Numerical Methods in Finance II

Lecture 8 — Short Rate Models

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Introduction

- In this module, we look at models for the dynamics of the instantaneous continuously compounded short rate.
- We mostly concentrate on the model of Vasicek with the corresponding extensions by Hull and White.
- Initially we focus on generating realisations of the short rate under this model.
- We then develop a general bond pricing formula.
- This is followed by a brief discussion of how the Hull-White extension may be calibrated to bond market data.
- Since discount factors are correlated with the short rate realisations, we then present a discretisation-error free way of generating both. These are compared to discount factors computed using quadrature.
- We end with a brief presentation of the Cox-Ingersoll-Ross model, showing how short rate realisations can be generated using a transition density approach. Discount factors are computed using quadrature.

Gaussian Short Rate Models

- We model the dynamics of the instantaneous continuously compounded short rate $r(t)$.
- A unit of currency at time 0, attracting the rate $r(t)$, grows in value to

$$\beta(t) = \exp\left(\int_0^t r(u)\,du\right)$$

at time t. We now assume that this is the numeraire (or reciprocal of the discount factor) used in derivative pricing.
- This means that the price of a derivative security at time 0, X_0, with payoff X_T at time T is given by the expectation

$$X_0 = \mathbb{E}\left[\frac{1}{\beta(T)} X_T\right],$$

assuming that the asset underlying X grows at the risk-neutral rate $r(t)$.
- In particular, the price of a "risk-free" bond that pays $X_T = 1$ is given by

$$B(0,T) = \mathbb{E}\left[\exp\left(-\int_0^T r(u)\,du\right)\right]. \tag{1}$$

The Vasicek and Hull-White Models

- The model of Vasicek describes the short rate using the Ornstein-Uhlenbeck process

$$dr(t) = \alpha(b - r(t))\,dt + \sigma\,dW_t,$$

where W_t is a standard Brownian motion, and $\alpha, b, \sigma \in \mathbb{R}^+$ represent the *rate of mean reversion*, the *mean reversion level* and the *volatility* respectively.
- To ensure exact calibration to the bond, cap and swaption prices, Hull and White generalised the above SDE to have time varying parameters

$$dr(t) = (g(t) + h(t)r(t))\,dt + \sigma(t)\,dW_t,$$

where $g(t)$, $h(t)$ and $\sigma(t)$ are deterministic functions of time.
- Applying Itô to this SDE gives the general solution

$$r(t) = e^{H(t)}r(0) + \int_0^t e^{H(t)-H(s)}g(s)\,ds + \int_0^t e^{H(t)-H(s)}\sigma(s)\,dW_s,$$

where

$$H(t) = \int_0^t h(s)\,ds.$$
- Hull and White later proposed that only the function $g(t) = \alpha b(t)$ should be time varying, with $h(t) = -\alpha$ and $\sigma(t) = \sigma$ constant. While this means that calibration only allows approximate matching of cap and swaption prices, pricing of non-standard interest rate derivatives is more robust.

- Under this parametrisation the general solution above specialises to

$$r(t_2) = e^{-\alpha(t_2-t_1)}r(t_1) + \alpha \int_{t_1}^{t_2} e^{-\alpha(t_2-s)} b(s)\,ds + \sigma \int_{t_1}^{t_2} e^{-\alpha(t_2-s)}\,dW_s, \quad (2)$$

  for $0 < t_1 < t_2$.
- Given $r(t_1)$, then $r(t_2)$ is distributed normally with mean

$$\mu_r(t_1,t_2) = e^{-\alpha(t_2-t_1)}r(t_1) + \alpha \int_{t_1}^{t_2} e^{-\alpha(t_2-s)} b(s)\,ds \quad (3)$$

  and variance

$$\sigma_r^2(t_1,t_2) = \sigma^2 \int_{t_1}^{t_2} e^{-2\alpha(t_2-s)}\,ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t_2-t_1)}\right). \quad (4)$$

- In the special case where $b(t) = b$ for all t (i.e. the classic Vasicek model), then (2) becomes

$$\mu_r(t_1,t_2) = e^{-\alpha(t_2-t_1)}r(t_1) + b\left(1 - e^{-\alpha(t_2-t_1)}\right).$$

  We can simplify, by defining

$$A(t_1,t_2) = \frac{1}{\alpha}\left(1 - e^{-\alpha(t_2-t_1)}\right),$$

  in which case

$$\mu_r(t_1,t_2) = r(t_1) + \alpha(b - r(t_1))A(t_1,t_2) \quad (5)$$

  and

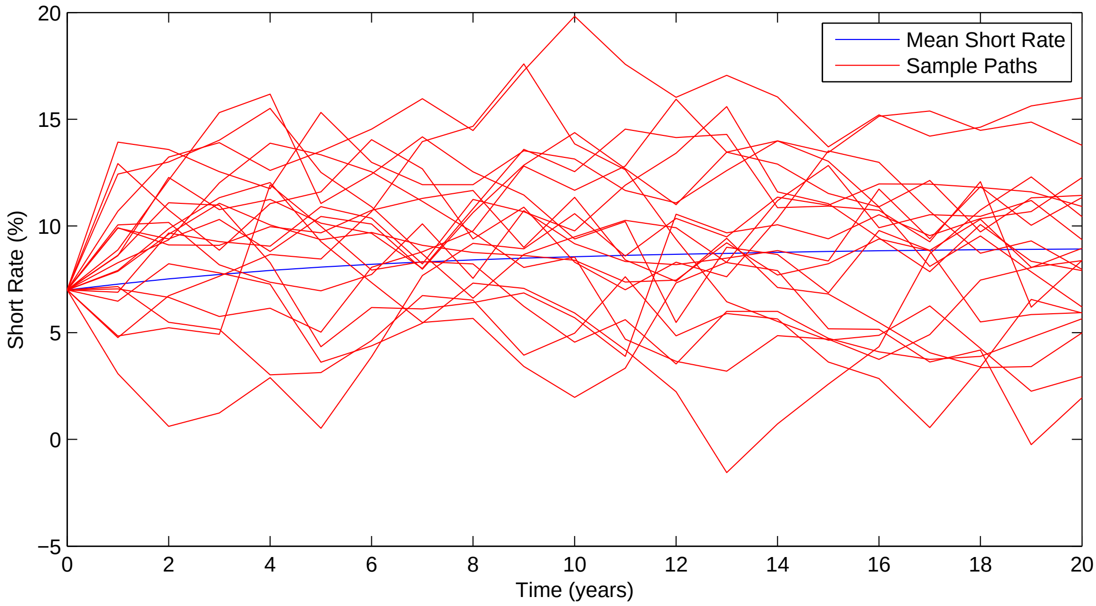
$$\sigma_r^2(t_1,t_2) = \frac{\sigma^2}{2}A(2t_1,2t_2). \quad (6)$$

Example: Vasicek Sample Paths

- Thus, to simulate r at times $0 = t_0 < t_1 < \ldots < t_n$ we set

$$r(t_{i+1}) = \mu_r(t_i,t_{i+1}) + \sigma_r(t_i,t_{i+1})Z_{i+1}$$

  for $Z_1, Z_2, \ldots, Z_n \sim N(0,1)$.
- The graph below shows 20 sample paths for the classic Vasicek model using (5) and (6) with the parameters $r(0) = 7\%$, $\alpha = 0.15$, $b = 9\%$ and $\sigma = 2\%$. Here, the rates are sampled on an annual basis over 20 years. Notice that, under this model, rates may become negative.



Bond Prices

- As seen above, $r(t)$ is a Gaussian process. Hence, the integral of $r(t)$ from 0 to T is a Gaussian process. Thus, using the bond pricing formula (1) and the fact that for $X \sim N(m, v^2)$ we have $\mathbb{E}(\exp(X)) = \exp(m + \frac{1}{2}v^2)$, the bond price at time t for maturity T is written as

$$\begin{aligned} B(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(u)\,du\right)\right] \\ &= \exp\left(-\mathbb{E}\left[\int_t^T r(u)\,du\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}\left[\int_t^T r(u)\,du\right]\right). \end{aligned} \tag{7}$$

- From (3), for the mean we have

$$\begin{aligned} \mathbb{E}\left[\int_t^T r(u)\,du\right] &= \int_t^T \mathbb{E}[r(u)]\,du \\ &= \int_t^T \mu_r(t,u)\,du \\ &= \int_t^T e^{-\alpha(u-t)}r(t)\,du + \alpha\int_t^T\int_t^u e^{-\alpha(u-s)}b(s)\,ds\,du \\ &= \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)r(t) + \alpha\int_t^T\int_s^T e^{-\alpha(u-s)}b(s)\,du\,ds. \\ &= A(t,T)r(t) + \alpha\int_t^T A(s,T)b(s)\,ds. \end{aligned} \tag{8}$$

- The variance is computed as

$$\begin{aligned} \mathbb{V}\mathrm{ar}\left[\int_t^T r(u)\,du\right] &= \mathbb{C}\mathrm{ov}\left[\int_t^T r(u)\,du, \int_t^T r(s)\,ds\right] \\ &= \int_t^T\int_t^T \mathbb{C}\mathrm{ov}_t[r(u), r(s)]\,ds\,du \\ &= 2\int_t^T\int_t^u \mathbb{C}\mathrm{ov}_t[r(u), r(s)]\,ds\,du, \end{aligned} \tag{9}$$

where the covariance on the interval $[t, u]$, for $t \le s \le u$, is calculated as

$$\begin{aligned} \mathbb{C}\mathrm{ov}_t[r(u), r(s)] &= \mathbb{C}\mathrm{ov}\left[\sigma\int_t^u e^{-\alpha(u-x)}\,dW_x, \sigma\int_t^s e^{-\alpha(s-y)}\,dW_y\right] \\ &= \sigma^2\mathbb{C}\mathrm{ov}\left[\int_t^s e^{-\alpha(u-x)}\,dW_x + \int_s^u e^{-\alpha(u-x)}\,dW_x,\right. \\ &\qquad\left.\int_t^s e^{-\alpha(s-y)}\,dW_y\right] \\ &= \sigma^2\int_t^s e^{-\alpha(u-v)}e^{-\alpha(s-v)}\,dv, \end{aligned} \tag{10}$$

where the last step follows by the independence of Brownian increments.

