1. Functions of several variables

Points, vectors and lines

1. Find cartesian and parametric equations for these lines:
   (a) The line through the points $(1, 2)$ i $(3, 1)$;
   (b) The line through $(1, 1)$ with direction vector equal to $(0, 1)$.
2. Provide parametrizations for the sides of the triangle determined by $x = 0$, $y = 0$ and $x + y = 6$.
3. Find cartesian and parametric equations for these lines:
   (a) The line through the points $(-1, 0, 2)$ i $(3, 3, 1)$;
   (b) The line through $(\sqrt{2}, 1, -\pi)$ with direction vector equal to $(0, 1, e)$.
4. Compute the measure of the angles given by:
   (a) The vectors $(2, 4)$ i $(4, 2)$,
   (b) The lines $x + 2y + 4 = 0$ i $4x + 3y + 1 = 0$,
   (c) The vectors $(2, 4, 7)$ i $(7, 4, -2)$,
   (d) The lines r_1 i r_2 given by these equations

$$\begin{aligned} r_1 : &\quad x + 2y + 3z + 4 = 0,\ 4x + 3y + 2z + 1 = 0 \\ r_2 : &\quad 5x + y - z + 1 = 0,\ -x - y + z + 3 = 0. \end{aligned}$$

5. (a) Compute the dot product of these plane vectors: $(\frac{1}{2}, \frac{1}{\sqrt{2}}) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{2})$ and of these 3-d vectors i $(-1, 0, 2) \cdot (3, 3, 1)$.
   (b) Find equations for the line in $\mathbb{R}^3$ which is orthogonal to the plane $x + y + z - 1 = 0$ and goes through the point $(3, 1, -4)$.
   (c) Find all vectors in $\mathbb{R}^3$ which are perpendicular to the line given by the equations

$$5x + y - z + 1 = 0,\ -x - y + z + 3 = 0.$$

   (d) Determine all pairs of vectors v, w satisfying $||v + w|| = ||v|| + ||w||$. Find also all vectors u, w such that $|\langle u, w\rangle| = ||u|| \cdot ||w||$.
6. Prove or disprove these claims:
   (a) $||v - w|| \geq \big| ||v|| - ||w|| \big|$.
   (b) $||u - v||^2 + ||u + v||^2 = 2(||u||^2 + ||v||^2)$.

7. Compute $v \times w$ for $v = (-1, 2, 3)$ i $w = (6, 1, 0)$. Find all vectors which are orthogonal to v. Provide a description of all vectors of modulus 1 in $\mathbb{R}^3$ as well as the subset of those which are orthogonal to w. (Here all vectors are supposed to start at $(0, 0, 0) \in \mathbb{R}^3$.)

8. Describe these sets of point of the plane $\mathbb{R}^2$:

   (a) Those at a fixed distance $r > 0$ to the point (a, b).

   (b) Those at the same distance to the point $(a, 0)$ as to the Y-axis.

   (c) The points (x, y) such that $x^2 + y^2 = 2x$.

   (d) The points (x, y) such that $x^2 - y^2 = 2x$.

9. Give a geometric description of this subset of $\mathbb{R}^3$: (Hint: It's a sphere.)

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2x + 2y\}.$$

Curves in the plane and in 3-dimensional space

10. Provide descriptions of the paths in the plane given by these curves:

$$(a)\ t \to (0, 0); \quad (b)\ t \to (t, t); \quad (c)\ t \to (t, t^2); \quad (d)\ t \to (t^2, t^3);$$

$$(e)\ t \to (\sin 2\pi t, \cos 2\pi t); \quad (f)\ t \to (t - \sin t, 1 - \cos t); \quad (g)\ t \to (t, \sqrt[4]{|t|});$$
$$(h)\ t \to (\cos 2\pi t, t); \quad (i)\ t \to (e^{-t} \cos t, e^{-t} \sin t); \quad (j)\ t \to (e^{-t} \cos t, e^{-t}).$$

11. Compute the tangent vector and the tangent line at the point $P = (1, 0)$ for the curve (i) in exercise 10. Is there any tangent line for the curve (c) in exercise 10 which passes through the point $Q = \left(\frac{13}{4}, 10\right)$?

