of European call and put options given by
$$C(S_0, K, \sigma, \alpha, T) = S_0[1 - \mathcal{P}(y; z, x)] - Ke^{-rT}\mathcal{P}(x; z - 2, y)$$
$$P(S_0, K, \sigma, \alpha, T) = -S_0\mathcal{P}(y; z, x) + Ke^{-rT}[1 - \mathcal{P}(x; z - 2, y)],$$
where $\mathcal{P}(\cdot; d, \lambda)$ is the non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ, and
$$x = \kappa S_0^{2(1-\alpha)} e^{2r(1-\alpha)T}, \quad y = \kappa K^{2(1-\alpha)}, \quad \text{and} \quad z = 2 + \frac{1}{1-\alpha}$$
where
$$\kappa = \frac{2r}{\sigma^2(1-\alpha)(e^{2r(1-\alpha)T} - 1)}.$$
- Using finite difference approximations for the partial derivatives in (6), and the above formulae as an (exact) pricing surface, it is now possible to illustrate the Dupire equation in action.

The Dupire Equation Applied to the CEV Model

- Consider the situation where $S_0 = 40$, $\sigma = 60\%$, $\alpha = 0.85$, $r = 6\%$ and $T = 2$. The left graph below shows a plot of the CEV local volatility (7) as a function of strike price. The corresponding Dupire estimates, using central differences with $\delta T = \delta K = 0.001$, are plotted. Note that these estimates were generated using call prices for $K < S_0$ and put prices for $K \geq S_0$. The right graph shows the difference between the two.





Finite Differences and Local Volatility

- In the second module of this course, we developed the Theta Finite-difference Scheme that produced solutions for the time-reversed Black-Scholes PDE. We now consider how this approach may be modified to compute prices for the local volatility version of the PDE

$$\frac{\partial U}{\partial \tau} - rS\frac{\partial U}{\partial S} - \frac{1}{2}\sigma^2(S, T-\tau)S^2\frac{\partial^2 U}{\partial S^2} + rU = 0. \tag{8}$$

- Recall, that the solution at time $m+1$ was given as

$$\mathbf{U}_{m+1} = (\theta\mathbf{G} + (1-\theta)\mathbf{I})^{-1}[((1-\theta)\mathbf{F} + \theta\mathbf{I})\mathbf{U}_m + (1-\theta)\mathbf{b}_m + \theta\mathbf{b}_{m+1}],$$

in terms of the identity matrix $\mathbf{I}$, the matrices $\mathbf{F}$ and $\mathbf{G}$, the boundary condition vectors $\mathbf{b}_m$ and the constant θ.

- Define the matrix

$$\mathbf{\Sigma}_m = \mathrm{diag}(\sigma^2(t_m, S_{\min}+\delta s), \sigma^2(t_m, S_{\min}+2\delta s), \ldots, \sigma^2(t_m, S_{\min}+(N-1)\delta s))$$

with $t_m = T - m\delta\tau$. We may then define

$$\mathbf{F}_m = (1 - r\delta\tau)\mathbf{I} + \frac{1}{2}r\delta\tau\mathbf{D}_1\mathbf{T}_1 + \frac{1}{2}\mathbf{\Sigma}_m\delta\tau\mathbf{D}_2\mathbf{T}_2$$

$$\mathbf{G}_{m+1} = (1 + r\delta\tau)\mathbf{I} - \frac{1}{2}r\delta\tau\mathbf{D}_1\mathbf{T}_1 - \frac{1}{2}\mathbf{\Sigma}_{m+1}\delta\tau\mathbf{D}_2\mathbf{T}_2.$$

