**Ejercicio 5** ★

Demostrar en deducción natural que las siguientes fórmulas son teoremas **sin usar principios de razonamiento clásicos** salvo que se indique lo contrario. Recordemos que una fórmula $\sigma$ es un teorema si y sólo si vale $\vdash \sigma$:

- I. *Modus ponens* relativizado: $(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau$
- II. Reducción al absurdo: $(\rho \Rightarrow \bot) \Rightarrow \neg\rho$
- III. Introducción de la doble negación: $\rho \Rightarrow \neg\neg\rho$
- IV. Eliminación de la triple negación: $\neg\neg\neg\rho \Rightarrow \neg\rho$
- V. Contraposición: $(\rho \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg\rho)$
- VI. Adjunción: $((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$
- VII. de Morgan (I): $\neg(\rho \vee \sigma) \Leftrightarrow (\neg\rho \wedge \neg\sigma)$
- VIII. de Morgan (II): $\neg(\rho \wedge \sigma) \Leftrightarrow (\neg\rho \vee \neg\sigma)$. Para la dirección $\Rightarrow$ es necesario usar principios de razonamiento clásicos.
- IX. Conmutatividad ($\wedge$): $(\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)$
- X. Asociatividad ($\wedge$): $((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$
- XI. Conmutatividad ($\vee$): $(\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)$
- XII. Asociatividad ($\vee$): $((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$

¿Encuentra alguna relación entre teoremas de adjunción, asociatividad y conmutatividad con algunas de las propiedades demostradas en la práctica 2?

i)

$\Gamma = (P \to Q \to T), (P \to Q), P$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (P \to Q \to T)}\,Ax \qquad \dfrac{}{\Gamma \vdash P \to Q}\,Ax}{(P \to Q \to T), (P \to Q), P \vdash T}\,{\Rightarrow}e}{(P \to Q \to T), (P \to Q) \vdash P \to T}\,{\Rightarrow}i}{(P \to Q \to T) \vdash (P \to Q) \to P \to T}\,{\Rightarrow}i}{\underbrace{(P \to Q \to T)}_{F2} \to (\underbrace{(P \to Q)}_{F1} \to (\underbrace{P}_{P1} \to (\underbrace{T}_{SAIDA})))}\,{\Rightarrow}i$$

ii)

$$\dfrac{\dfrac{\dfrac{}{(P \to \bot) \vdash P \to \bot}\,Ax \qquad \dfrac{\dfrac{}{(P \to \bot), \bot \vdash \bot}\,Ax}{(P \to \bot) \vdash \neg\bot}\,\neg i}{(P \to \bot) \vdash \neg P}\,MT}{\emptyset \vdash (P \to \bot) \Rightarrow \neg P}\,{\Rightarrow}i$$

iii)

$$\dfrac{\dfrac{\dfrac{}{P \vdash P}\,Ax}{P \vdash \neg\neg P}\,\neg\neg i}{P \to \neg\neg P}\,{\Rightarrow}i$$

iv)

$$\dfrac{\dfrac{\dfrac{}{\neg\neg\neg P, P \vdash \neg\neg\neg P}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg\neg P, P \vdash P}\,Ax}{\neg\neg\neg P, P \vdash \neg\neg P}\,\neg\neg i}{\neg\neg\neg P, P \vdash \neg\neg\neg\neg P}\,\neg\neg i}{\neg\neg\neg P, P \vdash \bot}\,\neg e}{\quad}\,\neg i$$

$$\frac{\neg\neg\neg P \vdash \neg P}{\neg\neg\neg P \Rightarrow \neg P} \Rightarrow i$$

v)

$$\frac{\dfrac{\dfrac{\overline{P \Rightarrow 6, \neg 6, P \vdash \neg 6}\,Ax \qquad \dfrac{\dfrac{\overline{P \Rightarrow 6, \neg 6, P \vdash P \Rightarrow 6}\,Ax \quad \overline{P \Rightarrow 6, \neg 6, P \vdash P}\,Ax}{P \Rightarrow 6, \neg 6, P \vdash 6} \Rightarrow e}{P \Rightarrow 6, \neg 6, P \vdash \neg\neg 6} \neg\neg i}{P \Rightarrow 6, \neg 6, P \vdash \bot} \neg e}{\dfrac{P \Rightarrow 6, \neg 6 \vdash \neg P}{\dfrac{P \Rightarrow 6 \vdash \neg 6 \Rightarrow \neg P}{(P \Rightarrow 6) \Rightarrow (\neg 6 \Rightarrow \neg P)} \Rightarrow i} \Rightarrow i} \neg i$$

T: 6

T: ¬6

vi)

$\Rightarrow$)

$$\frac{\overline{((P \wedge 6) \Rightarrow T), P, 6 \vdash (P \wedge 6) \Rightarrow T}\,Ax \qquad \dfrac{\overline{((P \wedge 6) \Rightarrow T), P, 6 \vdash P}\,Ax \quad \overline{((P \wedge 6) \Rightarrow T), P, 6 \vdash 6}\,Ax}{((P \wedge 6) \Rightarrow T), P, 6 \vdash P \wedge 6} \wedge i}{((P \wedge 6) \Rightarrow T), P, 6 \vdash T} \Rightarrow e$$

$$\frac{\dfrac{\dfrac{((P \wedge 6) \Rightarrow T), P, 6 \vdash T}{((P \wedge 6) \Rightarrow T), P \vdash 6 \Rightarrow T} \Rightarrow i}{((P \wedge 6) \Rightarrow T) \vdash (P \Rightarrow (6 \Rightarrow T))} \Rightarrow i}{((P \wedge 6) \Rightarrow T) \Rightarrow (P \Rightarrow 6 \Rightarrow T)} \Rightarrow i$$

T: (P∧6)

$\Leftarrow$)

$$\frac{\overline{(P \Rightarrow 6 \Rightarrow T), P \wedge 6 \vdash P \Rightarrow 6 \Rightarrow T}\,Ax \qquad \dfrac{\dfrac{\overline{(P \Rightarrow 6 \Rightarrow T), P \wedge 6, P \vdash P \wedge 6}\,Ax}{(P \Rightarrow 6 \Rightarrow T), P \wedge 6, P \vdash 6} \wedge e_2}{(P \Rightarrow 6 \Rightarrow T), P \wedge 6 \vdash P \Rightarrow 6} \Rightarrow i}{(P \Rightarrow 6 \Rightarrow T), P \wedge 6 \vdash T} \Rightarrow e$$

$$\frac{\dfrac{(P \Rightarrow 6 \Rightarrow T), P \wedge 6 \vdash T}{(P \Rightarrow 6 \Rightarrow T) \vdash ((P \wedge 6) \Rightarrow T)} \Rightarrow i}{(P \Rightarrow 6 \Rightarrow T) \Rightarrow ((P \wedge 6) \Rightarrow T)} \Rightarrow i$$

vii)

$\Rightarrow$)

$$\frac{\dfrac{\overline{\neg(P \vee 6), P \vdash P}\,Ax}{\neg(P \vee 6), P \vdash (P \vee 6)} \vee i_1}{\cdots} \neg\neg i \qquad \frac{\dfrac{\overline{\neg(P \vee 6), 6 \vdash 6}\,Ax}{\neg(P \vee 6), 6 \vdash P \vee 6} \vee i_2}{\cdots} \neg\neg i$$

$$\dfrac{\neg(P\vee G), P \vdash \neg(P\vee G) \qquad \neg(P\vee G), P \vdash \neg\neg(P\vee G)}{\neg(P\vee G), P \vdash \bot}\,\neg e \qquad \dfrac{\neg(P\vee G), G \vdash \neg(P\vee G) \qquad \neg(P\vee G), G \vdash \neg\neg(P\vee G)}{\neg(P\vee G), G \vdash \bot}\,\neg e$$

$$\dfrac{\neg(P\vee G), P \vdash \bot}{\neg(P\vee G) \vdash \neg P}\,\neg i \qquad \dfrac{\neg(P\vee G), G \vdash \bot}{\neg(P\vee G) \vdash \neg G}\,\neg i$$

$$\dfrac{\neg(P\vee G) \vdash \neg P \qquad \neg(P\vee G) \vdash \neg G}{\neg(P\vee G) \vdash \neg P \wedge \neg G}\,\wedge i$$

$$\dfrac{\neg(P\vee G) \vdash \neg P \wedge \neg G}{\neg(P\vee G) \Rightarrow (\neg P \wedge \neg G)}\,\Rightarrow i$$

