Computational Science:
Introduction to Finite-Difference Time-Domain

Formulation of Two-Dimensional FDTD Without PML

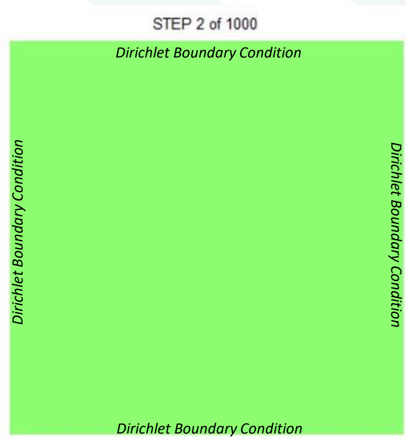
Lecture Outline

- Code development sequence for 2D
- Maxwell's equations
- Finite-difference approximations
- Reduction to two dimensions
- Update equations
- Boundary Conditions
- Revised FDTD Algorithm for 2D Simulations

Code Development Sequence

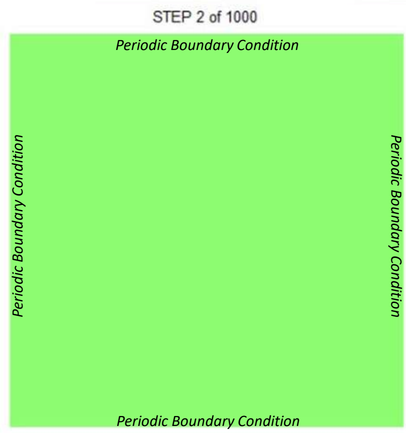
Step 1 – Basic Update Equations

The basic update equations are implemented along with simple Dirichlet boundary conditions.



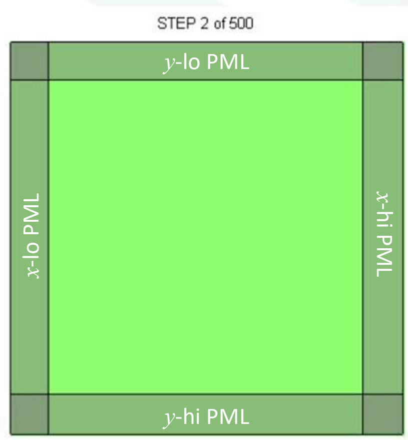
Step 2 – Incorporate Periodic Boundaries

Periodic boundary conditions are incorporated so that a wave leaving the grid reenters the grid at the other side.



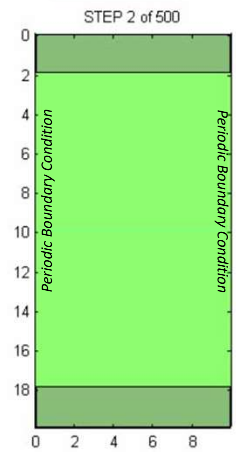
Step 3 – Incorporate a PML

The perfectly matched layer (PML) absorbing boundary condition is incorporated to absorb outgoing waves.



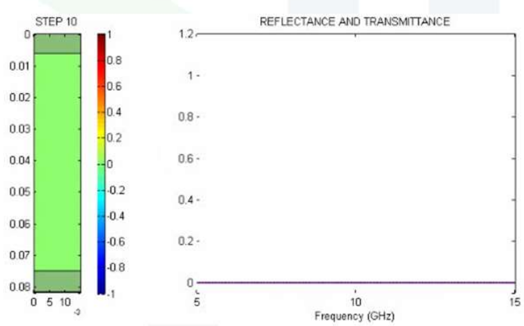
Step 4 – Total-Field/Scattered-Field

Most periodic electromagnetic devices are modeled by using periodic boundaries for the horizontal axis and a PML for the vertical axis. We then implement TF/SF at the vertical center of the grid for testing.



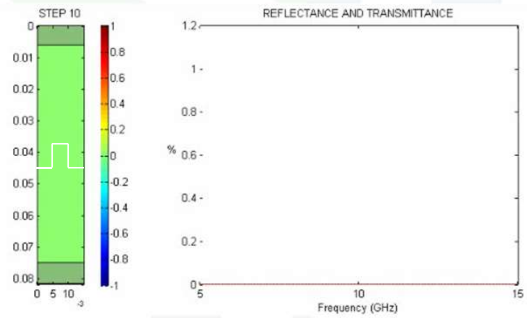
Step 5 – Calculate TRN, REF, and CON

We move the TF/SF interface to a unit cell or two outside of the top PML. We include code to calculate Fourier transforms and to calculate transmittance, reflectance, and conservation of power.



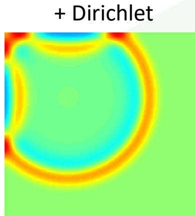
Step 6 – Model a Device to Benchmark

We build a device on the grid that has a known solution. We run the simulation and duplicate the known results to benchmark our new code.

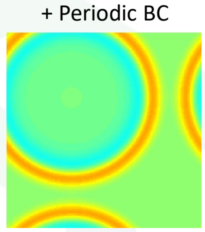


Summary of Code Development Sequence

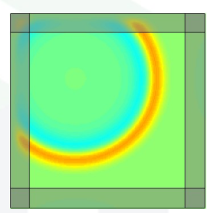
Step 1 – Basic Update + Dirichlet



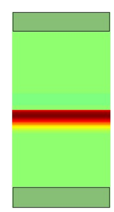
Step 2 – Basic Update + Periodic BC



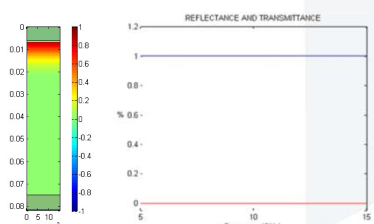
Step 3 – Add PML



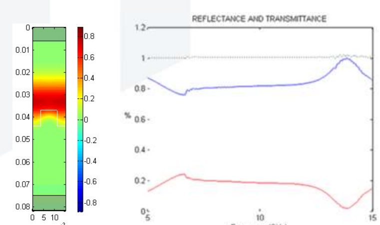
Step 4 – TF/SF



Step 5 – Calculate Response



Step 6 – Add a Device and Benchmark



Maxwell's Equations

Maxwell's Equations

Start with Maxwell's equations in the following form:

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

$$\vec{D}(t) = \varepsilon_0 \varepsilon_r \vec{E}(t)$$

$$\vec{B}(t) = \mu_0 \mu_r \vec{H}(t)$$

Normalize the Electric Fields

Now we will adopt the more conventional approach in FDTD and normalize the electric field according to:

$$\vec{\tilde{E}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{E} = \frac{1}{\eta_0}\vec{E}$$

To be consistent, other parameters related to the electric field must also be normalized.

$$\vec{\tilde{D}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\vec{D} = c_0\vec{D}$$

$$\vec{\tilde{P}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\vec{P} = c_0\vec{P}$$

$$\vec{\tilde{J}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\vec{J} = c_0\vec{J}$$

$\vec{J}$ and $\vec{P}$ won't be used in this course.

