Fundamentals of Math I | AI Degree
Second Partial Exam | January 9, 2023

Full Name: ....

NIU: ....

Exercise 1. (1,5 points + 1,5 points + 1 point)

Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 0 \\ -2 & -4 & 0 \\ 2 & 2 & 2 \end{pmatrix} \in M_3(\mathbb{R}).$$

(a) Compute all the eigenvalues of A.

(b) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) Find a basis of the image of f_A. Check that if v is any vector *in the image* of f_A, then $\|f_A(v)\| = 2\|v\|$.

Exercise 2. (1,5 points + 1,5 points + 1 point)

Let V be the linear subspace of $\mathbb{R}^4$ given by:

$$V = \langle (1,0,1,0), (1,-1,1,-1) \rangle$$

(a) Find an orthonormal basis of V. What is the dimension of V? What is the dimension of $V^\perp$?

(b) Find an orthonormal basis of $V^\perp$, the orthogonal subspace of V. Give an orthonormal basis of $\mathbb{R}^4$ containing the orthonormal basis of V found in (a).

(c) Find the matrix of the orthogonal projection proj_V onto the linear subspace V, with respect to the canonical basis of $\mathbb{R}^4$. Using this matrix, compute the vector $\mathrm{proj}_V(w)$, where $w = (5,0,2,1)$.

Theory. $(0,7$ points $+ \, 0,7$ points $+ \, 0,6$ points$)$

For each of the following assertions, say if the assertion is true or false. Justify your answer in each case.

(a) If A is an orthogonal matrix, then the determinant of A is either 1 or -1.

(b) If a square matrix $A \in M_n(\mathbb{R})$ is diagonalizable, and all the eigenvalues of A are strictly positive, then A is invertible.

(c) If $X \in M_{m\times n}(\mathbb{R})$ for some positive integers m, n, then the $n \times n$ matrix X^TX is diagonalizable. (Hint: Show that X^TX is a symmetric matrix.)

All answers must be carefully explained. You must specify the theoretical results used in your arguments and procedures.

(1) (a) $p_A(x) = \begin{vmatrix} 2-x & 4 & 0 \\ -2 & -4-x & 0 \\ 2 & 2 & 2-x \end{vmatrix} = (2-x) \begin{vmatrix} 2-x & 4 \\ -2 & -4-x \end{vmatrix}$ (1)

$= (2-x)\left[(2-x)(-4-x)+8\right] = -(x-2)\left[x^2+2x\right] = -x(x-2)(x+2).$

Hence $p_A(x) = -x(x-2)(x+2)$, and the eigenvalues of A are:

$\lambda_1 = 0,\ \lambda_2 = -2,\ \lambda_3 = 2.$

(b) We find eigenvectors for each eigenvalue:

$\underline{\lambda_1 = 0} \to \ker(A)$: $\begin{pmatrix} 2 & 4 & 0 \\ -2 & -4 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+2y=0 \\ x+y+z=0 \end{cases}$

$\to x = -2y \Rightarrow -2y+y+z=0 \Rightarrow -y+z=0 \Rightarrow y=z.$

We get $\begin{cases} x=-2y \\ z=y \end{cases}$ and an eigenvector $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ of eigenv. 0.

$\underline{\lambda_2 = -2} \to \ker(A+2I_3)$: $\begin{pmatrix} 4 & 4 & 0 \\ -2 & -2 & 0 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x=-y \\ z=0 \end{cases}$

and we get an eigenvector $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ of eigenvalue -2.

$\underline{\lambda_3 = 2} \to \ker(A-2I_3)$: $\begin{pmatrix} 0 & 4 & 0 \\ -2 & -6 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to x=y=0,$

and we get an eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ of eigenvalue 2.

Hence we have the diagonal matrix $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and

the invertible matrix $P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, so that $D = P^{-1} \cdot A \cdot P$.

(c) Since $[\vec{v}_1, \vec{v}_2, \vec{v}_3]$ is a basis of $\mathbb{R}^3$, for each $\vec{v} \in \mathbb{R}^3$ we have

$\vec{v} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for some $a,b,c \in \mathbb{R}$, so that

$f(\vec{v}) = af(\vec{v}_1) + bf(\vec{v}_2) + cf(\vec{v}_3) = -2b\vec{v}_2 + 2c\vec{v}_3.$

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Hence any vector in $\mathrm{Im}(f_A)$ is a linear combination of $\vec{v}_2$ and $\vec{v}_3$, and since $\vec{v}_2$ and $\vec{v}_3$ are linearly independent, we get that $[\vec{v}_2, \vec{v}_3]$ is a basis of $\mathrm{Im}(f_A)$.