$\Leftarrow)$

$$\dfrac{}{\Gamma', P \vdash \neg P \wedge \neg G}\,Ax \qquad \dfrac{}{\Gamma', G \vdash \neg P \wedge \neg G}\,Ax$$

$$\dfrac{}{\Gamma', P \vdash P}\,Ax \qquad \dfrac{\Gamma', P \vdash \neg P \wedge \neg G}{\Gamma', P \vdash \neg P}\,\wedge e_1 \qquad \dfrac{}{\Gamma', G \vdash G}\,Ax \qquad \dfrac{\Gamma', G \vdash \neg P \wedge \neg G}{\Gamma', G \vdash \neg G}\,\wedge e_2$$

$$\dfrac{}{\Gamma' \vdash P\vee G}\,Ax \qquad \dfrac{\Gamma', P \vdash P \qquad \Gamma', P \vdash \neg P}{\Gamma', P \vdash \bot} \qquad \dfrac{\Gamma', G \vdash G \qquad \Gamma', G \vdash \neg G}{\Gamma', G \vdash \bot}\,\neg e$$

$$\dfrac{\Gamma' \vdash P\vee G \qquad \Gamma', P \vdash \bot \qquad \Gamma', G \vdash \bot}{\Gamma' = (\neg P \wedge \neg G), (P\vee G) \vdash \bot}\,\vee e$$

$$\dfrac{\Gamma' = (\neg P \wedge \neg G), (P\vee G) \vdash \bot}{(\neg P \wedge \neg G) \vdash \neg(P\vee G)}\,\neg i$$

$$\dfrac{(\neg P \wedge \neg G) \vdash \neg(P\vee G)}{\emptyset \vdash (\neg P \wedge \neg G) \Rightarrow \neg(P\vee G)}\,\Rightarrow i$$

viii)

$\Rightarrow)$

$$\dfrac{}{\neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash (P\wedge G)}\,Ax \qquad \dfrac{}{\neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash \neg(P\wedge G)}\,Ax$$

$$\dfrac{\neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash (P\wedge G) \qquad \neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash \neg(P\wedge G)}{\neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash \bot}\,\neg e$$

$$\dfrac{\neg(P\wedge G), (\neg P\vee\neg G), (P\wedge G) \vdash \bot}{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash P\wedge G}\,PBC$$

-> tengo $\neg(P\wedge G)$. Busco negar $P\wedge G$ para $\neg e$.

$$\dfrac{}{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg(P\wedge G)}\,Ax \qquad \dfrac{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash P\wedge G}{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg\neg(P\wedge G)}\,\neg\neg i$$

$$\dfrac{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg(P\wedge G) \qquad \neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg\neg(P\wedge G)}{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \bot}\,\neg e$$

$$\dfrac{\neg(P\wedge G), \neg(\neg P\vee\neg G) \vdash \bot}{\neg(P\wedge G) \vdash (\neg P\vee\neg G)}\,\neg i$$

$$\dfrac{\neg(P\wedge G) \vdash (\neg P\vee\neg G)}{\neg(P\wedge G) \Rightarrow (\neg P\vee\neg G)}\,\Rightarrow i$$

$\Leftarrow)$

$$\dfrac{}{(\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash (\neg P\vee\neg G)}\,Ax \qquad \dfrac{}{(\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg(\neg P\vee\neg G)}\,Ax$$

$$\dfrac{(\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash (\neg P\vee\neg G) \qquad (\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash \neg(\neg P\vee\neg G)}{(\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash \bot}\,\neg i$$

$$\dfrac{(\neg P\vee\neg G), (P\wedge G), \neg(\neg P\vee\neg G) \vdash \bot}{(\neg P\vee\neg G), (P\wedge G) \vdash \neg(\neg P\vee\neg G)}\,PBC.$$

$$\dfrac{(\neg P\vee\neg G), (P\wedge G) \vdash (\neg P\vee\neg G) \qquad (\neg P\vee\neg G), (P\wedge G) \vdash \neg(\neg P\vee\neg G)}{(\neg P\vee\neg G), (P\wedge G) \vdash \bot}\,\neg e$$

$$\dfrac{(\neg P\vee\neg G), (P\wedge G) \vdash \bot}{(\neg P\vee\neg G) \vdash \neg(P\wedge G)}\,\neg i$$

$$\dfrac{(\neg P\vee\neg G) \vdash \neg(P\wedge G)}{(\neg P\vee\neg G) \Rightarrow \neg(P\wedge G)}\,\Rightarrow i$$

$(\neg P \vee \neg G) \Rightarrow \neg(P \wedge G)$

ix)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P\wedge G)\vdash P\wedge G}\,Ax}{(P\wedge G)\vdash G}\,\wedge e_2 \qquad \dfrac{\dfrac{}{(P\wedge G)\vdash P\wedge G}\,Ax}{(P\wedge G)\vdash P}\,\wedge e_1}{(P\wedge G)\vdash G\wedge P}\,\wedge i}{(P\wedge G)\Rightarrow(G\wedge P)}\,\Rightarrow i$$

x)

$\Rightarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((P\wedge G)\wedge T)\vdash(P\wedge G)\wedge T}\,Ax}{((P\wedge G)\wedge T)\vdash P\wedge G}\,\wedge e_2}{((P\wedge G)\wedge T)\vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{((P\wedge G)\wedge T)\vdash(P\wedge G)\wedge T}\,Ax}{((P\wedge G)\wedge T)\vdash(P\wedge G)}\,\wedge e_1}{((P\wedge G)\wedge T)\vdash G}\,\wedge e_2 \qquad \dfrac{\dfrac{}{((P\wedge G)\wedge T)\vdash(P\wedge G)\wedge T}\,Ax}{((P\wedge G)\wedge T)\vdash T}\,\wedge e_2}{((P\wedge G)\wedge T)\vdash(G\wedge T)}}{((P\wedge G)\wedge T)\vdash(P\wedge(G\wedge T))}\,\wedge i}{((P\wedge G)\wedge T)\Rightarrow(P\wedge(G\wedge T))}\,\Rightarrow i$$

$\Leftarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\wedge(G\wedge T)\vdash P\wedge(G\wedge T)}\,Ax}{P\wedge(G\wedge T)\vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{\dfrac{}{P\wedge(G\wedge T)\vdash P\wedge(G\wedge T)}\,Ax}{P\wedge(G\wedge T)\vdash G\wedge T}\,\wedge e_2}{P\wedge(G\wedge T)\vdash G}\,\wedge e_1}{P\wedge(G\wedge T)\vdash P\wedge G}\,\wedge i \qquad \dfrac{\dfrac{\dfrac{}{P\wedge(G\wedge T)\vdash P\wedge(G\wedge T)}\,Ax}{P\wedge(G\wedge T)\vdash(G\wedge T)}\,\wedge e_2}{P\wedge(G\wedge T)\vdash T}\,\wedge e_2}{P\wedge(G\wedge T)\vdash((P\wedge G)\wedge T)}\,\wedge i}{P\wedge(G\wedge T)\Rightarrow((P\wedge G)\wedge T)}\,\Rightarrow i$$

xi)

$$\dfrac{\dfrac{\dfrac{}{(P\vee G)\vdash P\vee G}\,Ax \qquad \dfrac{\dfrac{}{(P\vee G),P\vdash P}\,Ax}{(P\vee G),P\vdash G\vee P}\,\vee i_2 \qquad \dfrac{\dfrac{}{(P\vee G),G\vdash G}\,Ax}{(P\vee G),G\vdash G\vee P}\,\vee i_1}{(P\vee G)\vdash(G\vee P)}\,\vee e}{(P\vee G)\Rightarrow(G\vee P)}\,\Rightarrow i$$

xii)

$\Rightarrow)$

$$
\dfrac{
\dfrac{}{((\rho\vee\sigma)\vee\tau) \vdash (\rho\vee(\sigma\vee\tau))}\,\text{Ax}
\quad
\dfrac{\dfrac{}{((\rho\vee\sigma)\vee\tau), \rho \vdash \rho}\,\text{Ax}}{((\rho\vee\sigma)\vee\tau), \rho \vdash (\rho\vee(\sigma\vee\tau))}\,\vee i_1
\quad
\dfrac{\dfrac{}{((\rho\vee\sigma)\vee\tau), (\sigma\vee\tau) \vdash (\sigma\vee\tau)}\,\text{Ax}}{((\rho\vee\sigma)\vee\tau), (\sigma\vee\tau) \vdash (\rho\vee(\sigma\vee\tau))}\,\vee i_2
}{
\dfrac{((\rho\vee\sigma)\vee\tau) \vdash (\rho\vee(\sigma\vee\tau))}{\emptyset \vdash ((\rho\vee\sigma)\vee\tau) \Rightarrow (\rho\vee(\sigma\vee\tau))}\,\Rightarrow i
}\,\vee e
$$

