Fundamentals of Math II
First Partial Exam

AI Degree
April 17, 2024

Full Name: ....

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Exercise 1. (2 points + 2 points) Consider the following curves in $\mathbb{R}^3$:

$$r(t) = (2t - t^2, \cos(t), 2\sin(t)), \qquad s(t) = (t^3, t^2 - t, t^5)$$

(a) Find the tangent vectors to the curves $r(t)$ and $s(t)$ at $t = \pi/2$.

(b) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = r(t) \cdot s(t)$. Find the equation of the tangent line to the graphic of $f(t)$ at $t = \pi/2$.

Exercise 2. (1 point + 1 point + 2 points) Consider the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) = x^2 + y^2 - x - y + 1.$$

(a) Find the gradient $\nabla f(x, y)$ at each point $(x, y) \in \mathbb{R}^2$.

(b) Find all the local extremes of f. Use the Hessian to determine whether they are local maxima, local minima, or saddle points.

(c) Use the method of Lagrange multipliers to determine the absolute maxima and minima of $f(x, y)$ subject to $x^2 + y^2 \leq 1$.

Short questions. (0, 7 points + 0, 7 points + 0, 6 points)

(a) Compute the primitive $\int \ln(x)dx$

(b) Compute $\iint_R e^{x+y}dxdy$, where $R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

(c) Determine a parametric curve $r(t) = (r_1(t), r_2(t))$ such that $r(0) = (0, 0)$ and $r'(t) \cdot (\cos t, \sin t) = 1$ for all $t \in \mathbb{R}$.

All answers must be carefully explained. You must specify the theoretical results used in your arguments and procedures.

Solutions

① (a) We compute:

$r'(t) = (2-2t, -\sin t, 2\cos t)$, $s'(t) = (3t^2, 2t-1, 5t^4)$

Hence, the tangent vectors at $t=\pi/2$ are:

$r'(\pi/2) = (2-\pi, -1, 0)$, $s'(\pi/2) = \left(\frac{3\pi^2}{4}, \pi-1, \frac{5\pi^4}{16}\right)$.

(b) $f: \mathbb{R} \to \mathbb{R}$, $f(t) = r(t) \cdot s(t)$. First, we need to compute $f'(\pi/2)$. To compute $f'(t)$ we use the product rule:

$f'(t) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$. Hence at $t=\pi/2$, and using the computations in (a), we get:

$$f'(\pi/2) = r'(\pi/2) \cdot s(\pi/2) + r(\pi/2) \cdot s'(\pi/2) =$$

$$= \begin{pmatrix} 2-\pi \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \pi^3/8 \\ \pi^2/4 - \pi/2 \\ \pi^5/32 \end{pmatrix} + \begin{pmatrix} \pi - \pi^2/4 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3\pi^2/4 \\ \pi - 1 \\ \frac{5\pi^4}{16} \end{pmatrix}$$

$$= (2-\pi)\left(\frac{\pi^3}{8}\right) - \frac{\pi^2}{4} + \frac{\pi}{2} + \frac{3\pi^3}{4} - \frac{3\pi^4}{16} + \frac{5\pi^4}{8} =$$

$$= \frac{5}{16}\pi^4 + \pi^3 - \frac{\pi^2}{4} + \frac{\pi}{2}.$$

One can also compute $f(t)$, then $f'(t)$ and then substituting $t=\pi/2$.

Now the equation of the tangent line is:

$$\boxed{y - \left(\frac{\pi^5}{32} + \frac{\pi^4}{8}\right) = \left(\frac{5}{16}\pi^4 + \pi^3 - \frac{\pi^2}{4} + \frac{\pi}{2}\right)(x - \pi/2)}$$

② $f(x,y) = x^2 + y^2 - x - y + 1$

(a) $\nabla f(x,y) = \begin{bmatrix} 2x-1 \\ 2y-1 \end{bmatrix}$

(b) $\nabla f(x,y) = 0 \Rightarrow \begin{cases} 2x-1=0 \\ 2y-1=0 \end{cases} \Rightarrow x = y = 1/2 \Rightarrow (1/2, 1/2)$

Compute the Hessian $H(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. It follows that

$\left(\frac{1}{2},\frac{1}{2}\right)$ is a local minimum of $f(x,y)$.

(c) By the Lagrangian method, to find the extremes in the boundary $g(x,y)=0$, where $g(x,y)=x^2+y^2-1$, we have to find the solutions of: $\begin{cases} \nabla f(x,y)=\lambda \nabla g(x,y) \\ g(x,y)=0 \iff x^2+y^2=1. \end{cases}$

Hence $\begin{bmatrix} 2x-1 \\ 2y-1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \begin{cases} 2(1-\lambda)x = 1 \\ 2(1-\lambda)y = 1 \end{cases}$. We then have $x = \frac{1}{2(1-\lambda)} = y$ so $x=y$ and so $x^2+x^2=1$

$\Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow \boxed{x=\pm\frac{1}{\sqrt{2}}=y}$. We get two points $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$. (and $\lambda = 1\pm\frac{\sqrt{2}}{2}$)

on the boundary $g(x,y)=0$

We also have an interior point $\left(\frac{1}{2},\frac{1}{2}\right)$, since $\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{1}{2}<1$.

We compute the values of f at each of these three points:

$$f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2};\ f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=2-\sqrt{2}\simeq 0{,}6;\ f\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)=2+\sqrt{2}.$$

Hence the absolute minimum of f on the domain $S=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$ is $\left(\frac{1}{2},\frac{1}{2}\right)$, and the absolute maximum of f on S is $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$.

The minimum value of f on S is $f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$,

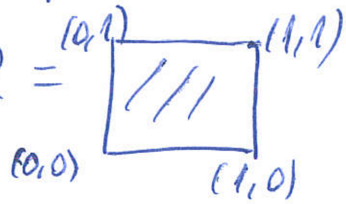
the maximum value of f on S is $f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=2+\sqrt{2}\simeq 3.4$.

Short questions

(a) $\int \ln(x)\,dx = x\ln(x) - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - x$.
$= x(\ln(x)-1)$

$u = \ln x \;\Big|\; du = \frac{1}{x}\,dx$
$dv = dx \;\Big|\; v = x$

(b) $R =$



$$\iint_R e^{x+y}\,dx\,dy = \int_0^1\left(\int_0^1 e^{x+y}\,dx\right)dy = \int_0^1 \left[e^{x+y}\right]_{x=0}^{x=1} dy$$
$$= \int_0^1 \left(e^{y+1} - e^{y}\right) dy = \left[e^{y+1} - e^{y}\right]_0^1 = e^2 - e^1 - (e^1 - e^0)$$
$$= e^2 - e - e + 1 = e^2 - 2e + 1.$$

(c) Determine $r(t) = (r_1(t), r_2(t))$: $r(0) = (0,0)$ and
$r'(t)\cdot(\cos t, \sin t) = 1$ for all $t \in \mathbb{R}$.

Since $\cos^2 t + \sin^2 t = 1$ for all $t \in \mathbb{R}$, we will find $r(t)$ such that $r(0) = (0,0)$ and $r'(t) = (\cos t, \sin t)$, that is

$$r(t) = (\sin t + C_1, -\cos t + C_2) \xrightarrow[r(0)=(0,0)]{} (C_1, -1 + C_2) = (0,0)$$

$\Rightarrow C_1 = 0,\ C_2 = 1$.

Hence $r(t) = (\sin t, 1 - \cos t)$.