Exercise: After substituting (10) into (9), perform the integration and simplification to arrive at the expression for the variance

$$\mathbb{V}\mathrm{ar}\left[\int_t^T r(u)\,du\right] = \frac{\sigma^2}{\alpha^2}\left[(T-t) + \frac{1}{2\alpha}\left(1-e^{-2\alpha(T-t)}\right) + \frac{2}{\alpha}\left(e^{-\alpha(T-t)}-1\right)\right]$$
$$= \frac{\sigma^2}{\alpha^2}\left[(T-t) - A(t,T) - \frac{\alpha}{2}A^2(t,T)\right]. \quad (11)$$

- Substituting (8) and (11) into (7) gives an expression for the bond price. To simplify notation, we introduce

$$C(t_1,t_2) = \frac{1}{2}\mathbb{V}\mathrm{ar}\left[\int_{t_1}^{t_2} r(u)\,du\right] - \alpha\int_{t_1}^{t_2} A(s,t_2)b(s)\,ds$$
$$= \frac{\sigma^2}{2\alpha^2}\left[(t_2-t_1) - A(t_1,t_2) - \frac{\alpha}{2}A^2(t_1,t_2)\right] - \alpha\int_{t_1}^{t_2} A(s,t_2)b(s)\,ds, \quad (12)$$

  in which case the bond price is given as

$$B(t,T) = e^{-A(t,T)r(t)+C(t,T)}.$$

- In the case where $b(t) = b$ for all t, i.e. the Vasicek case, the function $C(t_1,t_2)$ simplifies to

$$C(t_1,t_2) = \left(\frac{\sigma^2}{2\alpha^2} - b\right)\left[(t_2-t_1) - A(t_1,t_2)\right] - \frac{\sigma^2}{4\alpha}A^2(t_1,t_2).$$

- In the Hull-White case, this bond price equation allows a way of calibrating the short rate model in order to recover the bond prices in the market.

Calibration of the Hull White Model (Sketch)

- Suppose a yield curve has been bootstrapped for time 0. Let the associated zero coupon bonds be denoted by $B^*(0,T)$, $T \in \mathbb{R}+$. To calibrate the short rate model to recover these prices we require

$$B^*(0,T) = e^{-A(0,T)r^* + C(0,T)},$$

  where r^* is the bootstrapped short rate for time 0.

- Inverting this equation for $C(0,T)$ and equating with (12) we get

$$\alpha\int_0^T A(s,T)b(s)\,ds = \frac{\sigma^2}{2\alpha^2}\left[(T-0) - A(0,T) - \frac{\alpha}{2}A^2(0,T)\right] - \log(B^*(0,T)) - r^*A(0,T).$$

- Differentiating with respect to T twice, we get

$$b(t) = -\frac{1}{\alpha}\frac{\partial^2}{\partial t^2}\log(B^*(0,t)) - \frac{\partial}{\partial t}\log(B^*(0,t)) + \frac{\sigma^2}{2\alpha^2}\left(1-e^{-2\alpha t}\right).$$

- From this equation we find that $C(t,T)$ must be written as

$$C(t,T) = \log\left(\frac{B^*(0,T)}{B^*(0,t)}\right) + f^*(0,t)A(t,T) - \frac{\sigma^2}{4\alpha}\left(1-e^{-2\alpha t}\right)A(t,T)^2,$$

  where $f^*(0,t) = -\frac{\partial}{\partial t}\log(B^*(0,t))$ is the forward rate today for time t.

- The parameters α and σ must be estimated using other means.

Joint Simulation of Rates and Discount Factors

- In most cases we require not only the realisations of the short rate $r(t)$ but also realisations of the discount factor
$$\frac{1}{\beta(t)} = \exp\left(-\int_0^t r(u)\,du\right)$$
or, alternatively, of
$$Y(t) = \int_0^t r(u)\,du.$$
- Given realisations $r(t_0), r(t_1), \ldots, r(t_n)$ of the short rate, $Y(t_i)$ can be approximated using simple quadrature as
$$Y(t_i) \approx \sum_{j=1}^{i} r(t_{j-1})\Delta t_j,$$
where $\Delta t_j = t_j - t_{j-1}$, or more accurately using trapezoidal quadrature as
$$Y(t_i) \approx \sum_{j=1}^{i} \left(r(t_{j-1}) + r(t_j)\right)\frac{\Delta t_j}{2},$$
- Of course, these approaches suffer from discretisation error. But, due to the fact that $(r(t), Y(t))$ are jointly Gaussian, it is possible to simulate them simultaneously without discretisation error — obviously error due to finite Monte Carlo sample remains.

- We have already seen that the short rate
$$r(t_{i+1}) \sim N\left(\mu_r(t_i, t_{i+1}), \sigma_r^2(t_i, t_{i+1})\right).$$
- Our previous calculations in (8) and (11) show, given $r(t_i)$ and $Y(t_i)$, that
$$Y(t_{i+1}) \sim N\left(\mu_Y(t_i, t_{i+1}), \sigma_Y^2(t_i, t_{i+1})\right),$$
with
$$\begin{aligned}\mu_Y(t_i, t_{i+1}) &= \int_{t_0}^{t_i} r(u)\,du + \mathbb{E}\left[\int_{t_i}^{t_{i+1}} r(u)\,du\right] \\ &= Y(t_i) + A(t_i, t_{i+1})r(t_i) + \alpha\int_{t_i}^{t_{i+1}} A(s, t_{i+1})b(s)\,ds,\end{aligned}$$
from (8), and
$$\sigma_Y^2(t_i, t_{i+1}) = \frac{\sigma^2}{\alpha^2}\left[(t_{i+1} - t_i) - A(t_i, t_{i+1}) - \frac{\alpha}{2}A^2(t_i, t_{i+1})\right],$$
from (11).
- In the Vasicek case, where $b(t) = b$ for all t, we have
$$\mu_Y(t_i, t_{i+1}) = Y(t_i) + (t_{i+1} - t_i)b + (r(t_i) - b)A(t_i, t_{i+1}). \qquad (13)$$
- Thus, in order to simultaneously simulate these all that we require is to compute the correlation between them.

- Using (10), we can compute the covariance as follows

$$\begin{aligned}\sigma_{rY}(t_i,t_{i+1}) &= \mathbb{C}\mathrm{ov}_{t_i}[r(t_{i+1}),Y(t_{i+1})] \\ &= \int_{t_i}^{t_{i+1}} \mathbb{C}\mathrm{ov}_{t_i}[r(t_{i+1}),r(s)]\,ds \\ &= \sigma^2 \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} e^{-\alpha(t_{i+1}-v)} e^{-\alpha(s-v)}\,dv\,ds \\ &= \frac{\sigma^2}{2} A^2(t_i,t_{i+1}).\end{aligned}$$

- Thus, the correlation may be computed as

$$\rho_{rY}(t_i,t_{i+1}) = \frac{\sigma_{rY}(t_i,t_{i+1})}{\sigma_r(t_i,t_{i+1})\sigma_Y(t_i,t_{i+1})}.$$

- Finally, the pair $(r(t_i),Y(t_i))$ can be simulated using

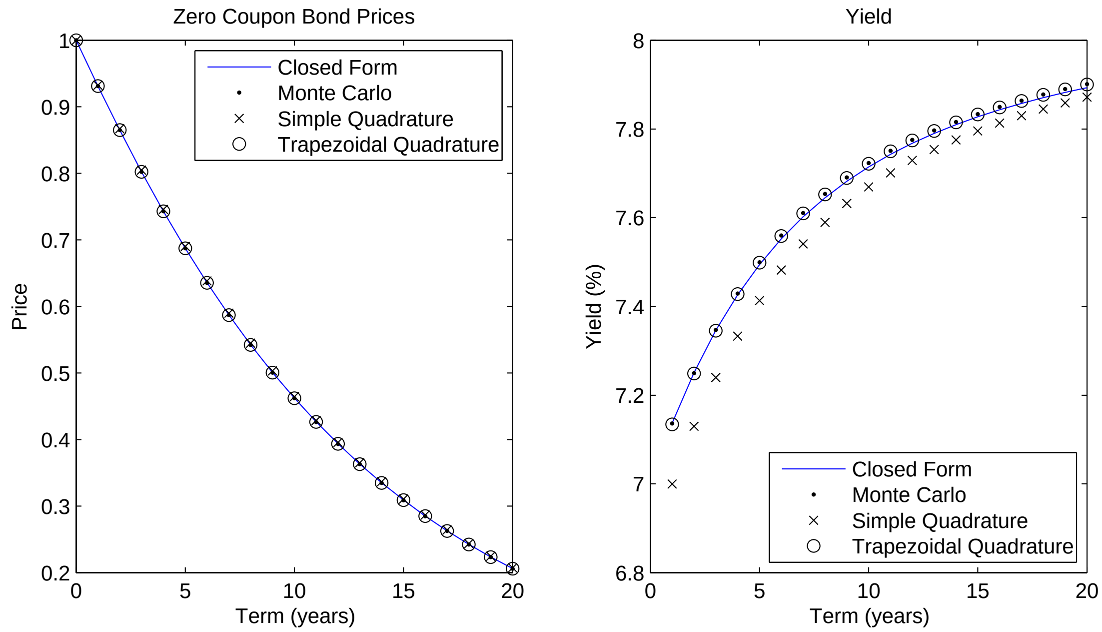
$$r(t_{i+1}) = \mu_r(t_i,t_{i+1}) + \sigma_r(t_i,t_{i+1})Z_1(i+1) \tag{14}$$

$$\begin{aligned}Y(t_{i+1}) = \mu_Y(t_i,t_{i+1}) + \sigma_Y(t_i,t_{i+1})\Big[&\rho_{rY}(t_i,t_{i+1})Z_1(i+1) \\ &+ \sqrt{1-\rho_{rY}^2(t_i,t_{i+1})}Z_2(i+1)\Big]\end{aligned} \tag{15}$$

  for $i = 0, 1, \ldots, n-1$, where $r(t_0) = r(0)$ and $Y(t_0) = 0$ and $(Z_1(i), Z_2(i))$ are independent standard normal bivariate normal random vectors.

