Generant Imatges amb un Ordinador Quàntic

TREBALL DE RECERCA DE BATXILLERAT

IES MIQUEL TARRADELL

Autor:

Tomàs Ockier Poblet
2nd Batxillerat
ockier1@gmail.com

Tutor:

Tomàs Ockier Poblet

Institut Miquel Tarradell

28 de setembre de 2021
Barcelona, Barcelona

Índex

Introducció

Desde hace más de un año, me he dedicado a estudiar computación cuántica durante mi tiempo libre. Buscaba investigar un campo relacionado con la mecánica cuántica, pero sin que sea muy complicado, que se pueda entender a un nivel teórico y que me entusiasme.

La Computación Cuántica encaja perfectamente con esos criterios. Es más sencilla que la mecánica cuántica debido a que no está basada en cálculo o ecuaciones diferenciales, se basa en la álgebra lineal, utilizando valores discretos, vectores y matrices. Además si se trabaja a un nivel teórico sencillo, no se tienen en consideración las interpretaciones físicas, lo cual simplifica mucho las cosas. Cuanto más me adentraba, más ganas tenía de seguir.

Mi parte favorita de este campo es el Quantum Machine Learning que consiste en diseñar y aplicar conceptos de Machine Learning a los ordenadores cuánticos, como por ejemplo implementar cuánticamente las famosas Redes Neuronales, que están detrás de la mayoría de inteligencias artificiales que vemos hoy en día [1].

QML es un campo de investigación joven y en crecimiento debido a que sus algoritmos son ideales para implementarlos con los ordenadores cuánticos actuales, los cuales no son muy potentes. Ejemplos de estas implementaciones serían [insertar aplicaciones aquí], etc.

De entre todos los tipos de algoritmos me he centrado en las Redes Neuronales Cuánticas, análogas cuánticas de las Redes Neuronales tan utilizadas hoy en día para hacer gran variedad de tareas. Me he interesado particularmente en ellas debido a que tenía experiencia en el pasado con las RNs clásicas y había visto que existen frameworks de software para trabajar con ellas como TensorFlow Quantum [2] que me podían ayudar.

Para adentrarme en el campo de QML, he tenido que adquirir conocimientos en álgebra lineal, cálculo y física. Dentro de QML en concreto me he dedicado a leer papers que me interesan y en un par de ocasiones intentar implementar los algoritmos detallados en esos papers. Puede parecer algo imposible en principio debido a que no tengo acceso directo a un ordenador cuántico, no obstante estos no son necesarios debido a que las operaciones cuánticas pueden ser simuladas en un ordenador corriente de escritorio (con ciertas limitaciones). Pero puedo tener acceso a ordenadores cuánticos ya que IBM permite acceder a los

suyos mediante IBM Quantum Experience [3], aunque nunca he dado uso de ello debido a que no lo veía necesario.

En este trabajo de investigación me he propuesto implementar mediante código uno de los algoritmos que he visto en un paper, una Red Adversaria Generativa Cuàntica (GAN, en inglés) [4] que genera imágenes a partir de un circuito cuántico [5]. Como objetivo tengo verificar una sugerencia que hacen los autores del paper: implementar una función no-lineal en una parte del algoritmo que podría mejorar el rendimiento de este. Mi hipótesis al igual que los autores (aunque ellos lo comentan muy brevemente) es que el algoritmo va a reducir ligeramente el número de interacciones que son necesarias para llegar a su punto óptimo. Es decir, el modelo con la función no-lineal va a necesitar menos operaciones que lo entren conseguir los mismos resultados que el modelo sin la función.

Part I

Marc Teòric

Capítol 1

Àlgebra Lineal

Quan vaig començar a buscar informació sobre computació quàntica, en vaig ràpidament donar compte que necessitava molt més coneixement matemàtic, degut a que no entenia gairebé res dels llibres sobre computació quàntica. Arran aquell temps, una serie de vídeos sobre àlgebra lineal en va captar l'atenció, que es justament la branca de les matemàtiques sobre la qual es basa la computació quàntica. Els vídeos son les lliçons que dona el Professor Gilbert Strang al Institut Tecnològic de Massachusetts (MIT en anglès) [6, 7]. Una vegada havia vist gairebé tots els vídeos, ja tenia bastants conceptes apresos.

Aquelles lliçons es van ajudar a entendre les matemàtiques de *Quantum Computation and Quantum Information* [8] i *Quantum Computing: A Gentle Introduction*. A poc a poc, vaig anar aprenent els fundaments matemàtics de la computació quàntica i mecànica quàntica.

En aquesta secció aniré explicant els conceptes bàsics de l'àlgebra lienal, per formar els coneixements en matemàtiques necessaris per poder comprendre aquest treball.