12. (a) Provide descriptions of the paths in $\mathbb{R}^3$ given by these curves:

$$(1)\ t \to (t, t, t); \quad (2)\ t \to (t, t^2, t^3); \quad (3)\ t \to (\sin 2\pi t, \cos 2\pi t, 1)$$
$$(4)\ t \to (\sin 2\pi t, \cos 2\pi t, t); \quad (5)\ t \to (e^{-t} \sin t, e^{-t} \cos t, e^{-t}).$$

   (b) Prove that the path given by the curve $\left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$ for $t \neq 0$ lies in a plane,

13. Compute the tangent vector to the curve (e) in exercise 12 at the point $P = (0, 1, 1)$, as well as cartesian equations of the tangent line to the curve through this same point.

Differentiable functions of several variables

14. Describe the largest subsets of $\mathbb{R}^2$ where these formulas define functions:

   (a) $f(x, y) = \sqrt{1 - x + y}$.

(b) $g(x, y) = \dfrac{2xy}{x^2 + y^2}$.

(c) $h(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$.

(d) $f_1(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$.

(e) $g_1(x, y) = |x - y|^{-1} + \sin(x + y) + e^{x - y^2}$

(f) $h_1(x, y) = \dfrac{1}{x^2 - y}$

(g) $f_2(x, y) = (\log(|x| - |y|))^{-1}$

(h) $g_2(x, y) = \log(f(x, y)^2) + \log(x^2 + y^2)$

15. Find the isolines of the functions f, g, h, f_1 in exercise 14.

16. Find the isolines/isosurfaces of these functions:

 a) $f(x, y) = x - y + 2$, b) $g(x, y) = x + y$, c) $h(x, y) = xy$.

 d) $F(x, y, z) = x^2 + y^2 + 4z^2$, e) $G(x, y, z) = 4x^2 + y^2 - z^2$.

17. Plot the functions f, g, h, f_1 in exercise 14.

18. Plot the functions f, g, h, in exercise 16.

19. Compute the gradient of these functions:

 (a) $f(x, y) = xy$.

 (b) $f(x, y) = e^{xy}$.

 (c) $f(x, y) = \dfrac{x^2 + y^2}{x^2 - y^2}$.

 (d) $g(x, y, z) = \dfrac{x}{y}$.

 (e) $g(x, y, z) = e^{zx} \cos(x + zy)$.

 (f) $g(x, y, z) = xe^{2y} \log(x^2 yz)$.

20. Compute the directional derivatives of the following functions at the given points in the indicated directions:

 a) $f(x, y) = x + 2xy - 3y^2$, $P = (1, 2)$, $v = (3/5, 4/5)$.

 b) $g(x, y) = \log \sqrt{x^2 + y^2}$, $P = (1, 0)$, $v = \dfrac{1}{\sqrt{5}}(2, 1)$.

 c) $h(x, y) = \sin x + \cos y$, $P = (0, \pi)$, $v = (0, 1)$.

 d) $k(x, y) = e^{2xy^2}$, $P = (0, 1)$, $v = (1, 0)$.

21. Find the cartesian equations of the tangent planes to the plots of the following functions at the given points.

   a) $f(x,y) = \arctan\left(\dfrac{x}{y}\right)$, $\quad P = \left(1, \sqrt{3}, \dfrac{\pi}{6}\right)$.

   b) $g(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$, $\quad P = (1,0,1)$.

22. For each of the following surfaces in $\mathbb{R}^3$ find the tangent plane and the normal line at the given points:

   a) $x^2+y^2+z^2 = 3$, $\quad P = (1,-1,1)$.

   b) $\cos z = \sin(x+y)$, $\quad P = \left(\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

   c) $x^2+z^2 = 4y$, $\quad P = (4,8,4)$.

   d) $xyz = 1$, $\quad P = (1,1,1)$.

   e) $z = \cos x \cos y$, $\quad P = (0, \pi/2, 0)$.

   f) $x^2-y^2 = z^2$, $\quad P = (-3,1,2\sqrt{2})$.

23. Compute the Jacobian matrix of each of these maps:

   a) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\quad (x,y) \mapsto (y,x)$.

   b) $g\colon \mathbb{R}^2 \to \mathbb{R}^3$, $\quad (x,y) \mapsto (xe^y + \cos y, x, x+e^y)$.