$\tau = \rho$

$\sigma = (\sigma\vee\tau)$

$\rho = (\rho\vee(\sigma\vee\tau))$

$\Leftarrow)$

$$
\dfrac{
\dfrac{}{\rho\vee(\sigma\vee\tau) \vdash \rho\vee(\sigma\vee\tau)}\,\text{Ax}
\quad
\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \rho}\,\text{Ax}}{\Gamma' \vdash (\rho\vee\sigma)}\,\vee i_1}{\Gamma' = \rho\vee(\sigma\vee\tau), \rho \vdash (\rho\vee\sigma)\vee\tau}\,\vee i_1
\quad
\dfrac{
\dfrac{}{\Gamma'' \vdash \sigma\vee\tau}\,\text{Ax}
\quad
\dfrac{\dfrac{\dfrac{}{\Gamma'', \sigma \vdash \sigma}\,\text{Ax}}{\Gamma'', \sigma \vdash \rho\vee\sigma}\,\vee i_2}{\Gamma'', \sigma \vdash (\rho\vee\sigma)\vee\tau}\,\vee i_1
\quad
\dfrac{\dfrac{}{\Gamma'', \tau \vdash \tau}\,\text{Ax}}{\Gamma'', \tau \vdash (\rho\vee\sigma)\vee\tau}\,\vee i_2
}{\Gamma'' = \rho\vee(\sigma\vee\tau), (\sigma\vee\tau) \vdash (\rho\vee\sigma)\vee\tau}\,\vee e
}{
\dfrac{\rho\vee(\sigma\vee\tau) \vdash (\rho\vee\sigma)\vee\tau}{\emptyset \vdash \rho\vee(\sigma\vee\tau) \Rightarrow ((\rho\vee\sigma)\vee\tau)}\,\Rightarrow i
}\,\vee e
$$

o sale $\sigma$ o sale $\tau$

**Ejercicio 6** ★

Demostrar en deducción natural que vale $\vdash \sigma$ para cada una de las siguientes fórmulas. Para estas fórmulas es imprescindible **usar lógica clásica**:

- I. Absurdo clásico: $(\neg\tau \Rightarrow \bot) \Rightarrow \tau$
- II. Ley de Peirce: $((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau$
- III. Tercero excluido: $\tau \vee \neg\tau$
- IV. Consecuencia milagrosa: $(\neg\tau \Rightarrow \tau) \Rightarrow \tau$
- V. Contraposición clásica: $(\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)$
- VI. Análisis de casos: $(\tau \Rightarrow \rho) \Rightarrow (\neg\tau \Rightarrow \rho) \Rightarrow \rho$
- VII. Implicación vs. disyunción: $(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$

i)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash \neg\tau \Rightarrow \bot}\,\text{Ax} \quad \dfrac{}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash \neg\tau}\,\text{Ax}}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash \bot}\,\Rightarrow e}{(\neg\tau \Rightarrow \bot) \vdash \tau}\,\text{PBC}}{(\neg\tau \Rightarrow \bot) \Rightarrow \tau}\,\Rightarrow i
$$

$(\tau : \bot)$

solo luego aparecen $\Rightarrow$ con $\Rightarrow e$.

ii)

$$
\dfrac{\dfrac{
\dfrac{}{\Gamma' = \tau\vee\neg\tau}\,\text{Ax}
\quad
\dfrac{}{\Gamma', \tau \vdash \tau}\,\text{Ax}
\quad
\dfrac{
\dfrac{}{\Gamma'' \vdash (\tau\Rightarrow\rho)\Rightarrow\tau}\,\text{Ax}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'', \tau \vdash \tau}\,\text{Ax} \quad \dfrac{}{\Gamma'', \tau \vdash \neg\tau}\,\text{Ax}}{\Gamma'', \tau \vdash \bot}\,\neg e}{\Gamma'', \tau \vdash \rho}\,\bot e}{\Gamma'' \vdash \tau\Rightarrow\rho}\,\Rightarrow i
}{\Gamma'' = \Gamma', \neg\tau \vdash \tau}\,\Rightarrow e
}{\Gamma' = ((\tau\Rightarrow\rho)\Rightarrow\tau) \vdash \tau}}{((\tau\Rightarrow\rho)\Rightarrow\tau) \Rightarrow \tau}\,\Rightarrow i
$$

$((\tau\Rightarrow\rho)\Rightarrow\tau), \neg\tau$ contradicción iii)

$$
\dfrac{
  \dfrac{}{\neg T \vdash \neg T}\,\text{Ax}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\neg T, T \vdash \neg T}\,\text{Ax}
        \quad
        \dfrac{}{\neg T, T \vdash T}\,\text{ax}
      }{\neg T, T \vdash \bot}\,\neg e\ (T = T)
    }{\neg T \vdash T}\,\text{PBC}
  }{\neg T \vdash \neg\neg T}\,\neg\neg i
}{
  \dfrac{\neg T \vdash \bot}{
    \dfrac{\vdash T}{\vdash (T \vee \neg T)}\,\vee i_1
  }\,\text{PBC}
}\,\neg e\ (T = \neg T)
$$

iv)

$$
\dfrac{
  \dfrac{
    \dfrac{}{(\neg T \Rightarrow T), \neg T \vdash \neg T \Rightarrow T}\,\text{Ax}
    \quad
    \dfrac{}{(\neg T \Rightarrow T), \neg T \vdash \neg T}\,\text{Ax}
  }{(\neg T \Rightarrow T), \neg T \vdash T}\,\Rightarrow e
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{(\neg T \Rightarrow T), \neg T, T \vdash \neg T \qquad (\neg T \Rightarrow T), \neg T, T \vdash T}\,\text{Ax}
    }{(\neg T \Rightarrow T), \neg T, T \vdash \bot}\,\neg e
  }{(\neg T \Rightarrow T), \neg T \vdash \neg T}\,\text{PBC}
}{
  \dfrac{(\neg T \Rightarrow T), \neg T \vdash \bot}{
    \dfrac{(\neg T \Rightarrow T) \vdash T}{(\neg T \Rightarrow T) \Rightarrow T}\,\Rightarrow i
  }\,\text{PBC}
}\,\neg e
$$

v)

$$
\dfrac{
  \dfrac{}{(\neg P \Rightarrow \neg T), T, \neg P \vdash T}\,\text{Ax}
  \qquad
  \dfrac{
    \dfrac{}{(\neg P \Rightarrow \neg T), T, \neg P \vdash (\neg P \Rightarrow \neg T)}\,\text{Ax}
    \quad
    \dfrac{}{(\neg P \Rightarrow \neg T), T, \neg P \vdash \neg P}\,\text{Ax}
  }{(\neg P \Rightarrow \neg T), T, \neg P \vdash \neg T}\,\Rightarrow e
}{
  \dfrac{(\neg P \Rightarrow \neg T), T, \neg P \vdash \bot}{
    \dfrac{(\neg P \Rightarrow \neg T), T \vdash P}{
      \dfrac{(\neg P \Rightarrow \neg T) \vdash T \Rightarrow P}{(\neg P \Rightarrow \neg T) \Rightarrow (T \Rightarrow P)}\,\Rightarrow i
    }\,\Rightarrow i
  }\,\text{PBC}
}\,\neg e
$$

PBC $\to$ para usar $(\neg P \Rightarrow \neg T)$ g $\dots \vdash \neg P$.

vii)

$$
\dfrac{}{(T \Rightarrow P), (\neg T \Rightarrow P), \neg P, \neg T \vdash \neg P}\,\text{Ax}
\qquad
\dfrac{
  \dfrac{}{(T \Rightarrow P), (\neg T \Rightarrow P), \neg P, \neg T \vdash \neg T \Rightarrow P}\,\text{Ax}
  \quad
  \dfrac{}{(T \Rightarrow P), (\neg T \Rightarrow P), \neg P, \neg T \vdash \neg T}\,\text{Ax}
}{(T \Rightarrow P), (\neg T \Rightarrow P), \neg P, \neg T \vdash P}\,\Rightarrow e
$$

$$\dfrac{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P,\neg T\vdash\bot}{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash T}\;PBC$$

$$\dfrac{\dfrac{}{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash T\Rightarrow P}\;Ax \qquad (T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash T}{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash P}\;\Rightarrow e$$

$$\dfrac{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash P}{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash\neg\neg P}\;\neg\neg i$$

$$\dfrac{\dfrac{}{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash\neg P}\;Ax \qquad (T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash\neg\neg P}{\dfrac{(T\Rightarrow P),(\neg T\Rightarrow P),\neg P\vdash\bot}{\dfrac{(T\Rightarrow P),(\neg T\Rightarrow P)\vdash P}{\dfrac{(T\Rightarrow P)\vdash(\neg T\Rightarrow P)\Rightarrow P}{(T\Rightarrow P)\Rightarrow((\neg T\Rightarrow P)\Rightarrow P)}\;\Rightarrow i}\;\Rightarrow i}\;PBC}\;\neg e$$

Vii)