1.1 Vectors i Espais Vectorials

Els objectes fonamentals de l'àlgebra lineal són els espais vectorials. Un espai vectorial es el conjunt de tots els vectors que tenen les mateixes dimensions. Per exemple $\mathbb{R}^3$ seria el espai vectorial de tots els vectors de 3 dimensions, aquests

vectors normalment s'utilitzen per representar punts en un espai tridimensional. En computació quàntica un tipus d'espais vectorials en concret són utilitzats: Els espais de Hilbert, en altres paraules, un espai vectorial amb un producte interior [9]. Els espais de Hilbert segueixen un conjunt de productes i compleixen unes certes normes, en aquest capítol presentaré una part d'aquestes normes i productes, la quantitat que és necessària. S'ha de tenir en compte que els espais de Hilbert són molt més complicats que el que es representa en aquest treball, també que d'aquí en endavant, quan mencioni espai vectorial hem referiré a un espai de Hilbert, d'ha no ser que s'especifiqui el contrari.

Els espais vectorial estan definits per les seves bases, un set de vectors $B = \{|v_1\rangle, \ldots, |v_n\rangle\}$ es una base vàlida per l'espai V, si cada vector $|v\rangle$ en l'espai es pot escriure com $|v\rangle = \sum_i a_i |v_i\rangle$ per $|v_i\rangle \in B$. Els vectors en B són linealment independent entre ells.

La notació estàndard pels conceptes de àlgebra lienal en mecànica quàntica es la notació de Dirac, en la qual es representa un vector com $|\psi\rangle$. On ψ es la etiqueta del vector. Un vector $|\psi\rangle$ amb n dimensions també pot ser representat com una matriu columna que te la forma:

$$|\psi\rangle = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix}$$

On els nombres complexes $(z_1, z_2, \ldots, z_{n-1}, z_n)$ són els seus elements. Un vector escrit com a $|\psi\rangle$ també s'anomena *ket*.

La adició d'un par de vectors en un espai de Hilbert es definida per [1]:

$$|\psi\rangle + |\varphi\rangle = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix} + \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix}$$

[1] Els vectors d'aquesta definició tenen els seus elements representats per la seva etiqueta i un subscrit e.g. el vector $|\psi\rangle$ te un element qualsevol ψ_1 i el seu primer element es ψ_1. Aquesta notació es seguirà utilitzant al llarg del treball.

A més a més, hi ha una multiplicació per un escalar[2] definida per:

$$z\left|\psi\right\rangle = z\begin{bmatrix}\psi_1\\ \vdots \\ \psi_n\end{bmatrix} = \begin{bmatrix}z\psi_1\\ \vdots \\ z\psi_n\end{bmatrix}$$

On z es un escalar i $|\psi\rangle$ un vector. Cal que notar que cada element del vector es multiplicar per el escalar.

Degut a que els espais de Hilbert son complexos tenen un conjugat complex definit per escalar com a: Per un escalar complex $z = a + bi$, el seu conjugat z^* es igual a $a - bi$.

Aquesta noció pot ampliar per a vectors i matrius, agafant el conjugat de totes les seves entrades/elements:

$$\left|\psi\right\rangle^* = \begin{bmatrix}\psi_1\\ \vdots \\ \psi_n\end{bmatrix} * = \begin{bmatrix}\psi_1^*\\ \vdots \\ \psi_n^*\end{bmatrix}$$

$$A^* = \begin{bmatrix}A_{11} & \cdots & A_{1n}\\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_mn\end{bmatrix}^* = \begin{bmatrix}A_{11}^* & \dots & A_{1n}^*\\ \vdots & \ddots & \vdots \\ A_{m1}^* & \cdots & A_mn^*\end{bmatrix}$$

Amb $|\psi\rangle$ sent un vector de dimensions n, i A sent una matriu de dimensions $m \times n$.

Un altre concepte important es la transposada, representada per el supercrit T que ”rotaün vector o una matriu. Un vector columna amb una dimensió $n, 1$ es transforma amb un vector fila amb una dimensió $1, n$[3]:

$$\left|\psi\right\rangle^T = \begin{bmatrix}\psi_1\\ \vdots \\ \psi_n\end{bmatrix}^T = \begin{bmatrix}\psi_1 & \dots & \psi_n\end{bmatrix}$$

El mateix és veritat per les matrius, una matriu $m \times n$ transposada es conver-

[2] Un numero qualsevol en $\mathbb{R}$.

[3] En realitat els vectors columna son matrius amb dimensió $n, 1$ però he estat ometent el 1. Quan hem refereixo a les dimensions de un vector qualsevol, només diré un numero, no obstant, especificaré si és un vector columna o un vector fila.

teix en una matriu $n \times m$. Per exemple:

$$A^T = \begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & 6 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

La combinació de un conjugat complex i la transposada s'anomena el conjugat Hermitià, la seva notació es una † supercrita. Per un vector $|\psi\rangle$ el seu conjugat Hermitià $|\psi\rangle^\dagger$ és:

$$|\psi\rangle^\dagger = (|\psi\rangle^*)^T = \begin{bmatrix} \psi_1^* & \dots & \psi_n^* \end{bmatrix} = \langle\psi|$$

El conjugat Hermitià compleix que $|\psi\rangle^\dagger = \langle\psi|$ i $\langle\psi|^\dagger = |\psi\rangle$.