24. Use the chain rule to compute $\dfrac{\partial h}{\partial x}$, where $h(x,y) = f(u(x,y), v(x,y))$ and f, u, v are given as follows:

   a) $f(u,v) = u^2+v^2$, $\quad u(x,y) = xy$, $\quad v(x,y) = x+y$.

   b) $f(u,v) = \dfrac{u^2+v^2}{u^2-v^2}$, $\quad u(x,y) = -x-y$, $\quad v(x,y) = xy$.

   c) $f(u,v) = \sqrt{u^2+2uv}$, $\quad u(x,y)) = \cos xy$, $\quad v(x,y) = \sin xy$.

   d) $f(u,v) = \sqrt{v^2+2uv}$, $\quad u(x,y) = \sin xy$, $\quad v(x,y) = \cos xy$.

   e) $f(u,v) = \log(u^2+v^2)$, $\quad u(x,y) = \sqrt{xy}$, $\quad v(x,y) = \sqrt{x^2+y^2-xy}$.

   f) $f(u,v) = e^{uv}$, $\quad u(x,y) = e^{xy}$, $\quad v(x,y) = x^2y - xy^2$.

25. Consider the maps $g : \mathbb{R}^2 \to \mathbb{R}^2$ i $h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x,y) = (x^2+y, x-y^2) \quad \text{i} \quad h(x,y) = (x^2+y)^3 - (x-y^2)^4.$$

   (a) Compute the differential matrix $Dg(1,0)$.

   (b) Find a function $f(u,v)$ such that $h(x,y) = f(g(x,y))$ and compute the gradient vector $\nabla f(1,1)$.

   (c) Check that $\nabla h(1,0) = (2,3)$.

   (d) Find the equation of the tangent plane to the plot $z = h(x,y)$ at the point $(1,0,0)$.

26. The *ideal gas law* is $PV = nRT$ where P is the pressure, V is the volume, T is the temperature and R is the *ideal gas constant* (the same for all gases), while n is a constant related to the number of particles in the gas divided by the Avogadro constant. Prove that the following formula holds:

$$\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}\frac{\partial P}{\partial V} = -1.$$

The Van der Waals equation extends the ideal gas law in the following way:

$$P = \frac{RT}{V-\beta} - \frac{\alpha}{V^2},$$

where α, β are constants. Check that the previous formula still holds for the Van der Waals equation.

27. Consider a square metallic plate $Q = \{(x, y) : 0 \leq x \leq 5,\ 0 \leq y \leq 5\}$ which is heated in a way that the temperature at the point (x, y) is given by the function $T(x, y) = x^2 + y^2$. Compute the direction of the thermal flux (in the plate) at the point $(2, 4)$. In which points of the plate can we find the highest temperature?

28. Compute $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ i $\dfrac{\partial^2 f}{\partial y^2}$ for each of the following functions:

$$f(x, y) = \cos(xy^2), \quad f(x, y) = e^{x^2+y^2}.$$

29. Is there any function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f(x, y) = (2xy + 1, x^2)$?

30. Prove that the origin is a critical point of the function $f(x, y) = ax^2+2bxy+y^2$. Describe its type for all values of a and b.

Extrema

31. Find the local extrema of these functions:

    (a) $f(x, y) = 8x^3 - 24xy + y^3$.
    (b) $f(x, y) = \log(2 + \sin(xy))$.
    (c) $f(x, y) = \sin x \cos y$.
    (d) $f(x, y) = (x^2 + y^2 + 1)^{-1}$.
    (e) $g(x, y, z) = x^2 + y^2 - z^2 - xy + xz - 2z$.
    (f) $g(x, y, z) = xyz(1 - x)(1 - y)(1 - z)$.

32. Find the absolute extrema of the following functions in the indicated sets:

a) $f(x, y) = x^2 + y^2$	along the line $3x + 2y = 6$.
b) $f(x, y) = 1 - x^2 - y^2$	along the line $x + y = 1$ with $x \geq 0$ i $y \geq 0$.
c) $f(x, y) = x^2y + 12y^2 + 2xy$	along the ellipse $x^2 + 2x + 16y^2 = 9$.
d) $f(x, y) = x - y$	along the hyperbola $x^2 - y^2 = 2$.
e) $f(x, y) = \cos^2 x + \cos^2 y$	along the line $x + y = \pi/4$.
f) $g(x, y, z) = 3x^2 + 3y^2 + z^2$	in the plane $x + y + z = 1$.
g) $g(x, y, z) = x^2 + y^2 + z^2$	in the set $4x^2 + 9y^2 + 16z^2 = 1$, $x = y$.