Example: Vasicek Bond Price Recovery

- Using the same parameters as in the previous example, we have generated realisations of the short rate and corresponding correlated discount factors using equations (14) and (15) for the Vasicek case (5) and (13). In the left panel, we plot the expected value of the bond prices using the stochastic discount factor produced by Monte Carlo, simple quadrature of the Monte Carlo rates and trapezoidal quadrature. The right panel shows the corresponding yields for these bond prices. Simple quadrature performs poorly, while trapezoidal quadrature performs relatively well.



The CIR Model

- To overcome the shortcoming that rates may become negative in the Vasicek model, Cox, Ingersoll and Ross proposed using a class of square-root diffusions introduced by Feller.
- The SDE proposed for the short rate is given by

$$dr(t) = \alpha(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t, \tag{16}$$

  where W_t is a standard Brownian motion, and $\alpha, b, \sigma \in \mathbb{R}^+$.
- The following result known as Feller's condition guarantees positivity for the short rate.

Theorem 8.1 (Feller's Square Root Condition) *Let $\alpha, b, \sigma \in \mathbb{R}^+$ and $r(0) > 0$. If Feller's condition*

$$2\alpha b > \sigma^2$$

is satisfied, then there exists a unique positive solution of the SDE (16) on each finite time interval $t \in [0, \infty)$.

Proof. See Gikhman (2011).

- The SDE can be generalised with α, b and σ time-dependant. Using the SDE

$$dr(t) = \alpha(b(t) - r(t))\,dt + \sigma\sqrt{r(t)}\,dW_t,$$

  allows calibration to bond prices in a similar manner to the Hull-White model. (See, for example, Brigo and Mercurio (2006)).

Simulation Based on the Transition Density

- Unfortunately, unlike the Vasicek model, the SDE (16) does not admit an explicit solution. Of course, one can perform an Euler or Milstein discretisation to provide approximate solutions.
- Since the transition density of the process is known, there is a numerical approach, albeit computationally expensive, that allows exact simulation.
- Based on results of Feller and Cox, Ingersoll and Ross, the distribution of $r(t_{i+1})$ given $r(t_i)$ for $t_i < t_{i+1}$ is, up to a scale factor, a noncentral chi-square distribution.
- A noncentral chi-square random variable $\chi_d'^2(\lambda)$ with d degrees of freedom and noncentrality parameter λ has cumulative distribution

$$P(\chi_d'^2(\lambda) \le x) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\lambda)^j / j!}{2^{(d/2)+j}\Gamma\left(\frac{d}{2} + j\right)} \int_0^x z^{(d/2)+j-1} e^{-z/2}\,dz,$$

  for $x > 0$.
- While it is difficult to generate noncentral chi-square distributed random variates (see Glasserman (2004)), the Matlab statistics toolbox has a function `ncx2rnd(d,lambda,m,n)` which generates `m`×`n` realisations.

- The transition law for $r(t_{i+1})$, given $r(t_i)$ is

$$r(t_{i+1}) = \frac{\sigma^2\left(1-e^{-\alpha(t_{i+1}-t_i)}\right)}{4\alpha}\chi_d'^2(\lambda(t_i,t_{i+1})), \tag{17}$$

where

$$d = \frac{4b\alpha}{\sigma^2} \quad \text{and} \quad \lambda(t_i,t_{i+1}) = \frac{4\alpha e^{-\alpha(t_{i+1}-t_i)}r(t_i)}{\sigma^2\left(1-e^{-\alpha(t_{i+1}-t_i)}\right)}.$$

- Bond prices can, however, be computed in closed form and are given as follows

$$B(t,T) = e^{-A(t,T)r(t)+C(t,T)},$$

where

$$A(t_1,t_2) = \frac{2(e^{\gamma(t_2-t_1)}-1)}{(\gamma+\alpha)(e^{\gamma(t_2-t_1)}-1)+2\gamma}$$

and

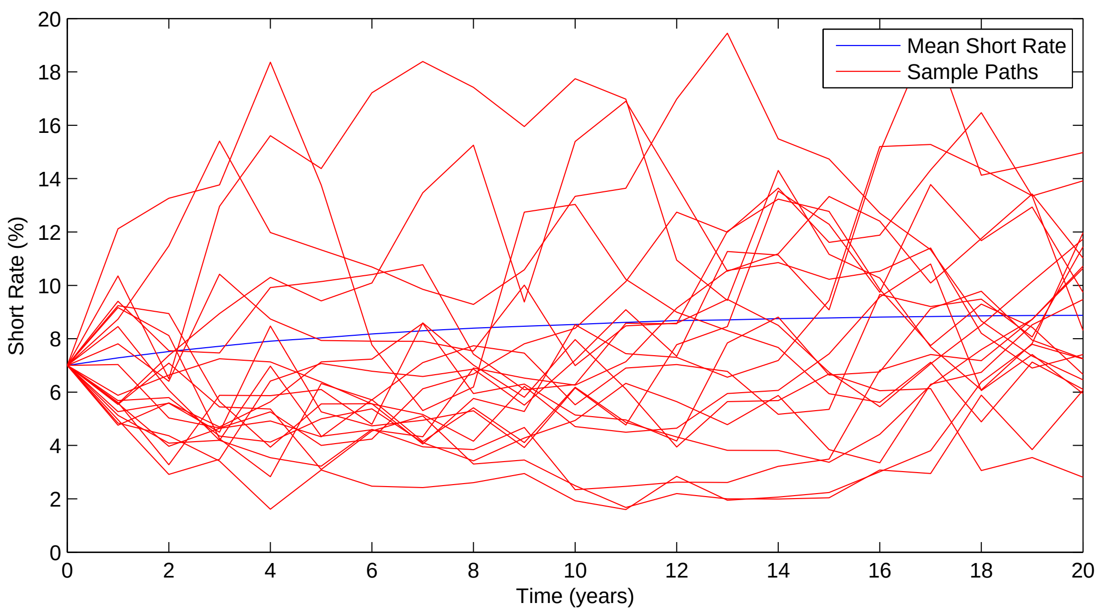
$$C(t_1,t_2) = \frac{2\alpha b}{\sigma^2}\log\left(\frac{2\gamma e^{(\alpha+\gamma)(t_2-t_1)/2}}{(\gamma+\alpha)(e^{\gamma(t_2-t_1)}-1)+2\gamma}\right),$$

with $\gamma = \sqrt{\alpha^2+2\sigma^2}$.

- Unfortunately joint simulation of rates and discount factors is not possible using an analytical approach. Numerical methods based on the Laplace transform are possible but are difficult (See Glasserman).

Example: CIR Sample Paths

- The graph below shows 20 sample paths for the CIR model using the transition law (17) and the parameters $r(0) = 7\%$, $\alpha = 0.15$, $b = 9\%$ and $\sigma = 7\%$. Here, the rates are sampled on an annual basis over 20 years.



Example: CIR Bond Price Recovery

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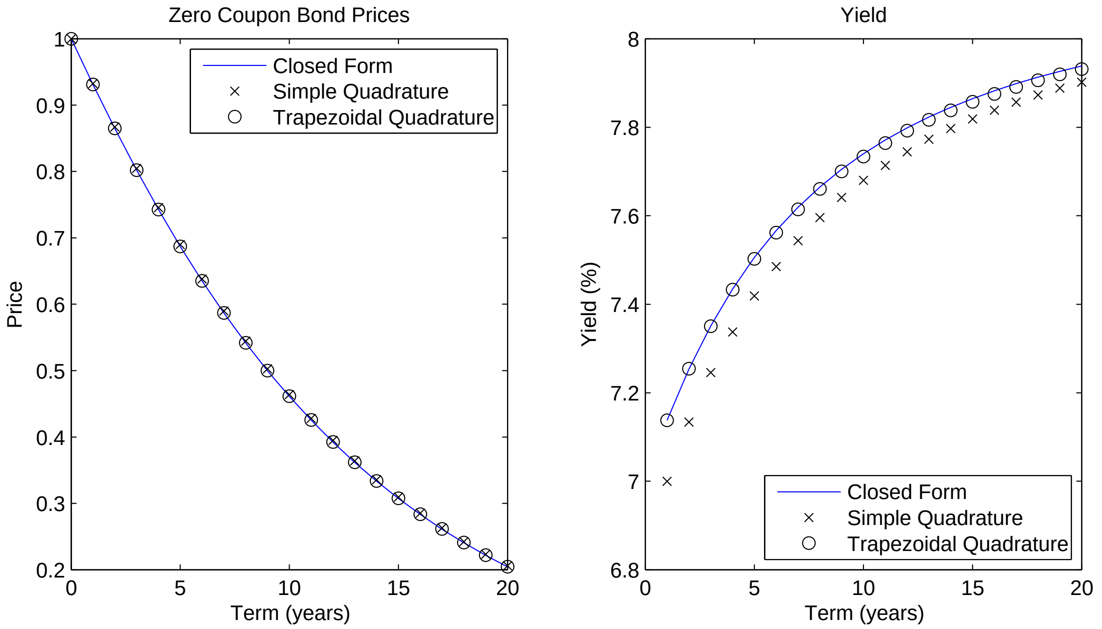

Numerical Methods in Finance II

Lecture 9 — The HJM Model

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M.Phil. in Financial Mathematics