El conjugat Hermitià de un vector columna $|\psi\rangle$ s'anomena *bra* o vector dual. En la notació de Dirac un vector dual s'escriu com $\langle\psi|$.

1.2 Operadors Lineals

Per poder operar amb vectors i fer operacions amb ells, s'utilitzen les matrius, que també son denominades mapes lineal o operadors lineals, que són noms que descriuen millor com funcionen aquests objectes. La definició formal de un operador lineal pot ser bastant complicada, per aquesta raó, utilitzaré termes més informals al en aquesta secció.

Bàsicament, un operador lineal transforma un vector en un altre vector, aquest vector poden o no ser de espais diferents [10]. Més formalment, per un vector $|v\rangle$ en un espai V i un vector $|w\rangle$ en un espai W, un operador lienal A entre els vectors, fa l'acció:

$$A\,|v\rangle = |w\rangle$$

En altres paraules, l'operador mapa un element del espai vectorial V cap a un espai vectorial W. Els operadors lineals han de complir les següents operacions:

1. Adició de Vectors:
   Per els vectors $|\psi\rangle$ i $|\varphi\rangle$ en un mateix espai vectorial, i un operador lienal A:

$$A(|\psi\rangle + |\varphi\rangle) = A\,|\psi\rangle + A\,|\varphi\rangle$$

2. Multiplicació Escalar:
   Per el vector $|\psi\rangle$, el escalar z i el operador lienal A:

$$A(z\,|\psi\rangle) = zA\,|\psi\rangle$$

Aquestes afirmacions tenen que ser veritat per tots els vectors i tots els escalars en els espais on els operadors actuen. Cal notar que un operador lineal no te perquè ser una matriu necessàriament, fer exemple, les derivades i les integrals son operadors lienals, això es pot provar fàcilment al veure que compleixen els criteris especificats posteriorment. No obstant, les derivades i les integrals usualment no s'apliquen a vectors, sinó a les funcions, però es possible aplicar-les a vectors [4].

Les matrius només son la representació matricial del operadors lienals [11].

1.1 Tipus d'Operadors Lineals

En la secció actual, exposaré els tipus bàsics d'operadors lienals que són indispensables en la teoria presentada en aquest capítol i la rest del treball.

1. **Operador Zero**
   Qualsevol espai vectorial té un vector zero expressat en notació de Dirac com a 0, degut a que $|0\rangle$ es un altre concepte totalment diferent en CQ i IQ[5]. El vector zero es aquell vector que per qualsevol vector $|\psi\rangle$ i qualsevol escalar z, es compleix que: $|\psi\rangle + 0 = |\psi\rangle$ i $z0 = 0$.
   El operador zero també s'escriu com a 0 i es defineix com l'operador que mapa qualsevol vector al vector zero: $0\,|\psi\rangle = 0$.

2. **Matriu inversa**
   Un matriu quadrada[6] A és invertible si existeix una matriu A^{-1} de manera que $AA^{-1} = A^{-1}A$. A^{-1} es la matriu inversa de A. La manera més ràpida de saber si una matriu es invertible es veient si el seu determinant no és zero.

[4] No et preocupis, que es clar que les aplicaré a vectors :D.
[5] Computació Quàntica i Informació Quàntica.
[6] Una matriu quadrada és una matriu amb dimensions $n \times n$, on $n \in \mathbb{N}$.

3. **Operador Identitat**
   Per a qualsevol espai vectorial V existeix un operador identitat I que es definit com $I|\psi\rangle = |\psi\rangle$, aquest operador no fa cap canvi al vectors als quals opera. Cal notar també que per qualsevol matriu A i la seva inversa és veritat que $AA^{-1} = I$

4. **Operador Unitari**
   Un operador unitari es qualsevol operador que no altera la norma dels vectors al quals es aplicat, per tant, una matriu es unitària si $AA^{\dagger} = I$ Per convertir qualsevol operador en unitari, es divideix les seves entrades entre la norma del operador.

5. **Operadors Hermitians**
   Un operador Hermitià o *self-adjoint operator* en anglès, es qualsevol operador que el seu conjugat Hermitià es ell mateix: $A = A^{\dagger}$
   Una altre cosa a tenir en compte es que existeix un operador únic A en un espai de Hilbert, de manera que per qualsevol vectors $|\psi\rangle$ i $|\varphi\rangle$, es compleix que:
   $$\langle\psi|\left(A|\varphi\rangle\right) = \left(A^{\dagger}\langle\psi|\right)|\varphi\rangle$$
   Aquest operador es conegut com el *adjoint* o conjugat Hermitià de A.