$\Rightarrow$)

$$\dfrac{(T\Rightarrow P),\neg T,\neg P\vdash\bot}{(T\Rightarrow P),\neg T\vdash P} \qquad \dfrac{(T\Rightarrow P),\neg T,P\vdash\bot}{(T\Rightarrow P),\neg T\vdash\neg P}$$

$$\dfrac{(T\Rightarrow P),\neg T\vdash P \qquad (T\Rightarrow P),\neg T\vdash\neg P}{(T\Rightarrow P),\neg T\vdash\bot}$$

$$\dfrac{\dfrac{}{(T\Rightarrow P)\vdash(T\Rightarrow P)}\;Ax \qquad \dfrac{(T\Rightarrow P),\neg T\vdash\bot}{(T\Rightarrow P)\vdash T}}{\dfrac{(T\Rightarrow P)\vdash P}{\dfrac{(T\Rightarrow P)\vdash(\neg T\vee P)}{((T\Rightarrow P)\Rightarrow(\neg T\vee P))}\;\Rightarrow i}\;\vee e_2}\;\Rightarrow e$$

$\Leftarrow$)

$$\dfrac{\dfrac{}{(\neg T\vee P),T,\neg T,\neg P\vdash T}\;Ax \qquad \dfrac{}{(\neg T\vee P),T,\neg T,\neg P\vdash\neg T}\;Ax}{\dfrac{(\neg T\vee P),T,\neg T,\neg P\vdash\bot}{(\neg T\vee P),T,\neg T\vdash P}\;PBC}\;\neg e$$

$$\dfrac{\dfrac{}{(\neg T\vee P),T\vdash(\neg T\vee P)}\;Ax \qquad (\neg T\vee P),T,\neg T\vdash P \qquad (\neg T\vee P),T,P\vdash P}{\dfrac{(\neg T\vee P),T\vdash P}{\dfrac{(\neg T\vee P)\vdash T\Rightarrow P}{(\neg T\vee P)\Rightarrow(T\Rightarrow P)}\;\Rightarrow i}\;\Rightarrow i}\;\vee e$$

$\tau:\neg T$

$\sigma: P$

$$\overline{\Gamma,\tau\vdash\tau}\;\text{ax}$$

$$\dfrac{\Gamma\vdash\tau\quad\Gamma\vdash\sigma}{\Gamma\vdash\tau\wedge\sigma}\;\wedge_i \qquad \dfrac{\Gamma\vdash\tau\wedge\sigma}{\Gamma\vdash\tau}\;\wedge_{e_1} \qquad \dfrac{\Gamma\vdash\tau\wedge\sigma}{\Gamma\vdash\sigma}\;\wedge_{e_2}$$

$$\dfrac{\Gamma,\tau\vdash\sigma}{\Gamma\vdash\tau\Rightarrow\sigma}\;\Rightarrow_i \qquad \dfrac{\Gamma\vdash\tau\Rightarrow\sigma\quad\Gamma\vdash\tau}{\Gamma\vdash\sigma}\;\Rightarrow_e$$

$$\vee_{i_1} \qquad \vee_{i_2} \qquad \Gamma\vdash\tau\vee\sigma\quad\Gamma,\tau\vdash\rho\quad\Gamma,\sigma\vdash\rho\;\;\vee_e$$

$$\frac{\Gamma, \tau \vdash \bot}{\Gamma \vdash \neg\tau}\ \neg_i \qquad \frac{\Gamma \vdash \tau \quad \Gamma \vdash \neg\tau}{\Gamma \vdash \bot}\ \neg_e$$

Lógica intuicionista $\qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \tau}\ \bot_e$

Lógica clásica $\qquad \frac{\Gamma \vdash \neg\neg\tau}{\Gamma \vdash \tau}\ \neg\neg_e$

**Ejercicio 9**

Probar los siguientes teoremas:

- I. $((P \Rightarrow Q) \Rightarrow Q) \Rightarrow ((Q \Rightarrow P) \Rightarrow P)$
- II. $(P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)$

i)

$$\frac{\overline{\Gamma, P \vdash P}\ Ax \qquad \overline{\Gamma, P \vdash \neg P}\ Ax}{\Gamma, P \vdash \bot}\ \neg e$$

$$\frac{\Gamma, P \vdash \bot}{\Gamma, P \vdash Q}\ \bot e$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash (P \Rightarrow Q)}\ \Rightarrow i$$

$$\frac{\overline{\Gamma \vdash (P \Rightarrow Q) \Rightarrow Q}\ Ax \qquad \Gamma \vdash (P \Rightarrow Q)}{\underbrace{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P), \neg P}_{\Gamma} \vdash P}\ \Rightarrow e$$

$$\frac{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P), \neg P \vdash P \qquad \overline{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P), \neg P \vdash \neg P}\ Ax}{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P), \neg P \vdash \bot}\ \neg e$$

$$\frac{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P), \neg P \vdash \bot}{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P) \vdash P}\ PBC$$

$$\frac{((P \Rightarrow Q) \Rightarrow Q), (Q \Rightarrow P) \vdash P}{((P \Rightarrow Q) \Rightarrow Q) \vdash (Q \Rightarrow P) \Rightarrow P}\ \Rightarrow i$$

$$\frac{((P \Rightarrow Q) \Rightarrow Q) \vdash (Q \Rightarrow P) \Rightarrow P}{\emptyset \vdash ((P \Rightarrow Q) \Rightarrow Q) \Rightarrow ((Q \Rightarrow P) \Rightarrow P)}\ \Rightarrow i$$

ii)

$$\frac{\overline{\Gamma, \neg P \vdash \neg P \Rightarrow Q}\ Ax \qquad \overline{\Gamma, \neg P \vdash \neg P}\ Ax}{\Gamma, \neg P \vdash Q}$$

$$\frac{\overline{\Gamma, \neg P \vdash \neg Q}\ Ax \qquad \Gamma, \neg P \vdash Q}{\Gamma, \neg P \vdash \bot}\ \neg e$$

$$\frac{\Gamma, \neg P \vdash \bot}{\Gamma \vdash P}\ PBC$$

$$\frac{\overline{\Gamma \vdash P \Rightarrow Q}\ Ax \qquad \Gamma \vdash P}{\Gamma \vdash Q}$$

$$\frac{\Gamma \vdash Q \qquad \overline{\Gamma \vdash \neg Q}\ Ax}{\Gamma : P \Rightarrow Q, \neg P \Rightarrow Q, \neg Q \vdash \bot}\ \neg e$$

$$\frac{P \Rightarrow Q, \neg P \Rightarrow Q, \neg Q \vdash \bot}{P \Rightarrow Q, \neg P \Rightarrow Q \vdash Q}\ PBC$$

$$\frac{P \Rightarrow Q, \neg P \Rightarrow Q \vdash Q}{(P \Rightarrow Q) \vdash (\neg P \Rightarrow Q) \Rightarrow Q}\ \Rightarrow i$$

$$\frac{(P \Rightarrow Q) \vdash (\neg P \Rightarrow Q) \Rightarrow Q}{\emptyset \vdash (P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)}\ \Rightarrow i$$

Lo q busqué

$\Gamma \vdash Q$ $\qquad \overline{\Gamma \vdash \neg Q}\ Ax$

$\Gamma \vdash \neg P \Rightarrow Q$ , $\Gamma \vdash \neg P$

**Ejercicio 10**

Demostrar las siguientes tautologías utilizando deducción natural.

- I. $(P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$
- II. $(R \Rightarrow \neg Q) \Rightarrow ((R \wedge Q) \Rightarrow P)$
- III. $((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \Rightarrow \neg(R \wedge Q)$

i)

$$\frac{\overline{\Gamma \vdash P \Rightarrow (P \Rightarrow Q)}\ Ax \qquad \overline{\Gamma \vdash P}\ Ax}{\Gamma \vdash P \Rightarrow Q}\ \Rightarrow e \qquad \overline{\Gamma \vdash P}\ Ax$$

$\Rightarrow_e$

$\Gamma = P \Rightarrow (P \Rightarrow Q), P \vdash Q$ $\quad \Rightarrow_i$

$P \Rightarrow (P \Rightarrow Q) \vdash (P \Rightarrow Q)$ $\quad \Rightarrow_i$

$$\emptyset \vdash (P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$$

ii)

$\overline{\Gamma, \neg R \vdash R \wedge Q}$ Ax $\quad \wedge_{e_1}$ $\qquad \overline{\Gamma, R \vdash R \wedge Q}$ Ax $\quad \wedge_{e_2}$ $\qquad \overline{\Gamma, R \vdash R \Rightarrow \neg Q}$ Ax $\qquad \overline{\Gamma, R \vdash R}$ Ax $\quad \Rightarrow_e$