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Normalized Maxwell's Equations

Using the normalized fields, Maxwell's equations become

$$\nabla \times \vec{\tilde{E}} = -\frac{\mu_r}{c_0}\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{1}{c_0}\frac{\partial \vec{\tilde{D}}}{\partial t}$$

These equations are independent of ε_r.

Update equations can be derived from these expressions without complicating them with what might need to be done to model ε_r.

$$\vec{\tilde{D}} = \varepsilon_r\vec{\tilde{E}}$$

This is a very simple equation that makes it much easier to model more sophisticated dielectrics that may be anisotropic, nonlinear, dispersive, all of the above, or something else altogether.

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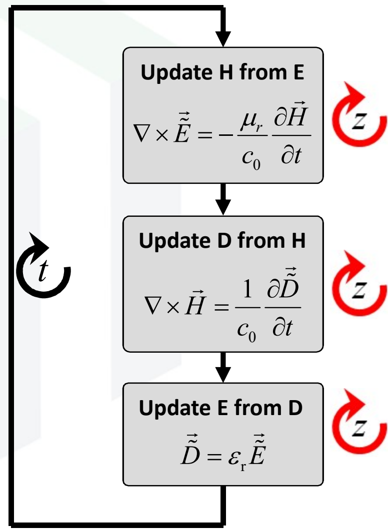
HDE Algorithm

To make FDTD more modular, modify the steps to what is called the *HDE algorithm*.



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Expand Maxwell's Equations

$$\nabla \times \vec{\tilde{E}} = -\frac{\mu_{\mathrm{r}}}{c_0}\frac{\partial \vec{H}}{\partial t} \quad \Rightarrow \quad
\begin{aligned}
\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} &= -\frac{1}{c_0}\left(\mu_{xx}\frac{\partial H_x}{\partial t} + \mu_{xy}\frac{\partial H_y}{\partial t} + \mu_{xz}\frac{\partial H_z}{\partial t}\right)\\
\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} &= -\frac{1}{c_0}\left(\mu_{yx}\frac{\partial H_x}{\partial t} + \mu_{yy}\frac{\partial H_y}{\partial t} + \mu_{yz}\frac{\partial H_z}{\partial t}\right)\\
\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} &= -\frac{1}{c_0}\left(\mu_{zx}\frac{\partial H_x}{\partial t} + \mu_{zy}\frac{\partial H_y}{\partial t} + \mu_{zz}\frac{\partial H_z}{\partial t}\right)
\end{aligned}$$

$$\nabla \times \vec{H} = \frac{1}{c_0}\frac{\partial \vec{\tilde{D}}}{\partial t} \quad \Rightarrow \quad
\begin{aligned}
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t}\\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t}\\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t}
\end{aligned}$$

$$\vec{\tilde{D}} = \varepsilon_{\mathrm{r}}\vec{\tilde{E}} \quad \Rightarrow \quad
\begin{aligned}
\tilde{D}_x &= \varepsilon_{xx}\tilde{E}_x + \varepsilon_{xy}\tilde{E}_y + \varepsilon_{xz}\tilde{E}_z\\
\tilde{D}_y &= \varepsilon_{yx}\tilde{E}_x + \varepsilon_{yy}\tilde{E}_y + \varepsilon_{yz}\tilde{E}_z\\
\tilde{D}_z &= \varepsilon_{zx}\tilde{E}_x + \varepsilon_{zy}\tilde{E}_y + \varepsilon_{zz}\tilde{E}_z
\end{aligned}$$

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Assume Only Diagonal Tensors

$$\nabla \times \vec{\tilde{E}} = -\frac{\mu_r}{c_0}\frac{\partial \vec{H}}{\partial t} \Rightarrow \begin{aligned} \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} &= -\frac{1}{c_0}\left(\mu_{xx}\frac{\partial H_x}{\partial t} + \cancel{\mu_{xy}\frac{\partial H_y}{\partial t}} + \cancel{\mu_{xz}\frac{\partial H_z}{\partial t}} \right) \\ \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} &= -\frac{1}{c_0}\left(\cancel{\mu_{yx}\frac{\partial H_x}{\partial t}} + \mu_{yy}\frac{\partial H_y}{\partial t} + \cancel{\mu_{yz}\frac{\partial H_z}{\partial t}} \right) \\ \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} &= -\frac{1}{c_0}\left(\cancel{\mu_{zx}\frac{\partial H_x}{\partial t}} + \cancel{\mu_{zy}\frac{\partial H_y}{\partial t}} + \mu_{zz}\frac{\partial H_z}{\partial t} \right) \end{aligned}$$

$$\nabla \times \vec{H} = \frac{1}{c_0}\frac{\partial \vec{\tilde{D}}}{\partial t} \Rightarrow \begin{aligned} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t} \end{aligned}$$

$$\vec{\tilde{D}} = \varepsilon_r \vec{\tilde{E}} \Rightarrow \begin{aligned} \tilde{D}_x &= \varepsilon_{xx}\tilde{E}_x + \cancel{\varepsilon_{xy}\tilde{E}_y} + \cancel{\varepsilon_{xz}\tilde{E}_z} \\ \tilde{D}_y &= \cancel{\varepsilon_{yx}\tilde{E}_x} + \varepsilon_{yy}\tilde{E}_y + \cancel{\varepsilon_{yz}\tilde{E}_z} \\ \tilde{D}_z &= \cancel{\varepsilon_{zx}\tilde{E}_x} + \cancel{\varepsilon_{zy}\tilde{E}_y} + \varepsilon_{zz}\tilde{E}_z \end{aligned}$$

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Governing Equations

These are the primary governing equations from which the 2D and 3D FDTD methods will be formulated.

$$\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} = -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t} \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t}$$

$$\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} = -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t} \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t}$$

$$\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} = -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t} \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t}$$

$$\tilde{D}_x = \varepsilon_{xx}\tilde{E}_x$$
$$\tilde{D}_y = \varepsilon_{yy}\tilde{E}_y$$
$$\tilde{D}_z = \varepsilon_{zz}\tilde{E}_z$$

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Final Analytical Equations

It will be beneficial in the formulation and implementation to calculate the curl terms separately. Start by expressing Maxwell's equations in the following form.

$$\boxed{\begin{aligned} C_x^E &= -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t} \\ C_y^E &= -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t} \\ C_z^E &= -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t} \end{aligned}}$$

$$\boxed{\begin{aligned} C_x^H &= \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t} \\ C_y^H &= \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t} \\ C_z^H &= \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t} \end{aligned}}$$

$$\boxed{\begin{aligned} \tilde{D}_x &= \varepsilon_{xx}\tilde{E}_x \\ \tilde{D}_y &= \varepsilon_{yy}\tilde{E}_y \\ \tilde{D}_z &= \varepsilon_{zz}\tilde{E}_z \end{aligned}}$$

$$\begin{aligned} C_x^E &= \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} \\ C_y^E &= \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} \\ C_z^E &= \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \end{aligned}$$

$$\begin{aligned} C_x^H &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ C_y^H &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ C_z^H &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}$$

