M.Phil. in Mathematical Finance

Numerical Methods in Finance II 2020

Tutorial 5 (Counterparty Credit Risk)

Problems

In this tutorial you will price the CVA for a call option written by a credit-risky counterparty using both structural and intensity models. We assume that we are default free and thus compute only the Unilateral CVA. The parameters for the call option are

$$S_0 = 50, \quad K = 55, \quad \sigma = 25\%, \quad r = 10\% \quad \text{and} \quad T = 2.$$

1. In this problem we demonstrate the phenomenon of wrong-way risk using a Structural Model (Merton) for default.

1a. Under the structural approach we model the assets of the risky counterparty as a geometric Brownian motion. To simplify things, we consider the case where default occurs only at maturity. Essentially, default occurs if the value of the counterparty's assets is less than the value of debt outstanding. Assume the company value at inception is $V_0 = 100$ and that it has an obligation to pay back the face value of $\bar{B} = 75$ in (zero coupon) corporate bonds at time T. The company's bonds are trading at a spread of 2.5% on the risk-free rate. Use the Merton spread formula and the Matlab command `fzero` to find an estimate for $\bar{\sigma}$.

Now, generate terminal stock realisations, S_T, using the stock parameters as above, along with correlated realizations of the terminal asset value of the counterparty, V_T. Since this is a two dimensional problem, use the explicit formula $z_v = \rho z_1 + \sqrt{1-\rho^2} z_2$, where z_1 and z_2 are independent normal random variables. Then z_1 can be used to drive the stock price while z_v is used to drive the company asset value.

As a function of the ρ in the range `-1:0.1:1` and for a sample size of $n = 100000$, plot Monte Carlo estimates for the probability of default

$$\mathbb{E}\left[\mathbb{1}_{\{V_T < \bar{B}\}}\right].$$

Compare this with three standard deviation bounds around the analytical value for the probability of default given by

$$\text{PD} = \Phi(-\bar{d}_2), \qquad \text{where} \qquad \bar{d}_2 = \frac{\ln(V_0/\bar{B}) + (r - \bar{\sigma}^2/2)T}{\bar{\sigma}\sqrt{T}}.$$

For debugging purposes, initialize your seed with the Matlab command `rng(0)` and generate your z values using `z=randn(2,n)`.

1b. Now in a separate figure compute Monte Carlo estimates for the CVA as a function of correlation by evaluating

$$\text{CVA} = \mathbb{E}\left[e^{-rT}(1-\delta)(S_T - K)^+ \mathbb{1}_{\{V_T < \bar{B}\}}\right],$$

where $\delta = 20\%$ is the recovery ratio. Plot three standard deviation bounds around these estimates (since, we do not have an analytical value for correlated assets). For reference, plot a line indicating the analytical value of the CVA for the case where the processes are independent ($\rho = 0$).

As you can see, for correlations close to -1 the CVA is high, indicating significant wrong-way risk, while for correlations close to 1 the CVA is low, indicating right-way risk. Note that the probability of default does not change as a function of correlation (as illustrated in part 1a.), the wrong-way (resp. right-way) risk occurs because the option is more likely to be in (resp. out) the money when default occurs.

2. In this problem you will price the CVA for the same call option (with the same stock parameters) using a hazard rate approach.

2a. Start by generating default times
$$\tau = -\frac{\ln(U)}{\lambda},$$
where $U \sim \mathscr{U}[0,1)$ is a uniformly distributed random variable and $\lambda = 0.125$ is the arrival rate.

As a function of sample size, for sample sizes in the range $1000, 2000, \dots, 50000$, plot a Monte Carlo estimate for the probability of default
$$\mathbb{E}\left[\mathbb{1}_{\{\tau \le T\}}\right].$$
Compare these estimates to the analytical value given by
$$\mathsf{PD} = \mathbb{P}(\tau \le T) = 1 - e^{-\lambda T}$$
and produce a three standard deviation error bound. For debugging purposes, initialize your seed with the Matlab command `rng(0)`.

2b. Now, using the algorithm in the notes, augment your code to compute an estimate of the CVA for the call option written by the default-prone counterparty. Compute the CVA as the discounted expected loss (under the risk-neutral measure)
$$\mathsf{CVA} = \mathbb{E}\left[e^{-r\tilde{t}}(1-\delta)\mathsf{BS}(S_{\tilde{t}}, K, \sigma, r, T, \tilde{t})\mathbb{1}_{\{\tau \le T\}}\right],$$
where $\mathsf{BS}(\cdot)$ is the Black-Scholes value of the call option, and $\delta = 20\%$ as before. **Hint:** Create an inline function for the Black-Scholes formula that can take stock and time vectors as input. Recall that
$$\mathsf{BS}(S_0, K, \sigma, r, T, t) = S_0\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$
with
$$d_1 = \frac{\log(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{(T-t)}.$$
In a new figure, plot Monte Carlo estimates of the CVA as a function of sample size with three standard deviation error bounds as compared to the analytical estimate given by
$$\mathsf{BS}(S_0, K, \sigma, r, T, 0)(1 - e^{-\lambda T})(1-\delta).$$

2c. (Homework) You can compute an alternative estimate for the CVA by computing an estimate for the price of the risky option and subtracting this from the Black-Scholes price of the option:
$$\mathsf{CVA} = \mathsf{BS}(S_0, K, \sigma, r, T, 0) - C,$$
where C is the Monte Carlo estimate of the price of the risky option given by
$$C = \mathbb{E}\left[e^{-r\tilde{t}}(S_{\tilde{t}} - K)^+\mathbb{1}_{\{\tau > T\}} + e^{-r\tilde{t}}\delta\mathsf{BS}(S_{\tilde{t}}, K, \sigma, r, T, \tilde{t})\mathbb{1}_{\{\tau \le T\}}\right].$$
Interestingly, if you plot the error bounds for this estimate, you will see that they are a lot larger than the error bounds for the estimate in 1b above. Can you think why?

2d. (Homework) Can you think of a way to evaluate the CVA in parts 2b and 2c above without making use of the Black-Scholes formula? How does this affect error estimates?