$\Gamma, \neg R \vdash R \qquad \overline{\Gamma, \neg R \vdash \neg R}$ Ax $\quad \neg e \qquad \Gamma, R \vdash Q \qquad \Gamma, R \vdash \neg Q \quad \neg e$

$\Gamma, \neg R \vdash \bot$ $\quad$ PBC $\qquad \Gamma, R \vdash \bot$ $\quad \neg i$

$\Gamma \vdash R \qquad \Gamma \vdash \neg R$

$\Gamma = (R \Rightarrow \neg Q), (R \wedge Q), \neg P \vdash \bot$ $\quad$ PBC

$(R \Rightarrow \neg Q), (R \wedge Q) \vdash P$ $\quad \Rightarrow_i$

$(R \Rightarrow \neg Q) \vdash (R \wedge Q) \Rightarrow P$ $\quad \Rightarrow_i$

$$\emptyset \vdash (R \Rightarrow \neg Q) \Rightarrow ((R \wedge Q) \Rightarrow P)$$

Estrategia

$R \Rightarrow \neg Q \quad \underline{R}$

$\underline{\neg Q}$

$\underline{Q}$

iii)

$\overline{\Gamma, R, P \vdash R \wedge Q}$ Ax $\quad \wedge_{e_2}$

$\overline{\Gamma, R \vdash (P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)}$ Ax $\qquad \Gamma, R, P \vdash Q \quad \Rightarrow_i$

$\Gamma, R \vdash (P \Rightarrow Q)$ $\quad \Rightarrow_e$

$\overline{\Gamma, R \vdash R \wedge Q}$ Ax $\quad \wedge_{e_2} \qquad \Gamma, R \vdash R \Rightarrow \neg Q \qquad \overline{\Gamma, R \vdash R}$ Ax $\quad \Rightarrow_e$

$\overline{\Gamma \vdash R \wedge Q}$ Ax $\quad \wedge_{e_1} \qquad \Gamma, R \vdash Q \qquad \Gamma, R \vdash \neg Q \quad \neg e$

$\Gamma, R \vdash \bot$ $\quad \neg i$

$\Gamma \vdash R \qquad \Gamma \vdash \neg R$ $\quad \neg e$

$\Gamma = ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)), (R \wedge Q) \vdash \bot$ $\quad \neg i$

$((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \vdash \neg (R \wedge Q)$ $\quad \Rightarrow_i$

$$\emptyset \vdash ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \Rightarrow \neg (R \wedge Q)$$

$(P \Rightarrow Q) \Rightarrow \ldots$

$\underline{R \wedge Q}$ ✓

$\underline{Q}$ $\Rightarrow_i$

$P \Rightarrow Q$

**Ejercicio 11**

Probar que los siguientes secuentes son válidos sin usar principios de razonamiento clásicos:

- I. $(P \wedge Q) \wedge R, S \wedge T \vdash Q \wedge S$
- II. $(P \wedge Q) \wedge R \vdash P \wedge (Q \wedge R)$
- III. $P \Rightarrow (P \Rightarrow Q), P \vdash Q$
- IV. $Q \Rightarrow (P \Rightarrow R), \neg R, Q \vdash \neg P$
- V. $\vdash (P \wedge Q) \Rightarrow P$
- VI. $P \Rightarrow \neg Q, Q \vdash \neg P$
- VII. $P \Rightarrow Q \vdash (P \wedge R) \Rightarrow (Q \wedge R)$
- VIII. $Q \Rightarrow R \vdash (P \vee Q) \Rightarrow (P \vee R)$
- IX. $(P \vee Q) \vee R \vdash P \vee (Q \vee R)$
- X. $P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)$
- XI. $(P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)$
- XII. $\neg P \vee Q \vdash P \Rightarrow Q$
- XIII. $P \Rightarrow Q, P \Rightarrow \neg Q \vdash \neg P$
- XIV. $P \Rightarrow (Q \Rightarrow R), P, \neg R \vdash \neg Q$

i) $\overline{\phantom{xxxxxx}}$ Ax

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash (P\wedge Q)\wedge R}{\Gamma \vdash P \wedge Q}\,\wedge_{e1}}{\Gamma \vdash Q}\,\wedge_{e2} \qquad \dfrac{\dfrac{}{\Gamma \vdash S \wedge T}\,Ax}{\Gamma \vdash S}\,\wedge_{e1}}{}\,\wedge_i$$

$$\Gamma = (P\wedge Q)\wedge R,\ S\wedge T \ \vdash Q \wedge S$$

ii)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (P\wedge Q)\wedge R}\,Ax}{\Gamma \vdash P \wedge Q}\,\wedge_{e1}}{\Gamma \vdash P}\,\wedge_{e1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (P\wedge Q)\wedge R}\,Ax}{\Gamma \vdash (P\wedge Q)}\,\wedge_{e1}}{\Gamma \vdash Q}\,\wedge_{e2} \qquad \dfrac{\dfrac{}{\Gamma \vdash (P\wedge Q)\wedge R}\,Ax}{\Gamma \vdash R}\,\wedge_{e2}}{\Gamma \vdash Q \wedge R}\,\wedge_i}{}\,\wedge_i$$

$$\Gamma = (P\wedge Q)\wedge R \vdash P \wedge (Q \wedge R)$$

iii)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow (P \Rightarrow Q)}\,Ax \qquad \dfrac{}{\Gamma \vdash P}\,Ax}{\Gamma \vdash P \Rightarrow Q}\,\Rightarrow e \qquad \dfrac{}{\Gamma \vdash P}\,Ax}{}\,\Rightarrow e$$

$$\Gamma = P \Rightarrow (P \Rightarrow Q),\ P \vdash Q$$

iv)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash Q \Rightarrow (P \Rightarrow R)}\,Ax \quad \dfrac{}{\Gamma, P \vdash Q}\,Ax}{\Gamma, P \vdash P \Rightarrow R}\,\Rightarrow e \qquad \dfrac{}{\Gamma, P \vdash P}\,Ax}{\Gamma, P \vdash R}\,\Rightarrow e \qquad \dfrac{}{\Gamma, P \vdash \neg R}\,Ax}{\Gamma, P \vdash \bot}\,\neg e}{}\,\neg i$$

$$\Gamma = Q \Rightarrow (P \Rightarrow R),\ \neg R,\ Q \vdash \neg P$$

$\Gamma \vdash T \Rightarrow G$ $\qquad$ $\Gamma \vdash T$ $\qquad$ ESTRATEGIA

v)

$$\dfrac{\dfrac{\dfrac{}{P \wedge Q \vdash P \wedge Q}\,Ax}{P \wedge Q \vdash P}\,\wedge_{e1}}{\emptyset \vdash (P \wedge Q) \Rightarrow P}\,\Rightarrow i$$

vi)

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash Q}\,Ax \qquad \dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow \neg Q}\,Ax \quad \dfrac{}{\Gamma \vdash P}\,Ax}{\Gamma \vdash \neg Q}\Rightarrow e}{\Gamma = P \Rightarrow \neg Q, Q, P \vdash \bot}\neg e}{P \Rightarrow \neg Q, Q \vdash \neg P}\neg i$$

vii)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \Rightarrow Q}\,Ax \quad \dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\,Ax}{\Gamma \vdash P}\wedge e_1}{\Gamma \vdash Q}\Rightarrow e \qquad \dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\,Ax}{\Gamma \vdash R}\wedge e_2}{\Gamma = P \Rightarrow Q, P \wedge R \vdash Q \wedge R}\wedge i}{P \Rightarrow Q \vdash (P \wedge R) \Rightarrow (Q \wedge R)}\Rightarrow i$$

viii)

"O BIEN P V, O BIEN Q V"