1.3 Producte Interior i Producte Exterior

1.1 Producte Interior

Un vector dual $\langle\psi|$ i un vector $|\varphi\rangle$ combinats formen el producte interior $\langle\psi|\varphi\rangle$, el qual efectua una operació que agafa els dos vectors com a input i produeix un nombre complex com a output:

$$\langle a|b\rangle = a_1b_1 + a_2b_2 + ... + a_{n-1}b_{n-1} + a_nb_n = z$$

Amb $z, a_i, b_i \in \mathbb{C}$. Quan hem refereixo a un producte interior, normalment diré ”el producte interior de dos vectors”, quan en realitat es una operació entre un vector dual i un vector.

El equivalent d'aquest producte en un espai real de dos dimensions $\mathbb{R}^2$ es el producte escalar, que es expressat com a :

$$\langle a|b\rangle = \||a\rangle\|_2 \cdot \||b\rangle\|_2 \cos\theta \tag{1.1}$$

Amb $\|\cdot\|_2$ sent la norma ℓ^2 definida com a $\||\psi\rangle\|_2 = \sqrt{\psi_1^2 + \cdots + \psi_n^2}$ amb θ sent l'angle entre els vectors $|a\rangle$ i $|b\rangle$. Com he dit l'equació (1.1) és equivalent al producte interior, no obstant, segons el que he vist, no es usada àmpliament ja que interpretar θ com un angle entre vectors de dimensions altes no té molt de sentit. En contrast, he vist aquest producte presentat en la seva interpretació geomètrica[7] com el producte entre un vector fila i un vector columna:

$$\langle a|b\rangle = \begin{bmatrix} a_1 \cdots a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Ja he definit la norma I have already defined the ℓ_2 norm as the square root of the sum of the squared entries of a vector:

$$\||a\rangle\|_2 = \sqrt{\sum_i |a_i|^2}$$

Nonetheless, the more common definition is based upon the inner product. As you can see the inner product of a vector by itself is the sum of the squared entries:

$$\langle a|a\rangle = a_1 a_1 + \cdots + a_n a_n = a_1^2 + \cdots + a_n^2 = \sum_i a_i^2$$

Thus, the norm can be defined as the square root of the inner product of a vector:

$$\||a\rangle\|_2 = \sqrt{\langle a|a\rangle} \tag{1.2}$$

When the norm is applied to a 2 dimensional vector you can see that is the same as the length of that vector, that is because norm and length are the same concepts, however, the norm is the generalized length that can be applied to a vector of any dimension.

[7] Els detalls exactes de l'interpretació geomètrica estan fora del domini d'aquest treball, malgrat que m'agradaria molt parlar sobre el tema.

From what I understand some properties of the length of a two dimensional vector do not hold with the norm of a vector that has more than 2 dimensions. In other words, the norm behaves in similar ways like the distance from the origin (which is the length), thus they are not the exact same thing. Moreover, there are different types of norm[8] that are used in different types of scenarios. That is why I am referring to a ℓ^2 norm, a specific type of norm that is also named Euclidean norm which is used to define the ℓ^2 distance or Euclidean distance, widely used to measure the distance of two points in a 2D space or a 3D space in high school [12].

1.1.1 Properties of the Inner Product

The basic properties of the inner product are as follows:

1. Is linear in the second argument $(z_1 \langle a| + z_2 \langle c|) |b\rangle = z_1 \langle a|b\rangle + z_2 \langle c|b\rangle$
2. Conjugate symmetry $\langle a|b\rangle = (\langle b|a\rangle)^*$
3. $\langle a|a\rangle$ is non-negative and real, except in the case of $\langle a|a\rangle = 0 \Leftrightarrow |a\rangle = 0$

1.2 Orthonormal and orthogonal vectors

From the concept of norm comes the concepts a pair of orthogonal vectors and a pair of orthonormal vectors[9].
Looking at the equation (1.2) we can see that if the inner product of a vector is one, the norm of this vector is also one. A vector that has norm one is named a unit vector. Therefore, if the inner product of a vector is one, that vector is a unit vector.

A pair of non-zero vectors are orthogonal if their inner product is zero. For two non-zero 2 dimensional vectors, if their inner product is equal to zero, you can

[8] But not different types of length, that I know of at least.
[9] Funny note, when I encountered these two terms for the first time in QC and QI [citation], I taught they were the same thing and a week passed until I realized. It was such a difficult mess to understand everything else with these two terms confused.

see that they are perpendicular to each other by looking at equation (1.1):

For $|a\rangle$ and $|b\rangle \neq 0$:

If $\langle a|b\rangle = 0$ then: $\||a\rangle\|_2 \cdot \||b\rangle\|_2 \cos\theta = 0$

Because $|a\rangle$ and $|b\rangle$ are non-zero vectors, their norms can't be zero.

Thus the remainder term $\cos\theta$ is equal to zero.

Therefore, the angle θ as to be $\frac{\pi}{2}$.

However, thinking that perpendicularity and orthogonality are the same concepts is a mistake, since, it only holds when looking at 2 dimensional vectors. As with norm and length, orthogonality is the generalized concept of perpendicularity that works for high dimensional vectors.