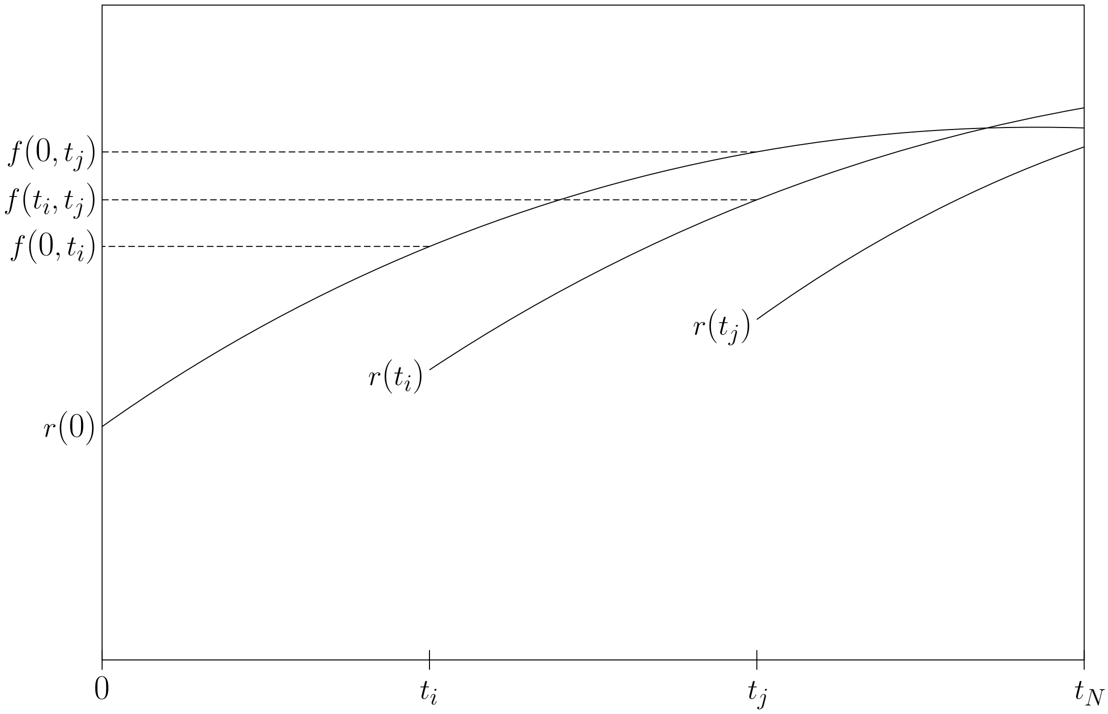
2020

Introduction

- In this module we turn our attention to modeling the dynamics of the entire forward rate curve.
- We start with the continuous-time theory of Heath, Jarrow and Morton (HJM).
- Specifying the dynamics of the forward rates using a single factor model, we derive an arbitrage free approach to the evolution of those rates.
- The advantage of this approach is that the initial term structure is an input to the model.
- Under simplifying assumptions, it is shown that one can recover certain short rate models (Ho-Lee and Vasicek/Hull-White).
- We then derive a discretisation of the system that preserves the arbitrage-free property.
- After presenting an algorithm for pricing claims, we provide examples. The first example shows that simulated bond prices are coherent with the initial term structure specified, while the second example shows the pricing of an option on a coupon bearing bond.

Instantaneous Forward Rates

- In contrast to short rate models, we now model the dynamics of the instantaneous forward rate, at time t_i for maturity t_j denoted $f(t_i, t_j)$ where $0 = t_0 \le t_i \le t_j \le t_N$.



- The forward rate represents the instantaneously compounded rate, agreed upon at time t, applicable at time T. Consequently, the (riskless) time t price of a bond with maturity T is given by

$$B(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right). \tag{1}$$

- This implies that

$$f(t,T) = -\frac{\partial}{\partial T}\log B(t,T), \tag{2}$$

and that the short and forward rates are linked by $r(t) = f(t,t)$.

- The HJM approach models the dynamics of the forward curve through the SDE

$$df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dW_t, \tag{3}$$

where the differential df is with respect to the current time t, not the maturity T. Note that this SDE holds for all $T > t$. In other words, it holds for the whole forward rate curve.

- To keep things simple we shall only consider a single factor model, but, in general, the above SDE may be specified using more than one factor. Moreover, it is possible to allow μ and σ to be stochastic. We shall only treat the case where they are deterministic functions of t and T.

- We have said nothing about the measure under which the forward rates are evolving. We shall assume that the evolution is under the risk-neutral measure, i.e., the measure that makes discounted bond prices martingales.
- To be consistent with the previous statement, this means that we are assuming the following dynamics for bond prices $B(t,T)$

$$dB(t,T) = r(t)B(t,T)\,dt + \nu(t,T)B(t,T)\,dW_t, \tag{4}$$

where $r(t)$ is the short rate and $\nu(t,T)$ is a volatility function that must be specified so as to be compatible with the forward rate dynamics. The compatibility is enforced using the relationship (2).
- Using the Itō formula on (4) above we have

$$d\log B(t,T) = \left(r(t) - \frac{1}{2}\nu(t,T)^2\right)dt + \nu(t,T)\,dW_t.$$

- Thus, using (2), and interchanging the order of differentiation we have

$$df(t,T) = -\frac{\partial}{\partial T}\left(r(t) - \frac{1}{2}\nu(t,T)^2\right)dt - \frac{\partial \nu(t,T)}{\partial T}\,dW_t.$$

- Comparing this with (3), consistency requires that

$$\sigma(t,T) = -\frac{\partial \nu(t,T)}{\partial T} \tag{5}$$

and that

$$\mu(t,T) = -\frac{\partial}{\partial T}\left(r(t) - \frac{1}{2}\nu(t,T)^2\right) = \frac{\partial \nu(t,T)}{\partial T}\nu(t,T). \tag{6}$$

- Integrating (5) gives

$$\nu(t,T) = \nu(t,t) - \int_t^T \sigma(t,s)\,ds.$$

- Now, since $B(t,T) \to 1$ as $t \to T$, i.e., the bond tends toward its principle value as it matures, we must require that $\nu(T,T) = \nu(t,t) = 0$.
- Thus, we can rewrite (6) as

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)\,ds, \tag{7}$$

in which case the SDE for our forward rates (under the risk-neutral measure) is given as

$$df(t,T) = \left(\sigma(t,T)\int_t^T \sigma(t,s)\,ds\right)dt + \sigma(t,T)\,dW_t. \tag{8}$$

- In particular, this means that once the volatility structure of the forward rates is determined, the drift in the forward rates (under the risk-neutral) measure is automatically determined.

Examples: Recovery of the Ho-Lee and Vasicek Models

- Consider the case of constant forward volatilities $\sigma(t,T) = \sigma$.
- By (7), this implies that $\mu(t,T) = \sigma \int_t^T \sigma\, du = \sigma^2(T-t)$.
- Thus, the solution of the SDE for the forward rates (3) may be written as

$$f(t,T) = f(0,T) + \frac{1}{2}\sigma^2(T^2 - (T-t)^2) + \sigma W_t. \tag{9}$$

- Noting the $r(t) = f(t,t)$ implies (through the multivariate Itō equation) that

$$dr(t) = df(t,T)\Big|_{T=t} + \frac{\partial f(t,T)}{\partial T}\Big|_{T=t} dt.$$

- Now, since $\mu(t,t) = 0$ the first term in this expression is just $\sigma\, dW_t$, while the second term can be computed using (9) to give

$$dr(t) = \left(\frac{\partial f(0,T)}{\partial T}\Big|_{T=t} + \sigma^2 t \right) dt + \sigma\, dW_t.$$

- We have thus recovered the calibrated Ho-Lee model (cf. SCFII notes).

Exercise: Assuming $\sigma(t,T) = \sigma \exp(-\alpha(T-t))$ for $\sigma, \alpha > 0$ show that

$$\mu(t,T) = \frac{\sigma^2}{\alpha}\left(e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right).$$

Hence, show using a similar argument to the one above that you can recover the Vasicek model (with a time-varying drift, i.e., Hull-White).

Simulation

- Except under very special circumstances (eg. constant σ) it is not possible to simulate (8) exactly. We must, therefore, provide a discrete-time version of the SDE for the forward rates.
- In principle it is possible to discretise both t and T using different grids, but using the same grid simplifies notation considerably.
- We shall assume a grid given by $0 = t_0 < t_1 < \ldots < t_N$ and assume that both t and T take on these values in our setup.
- The discretised forward rate at time t_i, for maturity t_j, is denoted by $\hat{f}(t_i, t_j)$, with the bond price (1) given by the discretisation

$$\hat{B}(t_i, t_j) = \exp\left(-\sum_{l=i}^{j-1} \hat{f}(t_i, t_l)(t_{l+1} - t_l) \right). \tag{10}$$