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \vee Q}\,Ax \qquad \dfrac{\dfrac{}{\Gamma, P \vdash P}\,Ax}{\Gamma, P \vdash P \vee R}\vee i_1 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, Q \vdash Q \Rightarrow R}\,Ax \quad \dfrac{}{\Gamma, Q \vdash Q}\,Ax}{\Gamma, Q \vdash R}\Rightarrow e}{\Gamma, Q \vdash P \vee R}\vee i_2}{\Gamma = (Q \Rightarrow R), (P \vee Q) \vdash P \vee R}\vee e}{Q \Rightarrow R \vdash (P \vee Q) \Rightarrow (P \vee R)}\Rightarrow i \ \checkmark$$

ix)

$$\dfrac{\dfrac{}{\Gamma \vdash (P \vee Q) \vee R}\,Ax \qquad \dfrac{\dfrac{}{\Gamma \vdash P \vee Q}\,Ax \quad \dfrac{\dfrac{}{\Gamma, P \vdash P}\,Ax}{\Gamma, P \vdash P \vee (Q \vee R)}\vee e_1 \quad \dfrac{\dfrac{}{\Gamma, Q \vdash Q}\,Ax}{\Gamma, Q \vdash Q \vee R}\vee e_1}{\Gamma, (P \vee Q) \vdash P \vee (Q \vee R)} \qquad \dfrac{\dfrac{}{\Gamma, R \vdash R}\,Ax}{\Gamma, R \vdash P \vee (Q \vee R)}\vee i_{2 \times 2}}{(P \vee Q) \vee R \vdash P \vee (Q \vee R)}$$

o vale P o vale Q. pero P y Q

x)

$$\dfrac{}{\quad}\,Ax$$

$$\dfrac{\Gamma' \vdash P \wedge (Q \vee R)}{\Gamma' \vdash P}\,\wedge_{e_1} \qquad \dfrac{}{\Gamma' \vdash Q}\,Ax$$

$$\dfrac{\Gamma' \vdash P \qquad \Gamma' \vdash Q}{\Gamma' = P \wedge (Q \vee R), Q \vdash (P \wedge Q)}\,\wedge_i$$

$$\dfrac{\dfrac{}{\Gamma'' \vdash P \wedge (Q \vee R)}\,Ax}{\Gamma'' \vdash P}\,\wedge_{e_1} \qquad \dfrac{}{\Gamma'' \vdash R}\,Ax$$

$$\dfrac{\Gamma'' \vdash P \qquad \Gamma'' \vdash R}{\Gamma'' \vdash P \wedge R}\,\wedge_i$$

$$\dfrac{\dfrac{}{\Gamma \vdash P \wedge (Q \vee R)}\,Ax}{\Gamma \vdash Q \vee R}\,\wedge_{e_2}$$

$$\dfrac{\Gamma' = P \wedge (Q \vee R), Q \vdash (P \wedge Q)}{P \wedge (Q \vee R), Q \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee_{i_1} \qquad \dfrac{\Gamma'' \vdash P \wedge R}{\Gamma'' = P \wedge (Q \vee R), R \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee_{i_2}$$

si valen P y Q. ME QUEDO CON $(P \wedge Q)$

SE Q VALEN P y R. ME QUEDO CON $P \wedge R$

$$\dfrac{\Gamma \vdash Q \vee R \qquad P \wedge (Q \vee R), Q \vdash (P \wedge Q) \vee (P \wedge R) \qquad \Gamma'' = P \wedge (Q \vee R), R \vdash (P \wedge Q) \vee (P \wedge R)}{P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee_e$$

P vale. Averiguo si vale Q o R.

X:)

$$\dfrac{\dfrac{}{\Gamma, (P \wedge Q) \vdash P \wedge Q}\,Ax}{\Gamma, (P \wedge Q) \vdash P}\,\wedge_{e_1} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, (P \wedge Q) \vdash (P \wedge Q)}\,Ax}{\Gamma, (P \wedge Q) \vdash Q}\,\wedge_{e_2}}{\Gamma, (P \wedge Q) \vdash Q \vee R}\,\vee_{i_1}$$

$$\dfrac{\Gamma, (P \wedge Q) \vdash P \qquad \Gamma, (P \wedge Q) \vdash Q \vee R}{\Gamma, (P \wedge Q) \vdash P \wedge (Q \vee R)}\,\wedge_i$$

VALEN P y Q. TENGO P, BUSCO Q

$$\dfrac{\dfrac{}{\Gamma, (P \wedge R) \vdash (P \wedge R)}\,Ax}{\Gamma, (P \wedge R) \vdash P}\,\wedge_{e_1} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, (P \wedge R) \vdash (P \wedge R)}\,Ax}{\Gamma, (P \wedge R) \vdash R}\,\wedge_{e_2}}{\Gamma, (P \wedge R) \vdash (Q \vee R)}\,\vee_{i_2}$$

$$\dfrac{\Gamma, (P \wedge R) \vdash P \qquad \Gamma, (P \wedge R) \vdash (Q \vee R)}{\Gamma, (P \wedge R) \vdash P \wedge (Q \vee R)}\,\wedge_i$$

VALEN P y R. TENGO P, BUSCO R.

$$\dfrac{\dfrac{}{\Gamma \vdash (P \wedge Q) \vee (P \wedge R)}\,Ax \qquad \Gamma, (P \wedge Q) \vdash P \wedge (Q \vee R) \qquad \Gamma, (P \wedge R) \vdash P \wedge (Q \vee R)}{(P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)}\,\vee_e$$

P vale en ambos.

Debo ver si valen Q o R.

Xii)

$$\dfrac{\dfrac{}{\Gamma' \vdash P}\,Ax \qquad \dfrac{}{\Gamma' \vdash \neg P}\,Ax}{\Gamma' \vdash \bot}\,\neg_e$$

Fuerzo Bot.

$$\dfrac{\dfrac{}{\neg P \vee Q, P \vdash \neg P \vee Q}\,Ax \qquad \dfrac{\Gamma' \vdash \bot}{\Gamma' = \neg P \vee Q, P, \neg P \vdash Q}\,\bot_e \qquad \dfrac{}{\neg P \vee Q, P, Q \vdash Q}\,Ax}{\neg P \vee Q, P \vdash Q}$$

$$\dfrac{\neg P \vee Q, P \vdash Q}{\neg P \vee Q \vdash P \Rightarrow Q}\,\Rightarrow_i$$

Si vale $\neg P$ o $Q$. Necesito dos casos.

xiii)

$$\dfrac{\dfrac{}{\Gamma' \vdash P \Rightarrow Q}\,Ax \qquad \dfrac{}{\Gamma' \vdash P}\,Ax}{\Gamma' = P \Rightarrow Q, P \Rightarrow \neg Q, P \vdash Q}\,\Rightarrow_e \qquad \dfrac{\dfrac{}{\Gamma'' \vdash P \Rightarrow \neg Q}\,Ax \qquad \dfrac{}{\Gamma'' \vdash P}\,Ax}{\Gamma'' = P \Rightarrow Q, P \Rightarrow \neg Q, P \vdash \neg Q}\,\Rightarrow_e$$

$\neg e$

$$P \Rightarrow Q,\ P \Rightarrow \neg Q,\ P \vdash \bot$$

$\neg i$

$$P \Rightarrow Q,\ P \Rightarrow \neg Q \vdash \neg P$$

VALE P, BUSCO METER P A CONTEXTO

X iv)

$$\dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash \neg R}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash P \Rightarrow (Q \Rightarrow R)}\,Ax \quad \dfrac{}{\Gamma'' \vdash P}\,Ax}{\Gamma'' \vdash (Q \Rightarrow R)}\,{\Rightarrow}e \qquad \dfrac{}{\Gamma'' \vdash Q}\,Ax}{\Gamma'' \vdash R}\,{\Rightarrow}e}{\Gamma'' = P \Rightarrow (Q \Rightarrow R),\ P,\ \neg R,\ Q \vdash \bot}\,\neg e}{\Gamma' = P \Rightarrow (Q \Rightarrow R),\ P,\ \neg R \vdash \neg Q}\,\neg i$$

VALE P

NO VALE R.

(Q = V)

(Q=>R). necesito Q xq gracias a él obtengo R.

**Ejercicio 12**

Probar que los siguientes secuentes son válidos:

- I. $(P \wedge \neg Q) \Rightarrow R, \neg R, P \vdash Q$
- II. $\neg P \Rightarrow Q \vdash \neg Q \Rightarrow P$
- III. $P \vee Q \vdash R \Rightarrow (P \vee Q) \wedge R$
- IV. $(P \vee (Q \Rightarrow P)) \wedge Q \vdash P$
- V. $P \Rightarrow Q, R \Rightarrow S \vdash (P \wedge R) \Rightarrow (Q \wedge S)$
- VI. $P \Rightarrow Q \vdash ((P \wedge Q) \Rightarrow P) \wedge (P \Rightarrow (P \wedge Q))$
- VII. $P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)$
- VIII. $(P \Rightarrow Q) \wedge (P \Rightarrow R) \vdash P \Rightarrow (Q \wedge R)$
- IX. $P \vee (P \wedge Q) \vdash P$
- X. $P \Rightarrow (Q \vee R), Q \Rightarrow S, R \Rightarrow S \vdash P \Rightarrow S$
- XI. $(P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)$

i)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash (P \wedge \neg Q) \Rightarrow R}\,Ax \qquad \dfrac{\dfrac{}{\Gamma' \vdash P}\,Ax \quad \dfrac{}{\Gamma' \vdash \neg Q}\,Ax}{\Gamma' \vdash P \wedge \neg Q}\,\wedge_i}{\Gamma' \vdash R}\,{\Rightarrow}e \qquad \dfrac{}{\Gamma' \vdash \neg R}\,Ax}{\Gamma' = (P \wedge \neg Q) \Rightarrow R,\ \neg R,\ P,\ \neg Q \vdash \bot}\,\neg e}{(P \wedge \neg Q) \Rightarrow R,\ \neg R,\ P \vdash Q}\,PBC$$

No vale R.

