Numerical Methods in Finance II
Lecture 5 - Counterparty Credit Risk

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Introduction

- There are generally two types of problem associated with counterparty risk. The first is related to risk-management and asks the question "What are the loss exposures that result from counterparty default?" The second, and the one we mainly deal with in this module, is "How do we price credit-sensitive instruments?"
- We start by introducing some definitions for counterparty risk.
- We then introduce the general notion of a Credit Valuation Adjustment (CVA). To start with, we do not specify a model of the default of the counterparty.
- The first model of default that we explore is the structural model of Merton. This relies on modeling the value of the firm in relation to its debt obligations.
- Based on this model, we price the CVA of a simple instrument (a call option) and illustrate the phenomenon of wrong-way and right-way risk.
- We next look at a simple hazard rate model based on exponentially distributed default times. A simplistic sketch of how this approach has been extended is given.

Definitions

- Counterparty credit risk is the risk that the counterparty to a financial contract will default prior to the expiration of the contract and will be unable to honour the payments required by the contract.
- Only privately negotiated contracts are subject to the risk (i.e., over-the-counter derivatives and security financing transactions). Exchange traded derivatives are not affected because risk is underwritten by the exchange.
- Counterparty risk is similar to other forms of credit risk in that the cause of the financial loss is the obligor's default. It differs in that the risk is bilateral in nature.
- When estimating counterparty credit exposure (risk-measurement), the following definitions are applicable (Canabarro and Duffie (2003)):
  - *Counterparty exposure*: The larger of zero and the market value of positions with a counterparty.
  - *Current exposure (CE)*: The current value of the counterparty exposure.
  - *Potential future exposure (PFE)*: The maximum exposure to occur on a future date with a certain degree of significance (quantile based).
  - *Expected exposure (EE)*: The mean exposure on a future date.
  - *Expected positive exposure (EPE)*: The mean EE over an interval (1 year).

  Credit exposure should be computed under the real-world measure.
- We will, however, deal with the issue of incorporating default exposure into the problem of pricing instruments. (NB: we only explore CVA, ignoring DVA, FVA, etc.)

Credit Valuation Adjustment

- *Credit Valuation Adjustment* (*CVA*) is defined as the difference between the value of an instrument (portfolio) free of counterparty credit risk and the true value of the instrument taking into account the possibility of counterparty default. In other words, CVA is the market value of counterparty risk.
- Let us assume that τ is a stopping time that indicates the event that a counterparty defaults on its obligations, and assume the pay-off of an instrument held with that party is given as X_T for some terminal time.
- Then the value of the claim (X_0) is the usual risk-neutral expectation of the discounted pay-off

$$\begin{aligned}
X_0 &= \mathbb{E}^{\mathbb{Q}}[e^{-rT}X_T] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T\mathbb{I}_{\{\tau>T\}} + e^{-rT}X_T\mathbb{I}_{\{\tau\leq T\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T\mathbb{I}_{\{\tau>T\}} + e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-\tau)}X_T\middle|\mathscr{F}_\tau\right]\mathbb{I}_{\{\tau\leq T\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T\mathbb{I}_{\{\tau>T\}} + e^{-r\tau}X_\tau\mathbb{I}_{\{\tau\leq T\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T\mathbb{I}_{\{\tau>T\}} + e^{-r\tau}\delta X_\tau\mathbb{I}_{\{\tau\leq T\}} + e^{-r\tau}(1-\delta)X_\tau\mathbb{I}_{\{\tau\leq T\}}\right],
\end{aligned}$$

  where $\delta \in [0,1]$ is the *recovery rate*, and correspondingly $\text{LGD} = (1-\delta)$ is the *loss given default*.

