Fundamentals of Math I
Recovery Exam

AI Degree
January 23, 2023

Full Name: ....

NIU: ....

Exercise 1. (1,5 points + 1,5 points + 1 point)
Consider the matrices

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 & -1 \\ 2 & -2 & 4 & 1 & -1 \\ -1 & 1 & -2 & -1 & 0 \end{pmatrix} \in M_{3\times 5}(\mathbb{R}), \qquad B = \begin{pmatrix} 1 \\ 2 \\ a-2 \end{pmatrix} \in M_{3\times 1}(\mathbb{R}),$$

where $a \in \mathbb{R}$.

(a) Find the Reduced Row Echelon Form of the augmented matrix $[A \mid B]$.

(b) Find the values of a for which the system of linear equations $AX = B$ is compatible, and give the solution of the system for these values of a, determining the degree of freedom of the system and the parametric form of the solutions.

(c) Find all the solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system such that $x_1 = x_4 = 0$. Does there exist a solution $(x_1, x_2, x_3, x_4, x_5)$ such that $x_1 = x_2 = x_3 = x_4 = 0$?

Exercise 2. (1,5 points + 1,5 points + 1 point)
Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \in M_3(\mathbb{R}).$$

(a) Compute all the eigenvalues of A.

(b) For each eigenvalue of A, compute a corresponding eigenvector. Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

(c) Find an orthonormal basis of $\mathbb{R}^3$ consisting entirely of eigenvectors of A. Find an orthogonal matrix Q such that $D = Q^{-1}AQ = Q^T AQ$. *(Recall that an orthogonal matrix is by definition a square matrix Q such that $Q^{-1} = Q^T$.)*

Theory. ($0,7$ points + $0,7$ points + $0,6$ points)
For each of the following assertions, say if the assertion is true or false. Justify your answer in each case.

(a) Any family of vectors of $\mathbb{R}^n$ can be extended to a basis of $\mathbb{R}^n$.

(b) If A is a matrix of size 4×3, then the determinant of AA^T is 0.

(c) For any square matrix A of size $n \times n$ and with coefficients in $\mathbb{R}$, the matrix $A + A^T$ is diagonalizable in an orthonormal basis of $\mathbb{R}^n$.

All answers must be carefully explained. You must specify the theoretical results used in your arguments and procedures.

(1)

(a) $\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -1 & 1 \\ 2 & -2 & 4 & 1 & -1 & 2 \\ -1 & 1 & -2 & -1 & 0 & a-2 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 + R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & a-1 \end{array}\right)$

$\xrightarrow[R_3 \mapsto R_3 + R_2]{} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a-1 \end{array}\right)$

If $\underline{a \neq 1}$, then the RREF of $[A|B]$ is $\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

If $\underline{a = 1}$, then the RREF of $[A|B]$ is $\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$.

(b) The system will be compatible $\iff a = 1$.

The solution for $a=1$ is:

$$\begin{cases} x_1 = 1 + x_2 - 2x_3 + x_5 \\ x_4 = -x_5 \end{cases}$$

($x_1, x_2, x_5 \in \mathbb{R}$ free variables.)

or with parameters λ, γ, δ:

$$\begin{cases} x_1 = 1 + \lambda - 2\gamma + \delta \\ x_2 = \lambda \\ x_3 = \gamma \\ x_4 = -\delta \\ x_5 = \delta \end{cases}$$

The degree of freedom of the system is 3.

(c) The solutions such that $x_1 = x_4 = 0$ are:

$$\begin{cases} x_1 = 0 \\ x_2 = 2x_3 - 1 \\ x_4 = 0 \\ x_5 = 0 \end{cases}$$

($x_3 \in \mathbb{R}$), with 1 degree of freedom.

There are $\underline{no}$ solutions with $x_1 = x_2 = x_3 = x_4 = 0$, since in this case we have: $0 = 1 + 0 - 2 \cdot 0 + x_5 \Rightarrow \boxed{x_5 = -1}$

and also $0 = x_4 = -x_5 \Rightarrow \boxed{x_5 = 0}$

Incompatible

2 (a) We compute the characteristic polynomial $p_A(x)$:

$$p_A(x) = \begin{vmatrix} -x & 2 & 2 \\ 2 & 1-x & 0 \\ 2 & 0 & -1-x \end{vmatrix} = -x(1-x)(-1-x) - 4(1-x) - 4(-1-x)$$

$$= -[\,x^3 - x - 4x - 4x) = -[x^3 - 9x] = -x[x^2-9] =$$

$$= -x(x-3)(x+3).$$

Hence the eigenvalues of A are $\lambda_1 = 0,\ \lambda_2 = -3,\ \lambda_3 = 3$.

(b) We compute eigenvectors for each eigenvalue:

$$\lambda_1 = 0 \longrightarrow \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y = -z \\ 2x + y = 0 \\ \cancel{2x - z = 0} \end{cases}$$

$$2x = -y = z \quad \xrightarrow[x=1]{} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -3 \longrightarrow \begin{pmatrix} \cancel{3} & \cancel{2} & \cancel{2} \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = -2y \\ x = -z \end{cases}$$

$$\xrightarrow[y=1]{} \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

$$\lambda_3 = 3 \longrightarrow \begin{pmatrix} \cancel{-3} & \cancel{2} & \cancel{2} \\ 2 & -2 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x = y \\ x = 2z \end{cases}$$

$$\xrightarrow[z=1]{} \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$ Now take $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ diagonal

and $P = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ invertible, so that $D = P^{-1} \cdot A \cdot P$.

(c) Observe that v_1, v_2, v_3 are mutually orthogonal, so that an orthonormal basis of $\mathbb{R}^3$ of eigenvectors of A is given by: $u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$, $u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$,

$$u_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

Hence $Q = 1/3 \cdot P = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$ is an orthogonal matrix

and $\boxed{D = Q^{-1} A Q = Q^T A Q}$.

Theory

(a) False. In order to be extendable to a basis, a family of vectors must be linearly independent. For instance, in $\mathbb{R}^2$, the family $(1,2), (8,16)$ cannot be extended to a basis of $\mathbb{R}^2$.

(b) True. To show that $AA^T \in M_{4\times 4}(\mathbb{R})$ has determinant 0, it suffices to show that $\operatorname{rank}(AA^T) < 4$. But we know that $\operatorname{rank}(A\cdot B) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$ for all matrices A, B that can be multiplied, so

$$\operatorname{rank}(A\cdot A^T) \leq \operatorname{rank}(A) \leq 3$$ (because A has only 3 columns).

(c) $A + A^T$ is a symmetric matrix: $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$, hence by the Spectral Theorem, A is diagonalizable in an orthonormal basis. Hence the statement is true.