Second partial exam of Fundamentals of Mathematics II

June 13, 2022

1. Say all you can say about the maximum and minimum value of the function $f(x,y) = x^2 + y^2 - xy$ on the disc $x^2 + y^2 \leq 1$.

Solution: First we shall compute the maxima and the minima of the function $f(x,y)$ in the whole plane $\mathbb{R}^2$ where this function is defined, and verify if some of this maxima or minima belongs to the closed disc $x^2 + y^2 \leq 1$ centered at the origin of coordinates and of radius 1, that we call in what follows the disc D. So we must look for the solutions of the system

$$\frac{\partial f}{\partial x} = 2x - y = 0, \qquad \frac{\partial f}{\partial y} = 2y - x = 0.$$

The unique solution of this system is the $(x,y) = (0,0)$. So if the function $f(x,y)$ has a maximum or a minimum it must be in the point $(0,0)$.

The Hessian matrix of the function $f(x,y)$ is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Since the eigenvalues of this matrix are 1 and 3 the origin $(0,0)$ is a minimum of the function $f(x,y)$, and this minimum is inside the disc D.

Now we must study if the function $f(x,y)$ has maxima and minima in the boundary of the disc D. Therefore we consider the Lagrange function

$$F(x,y,\lambda) = f(x,y) + \lambda(x^2 + y^2 - 1),$$

and the solutions of the system

(1)
$$\frac{\partial F}{\partial x} = 2x-y+2\lambda x = 0, \qquad \frac{\partial F}{\partial y} = 2y-x+2\lambda y = 0, \qquad \frac{\partial F}{\partial \lambda} = x^2+y^2-1 = 0,$$

provide the possible points in the boundary of the disc D where the function $f(x,y)$ can have maxima and minima.

From the first equation of (1) we obtain that $y = 2(1+\lambda)x$, substituting this expression of y into the second equation of (1) we obtain $x(3+8\lambda+4\lambda^2) = 0$, whose solutions are $x = 0$, $\lambda = -1/2$ and $\lambda = -3/2$.

The solution $x = 0$ implies $y = 0$, but then the third equation of (1) is not satisfied, consequently this is not a solution of system (1).

The solution $\lambda = -1/2$ reduces system (1) to the system

$$x - y = 0, \qquad x^2 + y^2 - 1 = 0,$$

and the solutions of this system are $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$.

The solution $\lambda = -3/2$ reduces system (1) to the system

$$x + y = 0, \qquad x^2 + y^2 - 1 = 0,$$

and the solutions of this system are $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$.

In summary, we have four points (x, y, λ) in the boundary of the disc D where can be maxima and minima, namely

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{3}{2}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{3}{2}\right).$$

We compute the Hessian matrix of the function $F(x, y, \lambda)$, i.e.

$$M(\lambda) = \begin{pmatrix} 2(\lambda + 1) & -1 \\ -1 & 2(\lambda + 1) \end{pmatrix}$$

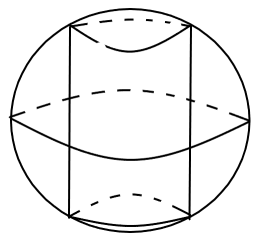
The eigenvalues of the matrix $M(-1/2)$ are 2 and 0. We know that if both eigenvalues are positive the function $f(x, y)$ has a minimum in the points $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$, but since one eigenvalues is zero, we cannot say this for the moment.

The eigenvalues of the matrix $M(-3/2)$ are -2 and 0. We know that if both eigenvalues are negative function $f(x, y)$ has a maximum in the points $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$, but since one eigenvalues is zero, we cannot say this for the moment.

This is the most that you can say according with the theory that we have seen, and this is the complete solution of this problem.

A comment, using other tools that you at this moment do not know it is possible to prove that the points $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (-1/\sqrt{2}, -1/\sqrt{2})$ are local minima, and the points $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$ are local maxima.

2. Compute the volume of the cylinder $x^2 + y^2 = 1$ inside the sphere $x^2 + y^2 + z^2 = 4$.



Solution: If we denote by V the volume of the cylinder $x^2 + y^2 = 1$ inside the sphere $x^2+y^2+z^2 = 4$. We must compute the triple integral

$$I = \int\int\int_V dxdydz,$$

but since we want to compute the volume of a piece of a cylinder, the best is to use cylindrical coordinates, i.e. (r, θ, z) where $x = r\cos\theta$ and $y = r\sin\theta$. So the previous triple integral in cylindrial coordinates becomes

$$\begin{aligned}
I &= \int\int\int_V r dr d\theta dz \\
&= \int_0^{2\pi} d\theta \int_0^1 r dr \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \\
&= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2} dr \\
&= \int_0^{2\pi} dt \cdot \left| -\frac{2}{3}(4-r^2)^{3/2} \right|_0^1 \\
&= \frac{16-6\sqrt{3}}{3} \int_0^{2\pi} dt \\
&= \frac{32-12\sqrt{3}}{3}\pi.
\end{aligned}$$

3. The Green Theorem states that the following equality between integrals holds:

$$\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_\gamma Pdx + Qdy,$$

where R is a region of the plane without holes and γ is the boundary curve of the region R oriented in counterclockwise sense. Show that this equality holds if $P = P(x, y) = x + 2y$, $Q = Q(x, y) = x^2 + y^2 + 1$ and $R = \{(x, y) : x^2 + y^2 \leq 1\}$.

Solution: We shall take polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$
\begin{aligned}
I_1 = & \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \\
= & \int_0^{2\pi} d\theta \int_0^1 (2r\cos\theta - 2) r dr \\
= & \int_0^{2\pi} d\theta \left| \frac{2}{3} r^3 \cos\theta - r^2 \right|_0^1 \\
= & \int_0^{2\pi} \left(\frac{2}{3}\cos\theta - 1\right) d\theta \\
= & \left| \frac{2}{3}\sin\theta - \theta \right|_0^{2\pi} = -2\pi.
\end{aligned}
$$

On the other hand since

$$dx = \cos\theta dr - r\sin\theta d\theta,\ dy = \sin\theta dr + r\cos\theta d\theta,\ r = 1 \text{ and } dr = 0,$$

it follows that

$$
\begin{aligned}
I_2 = & \int_\gamma P dx + Q dy \\
= & \int_\gamma (x + 2y) dx + (x^2 + y^2 + 1) dy \\
= & \int_0^{2\pi} \big((\cos\theta + 2\sin\theta)(-\sin\theta) + 2\cos\theta\big) d\theta \\
= & -\int_0^{2\pi} \sin\theta\cos\theta d\theta - 2\int_0^{2\pi} \sin^2\theta d\theta + 2\int_0^{2\pi} \cos\theta d\theta
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_0^{2\pi} \sin\theta\cos\theta d\theta = 0, \\
\int_0^{2\pi} \sin^2\theta d\theta = \pi, \\
\int_0^{2\pi} \cos\theta d\theta = 0,
\end{aligned}
$$

we have that $I_2 = -2\pi$.