M.Phil. in Mathematical Finance

Numerical Methods in Finance II 2020

Assignment 1

Instructions

- You are required to submit two Matlab functions, named `CalibrateHullWhite` and `PriceCaplet`. Please use these exact names. Do not submit any other files; all your code should be included in these files — if you wish to use other functions, use subfunctions.
- Submission is by email to `tom@analytical.co.za` or `Thomas.McWalter@gmail.com`.
- The due date for the assignment is 12 noon on Monday the 16th of November — marks will be deducted for late submission (5% for every half hour, or part thereof, that the assignment is late). The arrival time stamp of the email will be used to decide if submission is late or not. Absolutely no excuses accepted — so, rather submit early.
- Absolutely no coding together or copying is allowed. If such activity is detected, all individuals involved will receive a mark of zero and disciplinary action will be taken.
- To qualify for speed and accuracy marks (below) your calibration results must be within 1% of the real parameters (α and σ) used to generate the data given.
- Marks (20%) are allocated for speed and efficiency — The run-times of `PriceCaplet` will be ranked. The mark (out of 20) assigned for speed will be calculated as $(21 - n)$ where n is the ranking (e.g. if the code was ranked 3rd fastest then 18 marks will be allocated out of 20).
- Marks (20%) are allocated for accuracy. Accuracy (and run-times) will be determined by calling your function 1000 times with random time indices and strikes, and will be computed as the sum of absolute differences between your results and known values (using the real parameters α and σ).

The Hull White Model

Given the zero coupon bond and caplet prices for various dates (see the data section below) you will perform the tasks necessary to calibrate the Hull-White model, which entails finding estimates of the parameters α and σ. As per the notes, under this model, the time-t_1 price of a zero coupon bond maturing at t_2 is

$$B(t_1, t_2) = e^{-A(t_1,t_2)r_1 + C(t_1,t_2)}$$

with r_1 being the short rate at time t_1,

$$A(t_1, t_2) = \frac{1}{\alpha}\left(1 - e^{-\alpha(t_2 - t_1)}\right)$$

and

$$C(t_1, t_2) = \log\left(\frac{B^*(0, t_2)}{B^*(0, t_1)}\right) + f^*(0, t_1)A(t_1, t_2) - \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha t_1}\right)A(t_1, t_2)^2.$$

Here, $B^*(0, T)$ for $T \in \{T_0, T_1, \dots T_{N+1}\}$, with $T_0 = 0$, are the current prices of zero coupon bonds in the market and $f^*(0, t) = -\frac{\partial}{\partial t}\log(B^*(0, t))$ are the current forward rates for time t.

Now, consider the current price of a caplet with maturity T_i and strike K given by

$$\text{Caplet}(T_i, K, r_0, \alpha, \sigma) = (1 + K\delta_i)\text{Put}(T_i, T_{i+1}, 1/(1 + K\delta_i), r_0, \alpha, \sigma)$$

where $\delta_i = T_{i+1} - T_i$ and

$$\text{Put}(t_1, t_2, \kappa, r_0, \alpha, \sigma) = \kappa B(0, t_1)\Phi(-d_2) - B(0, t_2)\Phi(-d_1)$$

with

$$d_1 = \frac{\log\left(\frac{B(0,t_2)}{B(0,t_1)\kappa}\right) + \frac{\bar{\sigma}^2}{2}}{\bar{\sigma}} \qquad \text{and} \qquad d_2 = d_1 - \bar{\sigma}$$

and

$$\bar{\sigma} = \sigma\sqrt{\frac{1-e^{-2\alpha t_1}}{2\alpha}}A(t_1,t_2)$$

To calibrate the parameters α and σ in the Hull-White model, we can use the known values of caplets with an optimization procedure.

Tasks

- Write a Matlab script (or function, if you require sub-functions) named `CalibrateHullWhite` that computes estimates for α and σ using the following steps:

  1. Using the Bond prices, $B^*(0,T_i)$ for $i = 0, 1, \dots, N+1$, given in the data section below, compute numerical estimates for $f^*(0,T_i)$ for $i = 1, 2, \dots, N$. Note that $f^*(0,0) = r_0 = 0.07$.
  2. Set up closed-form functions for pricing bonds, put options on bonds and caplets. Ensure that these functions take α and σ as inputs. **Hint:** Since we will only be pricing instruments at times T_i, it may be beneficial to use time indices (i) as inputs to these functions, rather than the actual times, in order to compute the functions required—this way you can easily reference the values of $B^*(0,T_i)$ and $f^*(0,T_i)$. Remember that Matlab indexes start from 1, but i may be 0.
  3. Set up an objective function that takes a single vector, say `x`, containing α and σ as the first and second elements and returns the computed sum of the squared differences between the input caplet prices (given in the data section below, which have strike $K = 0.05$) and the caplet prices computed using the formulae above (in terms of α and σ).
  4. Use a Matlab minimization routine of your choice to minimize the objective function and thereby find estimates for α and σ. You can experiment with various optimization parameters and even change the objective function if you like in order to produce the lowest difference between the input and computed caplet prices.