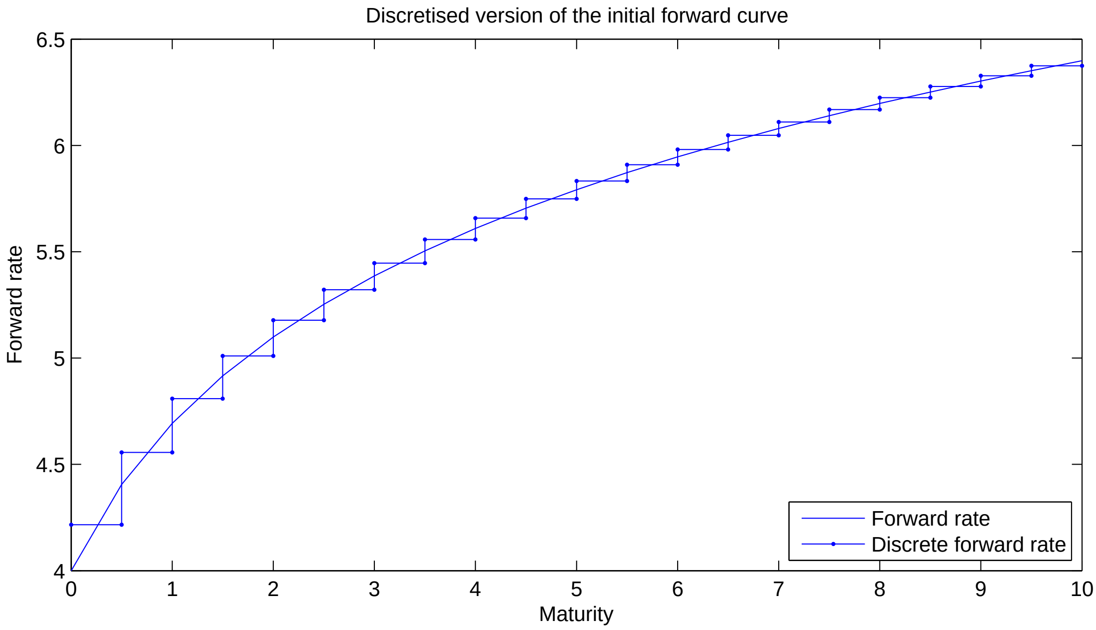
- To ensure as little discretisation error as possible, we initialise the $\hat{f}(t_0, t_j)$ values in such a way that the discretised bond prices correspond with the actual bond prices at inception, i.e., $\hat{B}(t_0, t_j) = B(t_0, t_j)$.
- From (10) we have $\hat{B}(t_0, t_{j+1}) = \hat{B}(t_0, t_j) \exp\left(-\hat{f}(t_0, t_j)(t_{j+1} - t_j)\right)$, thus for consistency we require, for $j = 0, 1, \ldots, N-1$, that

$$\hat{f}(t_0, t_j) = -\frac{\log\left(\hat{B}(t_0, t_{j+1})/\hat{B}(t_0, t_j)\right)}{t_{j+1} - t_j} = -\frac{\log\left(B(t_0, t_{j+1})/B(t_0, t_j)\right)}{t_{j+1} - t_j}. \tag{11}$$

- Note that this is equivalent to demanding that

$$\hat{f}(t_0, t_j) = \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} f(0, u)\, du,$$

which is depicted below.



- Having specified $\hat{f}(t_0, t_j)$ we now specify the discretised version of (3) as follows

$$\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)(t_i - t_{i-1}) + \hat{\sigma}(t_{i-1}, t_j)\sqrt{t_i - t_{i-1}} Z_i, \tag{12}$$

for $j = i, \ldots, N-1$, where Z_i are independent $\mathcal{N}(0,1)$ variates associated with time i and $\hat{\mu}$ and $\hat{\sigma}$ are the discrete-time counterparts of μ and σ.

- While $\hat{\sigma}$ must, in practice, be calibrated to market data to ensure consistency with market prices, $\hat{\mu}$ is constructed to ensure that the discretised model remains arbitrage free. This entails ensuring that the discounted bond prices remain martingales.

- Recall from the previous module that the numeraire asset may be written as

$$\beta(t) = \exp\left(\int_0^t r(u)\, du\right) = \exp\left(\int_0^t f(u,u)\, du\right).$$

The version of this in our discrete system is given by

$$\hat{\beta}_i = \exp\left(\sum_{k=0}^{i-1} \hat{f}(t_k, t_k)(t_{k+1} - t_k)\right).$$

- We require that all the bond prices (at time i, for maturities $i < j \leq N$) multiplied by the reciprocal of this quantity are martingales (under the risk-neutral measure).

- This martingale condition is expressed mathematically as

$$\mathbb{E}\left[\left.\frac{\hat{B}(t_i,t_j)}{\hat{\beta}_i}\right| t_{i-1}\right] = \frac{\hat{B}(t_{i-1},t_j)}{\hat{\beta}_{i-1}},$$

and, since the discount factor $\hat{\beta}_i$ is measurable at time t_{i-1}, we have

$$\begin{aligned}\mathbb{E}\left[\left.\hat{B}(t_i,t_j)\right| t_{i-1}\right] &= \frac{\hat{\beta}_i}{\hat{\beta}_{i-1}}\hat{B}(t_{i-1},t_j)\\ &= \exp\left(\hat{f}(t_{i-1},t_{i-1})(t_i - t_{i-1})\right)\hat{B}(t_{i-1},t_j).\end{aligned}$$

- Expressing the bond prices using (10) and simplifying gives

$$\mathbb{E}\left[\left.\exp\left(-\sum_{l=i}^{j-1}\hat{f}(t_i,t_l)(t_{l+1}-t_l)\right)\right| t_{i-1}\right] = \exp\left(-\sum_{l=i}^{j-1}\hat{f}(t_{i-1},t_l)(t_{l+1}-t_l)\right).$$

- Substituting the discretised version of the SDE (12) into the left side of this expression and simplifying gives

$$\mathbb{E}\left[\left.\exp\left(-\sum_{l=i}^{j-1}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l)\sqrt{t_i - t_{i-1}}Z_i\right)\right| t_{i-1}\right] = \\ \exp\left(\sum_{l=i}^{j-1}\hat{\mu}(t_{i-1},t_l)(t_i - t_{i-1})(t_{l+1}-t_l)\right).$$

- The conditional expectation in the relation above is given by

$$\exp\left(\frac{1}{2}\left[\sum_{l=1}^{j-1}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l)\right]^2 (t_i - t_{i-1})\right).$$

- Thus, we require

$$\frac{1}{2}\left[\sum_{l=i}^{j-1}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l)\right]^2 = \sum_{l=i}^{j-1}\hat{\mu}(t_{i-1},t_l)(t_{l+1}-t_l),$$

which is true if (show as an exercise)

$$\hat{\mu}(t_{i-1},t_j)(t_{j+1}-t_j) = \\ \frac{1}{2}\left[\sum_{l=i}^{j}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l)\right]^2 - \frac{1}{2}\left[\sum_{l=i}^{j-1}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l)\right]^2.$$

- Note that the right side of this expression may be expressed as a difference in the squares of successive terms of a `cumsum` command. That is,

$$\hat{\mu}(t_{i-1},t_j) = \frac{(s_j)^2 - (s_{j-1})^2}{2(t_{j+1}-t_j)}, \tag{13}$$

for $j = i, \ldots, N-1$, where

$$s_n = \sum_{l=i}^{n}\hat{\sigma}(t_{i-1},t_l)(t_{l+1}-t_l).$$

Notice that the term s_{i-1} is zero.

Implementation

- When implementing, it is possible to store the evolution of the forward curve through both time and maturity. We shall, however, choose to keep a single version of the forward curve which contains evolved values only.
- Values in this stored curve up to the evolved time, in position i corresponding to time t_i, say, represent the (average) short rates for that particular time, i.e., $\hat{r}_j = \hat{f}(t_j, t_j)$ for $j \leq i$. Values in the curve with $j > i$ represent the evolved forward rates $\hat{f}(t_i, t_j)$.
- Suppose we wish to price a derivative H with maturity T_M that produces cash-flows c_i for $i \leq M \leq N$. The algorithm to price (a single path of) this derivative proceeds as follows:

1. Initialise $f_i = \hat{f}(t_0, t_i)$, for $i = 0, 1, \ldots, N-1$, using (11) and observed bonds.
2. Set $\beta = 1$.
3. For $i = 1, 2, \ldots, M$ perform the following:
   - 3.1 Update any non-interest rate processes to time t_i, using drift $r = f_{i-1}$.
   - 3.2 Update numeraire by setting $\beta = \beta \exp(r(t_i - t_{i-1}))$.
   - 3.3 If $i < N$ perform the following:
     - 3.3.1 Compute $\sigma_j = \hat{\sigma}(t_{i-1}, t_j)$, for $j = i, \ldots, N-1$.
     - 3.3.2 Compute $\mu_j = \hat{\mu}(t_{i-1}, t_j)$, for $j = i, \ldots, N-1$ using (13).
     - 3.3.3 Generate $Z_i \sim \mathcal{N}(0, 1)$.
     - 3.3.4 Set $f_j = f_j + \mu_j(t_i - t_{i-1}) + \sigma_j \sqrt{t_i - t_{i-1}} Z_i$, for $j = i, \ldots, N-1$.
   - 3.4 Discount cash-flow c_i using $1/\beta$, (c_i may depend on updated forward rates).
4. Return H, as the sum of the discounted cash-flows.

Example: Recovery of Bond Prices

- In this example, we show that the discrete simulation of rates recovers the initial term structure by pricing zero coupon bonds for various maturities and comparing the simulated prices (and yields) to the bond prices used to generate the initial discretised forward curve.
- We use the Vasicek short rate model as a comparison case. We initially compute Vasicek bond prices with parameters

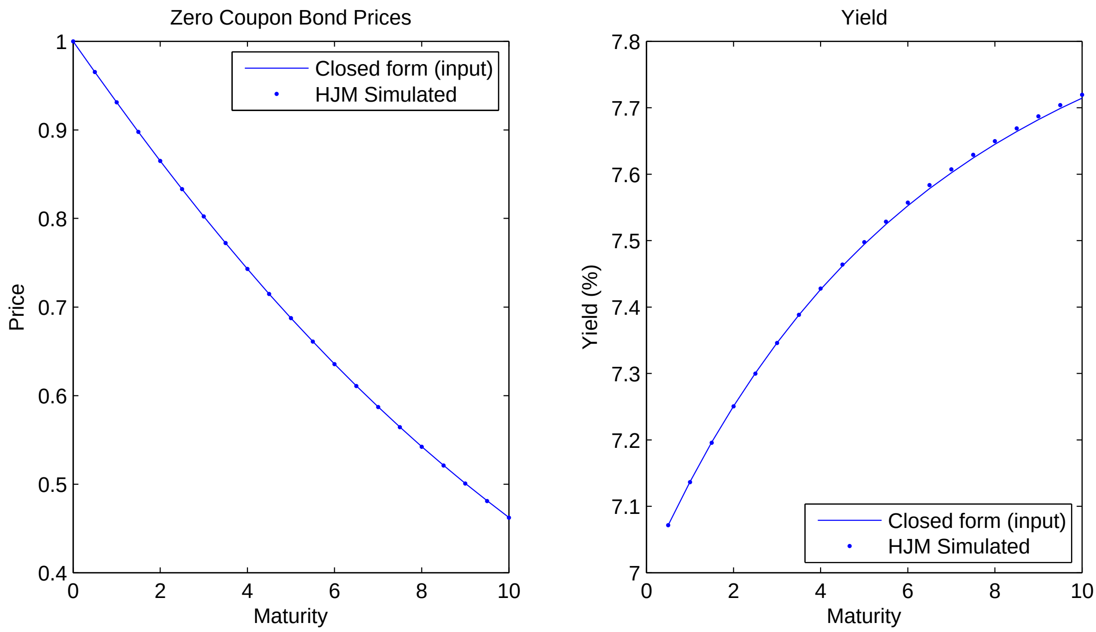
$$r_0 = 7\%, \quad \alpha = 0.15, \quad b = 9\% \quad \text{and} \quad \sigma = 2\%,$$

  for maturities at half year intervals.
- These bond prices are used to produce the initial discretised forward rate curve using (11)
- As discussed earlier, to be compatible with this short-rate model, the HJM volatility function to be used is

$$\hat{\sigma}(t_i, t_j) = \sigma e^{-\alpha(t_j - t_i)}.$$

- The simulated bond prices are computed as the mean of the corresponding discount factor at each time step (corresponding to a zero-coupon bond price with unit face value).