Finite-Difference Approximations

Finite-Difference Equations for $\vec{H}$

Curl Terms

$$C_x^E = \frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z} \quad\Rightarrow\quad C_x^E\Big|_t^{i,j,k} = \frac{\tilde{E}_z\big|_t^{i,j+1,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y\big|_t^{i,j,k+1} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta z}$$

$$C_y^E = \frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x} \quad\Rightarrow\quad C_y^E\Big|_t^{i,j,k} = \frac{\tilde{E}_x\big|_t^{i,j,k+1} - \tilde{E}_x\big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{E}_z\big|_t^{i+1,j,k} - \tilde{E}_z\big|_t^{i,j,k}}{\Delta x}$$

$$C_z^E = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \quad\Rightarrow\quad C_z^E\Big|_t^{i,j,k} = \frac{\tilde{E}_y\big|_t^{i+1,j,k} - \tilde{E}_y\big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{E}_x\big|_t^{i,j+1,k} - \tilde{E}_x\big|_t^{i,j,k}}{\Delta y}$$

Maxwell's Equations

$$C_x^E = -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t} \quad\Rightarrow\quad C_x^E\Big|_t^{i,j,k} = -\frac{\mu_{xx}\big|^{i,j,k}}{c_0}\frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$

$$C_y^E = -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t} \quad\Rightarrow\quad C_y^E\Big|_t^{i,j,k} = -\frac{\mu_{yy}\big|^{i,j,k}}{c_0}\frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$

$$C_z^E = -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t} \quad\Rightarrow\quad C_z^E\Big|_t^{i,j,k} = -\frac{\mu_{zz}\big|^{i,j,k}}{c_0}\frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\big|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$

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Slide 21

Finite-Difference Equations for $\vec{\tilde{D}}$

Curl Terms

$$C_x^H = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \quad\Rightarrow\quad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}$$

$$C_y^H = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \quad\Rightarrow\quad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} - \frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z\big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}$$

$$C_z^H = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \quad\Rightarrow\quad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x} - \frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}$$

Maxwell's Equations

$$C_x^H = \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t} \quad\Rightarrow\quad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{1}{c_0}\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_x\big|_t^{i,j,k}}{\Delta t}$$

$$C_y^H = \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t} \quad\Rightarrow\quad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{1}{c_0}\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_y\big|_t^{i,j,k}}{\Delta t}$$

$$C_z^H = \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t} \quad\Rightarrow\quad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{1}{c_0}\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j,k} - \tilde{D}_z\big|_t^{i,j,k}}{\Delta t}$$

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Finite-Difference Equations for $\vec{\tilde{E}}$

$$\tilde{D}_x = \varepsilon_{xx}\tilde{E}_x \quad \Rightarrow \quad \tilde{D}_x\Big|_t^{i,j,k} = \left(\varepsilon_{xx}\Big|^{i,j,k}\right)\tilde{E}_x\Big|_t^{i,j,k}$$

$$\tilde{D}_y = \varepsilon_{yy}\tilde{E}_y \quad \Rightarrow \quad \tilde{D}_y\Big|_t^{i,j,k} = \left(\varepsilon_{yy}\Big|^{i,j,k}\right)\tilde{E}_y\Big|_t^{i,j,k}$$

$$\tilde{D}_z = \varepsilon_{zz}\tilde{E}_z \quad \Rightarrow \quad \tilde{D}_z\Big|_t^{i,j,k} = \left(\varepsilon_{zz}\Big|^{i,j,k}\right)\tilde{E}_z\Big|_t^{i,j,k}$$

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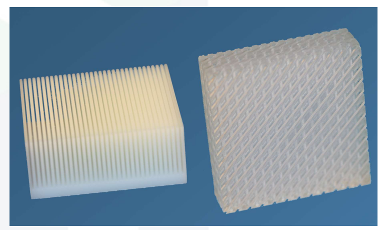
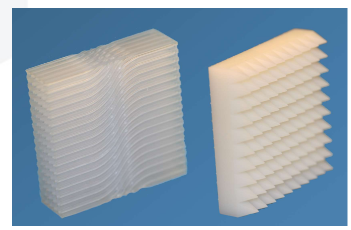
Reduction to Two Dimensions

Slide 24

Real Electromagnetic Devices

All physical devices are three-dimensional.





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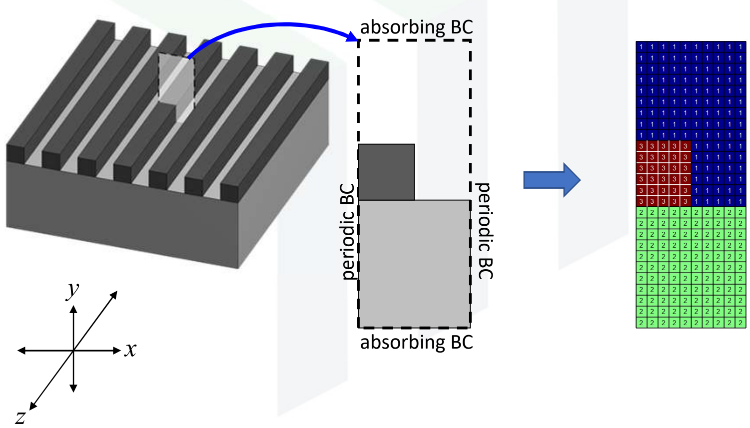
3D → 2D (Exact)

Sometimes it is possible to describe a physical device using just two dimensions. Doing so dramatically reduces the numerical complexity of the problem and is ALWAYS GOOD PRACTICE.



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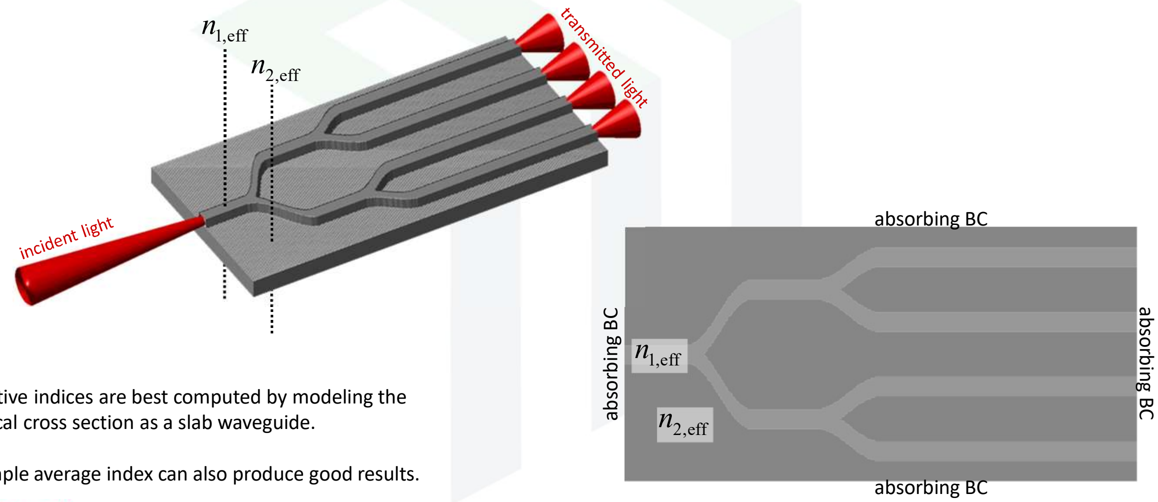
3D → 2D (Approximate)

Many times it is possible to approximate a 3D device in two dimensions. It is very good practice to at least perform the initial simulations in 2D and only moving to 3D to verify the final design.