33. Find the maximum of the function $f(x, y) = x^2y(4 - x - y)$ in the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 6$.

34. Consider the function $f(x, y, z) = x^3 + y^3 + z^3$ where n is a natural number. Find its absolute extrema in the ball $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

35. Let $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \geq x\}$ and let f be the function given by $f(x, y) = x^2 + y^2 + 2x$.

    (a) Is E compact?

    (b) Does f have maximum and minimum in E? If it does, find these extrema.

36. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x, y) = x^2 + y^2 - 2x - 2y$ and let D be the semi-disc $D = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 4,\ y \geq 0\}$.

    (a) Find the critical points of f in $\mathbb{R}^2$ and describe their type (local maximum, local minimum or saddle point).

    (b) Has f absolute extrema in D? If it does, find them.

37. Find the absolute maximum and minimum of the function $f(x, y) = 5x^2 + 5y^2 - 8xy$ in the compact set $\{(x, y) : x^2 + y^2 - xy \leq 1\}$.

38. Find the triangle with the largest area among the triangles with a fixed perimeter. (Hint: use Heron's formula $S^2 = p(p - a)(p - b)(p - c)$ where a, b, c are the lengths of the sides, S is the area and p is half the perimeter.)

39. Consider the function $T(x, y) = 20 + 2x + 2y - x^2 - y^2$. Prove that the value of T in any point of the disc $D = \{(x, y) : x^2 + y^2 \leq 2\}$ is higher that in any point of

$$R = \left\{(x, y) : x^2 - 8x + y^2 + 16 \leq \frac{1}{16}\right\}.$$

40. Find the rectangular parallelepiped with the largest volume among those with the same area.

41. Find the circular sector with the smaller perimeter among those with the same area.

42. Find a rectangular parallelepiped satisfying each of these conditions:

    (a) Length of the diagonal is fixed and volume is maximum.

    (b) Volume is 500 and area without the top face is minimum.

43. Consider the ellipsoid $16x^2+4y^2+9z^2 = 144$. Find the largest volume of any rectangular parallelepiped inscribed in this ellipsoid in such a way that its sides are parallel to the coordinate planes.

44. Find the points closest to the origin in the plane $2x - y + 2z = 16$ and in the surface $z^2 - xy = 1$.

45. Imagine the temperature at each point of a plane is given by

$$T(x, y) = \frac{100}{x^2 + y^2 - 2x - 2y + 6}.$$

    (a) What is the temperature at the origin and in which direction does the thermal flux flow? Find the isotherm curve through the origin.

    (b) Which is the hottest point and what is its temperature?

    (c) Which points in the disc $x^2 + y^2 \leq 1$ have highest and lowest temperature?

46. Consider the function $f(x, y) = x^3 - x + y^2 - y + 2xy$.

    (a) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1, 2)$.

    (b) Find the direction in which the function decreases most quickly at the point $(1, -1)$.

    (c) Find all critical points of f, as well as their type.

47. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = -x^3 - y^3 + \frac{3}{2}x^2 + 3y^2.$$

    (a) Find all critical points of f, as well as their type.

    (b) Find the maximum and minimum of f along the circle $\{(x, y) : x^2 + y^2 = 1\}$.

    (c) Does f have absolute extrema in the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$? If yes, find them.

2. Multiple integrals

48. Compute

    (a) $\displaystyle\int_0^1 \left(\int_0^1 xye^{(x+y)}\, dy \right) dx.$

    (b) $\displaystyle\int_{-1}^0 \left(\int_1^2 x \log y \, dy \right) dx.$

49. Compute the following double integrals:

    (a) $\displaystyle\iint_Q (xy)^2 \cos x \, dx\, dy$ where $Q = [0, \frac{\pi}{2}] \times [0, 1]$.

    (b) $\displaystyle\iint_T y \, dx\, dy; \quad \iint_T x \, dx\, dy$
    where T is the triangle with vertices $A = (0, 3)$, $B = (3, 0)$, $C = (3, 6)$. Assuming density is constant, find the center of mass of T.