- Now, define the new boundary condition vectors

$$\mathbf{b}_m = \begin{bmatrix} \frac{1}{2}\delta\tau(S_{\min}/\delta s + 1)(\sigma^2(t_m, S_{\min} + \delta s)(S_{\min}/\delta s + 1) - r)U_m^0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\delta\tau(S_{\max}/\delta s - 1)(\sigma^2(t_m, S_{\max} - \delta s)(S_{\max}/\delta s - 1) + r)U_m^N \end{bmatrix},$$

with $U_m^0 = U^0(m\delta\tau, S_{\min})$ and $U_m^N = U^\infty(m\delta\tau, S_{\max})$ specified as before.
- Then, under appropriate initial and boundary conditions, we may compute the solution for the PDE (8) at time $m+1$, in terms of the solution at time m, $(\mathbf{U}_m)$, as

$$\mathbf{U}_{m+1} = (\theta\mathbf{G}_{m+1} + (1-\theta)\mathbf{I})^{-1}[((1-\theta)\mathbf{F}_m + \theta\mathbf{I})\mathbf{U}_m + (1-\theta)\mathbf{b}_m + \theta\mathbf{b}_{m+1}].$$

Example: Finite-difference Solution for a CEV Put Option

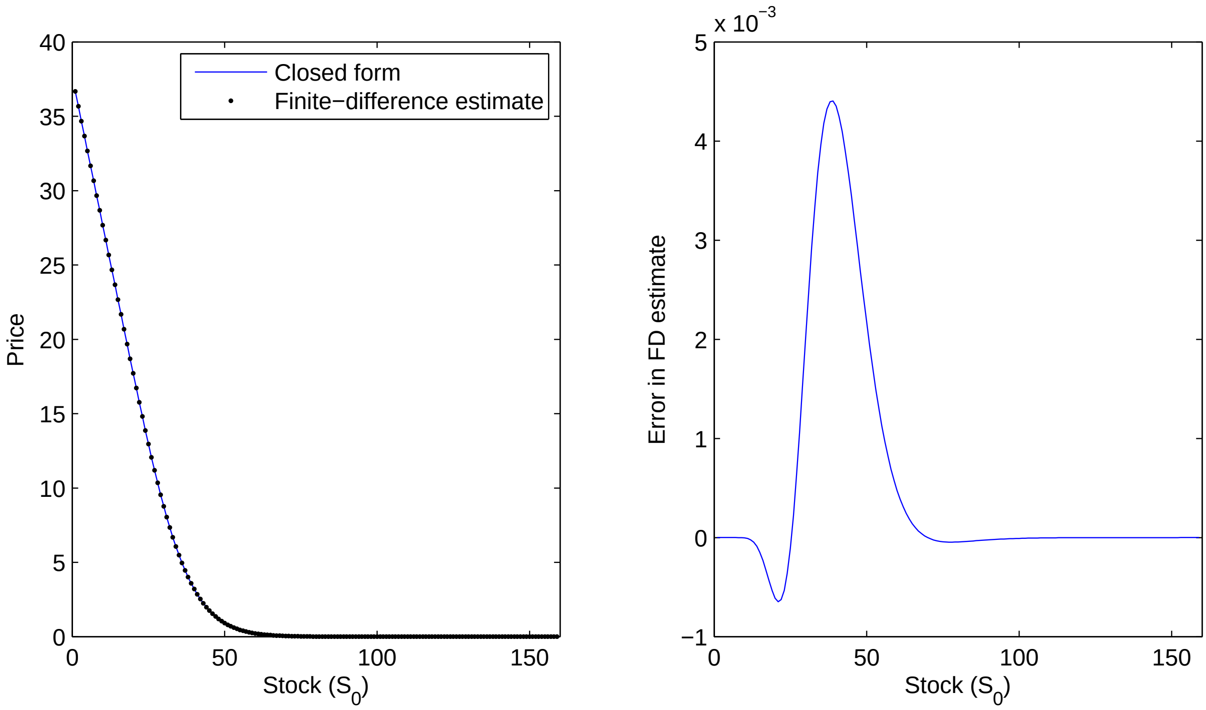
- Using the option related parameters

$$\sigma = 40\%, \quad \alpha = 0.9, \quad r = 6\%, \quad K = 40, \quad T = 1,$$

and the mesh related parameters

$$S_{\min} = 0, \quad S_{\max} = 160, \quad M = 80, \quad N = 160 \quad \text{and} \quad \theta = 0.5,$$

the finite-difference prices for the CEV put option were compared to analytical prices. The left graph below shows finite-difference and analytical prices as a function S_0, while the right graph shows the error.