  To verify that your caplet pricing function in step 2 above is working correctly, you can price the caplets with maturities and strikes given in Table 1.

- Now, using your calibrated parameters and the data provided, write a second function, named `PriceCaplet`, that returns the price of a caplet using Monte Carlo simulation (you may hard-code your estimates for α and σ as well as the values for B^* and f^* into this function). The inputs to the function should be the index i, which takes values in the range $\{1, 2, \dots, N-1\}$ corresponding to the maturity of the caplet (T_i), and the strike K. Your script should use the joint simulation technique in the notes, with certain modifications highlighted below. Although it is not required, you may use variance reduction or quasi-Monte Carlo techniques if you like, but your function may not use the analytical formula for a caplet. The caplet should be priced using risk-neutral pricing as

  $$\text{Caplet}(T_i, K) = \mathbb{E}\left[\frac{\delta_i}{\beta(T_{i+1})}\max\left(\frac{1}{\delta_i}\left(\frac{1}{B(T_i,T_{i+1})} - 1\right) - K\right)\right],$$

  where $B(T_i, T_{i+1})$ is the time-T_i realization of the bond maturing at T_{i+1} and $1/\beta(T_{i+1})$ is a corresponding realization of the discount factor at time T_{i+1}. This is because the caplet value is fixed at maturity, but paid in arrears. Thus, you will need to simulate jointly evolved quantities at both times T_i and T_{i+1}.

  Because we are simulating under the Hull-White model, the following two quantities used to perform joint simulation are different from those in the notes

  $$\mu_r(t_i, t_{i+1}) = e^{-\alpha(t_{i+1}-t_i)}r(t_i) + D(t_{i+1}) - e^{-\alpha(t_{i+1}-t_i)}D(t_i),$$

  with

  $$D(t) = \frac{\sigma^2}{2\alpha^2}\left[1 - e^{-\alpha t}(1 + \alpha A(0,t))\right] + f^*(0,t) - r_0 e^{-\alpha t}$$

  and

  $$\mu_Y(t_i, t_{i+1}) = Y_i + A(t_i, t_{i+1})r_i + \tfrac{1}{2}\sigma_Y^2(t_i, t_{i+1}) - C(t_i, t_{i+1});$$

  while $\sigma_r^2(t_i, t_{i+1})$ given by equation (4), $\sigma_Y^2(t_i, t_{i+1})$, $\sigma_{rY}(t_i, t_{i+1})$ and $\rho_{rY}(t_i, t_{i+1})$ are the same.

For the purposes of testing, you may assume that $i \in \{1, 2, \dots, N-1\}$ and $0.02 \le K \le 0.08$, and you can test the accuracy of your function using the closed-form functions you used to calibrate the model.

	$i = T_i = 1$	$i = T_i = 10$	$i = T_i = 19$
$K = 0.03$	0.04182	0.02553	0.01294
$K = 0.05$	0.02854	0.01995	0.00994
$K = 0.07$	0.01792	0.01522	0.00745

Table 1: Some example values of caplets (rounded), computed using the parameters $\alpha = 0.15$ and $\sigma = 0.05$. Note these are not the actual values of α and σ.

Bond and Caplet Data for Calibration

```
N=20;
T_i=0:N+1;    % Times T_0 to T_{N+1}

% Bond prices for T_0 to T_{N+1}
Bstar=[1.000000000000000 0.931733068514009 0.867336864194859 0.807180507606257 0.751390186399840 ...
       0.699924481765438 0.652631479135652 0.609290924427683 0.569644566898682 0.533417438303370 ...
       0.500332340416184 0.470119346466237 0.442521711052162 0.417299242183633 0.394229917259718 ...
       0.373110314043405 0.353755267713142 0.335997045644106 0.319684243642614 0.304680543400063 ...
       0.290863424934260 0.278122895071373];

% Caplet prices with strike K=0.05 and maturities T_1 to T_{N-1}
Caplets=[0.022922737518613 0.023535825778635 0.023194289857391 0.022294316394003 0.021086407641792 ...
         0.019728461392470 0.018320943715610 0.016927232296721 0.015586172322853 0.014320233052482 ...
         0.013140971196705 0.012052749925964 0.011055287463489 0.010145403965974 0.009318213062292 ...
         0.008567926766803 0.007888391002214 0.007273433858145 0.006717084341419];
```