1st partial examination - Sample paper - Solutions

(1) Two distinguishable dice are rolled.
   (a) What is the probability that the sum of the face values is 7?
   (b) What is the probability that no 5 appears?
   (c) What is the probability that the sum of the face values is 10 or less?

Solution Q1

In total we have 36 possible outcomes, each of which has the same chance to occur.

(a) 6 possible outcomes satisfy that the sum of the face values is 7, namely

$$(6,1),(5,2),(4,3),(3,4),(2,5),(1,6),$$

hence the probability of the event is $\frac{6}{36}=\frac{1}{6}$.

(b) The following 11 outcomes have at least one 5:

$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,5),(4,5),(3,5),(2,5),(1,5),$$

hence the probability that no 5 appears is

$$1-P(\text{at least one 5 appears})=1-\frac{11}{36}=\frac{25}{36}.$$

(c) There are 3 outcomes such that the sum of the face values is greater than 10, namely

$$(6,6),(6,5),(5,6),$$

hence the probability that the sum of the face values is 10 or less is

$$1-P(\text{sum of the face values }>10)=1-\frac{3}{36}=\frac{11}{12}.$$

(2) (a) How many 12-letter words can we write using 3 A's, 3 B's, 3 C's, and 3 D's?
   (b) If we choose randomly a word from the possible 12-letter words above, what is the probability that its last 3 letters are all A's?

Solution Q2

(a) The number of possible 12-letter words, if we had 12 different letters, would be 12!. We then permute independently each of the 3 A's, 3 B's, 3 C's and 3 D's to get in total

$$\frac{12!}{3!*3!*3!*3!}=369600.$$

(b) The number of 3-letter words ending with three A's is

$$\frac{9!}{3!*3!*3!}=1680,$$

therefore the probability we are looking for is

$$\frac{1680}{369600} = \frac{1}{220} \; (\approx 0.004545).$$

(3) Assume that X follows a binomial distribution with $n = 8$ and $p = 0.6$. Compute $P(3 \leq X \leq 5)$.

Solution Q3

$$\begin{aligned} P(3 \leq X \leq 5) =& P(X = 3) + P(X = 4) + P(X = 5) \\ =& \binom{8}{3} * 0.6^3 * 0.4^5 + \binom{8}{4} * 0.6^4 * 0.4^4 + \binom{8}{5} * 0.6^5 * 0.4^3 \\ =& 56 * 0.6^3 * 0.4^5 + 70 * 0.6^4 * 0.4^4 + 56 * 0.6^5 * 0.4^3 \approx 0.6346 \end{aligned}$$

(4) A toll road charges 2EUR for passenger cars and 5EUR for all other vehicles. Assume that during daytime hours 60% of all vehicles are passenger cars. If 4 vehicles cross the bridge during a particular daytime period, what is the resulting expected toll revenue?
[Hint: The total revenue will be a linear function of the number of passenger vehicles crossing denoted by X.]

Solution Q4

Let X denote the number of passenger cars crossing during the given daytime period, then X follows binomial distribution with $n = 4$ and $p = 0.6$, hence we have

$$E(X) = 4 * 0.6 = 2.4.$$

The total toll revenue will be

$$h(X) = 2 * X + 5 * (4 - X).$$

Using linearity of the expected value as a function we have

$$E(h(X)) = E(2*X+5*(4-X)) = 2*E(X)+5*(4-E(X)) = 4.8+5*1.6 = 12.8.$$

(5) Suppose that the number X of storms observed in a region during a one-year period follows a Poisson distribution with parameter $\lambda = 8$. Compute $P(X \leq 2)$ and $P(7 \leq X < 9)$.

Solution Q5

Since X follows Poison distribution with parameter $\lambda = 8$, we have

$$\begin{aligned} P(X \leq 2) =& P(X = 0) + P(X = 1) + P(X = 2) = \frac{e^{-8} * 8^0}{0!} + \frac{e^{-8} * 8^1}{1!} + \frac{e^{-8} * 8^2}{2!} \\ =& e^{-8} * \left(1 + 8 + \frac{64}{2}\right) \approx 0.01375. \end{aligned}$$

$$\begin{aligned} P(7 \leq X < 9) =& P(X = 7) + P(X = 8) = \frac{e^{-8} * 8^7}{7!} + \frac{e^{-8} * 8^8}{8!} \\ =& e^{-8} * \left(\frac{8^7}{7!} + \frac{8^8}{8!}\right) \approx 0.27917. \end{aligned}$$

(6) Suppose that X follows a normal distribution with $\mu = 25$ and $\sigma = 5$. What is the 90th percentile of the distribution?
[Hint: the 90th percentile of $Z \sim N(0,1)$ is approximately 1.285.]

Solution Q6

We have $X \sim N(25, 25)$, then

$$\begin{aligned} P(X \leq \alpha) = 0.9 &\longleftrightarrow P\left(\frac{X-25}{5} \leq \frac{\alpha-25}{5}\right) = 0.9 \\ &\longleftrightarrow P\left(Z \leq \frac{\alpha-25}{5}\right) = 0.9, \end{aligned}$$

where $Z \sim N(0,1)$, and therefore

$$\frac{\alpha-25}{5} \approx 1.285 \longrightarrow \alpha \approx 31.425.$$

(7) Suppose that X follows a normal distribution with $\mu = 3$ and $\sigma^2 = 4$. Find k such that $P(|X-3| \leq k) = 0.9$.
[Hint: $\Phi(1.65) \approx 0.95$.]

Solution Q7

We have

$$P(|X-3| \leq k) = P(-k \leq X-3 \leq k) = P\left(-\frac{k}{2} \leq \frac{X-3}{2} \leq \frac{k}{2}\right).$$

Then with $Z = \frac{X-3}{2} \sim N(0,1)$, and noting

$$P\left(-\frac{k}{2} \leq Z \leq \frac{k}{2}\right) = 2 * P\left(0 \leq Z \leq \frac{k}{2}\right) = 0.9,$$

and that

$$P(Z \leq 1.65) \approx 0.95,$$

we have that $\frac{k}{2} = 1.65$, therefore $k = 3.3$.