When we mix the concepts of unit vector and orthogonal vectors we arrive at the term orthonormality [9]. A pair of non-zero vectors are orthonormal when both are unit vectors and there are orthogonal to each other:

$$|a\rangle \text{ and } |b\rangle \text{ are othornormal if } \begin{cases} \langle a|b\rangle = 0 \\ \langle a|a\rangle = 1 \\ \langle b|b\rangle = 1 \end{cases}$$

Orthonormal vectors are important, they are broadly used in quantum computation as well as quantum mechanics because they form the basis for the vector spaces on which the quantum states are located.

maybe introduce them before if you talk about basis later on

One thing to point out is that I have been talking about a pair of vectors when referring to orthonormal vectors, however, orthonormality can be extended to a set of vectors. If a set has all unit vectors and the vectors are orthogonal to each other, the set is orthonormal. The set of vectors $B = \{|\beta_1\rangle, |\beta_2\rangle, ..., |\beta_{n-1}\rangle |\beta_n\rangle\}$ is orthonormal if $\langle\beta_i|\beta_i\rangle = \delta_{ij} \forall i, j$ [9] where δ_{ij} is the Kronecker delta defined as :

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

1.3 Outer product

The outer product is a function that takes two vectors -expressed as $|a\rangle\langle b|$, with $|a\rangle$ and $|b\rangle$ being vectors- and produces an linear operator as output. Unlike the inner product, there is no analog for the outer product on the mathematics taught in high school[10], and it is a bit difficult to understand it as it can take two vectors from different spaces as input. It is defined as follows:

For a vector $|v\rangle$ and $|v'\rangle$ of dimensions m and a vector $|w\rangle$ of dimension n. The output is a linear operator A of dimensions $m \times n$ in the space $M_{m\times n}$:

$$|v\rangle\langle w| = A \text{ with } A \in \mathrm{Mat}_{m\times n}.$$

Whose action is defined by:

$$(|v\rangle\langle w|)\,|v'\rangle \equiv |w\rangle\langle v|v'\rangle = \langle v|v'\rangle\,|w\rangle \tag{1.3}$$

From equation (1.3) the usefulness and meaning of the outer product are hard to comprehend, so I will look at the way to compute it next to clarify how it works. For two vectors $|a\rangle$ and $|b\rangle$ of dimensions m and n respectively, their outer product is computed multiplying each element of $|a\rangle$ by each element of $|b\rangle$ forming a matrix of size $m \times n$:

$$|a\rangle\langle b| = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_mb_1 & a_mb_2 & \cdots & a_mb_n \end{bmatrix}$$

The usefulness of the outer product will be shown in future sections.

1.4 Tensor product

The last product to mention is the tensor product, represented with the symbol $\otimes$. This product is used to create larger vector spaces by combining smaller vector spaces. The formal explanation of this concept is quite difficult, so I will focus

[10] The analog of the inner product would be the dot product.

on explaining the way to compute it by using the matrix representation of this product, named the Kronecker product.

For a $m \times n$ matrix A and a $p \times q$ matrix B the output of their Kronecker product [13] is a $pm \times qn$ matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$

Note that $a_{ij}B$ is a scalar multiplication by a matrix, with a_{ij} being the scalar and B being the matrix.

Here is a clearer example with two 2×2 matrices, note that each entry of the first matrix is multiplied by the second matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 0 & 1 \times 5 & 2 \times 0 & 2 \times 5 \\ 1 \times 6 & 1 \times 7 & 2 \times 6 & 2 \times 7 \\ 3 \times 0 & 3 \times 5 & 4 \times 0 & 4 \times 5 \\ 3 \times 6 & 3 \times 7 & 4 \times 6 & 4 \times 7 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

One important piece of notation to take into account is $\bigotimes$, used to represent the equivalent of the sum (noted with $\sum$), but instead of addition the Kronecker product is used. In other words $\bigotimes$ denotes the Kronecker product of a finite number of terms. To clarify here's an example with the identity matrix:

With $\mathbb{I}$ as the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and n as a power of 2:

$$\mathbb{I}_n = \bigotimes^{\log_2 n} \mathbb{I}$$

Here is the case for $n = 8$:

$$\mathbb{I}_8 = \bigotimes^{\log_2 8} \mathbb{I} = \bigotimes^{3} \mathbb{I} = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Kronecker product also works for vectors in the same way, with a scalar-vector multiplication:

For the vectors $|\psi\rangle$ and $|\varphi\rangle$ of dimensions n and m respectively:

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix} \psi_1 |\varphi\rangle \\ \psi_2 |\varphi\rangle \\ \vdots \\ \psi_m |\varphi\rangle \end{bmatrix} = \begin{bmatrix} \psi_1\varphi_1 \\ \psi_1\varphi_2 \\ \vdots \\ \psi_1\varphi_m \\ \vdots \\ \vdots \\ \psi_n\varphi_1 \\ \psi_n\varphi_2 \\ \vdots \\ \psi_n\varphi_m \end{bmatrix}$$

Note that the Kronecker product can also be taken between a vector and a matrix or vise-versa, however this form isn't as common as the other two.