- The following figure shows the initial zero-coupon bond price curve and the simulated prices, using $100\,000$ sample paths. The corresponding yield curve is also displayed.



- Note: We need not have used theoretical bond prices as our input parameters—we could have used a market curve in which case this discretised HJM model corresponds to a calibrated Hull-White model.

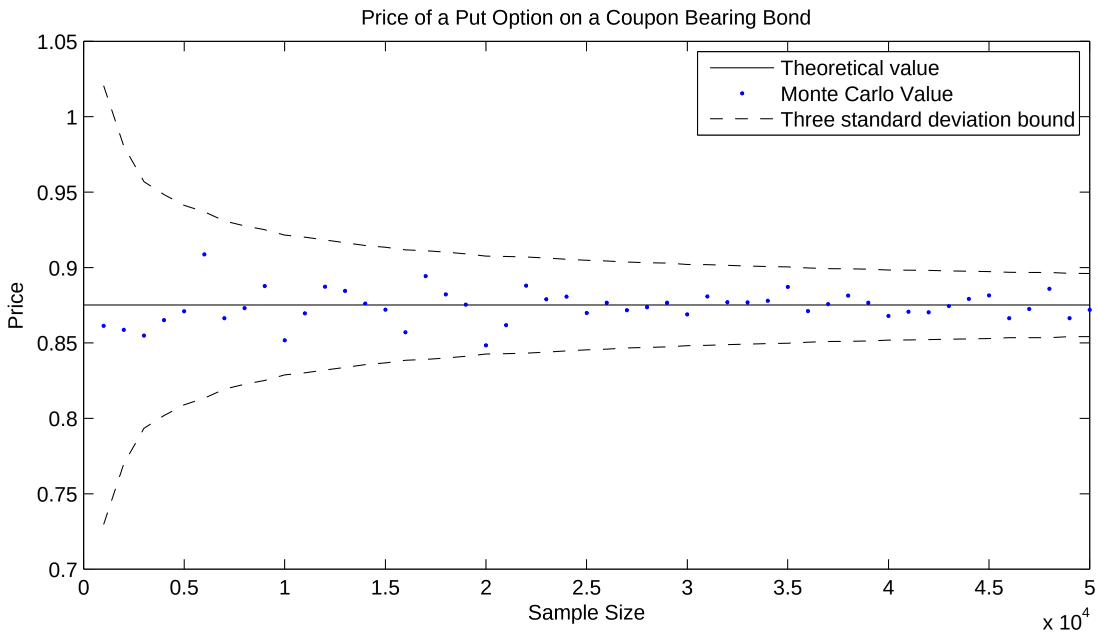
Example: Pricing an Option on a Coupon Bearing Bond

- Consider the problem of computing a put option on a bond with par 100 that pays coupons of 10% NACS (i.e., coupon value of 5). The strike price is 98 and the maturity of the option is 3 years with the maturity of the bond being 5 years.
- Under the Vasicek model with parameters

$$r_0 = 10\%, \quad \alpha = 0.1, \quad b = 10\% \quad \text{and} \quad \sigma = 2\%,$$

the theoretical price of a put option on this bond is 0.87513 (This can be computed using Jamshidian's trick, see the SCFII notes).

- We simulate on a time grid of half year intervals. The initial discretised forward rate curve is calibrated to Vasicek bond prices
- The figure below shows the Monte Carlo prices as a function of sample size compared to the theoretical value.



Price of a Put Option on a Coupon Bearing Bond
Theoretical value
Monte Carlo Value
Three standard deviation bound
Price
1.05
1
0.95
0.9
0.85
0.8
0.75
0.7
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Sample Size
x 10^4

Numerical Methods in Finance II

Lecture 10 - The LIBOR Model

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M.Phil. in Financial Mathematics

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Introduction

- The LIBOR model, also known as the BGM or BGM/J model after the authors who pioneered the approach (Brace, Gatarek and Musiela (1997) and Jamshidian (1997)), is a so-called *market model*.
- This means that it models the actual observed traded instruments in the market, as opposed to idealisations like the short rate or continuously compounded forward rates.
- In particular, all rates are now specified using simple compounding.
- By specifying a suitable numeraire asset and demanding that bond prices remain martingales when denominated using this asset, we use an analysis, similar to the approach taken for the HJM model, to develop a single factor model of LIBOR forward rates.
- The continuous-time SDE derived is then discretised and an algorithm very similar to that used for the HJM model is presented.
- Unfortunately, unlike in the HJM case, it is not possible to ensure the discretised system is arbitrage-free (by deriving an appropriate discrete drift).
- Therefore, we explore the use of a predictor-corrector method to ensure a more accurate discretised representation.
- To demonstrate the efficacy of the approaches, we show the recovery of bond prices. We also show how caplets may be priced and compared to the Black caplet pricing approach.

Simple Rates

- Previously we looked at modeling the continuously compounded forward rate curve. Continuously compounded rates are, however, idealisations.
- We now choose to specify all rates as simple rates. In particular, if R denotes the rate applicable for period of length δ, then the interest earned on a single unit of currency is δR.
- Forward rates work similarly. The forward rate at time t for maturity T_1 with accrual period between T_1 and T_2 is denoted $F(t; T_1, T_2)$. The interest earned per unit currency at time T_2 is $\delta F(t; T_1, T_2)$, where $\delta = T_2 - T_1$ is the accrual period (or year fraction).
- Exercise: Use a no-arbitrage argument to show that

$$F(t; T_1, T_2) = \frac{B(t, T_1) - B(t, T_2)}{\delta B(t, T_2)}, \tag{1}$$

  where $B(t, .)$ are the bond prices for a specific tenor at time t.
- Using a set of market instrument tenors $0 = T_0 < T_1 < \ldots < T_N$, we define the accrual periods between these tenors as

$$\delta_i = T_{i+1} - T_i,$$

  for $i = 0, 1, \ldots, N - 1$.
- We shall use the short-hand notation $F_i(t) = F(t; T_i, T_{i+1})$ and note that δ_i is the length of the accrual period associated with forward rate $F_i(\cdot)$.
- We also use the short-hand $B_i(t) = B(t, T_i)$ for the bonds.

- Expressing (1) in terms of the short-hand notation, we have

$$F_j(t) = \frac{B_j(t) - B_{j+1}(t)}{\delta_j B_{j+1}(t)}, \tag{2}$$

  for $0 \le t \le T_j$ and $j = 0, 1, \ldots, N - 1$.
- After time T_j the forward rate F_j becomes fixed and equal to the realised simple rate over the period, i.e., $F_j(t) = F_j(T_j)$ for $t \ge T_j$.
- The relation (2) may be inverted to give the bond prices at the tenor dates T_i, in terms of the forward rates,

$$B_j(T_i) = \prod_{k=i}^{j-1} \frac{1}{1 + \delta_k F_k(T_i)},$$

  for $j = i + 1, \ldots, N$.
- At times other than tenor dates, e.g. for $T_{i-1} \le t < T_i < T_j$, we have

$$B_j(t) = B_i(t) \prod_{k=i}^{j-1} \frac{1}{1 + \delta_k F_k(t)}.$$

- Thus, if we define a function $\mathcal{I}(t) = \{\min i | \, t < T_i\}$, which is the index of the first forward rate that has not yet expired, then we have

$$B_j(t) = B_{\mathcal{I}(t)}(t) \prod_{k=\mathcal{I}(t)}^{j-1} \frac{1}{1 + \delta_k F_k(t)}, \tag{3}$$

  for $0 \le t < T_j$.

Evolution of the LIBOR Forward Rates

- We shall model the evolution of the forward LIBOR rates using a single factor SDE of the form

$$dF_j(t) = F_j(t)\mu_j(t)\,dt + F_j(t)\sigma_j(t)\,dW_t \tag{4}$$

  for $0 \le t \le T_j$, $j = 1, \dots, N-1$.

- Notice that, in contrast to the HJM evolution, this is a log-normal model. In general the Brownian motion may be a multi-dimensional process, in which case σ_j must be specified appropriately, correlations may be specified and there may even be a factor for each tenor.

- As in the HJM model, for our model to remain arbitrage free, we require the deflated (discounted) bond prices to be martingales under the risk-neutral measure. This requires that we choose $\mu_j(t)$ carefully to ensure this feature.

- In the HJM model the numeraire asset, $\beta(t) = \exp\left(\int_0^t f(u,u)\,du\right)$, was defined in terms of the realised short rate. We now require a simply compounded numeraire asset.