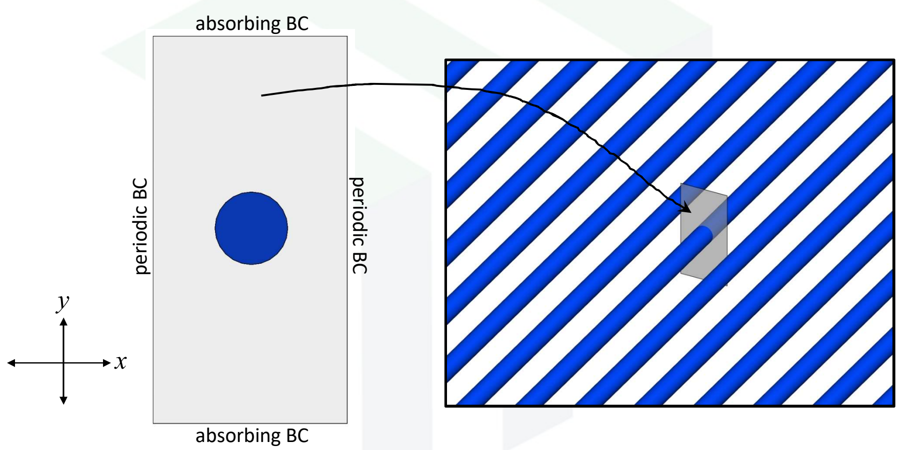


Effective indices are best computed by modeling the vertical cross section as a slab waveguide.

A simple average index can also produce good results.

2D Grids are Infinite in the 3rd Dimension

Anything represented on a 2D grid, is actually a device that is of infinite extent along the 3rd dimension.



Assuming the z direction is uniform and of infinite extent, then the field is also uniform and of infinite extent in the z direction.

$$\frac{\partial}{\partial z} = 0$$

Assume Uniform in z Direction

For 2D devices, $\partial/\partial z=0$ and Maxwell's equations reduce to

$$C_x^E=\frac{\partial\tilde{E}_z}{\partial y}-\cancel{\frac{\partial\tilde{E}_y}{\partial z}}\qquad C_x^E=-\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t}$$
$$C_y^E=\cancel{\frac{\partial\tilde{E}_x}{\partial z}}-\frac{\partial\tilde{E}_z}{\partial x}\qquad C_y^E=-\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t}$$
$$C_z^E=\frac{\partial\tilde{E}_y}{\partial x}-\frac{\partial\tilde{E}_x}{\partial y}\qquad C_z^E=-\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t}$$

$$\Rightarrow$$

$$\left.C_x^E\right|_t^{i,j,k}=\frac{\left.\tilde{E}_z\right|_t^{i,j+1,k}-\left.\tilde{E}_z\right|_t^{i,j,k}}{\Delta y}-\cancel{\frac{\left.\tilde{E}_y\right|_t^{i,j,k+1}-\left.\tilde{E}_y\right|_t^{i,j,k}}{\Delta z}}\qquad \left.C_x^E\right|_t^{i,j,k}=-\frac{\left.\mu_{xx}\right|^{i,j,k}}{c_0}\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_x\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$
$$\left.C_y^E\right|_t^{i,j,k}=\cancel{\frac{\left.\tilde{E}_x\right|_t^{i,j,k+1}-\left.\tilde{E}_x\right|_t^{i,j,k}}{\Delta z}}-\frac{\left.\tilde{E}_z\right|_t^{i+1,j,k}-\left.\tilde{E}_z\right|_t^{i,j,k}}{\Delta x}\qquad \left.C_y^E\right|_t^{i,j,k}=-\frac{\left.\mu_{yy}\right|^{i,j,k}}{c_0}\frac{\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_y\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$
$$\left.C_z^E\right|_t^{i,j,k}=\frac{\left.\tilde{E}_y\right|_t^{i+1,j,k}-\left.\tilde{E}_y\right|_t^{i,j,k}}{\Delta x}-\frac{\left.\tilde{E}_x\right|_t^{i,j+1,k}-\left.\tilde{E}_x\right|_t^{i,j,k}}{\Delta y}\qquad \left.C_z^E\right|_t^{i,j,k}=-\frac{\left.\mu_{zz}\right|^{i,j,k}}{c_0}\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$

$$C_x^H=\frac{\partial H_z}{\partial y}-\cancel{\frac{\partial H_y}{\partial z}}\qquad C_x^H=\frac{1}{c_0}\frac{\partial\tilde{D}_x}{\partial t}$$
$$C_y^H=\cancel{\frac{\partial H_x}{\partial z}}-\frac{\partial H_z}{\partial x}\qquad C_y^H=\frac{1}{c_0}\frac{\partial\tilde{D}_y}{\partial t}$$
$$C_z^H=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\qquad C_z^H=\frac{1}{c_0}\frac{\partial\tilde{D}_z}{\partial t}$$

$$\Rightarrow$$

$$\left.C_x^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}-\cancel{\frac{\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}}\qquad \left.C_x^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_x\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_x\right|_t^{i,j,k}}{\Delta t}$$
$$\left.C_y^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\cancel{\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}}-\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}\qquad \left.C_y^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_y\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_y\right|_t^{i,j,k}}{\Delta t}$$
$$\left.C_z^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}-\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}\qquad \left.C_z^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_z\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_z\right|_t^{i,j,k}}{\Delta t}$$

$$\tilde{D}_x=\varepsilon_{xx}\tilde{E}_x\qquad \tilde{D}_y=\varepsilon_{yy}\tilde{E}_y\qquad \tilde{D}_z=\varepsilon_{zz}\tilde{E}_z$$

$$\Rightarrow$$

$$\left.\tilde{D}_x\right|_t^{i,j,k}=\left(\left.\varepsilon_{xx}\right|^{i,j,k}\right)\left.\tilde{E}_x\right|_t^{i,j,k}\qquad \left.\tilde{D}_y\right|_t^{i,j,k}=\left(\left.\varepsilon_{yy}\right|^{i,j,k}\right)\left.\tilde{E}_y\right|_t^{i,j,k}\qquad \left.\tilde{D}_z\right|_t^{i,j,k}=\left(\left.\varepsilon_{zz}\right|^{i,j,k}\right)\left.\tilde{E}_z\right|_t^{i,j,k}$$

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Slide 29

2D Finite-Difference Equations

We eliminate the z-derivative terms.