- The final term in the expectation is the value that is lost due to default of the counterparty. Thus, by the definition of CVA, we have

$$\begin{aligned} \mathsf{CVA} &= X_0 - \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}X_T\mathbb{I}_{\{\tau>T\}} + e^{-r\tau}\delta X_\tau\mathbb{I}_{\{\tau\le T\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}(1-\delta)X_\tau\mathbb{I}_{\{\tau\le T\}}\right]. \end{aligned} \tag{1}$$

- Now that we have the general form for CVA, it remains for us to specify models for the default times τ.
- These models should, of course, be coherent with other market observables that give an indication of the probability of default, for example corporate bond spreads or the prices of credit default swaps.
- From now on, as a notational convenience, we shall assume a risk-neutral underlying and drop the superscript $\mathbb{Q}$ from all expectations.

The Merton Model

- Very shortly after the Black-Scholes formula was published, Merton made a first attempt to model corporate debt assuming that default is possible.
- While more complicated models are possible, we shall keep things simple and assume that default can only happen at terminal time, no coupons are issued, the risk-free rate, r, is constant and the value of the firm, V_t, follows the SDE

$$dV_t = rV_t\,dt + \bar{\sigma}V_t\,dW_t.$$

- If $\bar{B}$ is the face value of bonds that must be repaid at maturity T, then the price of the corporate bond incorporating the possibility of default is the discounted risk-neutral expectation of the payoff

$$\min(V_T, \bar{B}) = \bar{B} - (\bar{B} - V_T)^+.$$

- Thus, the corresponding corporate bond is priced at time $t < T$ as

$$B(t,T) = \bar{B}e^{-r(T-t)} - P(t, V_t, \bar{\sigma}, T, \bar{B}, r),$$

where $P(\cdot)$ is the Black-Scholes equation for a put option.

- Expanding this expression, we get
$$B(t,T) = V_t\Phi(-\bar{d}_1) + \bar{B}e^{-r(T-t)}(1-\Phi(-\bar{d}_2)) \\ = V_t\Phi(-\bar{d}_1) + \bar{B}e^{-r(T-t)}\Phi(\bar{d}_2), \tag{2}$$
where
$$\bar{d}_1 = \frac{\log(V_t/\bar{B}) + (r+\bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}} \quad \text{and} \quad \bar{d}_2 = \bar{d}_1 - \bar{\sigma}\sqrt{T-t}$$
as usual.
- Thus, letting
$$B(t,T) = \bar{B}e^{-\bar{r}(T-t)},$$
where $\bar{r}$ represents the yield of the corporate bond, we can use (2) to compute the spread on the corporate debt as
$$\bar{r} - r = \frac{-1}{T-t}\log\left(\frac{1}{D_t}\Phi(-\bar{d}_1) + \Phi(\bar{d}_2)\right),$$
where $D_t = \bar{B}e^{-r(T-t)}/V_t$ is the debt-equity ratio at time t.
- Note that the spread only depends on D_t, the time to maturity of the bond, the volatility and the risk-free rate. Thus, in this simplistic model, we can "calibrate" $\bar{\sigma}$ with a knowledge of the price of the corporate debt and the debt-equity ratio of the firm.
- Recall (or derive as an exercise) that the probability of default is
$$\text{PD} = \mathbb{E}\left[\mathbb{I}_{\{V_T<\bar{B}\}}\right] = \mathbb{P}(V_T < \bar{B}) = \Phi(-\bar{d}_2).$$