50. Compute the following double integrals:

    (a) $\displaystyle\iint_D (x^2 + y^2)\, dx\, dy$ where D is the closed unit disk.

    (b) $\displaystyle\iint_R (x^3 y)\, dx\, dy$
    where R is the region bounded by the y axis and the parabola $x = y^2 - 4$.

    (c) $\displaystyle\iint_S x^2 y \, dx\, dy$ where $S = \{(x, y) : 0 \le y \le \frac{1}{2},\ x^2 + y^2 \le 1\}$.

51. Plot the plane regions given by the integral limits and compute the integrals.

    (a) $\displaystyle\int_1^2 \left(\int_{2x}^{3x+1} y \, dy \right) dx$

    (b) $\displaystyle\int_0^1 \left(\int_{x^3}^{x^2} y^2 \, dy \right) dx$

    (c) $\displaystyle\int_0^{\pi/2} \left(\int_0^{\cos x} y \sin x \, dy \right) dx$

52. Use Fubini's theorem to compute the following iterated integrals.

    (a) $\displaystyle\int_0^2 \left(\int_{\frac{y}{2}}^1 e^{x^2}\, dx \right) dy$

    (b) $\displaystyle\int_0^1 \left(\int_{\sqrt[6]{x}}^1 \sin y^7 \, dy \right) dx$

(c) $\int_0^4 \left(\int_{\sqrt{y}}^2 \frac{ye^x}{x^4}\, dx \right) dy$

53. Use polar coordinates to find:

    (a) The area of the region bounded by

$$S = \{(x, y) \in \mathbf{R}^2 : (x^2 + y^2)^2 = 2a^2(x^2 - y^2)\},\ a > 0.$$

    (b) he area of the region bounded by the curves $r = b(1 + \cos\varphi)$ i $r = b\cos\varphi$, $b > 0$,

    (c) $\iint_D \sqrt{a^2 - x^2 - y^2}\, dx\, dy$ where $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq r^2\}$, $r > 0$,

    (d) $\iint_R \sqrt{x^2 + y^2}\, dx\, dy$ where $R = \{(x, y) \,|\, x^2 + y^2 \leq 2x\}$.

54. Compute the center of mass of this three-dimensional object (assuming constant density):

$$B = \{(x, y, z) \,|\, x^2 + y^2 \leq 1,\ 0 \leq z \leq (x^2 + y^2)^{1/2}\}.$$

55. Let B be the solid unit sphere. Using an appropriate change of variables compute

$$I = \iiint_B \frac{1}{\sqrt{2 - x^2 - y^2 - z^2}}\, dV.$$

56. Find the volume of the following three-dimensional bodies, as well as their surface area.

    (a) $\{\, (x, y, z) \ : \ x^2 + y^2 + z^2 \leq 36,\ z \geq 2 \,\}$.

    (b) $\{\, (x, y, z) \ : \ x^2 + y^2 + z^2 \leq 36,\ -1 \leq z \leq 3 \,\}$.

57. Find the center of mass of these bodies (assuming constant density):

    (a) A circular cone with base radius R and height h.

    (b) The body bounded by the conic surface $c^2z^2 = x^2 + y^2$ with $z \geq 0$, and the sphere with radius R and center at the origin. (c is a constant.)

58. Consider a plane metallic disc of radius π cm which has a variable density given by the function (polar coordinates, g/cm^2)

$$\rho(r, \theta) = r^2 \sin^2 4\theta + 2$$

    (a) Compute the weight of the disk.

    (b) What is the average density of the disk?

3. Curves and line integrals

Curves and line integrals

59. Plot the following curves and compute their length:

    (a) **Logarithmic spiral** $\gamma(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $b < 0$, $t \in [0, +\infty)$.

    (b) **Catenary** $y(x) = a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)/2$, $x \in [-a, a]$, $a > 0$.

    (c) **Cycloid** $c(t) = (rt - r\sin t, r - r\cos t)$, $0 \le t \le 2\pi$, $r > 0$.

    (d) $p(t) = (t, a\arcsin(t/a), (a/4)\log(a+t)/(a-t))$, $t \in [0, a/2]$, $a > 0$.

60. Compute the line integral of the vector field $\mathbf{F}(x, y, z) = (x, \cos z, y)$ along the curve $\gamma(t) = (t, t^2, 0)$ for $t \in [0, 1]$.