$$C_x^E=\frac{\partial\tilde{E}_z}{\partial y}\qquad C_x^E=-\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t}$$
$$C_y^E=-\frac{\partial\tilde{E}_z}{\partial x}\qquad C_y^E=-\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t}$$
$$C_z^E=\frac{\partial\tilde{E}_y}{\partial x}-\frac{\partial\tilde{E}_x}{\partial y}\qquad C_z^E=-\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t}$$

$$\Rightarrow$$

$$\left.C_x^E\right|_t^{i,j,k}=\frac{\left.\tilde{E}_z\right|_t^{i,j+1,k}-\left.\tilde{E}_z\right|_t^{i,j,k}}{\Delta y}\qquad \left.C_x^E\right|_t^{i,j,k}=-\frac{\left.\mu_{xx}\right|^{i,j,k}}{c_0}\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_x\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$
$$\left.C_y^E\right|_t^{i,j,k}=-\frac{\left.\tilde{E}_z\right|_t^{i+1,j,k}-\left.\tilde{E}_z\right|_t^{i,j,k}}{\Delta x}\qquad \left.C_y^E\right|_t^{i,j,k}=-\frac{\left.\mu_{yy}\right|^{i,j,k}}{c_0}\frac{\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_y\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$
$$\left.C_z^E\right|_t^{i,j,k}=\frac{\left.\tilde{E}_y\right|_t^{i+1,j,k}-\left.\tilde{E}_y\right|_t^{i,j,k}}{\Delta x}-\frac{\left.\tilde{E}_x\right|_t^{i,j+1,k}-\left.\tilde{E}_x\right|_t^{i,j,k}}{\Delta y}\qquad \left.C_z^E\right|_t^{i,j,k}=-\frac{\left.\mu_{zz}\right|^{i,j,k}}{c_0}\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t-\frac{\Delta t}{2}}^{i,j,k}}{\Delta t}$$

$$C_x^H=\frac{\partial H_z}{\partial y}\qquad C_x^H=\frac{1}{c_0}\frac{\partial\tilde{D}_x}{\partial t}$$
$$C_y^H=-\frac{\partial H_z}{\partial x}\qquad C_y^H=\frac{1}{c_0}\frac{\partial\tilde{D}_y}{\partial t}$$
$$C_z^H=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\qquad C_z^H=\frac{1}{c_0}\frac{\partial\tilde{D}_z}{\partial t}$$

$$\Rightarrow$$

$$\left.C_x^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}\qquad \left.C_x^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_x\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_x\right|_t^{i,j,k}}{\Delta t}$$
$$\left.C_y^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=-\frac{\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_z\right|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}\qquad \left.C_y^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_y\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_y\right|_t^{i,j,k}}{\Delta t}$$
$$\left.C_z^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_y\right|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}-\frac{\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j,k}-\left.H_x\right|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y}\qquad \left.C_z^H\right|_{t+\frac{\Delta t}{2}}^{i,j,k}=\frac{1}{c_0}\frac{\left.\tilde{D}_z\right|_{t+\Delta t}^{i,j,k}-\left.\tilde{D}_z\right|_t^{i,j,k}}{\Delta t}$$

$$\tilde{D}_x=\varepsilon_{xx}\tilde{E}_x\qquad \tilde{D}_y=\varepsilon_{yy}\tilde{E}_y\qquad \tilde{D}_z=\varepsilon_{zz}\tilde{E}_z$$

$$\Rightarrow$$

$$\left.\tilde{D}_x\right|_t^{i,j,k}=\left(\left.\varepsilon_{xx}\right|^{i,j,k}\right)\left.\tilde{E}_x\right|_t^{i,j,k}\qquad \left.\tilde{D}_y\right|_t^{i,j,k}=\left(\left.\varepsilon_{yy}\right|^{i,j,k}\right)\left.\tilde{E}_y\right|_t^{i,j,k}\qquad \left.\tilde{D}_z\right|_t^{i,j,k}=\left(\left.\varepsilon_{zz}\right|^{i,j,k}\right)\left.\tilde{E}_z\right|_t^{i,j,k}$$

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Slide 30

We No Longer Need Grid Index k

For 2D devices, we only retain grid indices i and j.

$$C_x^E = \frac{\partial \tilde{E}_z}{\partial y} \qquad C_x^E = -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t}$$
$$C_y^E = -\frac{\partial \tilde{E}_z}{\partial x} \qquad C_y^E = -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t}$$
$$C_z^E = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \qquad C_z^E = -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t}$$

$$\Rightarrow$$

$$C_x^E\Big|_t^{i,j} = \frac{\tilde{E}_z\Big|_t^{i,j+1} - \tilde{E}_z\Big|_t^{i,j}}{\Delta y} \qquad C_x^E\Big|_t^{i,j} = -\frac{\mu_{xx}\Big|^{i,j}}{c_0}\frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$
$$C_y^E\Big|_t^{i,j} = -\frac{\tilde{E}_z\Big|_t^{i+1,j} - \tilde{E}_z\Big|_t^{i,j}}{\Delta x} \qquad C_y^E\Big|_t^{i,j} = -\frac{\mu_{yy}\Big|^{i,j}}{c_0}\frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$
$$C_z^E\Big|_t^{i,j} = \frac{\tilde{E}_y\Big|_t^{i+1,j} - \tilde{E}_y\Big|_t^{i,j}}{\Delta x} - \frac{\tilde{E}_x\Big|_t^{i,j+1} - \tilde{E}_x\Big|_t^{i,j}}{\Delta y} \qquad C_z^E\Big|_t^{i,j} = -\frac{\mu_{zz}\Big|^{i,j}}{c_0}\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_x^H = \frac{\partial H_z}{\partial y} \qquad C_x^H = \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t}$$
$$C_y^H = -\frac{\partial H_z}{\partial x} \qquad C_y^H = \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t}$$
$$C_z^H = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad C_z^H = \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t}$$

$$\Rightarrow$$

$$C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_x\Big|_{t+\Delta t}^{i,j} - \tilde{D}_x\Big|_t^{i,j}}{\Delta t}$$
$$C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = -\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} \qquad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_y\Big|_{t+\Delta t}^{i,j} - \tilde{D}_y\Big|_t^{i,j}}{\Delta t}$$
$$C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} - \frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_z\Big|_{t+\Delta t}^{i,j} - \tilde{D}_z\Big|_t^{i,j}}{\Delta t}$$

$$\tilde{D}_x = \varepsilon_{xx}\tilde{E}_x \qquad \tilde{D}_y = \varepsilon_{yy}\tilde{E}_y \qquad \tilde{D}_z = \varepsilon_{zz}\tilde{E}_z$$

$$\Rightarrow$$

$$\tilde{D}_x\Big|_t^{i,j} = \left(\varepsilon_{xx}\Big|^{i,j}\right)\tilde{E}_x\Big|_t^{i,j} \qquad \tilde{D}_y\Big|_t^{i,j} = \left(\varepsilon_{yy}\Big|^{i,j}\right)\tilde{E}_y\Big|_t^{i,j} \qquad \tilde{D}_z\Big|_t^{i,j} = \left(\varepsilon_{zz}\Big|^{i,j}\right)\tilde{E}_z\Big|_t^{i,j}$$

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Slide 31

Two Distinct Modes: E_z and H_z

Maxwell's equations have decoupled into two distinct sets of equations.