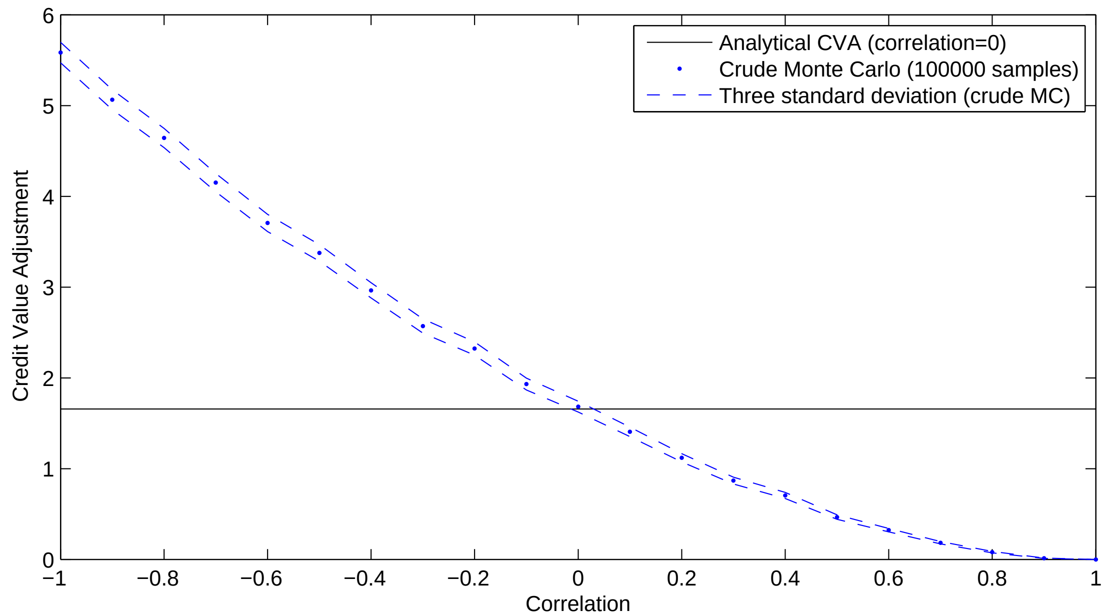
Computing CVA under the Merton Model

- As mentioned before, we only assume that default happens at maturity. Thus, we have the following relationship between default time and value of the firm
$$\mathbb{I}_{\{\tau\le T\}} = \mathbb{I}_{\{\tau=T\}} = \mathbb{I}_{\{V_T<\bar{B}\}}.$$
- Substituting this expression into (1) gives
$$\text{CVA} = e^{-rT}\mathbb{E}\left[(1-\delta)X_T\mathbb{I}_{\{V_T<\bar{B}\}}\right].$$
- Thus, if the process underlying X is independent of the process underlying the value of the firm (V), we can compute the CVA in closed form as
$$\text{CVA} = (1-\delta)e^{-rT}\mathbb{E}[X_T]\Phi(-\bar{d}_2) \\ = (1-\delta)X_0\text{PD},$$
where X_0 is the risk-neutral value of the claim.
- If, on the other hand, the processes are correlated, then we must compute the CVA using Monte Carlo evaluation.

Example: Wrong-way and Right-way Risk

- Wrong-way (resp. right-way) risk is the risk exposure that is negatively (resp. positively) correlated with the credit quality of the counterparty.
- Using the structural model, we now provide an example of the phenomenon.
- Consider an OTC call option that has been purchased from a risky counterparty. The call option is written on a stock with $S_0 = 50$, $\sigma = 0.25$, $K = 55$ and matures at $T = 2$.
- At time $t = 0$ the firm value of the counterparty is estimated to be $V_0 = 100$, while at time $T = 2$ the counterparty is obligated to pay back the face value of $\bar{B} = 75$ in (zero coupon) bonds. The counterpart's current bond yield is trading at a 2.5% spread over the risk-free rate $r = 10\%$.
- Based on these parameters, it is possible to estimate $\bar{\sigma} = 0.34$. The probability of default is then PD $= 0.22$ and the correlation-free CVA is 1.656.
- We now generate (100000) Monte Carlo realisations of the value of the firm and the stock underlying the option for a range of correlations and compute corresponding estimates for the CVA assuming that the recovery rate $\delta = 20\%$.

- The graph shows the Monte Carlo estimates for CVA, with corresponding three standard deviation bounds, as a function of correlation.
- For correlations close to -1 the CVA is high, indicating significant wrong-way risk, while for correlations close to 1 the CVA is low, indicating right-way risk. This risk occurs because the option is more likely to be in (resp. out) the money when default occurs.