61. Compute the line integral of the vector field $(x^3 z, -z^3 x, y^3)$ along the curve $(\sin t, t, \cos t)$ for $t \in [0, \frac{\pi}{2}]$.

62. Compute the work done by the force $\mathbf{F}(x, y) = (y, -x)$ acting on a particle which moves along the curve $\gamma(t) = (t^3, t^4)$ for $0 \le t \le 1$.

63. Compute the line integral of the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ along these paths:

    (a) $c_1(t) = (t, t, t)$, $t \in [0, 1]$.

    (b) $c_2(t) = (\cos \pi t, \sin \pi t, 0)$, $t \in [0, 1]$.

    (c) $c_3(t) = (\cos 2\pi t, \sin 2\pi t, t)$, $t \in [0, n]$, $n \in \mathbb{N}$.

4. Convex Optimization

Convex functions

64. Show that if $f\colon \mathbb{R}^d \to \mathbb{R}$ is a convex function, then it satisfies Jensen's inequality for all $n \geq 2$:
$$f(a_1x_1 + \cdots + a_nx_m) \leq a_1f(x_1) + \cdots + a_nf(x_n)$$
for $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_i \geq 0$ for all i and $\sum_{i=1}^n a_i = 1$, and all $x_1, \ldots, x_n \in \mathbb{R}^d$.

65. Show that for all $x_1, \ldots, x_n > 0$ in $\mathbb{R}$, we have
$$\frac{1}{n}(x_1 + \cdots x_n) \geq \sqrt[n]{x_1 \cdots x_n}.$$
Hint: Use Jensen's inequality and the fact that ln is concave.

66. Show that the functions $x^2, x^4, \ldots, x^{2n}$ are all convex.

67. Let $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Show that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by
$$f(x) = w^Tx + b$$
is both convex and concave.

68. Show that if $A \in M_n(\mathbb{R})$ is a symmetric positive semidefinite matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then
$$f(x) = \frac{1}{2}x^TAx + b^Tx + c$$
is convex.

69. Show that if $f_1, f_2\colon \mathbb{R}^d \to \mathbb{R}$ are convex and a_1, a_2 are non-negative real numbers, then $a_1f_1 + a_2f_2$ is also a convex function.

70. Show that if $f_1, f_2\colon \mathbb{R}^d \to \mathbb{R}$ are convex then so is the max function:
$$f(x) = \max\{f_1(x), f_2(x)\}.$$

71. Show that $f\colon \mathbb{R}^n_+ \to \mathbb{R}$,
$$f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i \ln(x_i)$$
is convex.

Convex sets

72. Let $f \colon \mathbb{R}^d \to \mathbb{R}$. The *epigraph* of f is:

$$E = \{(x, s) \in \mathbb{R}^d \times \mathbb{R} : f(x) \leq s\}.$$

Show that E is a convex set if and only if f is a convex function.

73. Suppose that $a_1, a_2, \dots, a_n > 0$. Show that the ellipsoid

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 \leq 1\}$$

is a convex set.

74. Consider whether the following statements are true or false:

    (a) The intersection of two convex subsets is convex.

    (b) The union of two convex subsets is convex.

    (c) The difference $A \setminus B = \{x \in A : x \notin B\}$ of a convex set A from another convex set B is convex.

Optimization. Duality.

75. Express the following optimization problem

$$\max\nolimits_{x \in \mathbb{R}^2, \xi \in \mathbb{R}} p^{\mathsf{T}} x + \xi$$

subject to the constraints $\xi \geq 0, x_0 \leq 0, x_1 \leq 3$ as a standard linear program in matrix notation.

76. Consider the linear program

$$\min\nolimits_{x \in \mathbb{R}^2} - \begin{bmatrix} 5 \\ 3 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 33 \\ 8 \\ -5 \\ -1 \\ 8 \end{bmatrix}.$$

Derive the dual program using Lagrange duality.

77. Consider the quadratic program

$$\min_{x\in\mathbb{R}^2} \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Derive the dual quadratic program using Lagrange duality.

78. Consider the convex optimization problem

$$\min_{w\in\mathbb{R}^d} \frac{1}{2} w^\top w \qquad \text{subject to} \quad w^\top x \geq 1.$$

Derive the Lagrangian dual by introducing Lagrange multipliers λ.