$$C_x^E = \frac{\partial \tilde{E}_z}{\partial y} \qquad C_x^E = -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t}$$
$$C_y^E = -\frac{\partial \tilde{E}_z}{\partial x} \qquad C_y^E = -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t}$$
$$C_z^E = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \qquad C_z^E = -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t}$$

$$\Rightarrow$$

$$C_x^E\Big|_t^{i,j} = \frac{\tilde{E}_z\Big|_t^{i,j+1} - \tilde{E}_z\Big|_t^{i,j}}{\Delta y} \qquad C_x^E\Big|_t^{i,j} = -\frac{\mu_{xx}\Big|^{i,j}}{c_0}\frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$
$$C_y^E\Big|_t^{i,j} = -\frac{\tilde{E}_z\Big|_t^{i+1,j} - \tilde{E}_z\Big|_t^{i,j}}{\Delta x} \qquad C_y^E\Big|_t^{i,j} = -\frac{\mu_{yy}\Big|^{i,j}}{c_0}\frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$
$$C_z^E\Big|_t^{i,j} = \frac{\tilde{E}_y\Big|_t^{i+1,j} - \tilde{E}_y\Big|_t^{i,j}}{\Delta x} - \frac{\tilde{E}_x\Big|_t^{i,j+1} - \tilde{E}_x\Big|_t^{i,j}}{\Delta y} \qquad C_z^E\Big|_t^{i,j} = -\frac{\mu_{zz}\Big|^{i,j}}{c_0}\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_x^H = \frac{\partial H_z}{\partial y} \qquad C_x^H = \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t}$$
$$C_y^H = -\frac{\partial H_z}{\partial x} \qquad C_y^H = \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t}$$
$$C_z^H = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad C_z^H = \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t}$$

$$\Rightarrow$$

$$C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_x\Big|_{t+\Delta t}^{i,j} - \tilde{D}_x\Big|_t^{i,j}}{\Delta t}$$
$$C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = -\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} \qquad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_y\Big|_{t+\Delta t}^{i,j} - \tilde{D}_y\Big|_t^{i,j}}{\Delta t}$$
$$C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} - \frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_z\Big|_{t+\Delta t}^{i,j} - \tilde{D}_z\Big|_t^{i,j}}{\Delta t}$$

$$\tilde{D}_x = \varepsilon_{xx}\tilde{E}_x \qquad \tilde{D}_y = \varepsilon_{yy}\tilde{E}_y \qquad \tilde{D}_z = \varepsilon_{zz}\tilde{E}_z$$

$$\Rightarrow$$

$$\tilde{D}_x\Big|_t^{i,j} = \left(\varepsilon_{xx}\Big|^{i,j}\right)\tilde{E}_x\Big|_t^{i,j} \qquad \tilde{D}_y\Big|_t^{i,j} = \left(\varepsilon_{yy}\Big|^{i,j}\right)\tilde{E}_y\Big|_t^{i,j} \qquad \tilde{D}_z\Big|_t^{i,j} = \left(\varepsilon_{zz}\Big|^{i,j}\right)\tilde{E}_z\Big|_t^{i,j}$$

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Slide 32

E_z Mode

$$C_x^E = \frac{\partial \tilde{E}_z}{\partial y} \qquad C_x^E = -\frac{\mu_{xx}}{c_0}\frac{\partial H_x}{\partial t} \quad \Rightarrow \quad C_x^E\Big|_t^{i,j} = \frac{\tilde{E}_z\Big|_t^{i,j+1} - \tilde{E}_z\Big|_t^{i,j}}{\Delta y} \qquad C_x^E\Big|_t^{i,j} = -\frac{\mu_{xx}\Big|^{i,j}}{c_0}\frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_y^E = -\frac{\partial \tilde{E}_z}{\partial x} \qquad C_y^E = -\frac{\mu_{yy}}{c_0}\frac{\partial H_y}{\partial t} \quad \Rightarrow \quad C_y^E\Big|_t^{i,j} = -\frac{\tilde{E}_z\Big|_t^{i+1,j} - \tilde{E}_z\Big|_t^{i,j}}{\Delta x} \qquad C_y^E\Big|_t^{i,j} = -\frac{\mu_{yy}\Big|^{i,j}}{c_0}\frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_z^H = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad C_z^H = \frac{1}{c_0}\frac{\partial \tilde{D}_z}{\partial t} \quad \Rightarrow \quad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} - \frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_z\Big|_{t+\Delta t}^{i,j} - \tilde{D}_z\Big|_t^{i,j}}{\Delta t}$$

$$\tilde{D}_z = \varepsilon_{zz}\tilde{E}_z \quad \Rightarrow \quad \tilde{D}_z\Big|_t^{i,j} = \left(\varepsilon_{zz}\Big|^{i,j}\right)\tilde{E}_z\Big|_t^{i,j}$$

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Slide 33

H_z Mode

$$C_z^E = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \qquad C_z^E = -\frac{\mu_{zz}}{c_0}\frac{\partial H_z}{\partial t} \quad \Rightarrow \quad C_z^E\Big|_t^{i,j} = \frac{\tilde{E}_y\Big|_t^{i+1,j} - \tilde{E}_y\Big|_t^{i,j}}{\Delta x} - \frac{\tilde{E}_x\Big|_t^{i,j+1} - \tilde{E}_x\Big|_t^{i,j}}{\Delta y} \qquad C_z^E\Big|_t^{i,j} = -\frac{\mu_{zz}\Big|^{i,j}}{c_0}\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_x^H = \frac{\partial H_z}{\partial y} \qquad C_x^H = \frac{1}{c_0}\frac{\partial \tilde{D}_x}{\partial t} \quad \Rightarrow \quad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \qquad C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_x\Big|_{t+\Delta t}^{i,j} - \tilde{D}_x\Big|_t^{i,j}}{\Delta t}$$

$$C_y^H = -\frac{\partial H_z}{\partial x} \qquad C_y^H = \frac{1}{c_0}\frac{\partial \tilde{D}_y}{\partial t} \quad \Rightarrow \quad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = -\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} \qquad C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_y\Big|_{t+\Delta t}^{i,j} - \tilde{D}_y\Big|_t^{i,j}}{\Delta t}$$

$$\tilde{D}_x = \varepsilon_{xx}\tilde{E}_x \quad \Rightarrow \quad \tilde{D}_x\Big|_t^{i,j} = \left(\varepsilon_{xx}\Big|^{i,j}\right)\tilde{E}_x\Big|_t^{i,j}$$

$$\tilde{D}_y = \varepsilon_{yy}\tilde{E}_y \quad \Rightarrow \quad \tilde{D}_y\Big|_t^{i,j} = \left(\varepsilon_{yy}\Big|^{i,j}\right)\tilde{E}_y\Big|_t^{i,j}$$

