FIRST PARTIAL EXAM

1 (a)

$$\left(\begin{array}{cccc|c} 1 & -2 & -2 & 3+a & 1+a \\ -1 & 2 & 1 & -2+3a & -1+3a \\ 0 & 0 & 0 & a & a \\ -2 & 4 & 0 & -2 & -2 \end{array}\right) \xrightarrow[\substack{R_2 \mapsto R_2+R_1 \\ R_3 \mapsto R_3+2R_1}]{} \left(\begin{array}{cccc|c} 1 & -2 & -2 & 3+a & 1+a \\ 0 & 0 & -1 & 1+4a & 4a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & -4 & 4+2a & 2a \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \mapsto -R_2 \\ R_4 \mapsto R_4+4R_2}]{} \left(\begin{array}{cccc|c} 1 & -2 & -2 & 3+a & 1+a \\ 0 & 0 & 1 & -1-4a & -4a \\ 0 & 0 & 0 & a & a \\ 0 & 0 & 0 & -14a & -14a \end{array}\right).$$

<u>Case 1</u> $a=0$:

$$\left(\begin{array}{cccc|c} 1 & -2 & -2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 \mapsto R_1+2R_2]{} \underbrace{\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{= \, rref(A_0|B)}$$

<u>Case 2</u> $a \neq 0$: after simplifying rows 3 and 4 dividing by a and $-14a$ respectively, we get:

$$\left(\begin{array}{cccc|c} 1 & -2 & -2 & 3+a & 1+a \\ 0 & 0 & 1 & -1-4a & -4a \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[\substack{R_4 \mapsto R_4-R_3 \\ R_2 \mapsto R_2+(1+4a)R_3 \\ R_1 \mapsto R_1-(3+a)R_3}]{}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 \mapsto R_1+2R_2]{} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In conclusion, $\boxed{\text{if } a \neq 0}$, then

$$rref(A|B) = \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$ which does not depend on a.

If $a=0$, then $rref(A|B) = \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

(b) The system $AX=B$ is <u>always</u> compatible. However, the solution depends on the values of a, as follows:

<u>Case 1</u> $\boxed{a=0}$. Then the parametric solution is:

$$\begin{cases} x_1 = 1+2\lambda-\mu \\ x_2 = \lambda \\ x_3 = \mu \\ x_4 = \mu \end{cases}$$

The degree of freedom is $2 = \underbrace{4}_{\text{number of variables}} - \underbrace{2}_{rank(A)}$

<u>Case 2</u> $\boxed{a\neq 0}$ Then the parametric solution is:

$$\begin{cases} x_1 = 2\lambda \\ x_2 = \lambda \\ x_3 = 1 \\ x_4 = 1 \end{cases}$$

The degree of freedom is:

$1 = \underbrace{4}_{\text{number of variables}} - \underbrace{3}_{rank(A)}$.

(2) (a) We solve the SLE: $AX=0$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -1 & 5 & 0 \\ 0 & -2 & -2 & 0 \\ 4 & -2 & 6 & 0 \end{array}\right) \xrightarrow[R_4 \mapsto R_4 - 4R_1]{R_2 \mapsto R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \mapsto R_1 + R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = rref(A|0)$$

The solution of the system is

$$\begin{cases} x_1 = -2\lambda \\ x_2 = -\lambda \\ x_3 = \lambda \end{cases}$$

It has degree of freedom $=1$, and $rank(A)=2$.

We have

- $\dim(\mathrm{Ker}(f_A)) = 1$ (= degree of freedom of the HSLE $AX=0$)
- $\dim(\mathrm{Im}(f_A)) = 2$ $(= \mathrm{rank}(A))$,

and the formula $n = \dim(\mathrm{Ker}(f_A)) + \dim(\mathrm{Im}(f_A))$ in this case gives $3 = 1 + 2$.

- Basis of $\mathrm{Ker}(f_A)$: Take $\lambda = 1$, and obtain $\vec{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$.

$\mathcal{B}_1 = [\vec{v}_1)$ is a basis of $\mathrm{Ker}(f_A)$.

- Basis of $\mathrm{Im}(f_A)$: Take the columns of A corresponding to the pivots of $\mathrm{rref}(A)$, that is, take:

$\vec{w}_1 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} -1 \\ -1 \\ -2 \\ -2 \end{pmatrix}$. Then $\mathcal{B}_2 = [\vec{w}_1, \vec{w}_2]$ is a basis of $\mathrm{Im}(f_A)$.

(b) Enlarge the basis $\mathcal{B}_2 = [\vec{w}_1, \vec{w}_2]$ to a basis of $\mathbb{R}^4$: $[\vec{w}_1, \vec{w}_2, \vec{e}_3, \vec{e}_4]$ is a basis of $\mathbb{R}^4$, because for instance

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{bmatrix} \xrightarrow[\text{transformations}]{\text{Column}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has clearly rank 4.

(c) Enlarge the basis $\mathcal{B}_1 = [\vec{v}_1)$ to a basis of $\mathbb{R}^3$: $[\vec{e}_1, \vec{e}_2, \vec{v}_1]$ is a basis of $\mathbb{R}^3$, because for instance

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{column transformation}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which has rank 3.

THEORY:

(a) Take $E_r = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$. (If $r=0$, $E_0 = O_{n\times n}$).

Then $E_r^2 = E_r$, and $\mathrm{rank}(E_r) = r$. $\boxed{\text{TRUE}}$

(b) This is not true in general. Indeed, for any $n \geq 1$, consider the SLE:

$$\begin{cases} x_1 & = 0 \\ x_2 & = 0 \\ \cdots & \\ x_n & = 0 \\ a_1 x_1 + \cdots + a_n x_n & = 1 \end{cases}$$

The system has $n+1$ equations and n unknowns $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

In matricial form $\left(\begin{array}{ccc|c} & I_n & & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline a_1 & a_2 \cdots & a_n & 1 \end{array}\right)$. The system is incompatible, and the homogeneous system $\left(\begin{array}{ccc|c} & I_n & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline a_1 & a_2 \cdots & a_n & 0 \end{array}\right)$ has a unique solution $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

If $m < n$, then the HSLE $AX=0$ has infinitely many solutions independently of the fact that $AX=B$ is compatible or not.

If $m=n$, then it is true that if $AX=B$ is incompatible, then $AX=0$ has infinitely many solutions, because $\mathrm{rank}(A) \underset{\text{incompatible}}{<} \mathrm{rank}(A|B) \leq n$, so $\mathrm{rank}(A) < n$ and the degree of freedom of $AX=0$ is $n - \mathrm{rank}(A) > 0$, so $AX=0$ has infinitely many solutions.

(c) $\boxed{\text{FALSE}}$ Any basis of $\mathbb{R}^3$ must have exactly 3 vectors. Hence $\mathcal{B}$ is not a basis of $\mathbb{R}^3$, because it has 4 vectors.