- The analogous counterpart, would be an instrument that starts with a value of one at $t = 0$ and accrues interest by investing all its value in the bond with the shortest tenor. All proceeds are reinvested in each successive bond of shortest tenor until the current time.

- At time t, this instrument would have a value of

$$\bar{\beta}(t) = B_{\mathcal{I}(t)}(t) \prod_{k=0}^{\mathcal{I}(t)-1} (1 + \delta_k F_k(T_k)).$$

  Recall that $\delta_k F_k(T_k)$ represents the simple interest earned over the period $[T_k, T_{k+1}]$. Since the product represents interest accrued up to time $T_{\mathcal{I}(t)} > t$, it must be deflated to t by multiplying by the current value of the shortest dated bond $B_{\mathcal{I}(t)}(t)$, hence the term preceding the product.

- We require an arbitrage free model, which means that the deflated bond prices, given by $D_j(t) = B_j(t)/\bar{\beta}(t)$, must be martingales.

- We use the term 'deflated' to distinguish from the case where a continuously compounded numeraire is used.

- Using (3) and the above expression, the deflated bond prices are given by

$$D_j(t) = \left(\prod_{k=\mathcal{I}(t)}^{j-1} \frac{1}{1 + \delta_k F_k(t)}\right) \prod_{k=0}^{\mathcal{I}(t)-1} \frac{1}{1 + \delta_k F_k(T_k)}, \tag{5}$$

  which conveniently cancels the factor $B_{\mathcal{I}(t)}(t)$, allowing D to be specified exclusively using the LIBOR forward rates.

- Given that the deflated bonds are required to be (positive) martingales, we may assume that they have dynamics given by
$$dD_j(t) = D_j(t)\nu_j(t)\,dW_t, \tag{6}$$
for $j = \mathcal{I}(t), \dots, N$ and some process ν_j.
- Notice that $D_1(t) = (1 + \delta_0 F_0(T_0))^{-1}$ is constant.
- Now, from (5) we have the following recursive relation
$$D_{j+1}(t) = D_j(t)\frac{1}{1 + \delta_j F_j(t)}$$
which means that
$$\delta_j F_j(t) D_{j+1}(t) = D_j(t) - D_{j+1}(t).$$
- Since $D_1(t)$ is a (constant) martingale. By induction, given that $D_j(t)$ is a martingale, for $D_{j+1}(t)$ to be a martingale we require that $F_j(t)D_{j+1}(t)$ be a martingale. By the multidimensional Itô Formula
$$d(F_j(t)D_{j+1}(t)) = F_j(t)D_{j+1}(t)\big[(\mu_j(t) + \nu_{j+1}(t)\sigma_j(t))\,dt + (\sigma_j(t) + \nu_{j+1}(t))\,dW_t\big].$$
Thus, we require
$$\mu_j(t) = -\nu_{j+1}(t)\sigma_j(t). \tag{7}$$
- It remains for us to find an explicit form for ν_j. To do this, note that an application of the Itô formula on (6) gives
$$d\log D_j(t) = -\tfrac{1}{2}(\nu_j(t))^2\,dt + \nu_j(t)\,dW_t. \tag{8}$$

- Taking logs of both sides of (5) and recognising that the term on the right, in the first line below, is a constant at t, we have
$$\log D_j(t) = -\sum_{k=\mathcal{I}(t)}^{j-1} \log(1 + \delta_k F_k(t)) - \sum_{k=0}^{\mathcal{I}(t)-1} \log(1 + \delta_k F_k(T_k))$$
$$\Rightarrow \quad d\log D_j(t) = -\sum_{k=\mathcal{I}(t)}^{j-1} d\log(1 + \delta_k F_k(t))$$
$$= -\sum_{k=\mathcal{I}(t)}^{j-1}\left[\left(\frac{\delta_k F_k(t)\mu_k(t)}{1 + \delta_k F_k(t)} - \frac{1}{2}\left(\frac{\delta_k F_k(t)\sigma_k(t)}{1 + \delta_k F_k(t)}\right)^2\right)dt + \frac{\delta_k F_k(t)\sigma_k(t)}{1 + \delta_k F_k(t)}\,dW_t\right], \tag{9}$$
where the last expression follows using (4) and the Itô Formula.
- Comparing the dW_t terms in (8) and (9), this means that
$$\nu_j(t) = -\sum_{k=\mathcal{I}(t)}^{j-1} \frac{\delta_k F_k(t)\sigma_k(t)}{1 + \delta_k F_k(t)}.$$
- Using this and (7), the SDE for the forward rates (4) may be written as
$$dF_j(t) = F_j(t)\sum_{k=\mathcal{I}(t)}^{j} \frac{\delta_k F_k(t)\sigma_k(t)\sigma_j(t)}{1 + \delta_k F_k(t)}\,dt + F_j(t)\sigma_j(t)\,dW_t.$$

Simulation

- As in the discrete HJM model, we assume a discrete time grid given by $0 = t_0 < t_1 < \ldots < t_N$ includes the tenors and times that we are interested in. Most often, we just assume that $t_i = T_i$. Note that from now on δ_i are defined with respect to these t_i.
- To simplify things further, we shall assume that $\sigma_i(t) = \sigma_i$ are constant.
- We initialise $\hat{F}_j(t_0) = F_j(0)$ for all $0 \leq j < N$.
- Then, the obvious approach would be to use the Euler-Maruyama scheme

$$\hat{F}_j(t_i) = \hat{F}_j(t_{i-1}) + \hat{\mu}_j(t_{i-1})\hat{F}_j(t_{i-1})\delta_{i-1} + \sigma_j \hat{F}_j(t_{i-1})\sqrt{\delta_{i-1}}Z_i,$$

  and, as in the HJM case, try to construct $\hat{\mu}$ so as to ensure that the discounted bonds are martingales. Unfortunately, this is not possible.
- We instead proceed as follows: working with the log of the process, we rather use the more accurate discretisation

$$\hat{F}_j(t_i) = \hat{F}_j(t_{i-1}) \exp\left(\left(\hat{\mu}_j(t_{i-1}) - \tfrac{1}{2}\sigma_j^2\right)\delta_{i-1} + \sigma_j\sqrt{\delta_{i-1}}Z_i\right),$$

  where

$$\hat{\mu}_j(t_{i-1}) = \sum_{k=i}^{j} \frac{\delta_k \hat{F}_k(t_{i-1})\sigma_k\sigma_j}{1 + \delta_k \hat{F}_k(t_{i-1})}, \tag{10}$$

  for $j = i, \ldots, N-1$, where $Z_i \sim \mathcal{N}(0,1)$. This is the exact solution for GBM, where we have fixed the drift over the interval to be the initial drift.

Implementation

- As we did in the HJM model, we shall only keep a single version of the forward curve which contains evolved values in an array called F.
- The algorithm to price a derivative H with maturity T_M that produces cash-flows c_i for $i \leq M \leq N$ is very similar to the one for HJM. For a single path it proceeds as follows:

1. Initialise $F_j = F_j(t_0)$, for $j = 0, 1, \ldots, N-1$, using (2) and observed bonds.
2. Set $\bar{\beta} = 1$.
3. For $i = 1, 2, \ldots, M$ perform the following:
   - 3.1 Update any non-interest rate processes to time t_i, using drift $r = \log(1 + \delta_{i-1}F_{i-1})/\delta_{i-1}$.
   - 3.2 Update numeraire by setting $\bar{\beta} = \bar{\beta}(1 + \delta_{i-1}F_{i-1})$.
   - 3.3 If $i < N$ perform the following:
     - 3.3.1 Compute $\mu_j = \hat{\mu}_j(t_{i-1})$, for $j = i, \ldots, N-1$ using (10).
     - 3.3.2 Generate $Z_i \sim \mathcal{N}(0,1)$.
     - 3.3.3 Set $F_j = F_j \exp\left((\mu_j - \tfrac{1}{2}\sigma_j^2)\delta_{i-1} + \sigma_j\sqrt{\delta_{i-1}}Z_i\right)$, for $j = i, \ldots, N-1$.
   - 3.4 Discount cash-flow c_i using $1/\bar{\beta}$, (c_i may depend on updated forward rates).
4. Return H, as the sum of the discounted cash-flows.

- Because we are only keeping a single version of the forward curve instead of storing the evolution through time, the values stored in F just before the execution of step 3.3.3 are $F_j = \hat{F}_j(t_{i-1})$ for $j \geq i$.
- Just after the execution of step 3.3.3, F stores the realised simple rates, $F_j = \hat{F}_j(t_j)$ for $j \leq i$, and the newly evolved forward rates, $F_j = \hat{F}_j(t_i)$ for $j > i$, which are still alive at time t_i.

- In the log discretisation proposed, discretisation error arises due to the fact that the drift is state dependent. In particular the evolution takes place assuming that the drift is constant and equal to the drift at the beginning of the time period (the initial time).
- Following an approach proposed by Hunter, Jäckel and Joshi (2001), it is possible to produce a more accurate estimate of the drift required over the update period.
- The idea is to evolve the rates to the end of the period and then compute the terminal drift using the evolved rates. Then, using the same random variates used to estimate the terminal drift, the initial rates are evolved using a drift computed as the average of the initial and terminal drift.
- To implement this approach, initialise F as before and change steps 3.3.1 and 3.3.3 in the algorithm in the following manner.