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Slide 34

Update Equations

Slide 35

Update Equations for E_z Mode

Solving the finite-difference equations for the future time values of the fields associated with the E_z mode leads to:

$$C_x^E\Big|_t^{i,j} = -\frac{\mu_{xx}\big|^{i,j}}{c_0}\frac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t} \qquad H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} = H_x\Big|_{t-\frac{\Delta t}{2}}^{i,j} + \left(-\frac{c_0\Delta t}{\mu_{xx}\big|^{i,j}}\right)C_x^E\Big|_t^{i,j}$$

$$C_y^E\Big|_t^{i,j} = -\frac{\mu_{yy}\big|^{i,j}}{c_0}\frac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t} \qquad H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} = H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j} + \left(-\frac{c_0\Delta t}{\mu_{yy}\big|^{i,j}}\right)C_y^E\Big|_t^{i,j}$$

$$\Rightarrow$$

$$C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_z\big|_{t+\Delta t}^{i,j} - \tilde{D}_z\big|_t^{i,j}}{\Delta t} \qquad \tilde{D}_z\Big|_{t+\Delta t}^{i,j} = \tilde{D}_z\Big|_t^{i,j} + \left(c_0\Delta t\right)C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j}$$

$$\tilde{D}_z\Big|_t^{i,j} = \left(\varepsilon_{zz}\big|^{i,j}\right)\tilde{E}_z\Big|_t^{i,j} \qquad \tilde{E}_z\Big|_{t+\Delta t}^{i,j} = \left(\frac{1}{\varepsilon_{zz}\big|^{i,j}}\right)\left(\tilde{D}_z\Big|_{t+\Delta t}^{i,j}\right)$$

Slide 36

Update Equations for H_z Mode

Solving the finite-difference equations for the future time values of the fields associated with the H_z mode leads to:

$$C_z^E\Big|_t^{i,j} = -\frac{\mu_{zz}\big|^{i,j}}{c_0}\frac{H_z\big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\big|_{t-\frac{\Delta t}{2}}^{i,j}}{\Delta t}$$

$$C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_x\big|_{t+\Delta t}^{i,j} - \tilde{D}_x\big|_t^{i,j}}{\Delta t}$$

$$C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{1}{c_0}\frac{\tilde{D}_y\big|_{t+\Delta t}^{i,j} - \tilde{D}_y\big|_t^{i,j}}{\Delta t}$$

$$\tilde{D}_x\Big|_t^{i,j} = \left(\varepsilon_{xx}\big|^{i,j}\right)\tilde{E}_x\Big|_t^{i,j}$$

$$\tilde{D}_y\Big|_t^{i,j} = \left(\varepsilon_{yy}\big|^{i,j}\right)\tilde{E}_y\Big|_t^{i,j}$$

$\Rightarrow$

$$H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} = H_z\Big|_{t-\frac{\Delta t}{2}}^{i,j} + \left(-\frac{c_0\Delta t}{\mu_{zz}\big|^{i,j}}\right)C_z^E\Big|_t^{i,j}$$

$$\tilde{D}_x\Big|_{t+\Delta t}^{i,j} = \tilde{D}_x\Big|_t^{i,j} + \left(c_0\Delta t\right)C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j}$$

$$\tilde{D}_y\Big|_{t+\Delta t}^{i,j} = \tilde{D}_y\Big|_t^{i,j} + \left(c_0\Delta t\right)C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j}$$

$$\tilde{E}_x\Big|_{t+\Delta t}^{i,j} = \left(\frac{1}{\varepsilon_{xx}\big|^{i,j}}\right)\left(\tilde{D}_x\Big|_{t+\Delta t}^{i,j}\right)$$

$$\tilde{E}_y\Big|_{t+\Delta t}^{i,j} = \left(\frac{1}{\varepsilon_{yy}\big|^{i,j}}\right)\left(\tilde{D}_y\Big|_{t+\Delta t}^{i,j}\right)$$

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Slide 37

Boundary Conditions

Slide 38

Where Are the Boundary Conditions?

All of the spatial derivatives appear only the curl calculations.

$$C_x^E\Big|_t^{i,j} = \frac{\tilde{E}_z\Big|_t^{i,j+1} - \tilde{E}_z\Big|_t^{i,j}}{\Delta y} \quad (y\text{-hi})$$

$$C_y^E\Big|_t^{i,j} = -\frac{\tilde{E}_z\Big|_t^{i+1,j} - \tilde{E}_z\Big|_t^{i,j}}{\Delta x} \quad (x\text{-hi})$$

$$C_z^E\Big|_t^{i,j} = \frac{\tilde{E}_y\Big|_t^{i+1,j} - \tilde{E}_y\Big|_t^{i,j}}{\Delta x} - \frac{\tilde{E}_x\Big|_t^{i,j+1} - \tilde{E}_x\Big|_t^{i,j}}{\Delta y} \quad (x\text{-hi and } y\text{-hi})$$

$$C_x^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \quad (y\text{-lo})$$

$$C_y^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = -\frac{H_z\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_z\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} \quad (x\text{-lo})$$

$$C_z^H\Big|_{t+\frac{\Delta t}{2}}^{i,j} = \frac{H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\Big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} - \frac{H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} \quad (x\text{-lo and } y\text{-lo})$$

Boundary conditions are handled in the curl computations.

We have “modularized” the boundary conditions by isolating the curl calculations.

Hint About Implementing Boundary Conditions

DO NOT USE IF STATEMENTS!!!

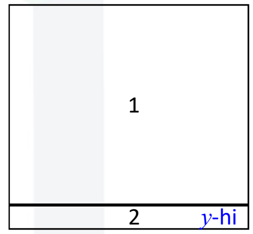
Your code will not be much shorter and it will run slower if you use `if` statements.

Dirichlet Boundary Conditions for CEx

For `CEx`, the problem occurs at the y-hi side of the grid.

It is fixed explicitly.

$$C_x^E\Big|_t^{i,j} = \begin{cases} \dfrac{\tilde{E}_z\big|_t^{i,j+1} - \tilde{E}_z\big|_t^{i,j}}{\Delta y} & 1.\ j < N_y \\[2ex] \dfrac{0 \quad - \tilde{E}_z\big|_t^{i,N_y}}{\Delta y} & 2.\ j = N_y \end{cases}$$




```
% Compute CEx
for nx = 1 : Nx
    for ny = 1 : Ny-1
        CEx(nx,ny) = (Ez(nx,ny+1) - Ez(nx,ny))/dy;
    end
    CEx(nx,Ny) = (0 - Ez(nx, Ny))/dy;
end
```

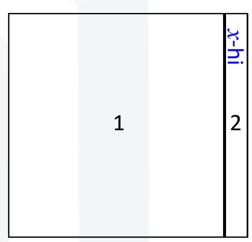
EMPossible Slide 41

Dirichlet Boundary Conditions for CEy

For `CEy`, the problem occurs at the x-hi side of the grid.

It is fixed explicitly.

$$C_y^E\Big|_t^{i,j} = \begin{cases} -\dfrac{\tilde{E}_z\big|_t^{i+1,j} - \tilde{E}_z\big|_t^{i,j}}{\Delta x} & 1.\ i < N_x \\[2ex] -\dfrac{0 \quad - \tilde{E}_z\big|_t^{i,j}}{\Delta x} & 2.\ i = N_x \end{cases}$$




```
% Compute CEy
for ny = 1 : Ny
    for nx = 1 : Nx-1
        CEy(nx,ny) = - (Ez(nx+1,ny) - Ez(nx,ny))/dx;
    end
    CEy(Nx,ny) = - (0 - Ez(Nx,ny))/dx;
end
```

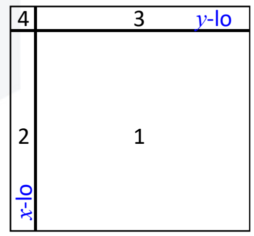
EMPossible Slide 42

Dirichlet Boundary Conditions for `CHz`

For `CHz`, the problem occurs at both the x-lo side of the grid and y-lo.

