1st partial examination - solutions

(1) Two distinguishable dice are rolled.
   (a) What is the probability that the sum of the face values is 5?
   (b) What is the probability that a 2 appears?
   (c) What is the probability that the product of the face values is 24 or less?

1.5

Solution Q1:

(a) The favorable cases are $\{(1,4),(2,3),(3,2),(4,1)\}$, which amounts to 4 favorable cases. The possibilities are $6*6=36$, i.e. the sample space consists of 36 elements, each of which has the same likelihood to occur. Thus,

$$P(5 \text{ is the sum}) = \frac{4}{36} = \frac{1}{9}.$$

0.5

(b) The favorable cases are

$$\{(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(2,1),(2,3),(2,4),(2,5),(2,6)\},$$

which amounts to 11 favorable cases. Thus,

$$P(2 \text{ appears}) = \frac{11}{36}.$$

0.5

(c) Note that there are 4 pairs $\{(5,5),(5,6),(6,5),(6,6)\}$, such that the product of the face values is 25 or more. Thus we have,

$$P(\text{product is 24 or less}) = 1 - P(\text{product is 25 or more}) = 1 - \frac{4}{36} = \frac{8}{9}.$$

0.5

(2) (a) How many 11-letter words can we write using the letters ABRACADABRA?
   (b) If we choose randomly a word from the possible words above, what is the probability that its last 3 letters are all A's?

1.5

Solution Q2:

(a) There are 5 A's, 2 B's, 1 C, 1 D and 2 R's. Then the total number of permutations with eleven (different) elements would be 11!, but some of the letters are repeated, which means that each word is multicounted $5! \cdot 2! \cdot 2!$ times. Thus, the amount of words is

$$\frac{11!}{5!2!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4} = 83160.$$

We can create 83160 different words.

0.5

(b) Fixing the last three letters to be A's, we have 8 remaining letters with 2 A's, 2 B's, 1 C, 1 D and 2 R's. Thus, the total number of such words is

$$\frac{8!}{2!2!2!} = \frac{8!}{8} = 7! = 5040.$$

Therefore the required probability is

$$\frac{5040}{83160} = \frac{504}{8316} = \frac{252}{4158} = \frac{126}{2079} = \frac{14}{231} = \frac{2}{33} \approx 0.06060606$$

1

(3) Assume that X follows a binomial distribution with $n = 5$ and $p = 0.7$. Compute $P(1 \leq X \leq 3)$.

1

Solution Q3:

Since X follows a binomial distribution with the given parameters we have the probabilities computed as

$$P(X = k) = \binom{5}{k} 0.7^k 0.3^{5-k}, \qquad 0 \leq k \leq 5.$$

Hence we have

$$\begin{aligned} P(1 \leq X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \binom{5}{1} \cdot 0.7 \cdot 0.3^4 + \binom{5}{2} 0.7^2 \cdot 0.3^3 + \binom{5}{3} \cdot 0.7^3 \cdot 0.3^2 \\ &= 5 \cdot 0.7 \cdot 0.0081 + 10 \cdot 0.49 \cdot 0.027 + 10 \cdot 0.343 \cdot 0.09 \approx 0.46935. \end{aligned}$$

1

(4) A post company charges 1EUR for small packages and 3EUR for all other packages. Assume that 75% of the deliveries are small packages. If 80 deliveries are posted a particular day, what is the resulting expected revenue?
[Hint: The total revenue will be a linear function of the number of small packages being sent, denoted by X.]

2

Solution Q4:

First note that if we denote by X the number of small packages sent on a particular day then X follows a binomial distribution with $n = 80$ and $p = 3/4$, hence we have

$$E(X) = 80 \cdot 3/4 = 60.$$

The total revenue can be calculated as

$$h(X) = 1 * X + 3 * (80 - X),$$

a linear function of the random variable X.

Using linearity of the expected value as a function we have

$$E(h(X)) = E(X + 3 * (80 - X)) = E(X) + 3 * (80 - E(X)) = 60 + 3 \cdot 20 = 120.$$

2

(5) Suppose that the number X of dead Prussian warriors dying when falling from their horse during a one-year period follows a Poisson distribution with parameter $\lambda = 5$. Compute $P(X = 0)$ and $P(5 \le X < 10)$.

2

Solution Q5:
Since X follows a Poisson distribution with parameter $\lambda = 5$, we have

$$P(X = 0) = \frac{e^{-5} * 5^0}{0!} = e^{-5} \approx 0.0067379.$$

1

Similarly we compute

$$\begin{aligned} P(5 \le X < 10) &= P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) \\ &= \frac{e^{-5} * 5^5}{5!} + \frac{e^{-5} * 5^6}{6!} + \frac{e^{-5} * 5^7}{7!} + \frac{e^{-5} * 5^8}{8!} + \frac{e^{-5} * 5^9}{9!} \\ &= e^{-8} * \left(\frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!} + \frac{5^8}{8!} + \frac{5^9}{9!} \right) \approx 0.5276787. \end{aligned}$$

1

(6) Suppose that X follows a normal distribution with $\mu = 15$ and $\sigma = 2$. What is the 90th percentile of the distribution?
[Hint: the 90th percentile of $Z \sim N(0, 1)$ is approximately 1.285.]

1

Solution Q6:
We have $X \sim N(15, 4)$, then

$$\begin{aligned} P(X \le \alpha) = 0.9 &\longleftrightarrow P\left(\frac{X - 15}{2} \le \frac{\alpha - 15}{2} \right) = 0.9 \\ &\longleftrightarrow P\left(Z \le \frac{\alpha - 15}{2} \right) = 0.9, \end{aligned}$$

where $Z \sim N(0, 1)$, and therefore

$$\frac{\alpha - 15}{2} \approx 1.285 \longrightarrow \alpha \approx 17.57.$$

1

(7) In a drawer you have 7 different rings, 3 of which were given to you by your partner. On a sunny morning you pull out randomly 3 rings from the drawer and put each of them randomly (with equal probability) to any of your 10 fingers. What is the probability that you put the 3 rings given to you by your partner on your left hand?

1

Solution Q7

The total number of possible ways to select 3 rings out of the 7 is $\binom{7}{3} = 35$. Each of these selections has the same likelihood to appear, hence the probability that you randomly select the ones given to you by your partner is $\frac{1}{35}$. Each ring you select goes independently to your left hand with probability $\frac{1}{2}$. Hence the required probability is

$$\frac{1}{35} * \frac{1}{8} = \frac{1}{280}.$$

1