A Recap on Exponential Random Variables

- A random variable τ with the density
$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$
where $\lambda \in \mathbb{R}^+$, is known as an *exponential random variable*.
- The expected value of τ can be computed using integration by parts
$$\begin{aligned}\mathbb{E}[\tau] &= \int_0^\infty t f(t)\, dt = \lambda \int_0^\infty t e^{-\lambda t}\, dt \\ &= -te^{-\lambda t}\Big|_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t}\, dt = 0 - \frac{1}{\lambda} e^{-\lambda t}\Big|_{t=0}^{t=\infty} = \frac{1}{\lambda}.\end{aligned}$$
- The cumulative distribution function (on $t \geq 0$) is given by
$$F(t) = \mathbb{P}(\tau \leq t) = \int_0^t \lambda e^{-\lambda s}\, ds = -e^{\lambda s}\Big|_{s=0}^{s=t} = 1 - e^{-\lambda t}.$$
- The survival function is given by $L(t) = \mathbb{P}(\tau > t) = 1 - F(t) = e^{-\lambda t}$.
- Using the inverse transform method, exponential random variables may be generated with
$$\tau = -\frac{\log(U)}{\lambda},$$
where $U \sim \mathcal{U}[0,1)$ is a uniformly distributed random variable.

A Hazard Rate Model

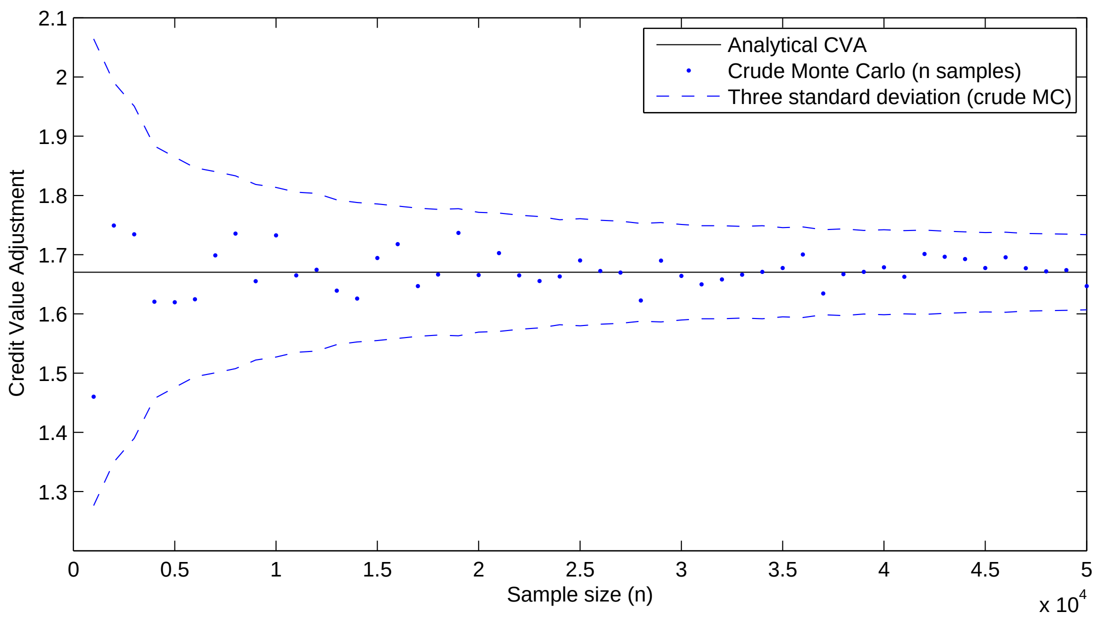
- There are a number of problems with structural models, including the fact that they arbitrarily assume that the value of the firm follows a continuous process and, as a result, the default characteristics of the model behave quite differently from those that are experienced in reality. In particular, corporate bond spreads do not tend to zero as time to maturity tends to zero, indicating that the market always assumes some intrinsic risk of default.
- Hazard rate models (also called reduced form models) are characterised by the fact that they attempt to model the default process directly using exponential times (Poisson processes) without reference to the value of the firm.
- The idea is to estimate the hazard of default (λ) and then model default times as exponential times τ.
- The CVA of an instrument can then be computed by directly using (1).
- The probability of default in this case is
$$\text{PD} = \mathbb{E}\left[\mathbb{I}_{\{\tau \leq T\}}\right] = \mathbb{P}(\tau \leq T) = 1 - e^{-\lambda T}.$$
- This approach can be easily integrated into the Monte Carlo approach.