1.1 Properties of the Tensor Product

The basic properties of the tensor product are as follows [14, 15]:

1. Associativity:
   $A \otimes (B + C) = A \otimes B + A \otimes C$
   $(zA) \otimes B = A \otimes (zB) = z(A \otimes B)$
   $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
   $A \otimes 0 = 0 \otimes A = 0$
2. Non-commutative [11]:
   $A \otimes B \neq B \otimes A$

1.5 Trace

The trace of a matrix is just the sum of the elements on the main diagonal, the one that goes from top to bottom and left to right.

Here's a matrix A with its main diagonal marked:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

And its trace, denoted by $\mathrm{Tr}[A]$ is:

$$\mathrm{Tr}[A] = 1 + 1 + 1 = 3$$

More formally, the trace of a n-dimensional squared matrix A is:

$$\mathrm{Tr}[A] = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

[11] One cool thing is that $A \otimes B$ and $B \otimes A$ are permutation equivalent: $\exists P, Q \Rightarrow A \otimes B = P(B \otimes A)Q$ where P and Q are permutation matrices.

The trace of a matrix as the following properties:

1. Linear operator:
   Because the trace is a linear mapping, it follows that:
   $\mathrm{Tr}[A+B] = \mathrm{Tr}[A] + \mathrm{Tr}[B]$ and $\mathrm{Tr}[zA] = z\,\mathrm{Tr}[A]$, for all squared matrices A and B and all scalars z.

2. Trace of a Kronecker product:
   $\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\,\mathrm{Tr}[B]$

3. Transpose has the same trace:
   $\mathrm{Tr}[A] = \mathrm{Tr}\left[A^T\right]$

4. Trace of a product is cyclic:
   For a $m \times n$ matrix A and a $n \times m$ matrix B:
   $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$

One very useful way to compute the trace of an operator is through the Gram-Schmidt procedure[12] and an outer product. Using Gram-Schmidt to represent the unit vector $|\psi\rangle$ with an orthonormal basis $|i\rangle$ which includes $|\psi\rangle$ as the first element, is true that:

$$\mathrm{Tr}[A\,|\psi\rangle\,\langle\psi|] = \sum_i \langle i|\,A\,|\psi\rangle\,\langle\psi|i\rangle = \langle\psi|\,A\,|\psi\rangle$$

[12] See A.1 for the definition of the Gram-Schmidt procedure.

Capítol 2

Quantum Computation

After some amount of math theory, it is time to start talking about quantum mechanics, in this chapter I am going to introduce the basic concepts of quantum computation and quantum information.

Quantum mechanics is a mathematical framework or rather a set of theories used to describe and explain the physical properties of atoms, molecules, and subatomic particles. It is the framework of all quantum physics including quantum information science. The right way of presenting quantum computation is through the formal quantum mechanics postulates because, with them, the statements made in quantum computation do not seem to come from anywhere [16]. However, to not complicate this section more than it is, I will do my best to explain the concepts and math of quantum computing just on their own, without presenting more generalized concepts from quantum mechanics, unless it is totally necessary to do so.

2.1 Quantum State and Superpositions

To describe how physical systems evolution through time, you need to represent them in some way. On quantum computing those quantum physical systems are represented through quantum states, which are some form of probably distribution for the possible outcomes of a measurement on a quantum system [17].

Imagine that you have a pen, however, you do not know which color is it, but you know that it can be red or blue. To figure out the color, you would do a

measurement, in other words, write something to see the color of the ink. You also know that there is a 50% change of being blue, and a 50% of being red[1]. So far you have your quantum system (the pen), a way to measured it (writing), and a list of all the possible outcomes (50% blue and 50% red). Now you need your quantum state, the way to represent everything mathematically. Let's try to store the information that we know about the pen in a vector.

If we put the probability of each possible outcome in an entry, we have the following vector:

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Where the first entry is the probability of painting blue and the second one of painting red, to make it more easy let me color the numbers:

$$\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Note that this vector is normalized went using the ℓ_1 norm, defined as the sum of the entries of a vectors[2], in other words, the ℓ_1 norm of this vector is 1.

So, how can we extract the probability about a possible outcome from the vectors, by a mathematical operation; I mean. We how that we need to get a real number out of this operation, so let's try with an inner product. As we know a vector in a space is a linear combination of space basis:

$$0.5\,|0\rangle + 0.5\,|1\rangle = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Where $|0\rangle$ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and, $|1\rangle$ is the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus to get the scalar, you take the inner product between the basis corresponding to that scalar and the vector, like so:

$$\langle 0|v\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 0.5$$

$$\langle 1|v\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 0.5$$

[1] This is starting to sound a lot like US politics.