Both of these are fixed explicitly.

$$
C_z^H\Big|_{t+\Delta t/2}^{i,j} = \begin{cases}
\dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j} - H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x} - \dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j} - H_x\big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} & \text{1. for } i>1 \text{ and } j>1 \\[2ex]
\dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{1,j} - 0}{\Delta x} - \dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{1,j} - H_x\big|_{t+\frac{\Delta t}{2}}^{1,j-1}}{\Delta y} & \text{2. for } i=1 \text{ and } j>1 \\[2ex]
\dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,1} - H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,1}}{\Delta x} - \dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,1} - 0}{\Delta y} & \text{3. for } i>1 \text{ and } j=1 \\[2ex]
\dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{1,1} - 0}{\Delta x} - \dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{1,1} - 0}{\Delta y} & \text{4. for } i=1 \text{ and } j=1
\end{cases}
$$



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Slide 43

MATLAB Code for CHz (1 of 3)

First, just blindly implement the curl calculation…

```
% Compute CHz
for ny = 1 : Ny
    for nx = 1 : Nx
        CHz(nx,ny) = (Hy(nx,ny) - Hy(nx-1,ny))/dx ...
                   - (Hx(nx,ny) - Hx(nx,ny-1))/dy;
    end
end
```

As expected, this will produce an error when trying to access `Hy(0,ny)` or `Hx(nx,0)`.

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Slide 44

MATLAB Code for CHz (2 of 3)

Second, the problem at nx = 1 is handled explicitly by copying the code inside the nx loop, pasting it above, and handling the problem.

```
% Compute CHz
for ny = 1 : Ny
    CHz(1,ny) = (Hy(1,ny) - 0)/dx ...
              - (Hx(1,ny) - Hx(1,ny-1))/dy;
    for nx = 2 : Nx
        CHz(nx,ny) = (Hy(nx,ny) - Hy(nx-1,ny))/dx ...
                   - (Hx(nx,ny) - Hx(nx,ny-1))/dy;
    end
end
```

There is still an error at Hx(1,0) and Hx(nx,0)

MATLAB Code for CHz (3 of 3)

Third, the problem at ny = 1 is handled explicitly by copying the code inside the ny loop, pasting it above, and handling the problem.

```
% Compute CHz
CHz(1,1) = (Hy(1,1) - 0)/dx ...
         - (Hx(1,1) - 0)/dy;
for nx = 2 : Nx
    CHz(nx,1) = (Hy(nx,1) - Hy(nx-1,1))/dx ...
              - (Hx(nx,1) - 0)/dy;
end
for ny = 2 : Ny
    CHz(1,ny) = (Hy(1,ny) - 0)/dx ...
              - (Hx(1,ny) - Hx(1,ny-1))/dy;
    for nx = 2 : Nx
        CHz(nx,ny) = (Hy(nx,ny) - Hy(nx-1,ny))/dx ...
                   - (Hx(nx,ny) - Hx(nx,ny-1))/dy;
    end
end
```

Revised FDTD Algorithm for 2D Simulations

FDTD Algorithm for E_z Mode

time

$$C_x^E\Big|_t^{i,j}=\begin{cases}\dfrac{\tilde{E}_z\big|_t^{i,j+1}-\tilde{E}_z\big|_t^{i,j}}{\Delta y} & j<N_y\\[2ex] \dfrac{0-\tilde{E}_z\big|_t^{i,N_y}}{\Delta y} & j=N_y\end{cases}\qquad C_y^E\Big|_t^{i,j}=\begin{cases}-\dfrac{\tilde{E}_z\big|_t^{i+1,j}-\tilde{E}_z\big|_t^{i,j}}{\Delta x} & i<N_x\\[2ex] -\dfrac{0-\tilde{E}_z\big|_t^{i,j}}{\Delta x} & i=N_x\end{cases}$$

x & y

$$H_x\Big|_{t+\frac{\Delta t}{2}}^{i,j}=H_x\Big|_{t-\frac{\Delta t}{2}}^{i,j}+\left(-\frac{c_0\Delta t}{\mu_{xx}\big|^{i,j}}\right)C_x^E\Big|_t^{i,j}\qquad H_y\Big|_{t+\frac{\Delta t}{2}}^{i,j}=H_y\Big|_{t-\frac{\Delta t}{2}}^{i,j}+\left(-\frac{c_0\Delta t}{\mu_{yy}\big|^{i,j}}\right)C_y^E\Big|_{t+\frac{\Delta t}{2}}^{i,j}$$

x & y

$$C_z^H\Big|_{t+\Delta t/2}^{i,j}=\begin{cases}\dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,j}-H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,j}}{\Delta x}-\dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,j}-H_x\big|_{t+\frac{\Delta t}{2}}^{i,j-1}}{\Delta y} & \text{for } i>1 \text{ and } j>1\\[2ex] \dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{1,j}-0}{\Delta x}-\dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{1,j}-H_x\big|_{t+\frac{\Delta t}{2}}^{1,j-1}}{\Delta y} & \text{for } i=1 \text{ and } j>1\\[2ex] \dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{i,1}-H_y\big|_{t+\frac{\Delta t}{2}}^{i-1,1}}{\Delta x}-\dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{i,1}-0}{\Delta y} & \text{for } i>1 \text{ and } j=1\\[2ex] \dfrac{H_y\big|_{t+\frac{\Delta t}{2}}^{1,1}-0}{\Delta x}-\dfrac{H_x\big|_{t+\frac{\Delta t}{2}}^{1,1}-0}{\Delta y} & \text{for } i=1 \text{ and } j=1\end{cases}$$

x & y

$$\tilde{D}_z\Big|_{t+\Delta t}^{i,j}=\tilde{D}_z\Big|_t^{i,j}+\left(c_0\Delta t\right)C_z^H\Big|_{t+\Delta t/2}^{i,j}$$

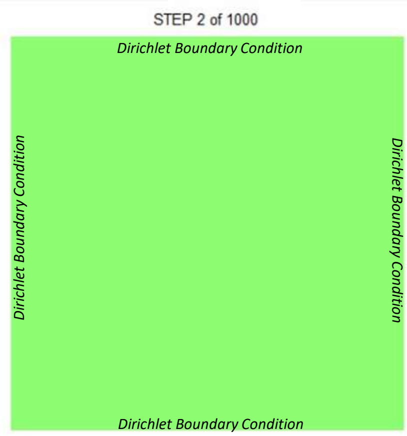
x & y

Simple soft source $\tilde{D}_z\Big|_{t+\Delta t}^{i_{src},j_{src}}=\tilde{D}_z\Big|_{t+\Delta t}^{i_{src},j_{src}}+g(T)$

$$\tilde{E}_z\Big|_{t+\Delta t}^{i,j}=\left(\frac{1}{\varepsilon_{zz}\big|^{i,j}}\right)\left(\tilde{D}_z\Big|_{t+\Delta t}^{i,j}\right)$$

x & y

Animation of Basic 2D FDTD Algorithm



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