Local Volatility as a Function of Implied Volatility

- In this section, we follow the approach of Gatheral (2006) very closely.
- Suppose the market prices of options are quoted in terms of the current spot price S_0 and the Black-Scholes implied volatilities $\sigma_{\mathsf{BS}}(T,K)$

$$C(T,K) = C_{\mathsf{BS}}(S_0, K, \sigma_{\mathsf{BS}}(T,K), T)$$

- Using a change of variables

$$w(T,K) = \sigma^2_{\mathsf{BS}}(T,K)T \qquad \text{and} \qquad y = \log\left(\frac{K}{F_T}\right),$$

where w is called the Black-Scholes implied total variance, we can write the Black-Scholes equation as

$$\begin{aligned} C_{\mathsf{BS}}(F_T, y, w) &= F_T[\Phi(d_1) - e^y\Phi(d_2)] \\ &= F_T\left[\Phi\left(-\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right) - e^y\Phi\left(-\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\right], \end{aligned}$$

and the Dupire equation (dropping arguments) as

$$\frac{\partial C}{\partial T} = \frac{v_L}{2}\left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right] + rC,$$

where $v_L = \sigma^2(T,K)$ is the local variance.

- Now, the partial derivatives in the Dupire equation can be expressed in terms of partial derivatives of the Black-Scholes equation as follows

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial C_{\mathsf{BS}}}{\partial y} + \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial y} \\ \frac{\partial^2 C}{\partial y^2} &= \frac{\partial^2 C_{\mathsf{BS}}}{\partial y^2} + 2\frac{\partial^2 C_{\mathsf{BS}}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial^2 C_{\mathsf{BS}}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial^2 w}{\partial y^2} \\ \frac{\partial C}{\partial T} &= \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial T} + \frac{\partial C_{\mathsf{BS}}}{\partial F_T}\frac{\partial F_T}{\partial T} = \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial T} + rC_{\mathsf{BS}}. \end{aligned}$$

- Thus, the Dupire equation can be written as

$$\begin{aligned} \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial T} = \frac{v_L}{2}\Bigg[&\frac{\partial^2 C_{\mathsf{BS}}}{\partial y^2} - \frac{\partial C_{\mathsf{BS}}}{\partial y} - \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial y} \\ &+ 2\frac{\partial^2 C_{\mathsf{BS}}}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial^2 C_{\mathsf{BS}}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial^2 w}{\partial y^2}\Bigg]. \end{aligned}$$

- Taking partial derivatives of the Black-Scholes equation (verify) we have

$$\begin{aligned} \frac{\partial^2 C_{\mathsf{BS}}}{\partial w^2} &= \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\frac{\partial C_{\mathsf{BS}}}{\partial w} \\ \frac{\partial^2 C_{\mathsf{BS}}}{\partial y \partial w} &= \left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial C_{\mathsf{BS}}}{\partial w} \\ \frac{\partial^2 C_{\mathsf{BS}}}{\partial y^2} - \frac{\partial C_{\mathsf{BS}}}{\partial y} &= 2\frac{\partial C_{\mathsf{BS}}}{\partial w} \end{aligned}$$

- Substituting these expressions into the equation above, we get

$$\frac{\partial C_{\mathsf{BS}}}{\partial w}\frac{\partial w}{\partial T} = \frac{v_L}{2}\frac{\partial C_{\mathsf{BS}}}{\partial w}\left[2 - \frac{\partial w}{\partial y} + 2\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial w}{\partial y} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2}\right].$$

- Solving for v_L, we get

$$v_L = \frac{\frac{\partial w}{\partial T}}{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}}.$$

- This specification allows the computation of local volatility in terms of Black-Scholes implied volatilities.
- As was the case with the application of the Dupire equation, usually it is applied to either interpolated/extrapolated data or to a fitted parametric model of the implied volatility surface.
- Again, care must be taken to ensure that such interpolations or parametrisations are arbitrage free.