Now take any vector $\vec{v} \in \mathrm{Im}(f_A)$. We can write

$$\vec{v} = a\vec{v}_2 + b\vec{v}_3 = \begin{pmatrix} -a \\ a \\ b \end{pmatrix}, \text{ for } a, b \in \mathbb{R}.$$

Hence $f(\vec{v}) = a f(\vec{v}_2) + b f(\vec{v}_3) = -2a\vec{v}_2 + 2b\vec{v}_3 = \begin{pmatrix} 2a \\ -2a \\ 2b \end{pmatrix}$

$= 2\begin{pmatrix} a \\ -a \\ b \end{pmatrix}$.

Therefore:

$$\|f_A(\vec{v})\| = \left\| 2 \cdot \begin{pmatrix} a \\ -a \\ b \end{pmatrix} \right\| = 2\sqrt{a^2 + (-a)^2 + b^2} =$$

$= 2\sqrt{(-a)^2 + a^2 + b^2} = 2 \cdot \|\vec{v}\|$, and we get $\|f_A(\vec{v})\| = 2 \cdot \|\vec{v}\|$.

(2) $V = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^4$.

(a) We use the Gram-Schmidt method: Take $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

then $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $u_2' = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 =$

$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} - \sqrt{2} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$. Hence $\vec{u}_2 = \dfrac{\vec{u}_2'}{\|\vec{u}_2'\|} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$.

We have $\dim(V) = 2$, and $\dim(V^\perp) = 4 - \dim(V) = 4 - 2 = 2$.

(b) Observe that a vector $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$ belongs to $V^\perp$ if and only if:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ We solve this HSLE:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix} \xrightarrow[R_2 \mapsto R_2 - R_1]{} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

Hence the solutions are given by: $\begin{cases} x = -z \\ y = -t \end{cases}$

We get the orthonormal basis of $V^{\perp}$:

$$u_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Observe that $[\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4]$ is an orthonormal basis of $\mathbb{R}^4$, obtained by joining the basis $[\vec{u}_1, \vec{u}_2]$ of V and the basis $[\vec{u}_3, \vec{u}_4]$ of $V^{\perp}$.

(c) Let $Q = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix}$. We know from theory that $[\text{proj}_V]_{Can} = Q \cdot Q^T$. Hence we compute:

$$[\text{proj}_V]_{Can} = QQ^T = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} = 1/2 \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Now $\text{proj}_V(\vec{w}) = \text{proj}_V \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 1 \\ 7 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 7/2 \\ 1/2 \\ 7/2 \\ 1/2 \end{pmatrix}.$$

THEORY:

(a) True: If A is an orthogonal matrix, then $A^{-1}=A^t$, and so from $A\cdot A^t = I_n$, we get

$$\det(A)\cdot\underbrace{\det(A^t)}_{=\det(A)} = \det(A\cdot A^t) = \det(I_n) = 1$$

$$\det(A)\cdot\det(A) = \det(A)^2.$$

Hence $\det(A)^2 = 1$, and thus $\det(A) = \pm 1$.

(b) True: Suppose that $A \in M_n(\mathbb{R})$ is diagonalizable, and all the eigenvalues of A are strictly positive. We can write

$$D = P^{-1}\cdot A\cdot P, \text{ where } D = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}, \text{ the elements}$$

$a_1, a_2 \ldots a_n$ are the eigenvalues of A, so that $a_i > 0$ for all i, and P is an invertible matrix.

Hence $A = P\cdot D\cdot P^{-1}$ is a product of invertible matrices, and so invertible, because D is invertible with inverse $D^{-1} = \begin{pmatrix} a_1^{-1} & & & 0 \\ & a_2^{-1} & & \\ & & \ddots & \\ 0 & & & a_n^{-1} \end{pmatrix}$

$= \begin{pmatrix} 1/a_1 & & & 0 \\ & 1/a_2 & & \\ & & \ddots & \\ 0 & & & 1/a_n \end{pmatrix}$. You can also obtain the same result taking determinants in the expression $A = P\cdot D\cdot P^{-1}$:

$$\det(A) = \det(P)\cdot\det(D)\cdot\det(P)^{-1} = \det(D) = a_1\cdot a_2 \cdots a_n \neq 0,$$

so $\det(A)\neq 0$ and hence A is invertible.

(c) True: If $X \in M_{m\times n}(\mathbb{R})$, then we have $X^TX \in M_{n\times n}(\mathbb{R})$ and $(X^T\cdot X)^T = X^T\cdot(X^T)^T = X^T\cdot X$, hence X^TX is a symmetric matrix. Hence by the Spectral Theorem, X^TX is diagonalizable in an orthonormal basis.