VALE P.

Ya vale Q ~> Busco meter en contexto

ii)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \neg P \Rightarrow Q}\,\text{Ax} \qquad \dfrac{}{\Gamma' \vdash \neg P}\,\text{Ax}}{\Gamma' \vdash Q}\,{\Rightarrow_e} \qquad \dfrac{}{\Gamma' \vdash \neg Q}\,\text{Ax}}{\Gamma' = \neg P \Rightarrow Q,\ \neg Q,\ \neg P \vdash \bot}}{\dfrac{\neg P \Rightarrow Q,\ \neg Q \vdash P}{\neg P \Rightarrow Q \vdash \neg Q \Rightarrow P}\,{\Rightarrow_i}}\,\text{PbC}
$$

(v=>?) si no tengo ¬P es trivial v.

No vale P hace valer Q ~> necesito ¬P

iii)

$$
\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P \vee Q}\,\text{Ax} \qquad \dfrac{}{\Gamma' \vdash R}\,\text{Ax}}{\Gamma' = P \vee Q, R \vdash (P \vee Q) \wedge R}\,{\wedge_i}}{P \vee Q \vdash R \Rightarrow (P \vee Q) \wedge R}\,{\Rightarrow_i}
$$

iv)

$$
\dfrac{\dfrac{}{\Gamma \vdash (P \vee (Q \Rightarrow P))}\,\text{Ax} \qquad \dfrac{}{\Gamma, P \vdash P}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma'' \vdash Q \Rightarrow P}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma'' \vdash (P \vee (Q \Rightarrow P)) \wedge Q}\,\text{Ax}}{\Gamma'' \vdash Q}\,{\wedge_{e_2}}}{\Gamma'' = \Gamma, (Q \Rightarrow P) \vdash P}\,{\Rightarrow_e}}{\Gamma = (P \vee (Q \Rightarrow P)) \wedge Q \vdash P}\,{\vee_e}
$$

VAU= Q. Si vale Q vale P.

v)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P \Rightarrow Q}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma' \vdash P \wedge R}\,\text{Ax}}{\Gamma' \vdash P}\,{\wedge_{e_1}}}{\Gamma' \vdash Q}\,{\Rightarrow_e} \qquad \dfrac{\dfrac{}{\Gamma' \vdash R \Rightarrow S}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma' \vdash P \wedge R}\,\text{Ax}}{\Gamma' \vdash R}\,{\wedge_{e_2}}}{\Gamma' \vdash S}\,{\Rightarrow_e}}{\Gamma' = P \Rightarrow Q,\ R \Rightarrow S,\ (P \wedge R) \vdash (Q \wedge S)}\,{\wedge_i}}{P \Rightarrow Q,\ R \Rightarrow S \vdash (P \wedge R) \Rightarrow (Q \wedge S)}\,{\Rightarrow_i}
$$

Si vale P, vale Q. (necesito P y R en contexto, vale)

P vale y R vale.

si vale R, vale S.

vi)

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{P\Rightarrow Q,\, P\wedge Q \vdash P\wedge Q}\;Ax}{P\Rightarrow Q,\, P\wedge Q \vdash P}\;\wedge_{e_1}
}{P\Rightarrow Q \vdash (P\wedge Q)\Rightarrow P}\;\Rightarrow_i
\qquad
\dfrac{
\dfrac{
\dfrac{}{P\Rightarrow Q,\, P\vdash P}\;Ax
\quad
\dfrac{\dfrac{}{P\Rightarrow Q,\, P\vdash P\wedge Q}\;Ax}{P\Rightarrow Q,\, P\vdash Q}\;\wedge_{e_2}
}{P\Rightarrow Q,\, P \vdash P\wedge Q}\;\wedge_i
}{P\Rightarrow Q \vdash P\Rightarrow (P\wedge Q)}\;\Rightarrow_i
}{P\Rightarrow Q \vdash ((P\wedge Q)\Rightarrow P)\wedge(P\Rightarrow(P\wedge Q))}\;\wedge_i
$$

Si vale P vale Q. necesito P si o si

vii)

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Gamma'\vdash P\Rightarrow(Q\wedge R)}\;Ax \quad \dfrac{}{\Gamma'\vdash P}\;Ax}{\Gamma'\vdash Q\wedge R}\;\Rightarrow_e
}{\Gamma' = P\Rightarrow(Q\wedge R),\, P\vdash Q}\;\wedge_{e_1}
}{P\Rightarrow(Q\wedge R)\vdash P\Rightarrow Q}\;\Rightarrow_i
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Gamma''\vdash P\Rightarrow(Q\wedge R)}\;Ax \quad \dfrac{}{\Gamma''\vdash P}\;Ax}{\Gamma''\vdash Q\wedge R}\;\Rightarrow_e
}{\Gamma'' = P\Rightarrow(Q\wedge R),\, P\vdash R}\;\wedge_{e_2}
}{P\Rightarrow(Q\wedge R)\vdash P\Rightarrow R}\;\Rightarrow_i
}{P\Rightarrow(Q\wedge R)\vdash (P\Rightarrow Q)\wedge(P\Rightarrow R)}\;\wedge_i
$$

⇒Q APARECE FÁCIL.
CON P ES FÁCIL

Si vale P, valen Q y R. necesito P seguro

viii)

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Gamma'\vdash P\Rightarrow Q}\;Ax \quad \dfrac{}{\Gamma'\vdash P}\;Ax}{\Gamma' = \Gamma, P\vdash Q}\;\Rightarrow_e
\qquad
\dfrac{\dfrac{}{\Gamma''\vdash P\Rightarrow R}\;Ax \quad \dfrac{}{\Gamma''\vdash P}\;Ax}{\Gamma'' = \Gamma, P\vdash R}\;\Rightarrow_e
}{\Gamma, P\vdash Q\wedge R}\;\wedge_i
}{\Gamma = (P\Rightarrow Q)\wedge(P\Rightarrow R)\vdash P\Rightarrow(Q\wedge R)}\;\Rightarrow_i
$$

P vale seguro. necesito en CONTEXTO P. Los demás valen de él.

ix)

$$\dfrac{\dfrac{}{\Gamma \vdash P \vee (P \wedge Q)}\,\text{Ax} \qquad \dfrac{}{\Gamma, P \vdash P}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma, (P\wedge Q) \vdash P \wedge Q}\,\text{Ax}}{\Gamma, (P \wedge Q) \vdash P}\,\wedge e_1}{\Gamma = P \vee (P \wedge Q) \vdash P}\,\vee e$$

O vale P o vale P∧Q. Con P me BASTA (en AMBAS vale). (¬ vale P.)

x)

$$\dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash P \Rightarrow (Q \vee R)}\,\text{Ax} \quad \dfrac{}{\Gamma'' \vdash P}\,\text{Ax}}{\Gamma'' = \Gamma', P \vdash (Q \vee R)}\Rightarrow e \qquad \dfrac{\dfrac{}{\Gamma''' \vdash Q \Rightarrow S}\,\text{Ax} \quad \dfrac{}{\Gamma''' \vdash Q}\,\text{Ax}}{\Gamma''' = \Gamma', P, Q \vdash S}\Rightarrow e \qquad \dfrac{\dfrac{}{\Gamma'''' \vdash R \Rightarrow S}\,\text{Ax} \quad \dfrac{}{\Gamma'''' \vdash R}\,\text{Ax}}{\Gamma'''' = \Gamma', P, R \vdash S}\Rightarrow e}{\dfrac{\Gamma', P \vdash S}{\Gamma' = P \Rightarrow (Q \vee R), Q \Rightarrow S, R \Rightarrow S \vdash P \Rightarrow S}\Rightarrow i}\,\vee e$$

Si vale P, vale Q o R.

Si vale Q vale S. ] Necesito Q y R si o si.

Si vale R vale S. ] En un O, o vale Q o R.