- Initialise $\bar{F}_j(t_0) = F_j(0)$ for all $0 \leq j < N$.
- Compute

$$\tilde{F}_j(t_i) = \bar{F}_j(t_{i-1}) \exp\left(\left(\bar{\mu}_j^{\text{init}}(t_{i-1}) - \tfrac{1}{2}\sigma_j^2\right)\delta_{i-1} + \sigma_j\sqrt{\delta_{i-1}}Z_i\right),$$

  where

$$\bar{\mu}_j^{\text{init}}(t_{i-1}) = \sum_{k=i}^{j} \frac{\delta_k \bar{F}_k(t_{i-1})\sigma_k\sigma_j}{1+\delta_k\bar{F}_k(t_{i-1})},$$

  for $j = i, \ldots, N-1$, where $Z_i \sim \mathscr{N}(0,1)$.
- Then, using the intermediate values $\tilde{F}_j(t_i)$, compute

$$\bar{\mu}_j^{\text{term}}(t_{i-1}) = \sum_{k=i}^{j} \frac{\delta_k \tilde{F}_k(t_i)\sigma_k\sigma_j}{1+\delta_k\tilde{F}_k(t_i)}.$$

- Finally, compute the new rates

$$\bar{F}_j(t_i) = \bar{F}_j(t_{i-1}) \exp\left(\tfrac{1}{2}\left(\bar{\mu}_j^{\text{init}}(t_{i-1}) + \bar{\mu}_j^{\text{term}}(t_{i-1}) - \sigma_j^2\right)\delta_{i-1} + \sigma_j\sqrt{\delta_{i-1}}Z_i\right),$$

  where the Z_i are the same normal random realizations used to compute $\tilde{F}_j(t_i)$.

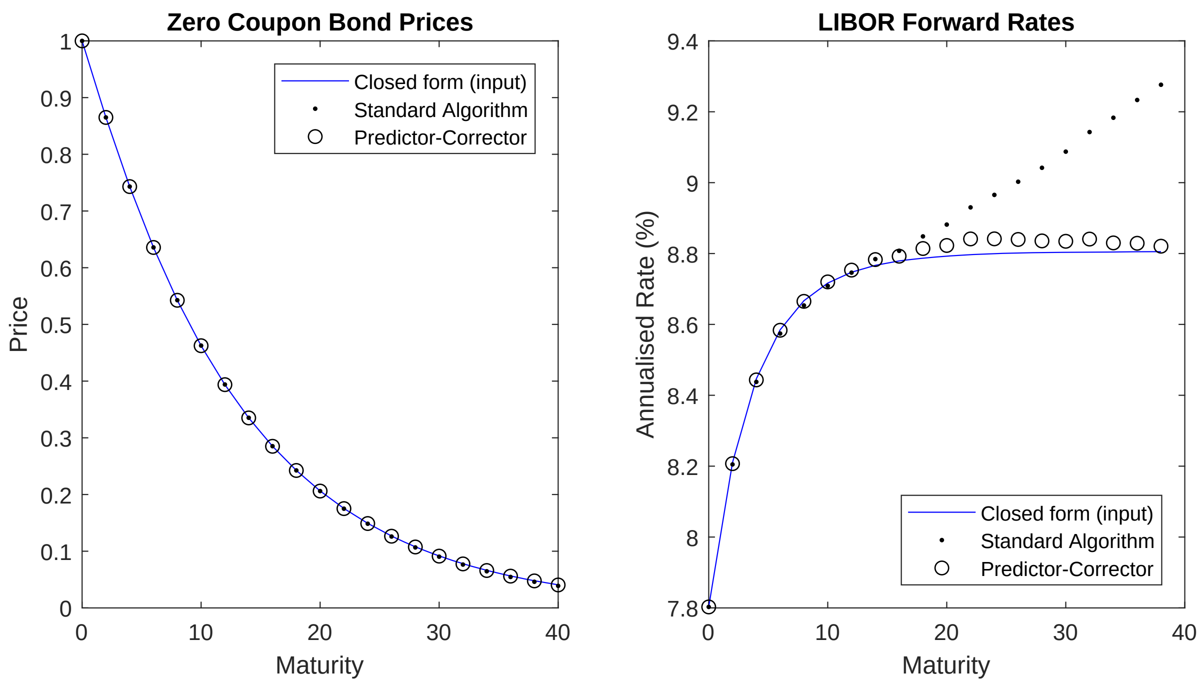
Example: Recovery of Bond Prices

- Again, we show the recovery of the initial term structure by pricing zero coupon bonds for various maturities. The simulated values are compared with the bond prices used to generate the initial LIBOR forward rates.
- As before, we use the closed form Vasicek short rate model to generate the zero compound bond prices with parameters

$$r_0 = 7\%, \quad \alpha = 0.15, \quad b = 9\%, \quad \text{and} \quad \sigma_v = 2\%,$$

  for maturities at two year intervals out to 40 years. Note that in practice market prices for the bonds (or quoted rates) should be used.
- Bond prices are simulated using the algorithm above and a Predictor-Corrector modified algorithm. The volatility used for all tenors was $\sigma_j = 20\%$.
- The simulated bond prices were computed as the mean of the corresponding deflator, $1/\bar{\beta}$, at each time step. Using the simulated bond prices, forward LIBOR rates were computed using (2) and compared with the input values.

- The following figure shows the results using $100\,000$ sample paths. Note that the original algorithm displays long-term bias as a result of cumulative discretisation error, while the Predictor-Corrector method gives accurate results (within Monte-Carlo sample error).



The Black Formula and Caplet Pricing

- In contrast to the short rate models and the HJM model, the LIBOR model allows log-normal dynamics of the forward rates using deterministic volatilities σ_i. Thus, it is a model that is compatible with the pricing of caplets under the Black Model.
- Consider a caplet with maturity T_i which has accrual period $[T_i, T_{i+1}]$. The applicable rate is F_i is fixed at T_i. The caplet payoff is $\delta_i(F_i(T_i) - K)^+$ but is paid in arrears (at time T_{i+1}).
- The risk-neutral price (under the spot measure) is thus

$$C_i(t_0) = \mathbb{E}\left[\frac{\delta_i(F_i(T_i) - K)^+}{\bar{\beta}(T_{i+1})}\right].$$

- If, however, we express this price under the forward measure associated with maturity T_{i+1} (see SCFII notes for details), we may price the caplet using

$$C_i(t_0) = B_{i+1}(t_0)\mathbb{E}^{i+1}\left[\frac{\delta_i(F_i(T_i) - K)^+}{B_{i+1}(T_{i+1})}\right],$$

  with $\mathbb{E}^{i+1}$ denoting expectation under the T_{i+1}-forward measure.
- Since the denominator of this expression is unity, if F_i is assumed log-normal with volatility σ_i the price is given using the Black Formula.

- The Black Formula for the price of the caplet is

$$C_i(t_0) = B_{\text{call}}(F_i(t_0), \sigma_i, T_i, K, \delta_i B_{i+1}(t_0)),$$

  where

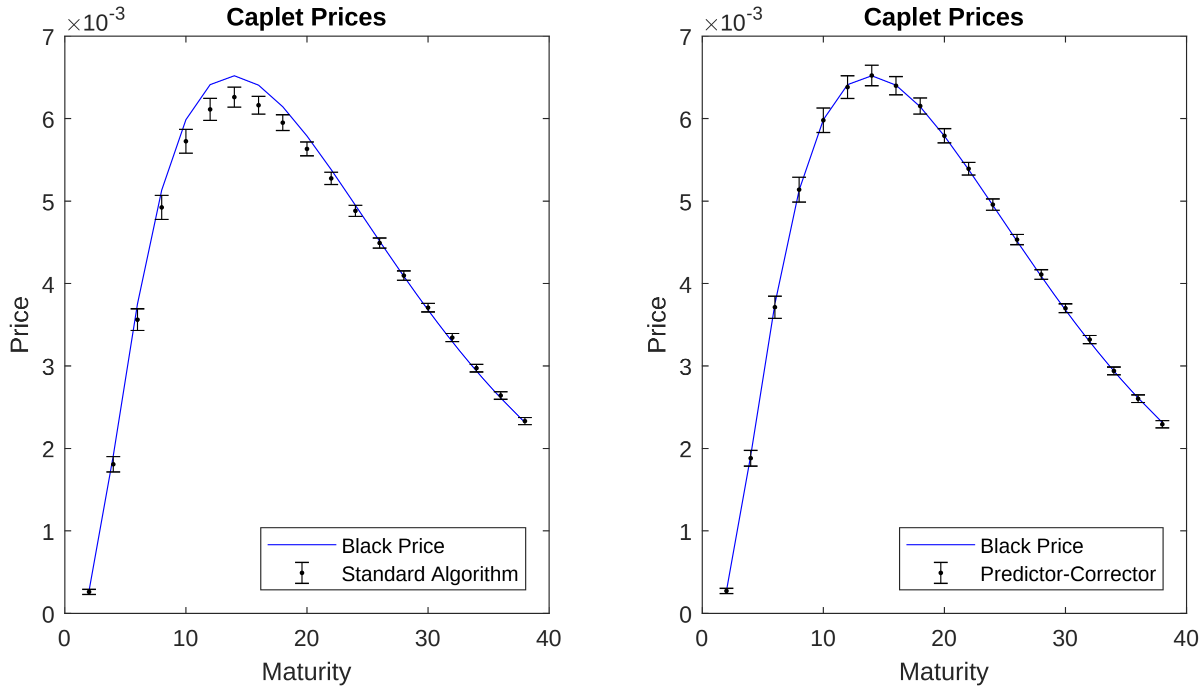
$$B_{\text{call}}(F, \sigma, T, K, B) = B(F\Phi(d_1) - K\Phi(d_2)),$$

  with

$$d_1 = \frac{\log(F/K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Example: Caplet Pricing

- Using the same input bond prices as in the previous example, caplets are priced and compared to the Black Formula. The strike used was $K = 15\%$. The results are shown below — Notice how the standard algorithm produces some results that are significantly biased (more than three standard deviations).
- While we have used a constant volatility in our model and shown that the model recovers the Black caplet price (using that volatility), usually the Black formula is used in reverse. The Black formula is inverted to find the volatility implied by market caplet prices. These volatilities are then used to calibrate the LIBOR model, thus allowing accurate pricing of more exotic interest derivatives.



Caplet Prices
×10^-3
Price
Maturity
Black Price
Standard Algorithm
0
1
2
3
4
5
6
7
0
10
20
30
40
Caplet Prices
×10^-3
Price
Maturity
Black Price
Predictor-Corrector
0
1
2
3
4
5
6
7
0
10
20
30
40