Monte Carlo Simulation of a Hazard Model

- Monte Carlo simulation requires the following steps
  1. Simulate the times τ at which a default occurs using
  $$\tau = -\frac{\log(U)}{\lambda},$$
  where $U \sim \mathcal{U}[0,1)$ and λ is the hazard rate.
  2. Let $\tilde{t} = \min(\tau, T)$ and simulate risk-neutral realisations of the value of the instrument at $X_{\tilde{t}}$ using these times.
  3. Now, compute the CVA using the following version of (1)
  $$\text{CVA} = \mathbb{E}\left[e^{-r\tilde{t}}(1-\delta)X_{\tilde{t}}\mathbb{I}_{\{\tau \le T\}}\right],$$
  or, alternatively, using
  $$\text{CVA} = X_0 - \mathbb{E}\left[e^{-r\tilde{t}}X_{\tilde{t}}\mathbb{I}_{\{\tau > T\}} + e^{-r\tilde{t}}\delta X_{\tilde{t}}\mathbb{I}_{\{\tau \le T\}}\right],$$
  where X_0 is the risk-neutral value of the instrument excluding any counterparty credit risk.
- As an example, we now consider the OTC call option with the same parameters as before ($S_0 = 100$, $\sigma = 0.25$, $K = 100$ and $r = 0.10$). In this case
$$X_{\tilde{t}} = (S_{\tilde{t}} - K)^+\mathbb{I}_{\{\tau > T\}} + \text{BS}(S_{\tilde{t}}, K, \sigma, r, T, \tilde{t})\mathbb{I}_{\{\tau \le T\}},$$
where BS is the Black-Scholes value of the option and $X_0 = \text{BS}(S_0, K, \sigma, r, T, 0)$.

Example CVA under a Hazard Model

- Assume that our default-prone counterparty has a hazard rate of $\lambda = 0.125$ and that the recovery rate is 20%.
- The following graph shows the CVA for the call option as a function of Monte Carlo sample size.



Extension to a Non-Homogeneous Hazard Rate

- In the previous example, we showed how we might generate default times using an exponential time, where the hazard rate λ is a constant.
- To make things more realistic, we would like to extend this model to the case where the hazard rate is a function of time or even a stochastic process.
- While we do not give proofs here, it can be shown that the survival function in terms of $\lambda(t)$ is given by

$$L(t) = \mathbb{P}(\tau > t) = \exp\left(-\int_0^t \lambda(s)\,ds\right).$$

- This, of course, gives us a way of computing random times, as long as the integral can be computed analytically or numerically.
- More importantly, it gives us a way of calibrating the hazard rate. Notice that the survival function looks like the price of a bond.

- Under the simplistic assumptions that the recovery rate is zero and that the hazard rate $\lambda(t)$ and short rate are independent, then the value of a corporate bond is

$$\begin{aligned}
B(0,T) &= \mathbb{E}\left[\mathbb{I}_{\{\tau>T\}}\exp\left(-\int_0^T r(t)\,dt\right)\right] \\
&= \mathbb{P}(\tau > T)\mathbb{E}\left[\exp\left(-\int_0^T r(t)\,dt\right)\right] \\
&= \exp\left(-\int_0^T \lambda(t)\,dt\right)\mathbb{E}\left[\exp\left(-\int_0^T r(t)\,dt\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\int_0^T r(t)+\lambda(t)\,dt\right)\right].
\end{aligned}$$

  These assumptions can be relaxed, but at the cost of added complexity.
- Thus the hazard rate function may be calibrated by looking at the spread associated with corporate debt

$$\bar{r}(t) - r(t) = \lambda(t).$$

- Moreover, because of the form of $\lambda(t)$, we may even produce a stochastic model for the "term-structure" of the hazard rate. Of course, we would need to use models that guarantee positivity like the CIR model (default probabilities must, after all, be positive).