Fundamentals of Math I **AI Degree**
Second Partial Exam **January 17, 2022**

Full Name: ....
NIU: ....

Exercise 1. (1,5 points + 1,5 points + 1 point)

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R}).$$

(a) Compute all the eigenvalues of A.

(b) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(c) Find all invariant directions of the endomorphism f_A of $\mathbb{R}^3$ associated to the matrix A. Find a basis of the image of f_A. Check that if v is a non-zero vector *in the image* of f_A, then $f_A(v)$ is also a non-zero vector in the image of f_A.

Exercise 2. (1,5 points + 1 point + 1,5 points)

Let V be the linear subspace of $\mathbb{R}^4$ given by:

$$V = \langle (1,1,1,1), (1,2,1,2) \rangle$$

(a) Find an orthonormal basis of V. What is the dimension of V? What is the dimension of $V^\perp$?

(b) For a subspace W of $\mathbb{R}^4$, denote by $\mathrm{proj}_W \colon \mathbb{R}^4 \to \mathbb{R}^4$ the orthogonal projection onto the subspace W. Compute $\mathrm{proj}_V(v)$ and $\mathrm{proj}_{V^\perp}(v)$, where $v \in \mathbb{R}^4$ is the vector $v = (0,0,1,1)$. Compute $\|v\|$, $\|\mathrm{proj}_V(v)\|$ and $\|\mathrm{proj}_{V^\perp}(v)\|$, and check Pitagoras Theorem:

$$\|v\|^2 = \|\mathrm{proj}_V(v)\|^2 + \|\mathrm{proj}_{V^\perp}(v)\|^2.$$

(c) Find the matrix of proj_V (with respect to the canonical basis of $\mathbb{R}^4$). What is the matrix of $\mathrm{proj}_{V^\perp}$?

Theory. $(0,7$ points $+\ 0,7$ points $+\ 0,6$ points$)$

For each of the following assertions, say if the assertion is true or false. Justify your answer in each case.

(a) Not all square matrices over $\mathbb{R}$ are diagonalizable, but all square matrices over $\mathbb{R}$ have a real eigenvalue.

(b) If a matrix $A \in M_n(\mathbb{R})$ is diagonalizable, then A^2 is also diagonalizable.

(c) All subspaces of $\mathbb{R}^n$ have an orthonormal basis.

All answers must be carefully explained. You must specify the theoretical results used in your arguments and procedures.