Xi)

$$\dfrac{\dfrac{}{\Gamma' \vdash (P\wedge Q) \vee (P \wedge R)}\,\text{Ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma', (P\wedge Q) \vdash P \wedge Q}\,\text{Ax}}{\Gamma', (P \wedge Q) \vdash P}\,\wedge e_1 \quad \dfrac{\dfrac{\dfrac{}{\Gamma', (P\wedge Q) \vdash P \wedge Q}\,\text{Ax}}{\Gamma', (P\wedge Q) \vdash Q}\,\wedge e_2}{\Gamma', (P \wedge Q) \vdash (Q \vee R)}\,\vee i_1}{\Gamma', (P \wedge Q) \vdash P \wedge (Q \vee R)}\,\wedge i \qquad \dfrac{\dfrac{\dfrac{}{\Gamma', (P\wedge R) \vdash P \wedge R}\,\text{Ax}}{\Gamma', (P \wedge R) \vdash P}\,\wedge e_1 \quad \dfrac{\dfrac{\dfrac{}{\Gamma', (P\wedge R) \vdash P \wedge R}\,\text{Ax}}{\Gamma', (P\wedge R) \vdash R}\,\wedge e_2}{\Gamma', (P \wedge R) \vdash Q \vee R}\,\vee i_2}{\Gamma', (P \wedge R) \vdash P \wedge (Q \vee R)}\,\wedge i}{\Gamma' = (P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)}\,\vee e$$

O valen P∧Q o valen P∧R

**Ejercicio 13**

Probar que los siguientes secuentes son válidos:

- I. $\neg P \Rightarrow \neg Q \vdash Q \Rightarrow P$
- II. $\neg P \vee \neg Q \vdash \neg(P \wedge Q)$
- III. $\neg P, P \vee Q \vdash Q$
- IV. $P \vee Q, \neg Q \vee R \vdash P \vee R$
- V. $P \wedge \neg P \vdash \neg(R \Rightarrow Q) \wedge (R \Rightarrow Q)$
- VI. $\neg(\neg P \vee Q) \vdash P$
- VII. $\vdash \neg P \Rightarrow (P \Rightarrow (P \Rightarrow Q))$
- VIII. $P \wedge Q \vdash \neg(\neg P \vee \neg Q)$
- IX. $\vdash (P \Rightarrow Q) \vee (Q \Rightarrow R)$

i)

$$\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash Q}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Gamma' \vdash \neg P \Rightarrow \neg Q}\,\text{Ax} \quad \dfrac{}{\Gamma' \vdash \neg P}\,\text{Ax}}{\Gamma' \vdash \neg Q}\Rightarrow e}{\Gamma' = \neg P \Rightarrow \neg Q, Q, \neg P \vdash \bot}\,\neg e}{}\,\text{PBC}$$

$$\dfrac{\neg P \Rightarrow \neg Q,\ Q \vdash P}{\neg P \Rightarrow \neg Q \vdash Q \Rightarrow P}\ {\Rightarrow_i}$$

Si no vale P, no vale Q.

Busco $\neg P$ en contexto y Q. Para $\neg Q, Q$ $\neg$ Ax

↳ $\neg P \Rightarrow \neg Q, \neg P$

ii)

$$\dfrac{\dfrac{}{\Gamma \vdash \neg P \vee \neg Q}\,Ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P \wedge Q}\,Ax}{\Gamma' \vdash P}\,\wedge_{e_1} \qquad \dfrac{}{\Gamma' \vdash \neg P}\,Ax}{\Gamma' = \Gamma, \neg P, (P \wedge Q) \vdash \bot}\,\neg e}{\Gamma, \neg P \vdash \neg(P \wedge Q)}\,\neg i \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash P \wedge Q}\,Ax}{\Gamma'' \vdash Q}\,\wedge_{e_2} \qquad \dfrac{}{\Gamma'' \vdash \neg Q}\,Ax}{\Gamma'' = \Gamma, \neg Q, (P \wedge Q) \vdash \bot}\,\neg e}{\Gamma, \neg Q \vdash \neg(P \wedge Q)}\,\neg i}{\neg P \vee \neg Q \vdash \neg(P \wedge Q)}\,\vee e$$

O vale $\neg P$ o vale $\neg Q$, después $\neg i$. Desp vale con $\neg e$ con $\neg P$ y $\neg Q$ y ahora el $\wedge$.

iii)

$$\dfrac{\dfrac{}{\neg P, P \vee Q \vdash P \vee Q}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P}\,Ax \quad \dfrac{}{\Gamma' \vdash \neg P}\,Ax}{\Gamma' \vdash \bot}\,\neg e}{\Gamma' = \neg P, P \vee Q, P \vdash Q}\,\bot e \qquad \dfrac{}{\neg P, P \vee Q, Q \vdash Q}\,Ax}{\neg P, P \vee Q \vdash Q}\,\vee e$$

No vale P

Vale P o vale Q ~> Busco contradicción entre vale P y no vale P.

Vale Q. Vale mismo

iv)

$$\dfrac{\dfrac{}{\Gamma \vdash P \vee Q}\,Ax \qquad \dfrac{\dfrac{}{\Gamma', P \vdash P}\,Ax}{\Gamma', P \vdash P \vee R}\,\vee_{i1} \qquad \dfrac{\dfrac{}{\Gamma'' \vdash \neg Q \vee R}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma'', \neg Q \vdash \neg Q}\,Ax \quad \dfrac{}{\Gamma'', \neg Q \vdash Q}\,Ax}{\Gamma'', \neg Q \vdash \bot}\,\neg e}{\Gamma'', \neg Q \vdash R}\,\bot e \qquad \dfrac{}{\Gamma'', R \vdash R}\,Ax}{\Gamma'' = \Gamma', Q \vdash R}\,\vee e}{\Gamma = P \vee Q, \neg Q \vee R \vdash P \vee R}\,\vee e$$

$P \vee Q, \neg Q \vee R, Q \vdash R$

O bien vale $\neg Q$ ~> contradicción

o bien vale R ~> Ax.

$\theta$ vale $P$ o vale $Q$

$\theta$ vale $\neg Q$ o vale $R$. Posible contradicción.

v)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P \wedge \neg P}\,Ax}{\Gamma' \vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{}{\Gamma' \vdash P \wedge \neg P}\,Ax}{\Gamma' \vdash \neg P}\,\wedge e_2}{\Gamma' \vdash \bot}\,\neg e}{\Gamma' = P \wedge \neg P \vdash \neg(R \Rightarrow Q) \wedge (R \Rightarrow Q)}\,\bot e$$

Busco Bottom de una. No hay relación entre $P \wedge \neg P$ y lo q' concluye.

vi)

$$\dfrac{\dfrac{}{\neg(\neg P \vee Q) \vdash \neg P \vee Q} \qquad \neg(\neg P \vee Q), \neg P \vdash P \qquad \dfrac{\neg(\neg P \vee Q), Q, \neg P \vdash \bot}{\neg(\neg P \vee Q), Q \vdash P}\,PBC}{\neg(\neg P \vee Q) \vdash P}\,\vee e$$

$\theta$ vale $\neg P$ o vale $Q$.

$(P \wedge \neg Q)$

vii)

$$\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \neg P}\,Ax \qquad \dfrac{}{\Gamma' \vdash P}\,Ax}{\Gamma' = \neg P, P \vdash Q}\,\bot e}{\neg P \Rightarrow (P \Rightarrow (P \Rightarrow Q))}\,\Rightarrow i_3$$

Contradicción

viii)

$$\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash P \wedge Q}\,Ax}{\Gamma' \vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{}{\Gamma' \vdash \neg P \vee \neg Q}\,Ax \qquad \dfrac{}{\Gamma', \neg P \vdash \neg P}\,Ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma'' \vdash P \wedge Q}\,Ax}{\Gamma'' \vdash Q}\,\wedge e_2 \qquad \dfrac{}{\Gamma'' \vdash \neg Q}\,Ax}{\Gamma'' = \Gamma', \neg Q \vdash \neg P}\,\bot e}{\Gamma' \vdash \neg P}}{\Gamma' = (P \wedge Q), \neg P \vee \neg Q \vdash \bot}\,\neg e$$

$(P \wedge Q), \neg P \vee \neg Q, \neg Q$

$\theta$ vale $\neg P$ o vale $\neg Q$

$\neg i$

$P \land Q \vdash \neg(\neg P \lor \neg Q)$

VALE P y Q.

ix)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$\vdash Q \lor \neg Q$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$Q \vdash Q \Rightarrow R$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$Q \vdash Q$}
\RightLabel{$\Rightarrow e$}
\BinaryInfC{$Q \vdash R$}
\RightLabel{$\vee_{i_1}, \Rightarrow i$}
\UnaryInfC{$Q \vdash (P \Rightarrow Q) \lor (Q \Rightarrow R)$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\neg Q, Q \vdash \neg Q$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\neg Q, Q \vdash Q$}
\RightLabel{$\neg e$}
\BinaryInfC{$\neg Q, Q \vdash \bot$}
\RightLabel{$\bot e$}
\UnaryInfC{$\neg Q, Q \vdash R$}
\RightLabel{$\Rightarrow i$}
\UnaryInfC{$\neg Q \vdash Q \Rightarrow R$}
\RightLabel{$\vee_{i_2}$}
\UnaryInfC{$\neg Q \vdash (P \Rightarrow Q) \lor (Q \Rightarrow R)$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\vdash (P \Rightarrow Q) \lor (Q \Rightarrow R)$}
\end{prooftree}
$$

Si vale P, vale Q.

Si vale Q vale R.

necesito Q en contexto.

No tengo nada contexto. Fuerzo LEM.