[2] With $|a\rangle$ being a vector, the ℓ_1 norm, denoted as $\|\cdot\|_1$ is $\||a\rangle\|_1 = \sum_i a_i$.

You can see that this is quite easy. For example, to represent a pen with 6 colors with a random possibility of writing in a specific color, a valid vector to represent this pen would be[3]:

$$|w\rangle = \begin{bmatrix} 0.25 \\ 0.3 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.05 \end{bmatrix}$$

Then we can measure the state with an inner product:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.3 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.05 \end{bmatrix} = 0.1$$

Note that we use the corresponding basis for the 3th element, the green number.

Leaving the pens aside, now we are going to substitute the pens by quantum systems, for example, with a photon. Photons have certain properties that can be measured, like polarization[4], when looking at photon as waves in the electromagnetic field, the polarization is the geometrical orientation of the oscillation of the wave. This can be interpreted as an angle respect to the propagation direction. insert figure here

We can set the basis states of the polarization as vertical and horizontal, denoted with the vectors $|\rightarrow\rangle$ and $|\uparrow\rangle$, respectively. And we can set a state that is a superposition of both, denoted as $|\nearrow\rangle$ [18]. The superposed state will be:

$$|\nearrow\rangle = \alpha\,|\rightarrow\rangle + \beta\,|\uparrow\rangle$$

Where α and β are complex numbers. The superpose state simply is an angle that is not 0 nor $\frac{\pi}{2}$. Note that we do not have to worry about the exact math of all these vectors[5], because we are only worry about the information that they carry.

[3] The entries in the vector are colored for clarity, but don't take it to seriously.

[4] More concretely, its a property of traverse waves, which electromagnetic waves are.

[5] The mathematical description of the polarization of the photon is not necessary to know went talking about quantum computation, unless you are working with a quantum computer that represents the quantum states as photons (using their polarization or something else.). Then you

Which is taken through the measurements, like with the pen. To know[6] the state of polarization, e.i. to measure it, you need to pass the photon through several polarizing filters, which let a photon pass completely through it or absorbed completely on a probabilistic matter.

Let me explain, what the filter does is to collapse the photon in two possible states of polarization in either the state on which the filter is oriented or on the state that is perpendicular to it. For example, a filter oriented in the horizontal direction will either pass a photon oriented vertically or block entirely. The matter on which the collapse is decided is probabilistic, a photon oriented on the horizontal direction has a 100% probability of being absorbed, while a photon polarized vertically has a 0% chance of being absorbed, and finally, a photon that has a $\frac{\pi}{4}$ angle of polarization respect to the filter, meaning that is exactly in between these two perpendicular states, has a 50% probability of being absorbed by the filter, thus it has a 50% chance of passing through.

However, the polarization of the photons are a concrete physical system, in quantum information everyone works with qubits, instead of physical systems[7]. Went presenting algorithms for example, researchers use qubits.

2.2 Qubits

Modern computers represent information through strings of zeros and ones named bits. Everything, from images to letters, characters and instructions. For example, the letter t is represented by the bitstring 01110100. Everything you do on a computer is on binary code.

Since we are so used to binary code, in the field of quantum computing is also used, however, instead of bits, qubits are used. A qubit is the analog of a bit, in other words the information unit used by quantum computers. We can apply quantum superposition and quantum entanglement to qubits. If a bit has

would be talking about quantum optics which is itself a complex part of quantum mechanics, that I do not want to learn about yet.

[6] Kind of, you will see later.

[7] Unless you are working with a concrete quantum system, and you want to represent the information in the system's terms.

two states 0 and 1, a qubit has a state that is a linear combination of the two fundamental states $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

Where α and β are complex numbers and $|\psi\rangle$ is a vector in a 2-dimensional Hilbert space[8]. The vectors $|0\rangle$ and $|1\rangle$ are called the computation basis vectors. They are represented as follows:

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus the vector $|\psi\rangle$ is:

$$|\psi\rangle = \alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

This vector is a valid quantum state to represent a qubit, however, there is an important factor to take into account. The vector has to be normalized according to the ℓ_2 norm, therefore α and β can not be any complex number, they need to form a vector that a norm of 1:

$$\||\psi\rangle\| = 1$$

Therefore:

$$\|\psi\| = \sqrt{|\alpha|^2 + |\beta|^2} = 1$$
$$\Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

Saying that a qubit is ïs a linear combination of the two fundamental statesï's not that meaningful, so let me elaborate more on that: A n-bitstring can represent just one possible combination at a time, while a n-qubit system can represent a combination of all the possible combinations. Consider a qubit as a mixture of the possible states, each coefficient on the linear combination is the number that indicates how much a state is part of that mixture. For example, if you want an equal mixture of the states $|0\rangle$ and $|1\rangle$

Something interesting happens as you increase the number of qubits, the "amount of informationïncreases exponentially. When you go from a 7-bitstring to a 8-string the amount of numbers[9] that can be represent increases by one,

[8] A space with an inner product.
[9] Binary numbers.

from 7 to 8. But n qubits can represent 2^n complex numbers. Note that the qubit represented before represents two complex numbers. All this is because the computational basis used by the qubits, it grows exponentially with the number of qubits.

Two qubits are represented by the vector:

$$\begin{aligned} |\psi\rangle &= \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle \\ &= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \end{aligned}$$

Note that the computational basis vectors are the columns of an identity matrix of dimensions $2^n \times 2^n$, where n is the number of qubits.

The quantity of information that a qubit has is the information of all possible combination that the qubit has. One qubit has two possible combination $|0\rangle$ and $|1\rangle$, meanwhile two qubits have four possible combinations $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and four qubits have 2^4 possible combinations, that is a total of 16. And the information about each combination is a complex number, represent in the examples above has α_i. The complex numbers are used to specify the quantum superposition that the system has.

2.3 Quantum Measurement

Turning back to the photons and their polarization, to predict on which basis the photon is going to collapse, we need to employ quantum measurements. On this case to know the probability of the collapse to a specific basis is enough. Where the vector $|\rightarrow\rangle$ expresses the polarization of the filter and $|\uparrow\rangle$ its perpendicular complement. A photon that is polarized in the following state [18]:

$$|\nearrow\rangle = \alpha |\rightarrow\rangle + \beta |\uparrow\rangle$$

The probability of being absorbed is α^2, note that α is the coefficient that corresponds to the basis $|\rightarrow\rangle$, which is the polarization of the filter. In other words, α^2 indicates the probability of the collapse of the polarization onto the state $|\rightarrow\rangle$.

The coefficients α and β can be expressed as function of an angle θ[10]:

$$|\nearrow\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle \tag{2.1}$$

And the expected probabilities become $\cos^2\theta$ and $\sin^2\theta$, respectively [18]. Now we can prove the previous statements. In the case of a $\pi/2$ angle we see that:

$$\begin{aligned} p(|\rightarrow\rangle) &= \cos^2\theta = \cos^2\frac{\pi}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= 0.5 \\ p(|\uparrow\rangle) &= \sin^2\theta = \sin^2\frac{\pi}{2} = \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= 0.5 \end{aligned}$$

[10] For more information on the polarization of a photon, see appendix C.

Part II

Part Experimental

Part III

Conclusions

Appendices

Apèndix A

More Linear Algebra

I have wrote many pages of linear algebra theory, but that wasn't enough. So here we go, I guess.

A.1 Gram–Schmidt Procedure

The Gram–Schmidt procedure is a method used to produce orthonormal basis for a vectors space [9]. For a d-dimensional inner product vector space V with a basis vectors set $|v_1\rangle, \cdots, |v_d\rangle$, we can define a new orthonormal basis set $\{|u\rangle\}$. The first element of that set is $|u_1\rangle = |v_1\rangle / \||v_1\rangle\|$, with the following element $|v_{k+1}\rangle$ being:

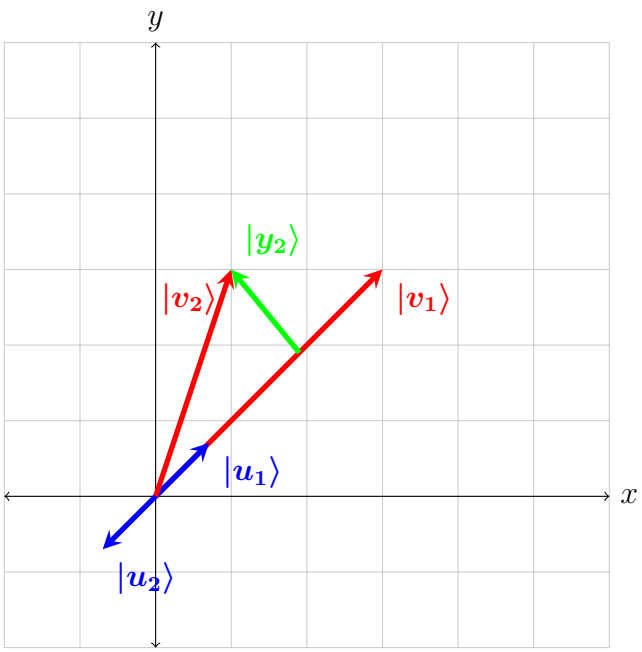
$$|u_{k+1}\rangle = \frac{|v_{k+1}\rangle - \sum_{i=1}^{k} \langle u_i|v_{k+1}\rangle |u_i\rangle}{\left\||v_{k+1}\rangle - \sum_{i=1}^{k} \langle u_i|v_{k+1}\rangle |u_i\rangle\right\|}$$

For k in the interval $1 \leq k \leq d-1$.

If we follow the above for each k in $1 \leq k \leq d-1$, we obtain the new vector set $|u_1\rangle, \cdots, |u_d\rangle$ that is a valid orthonormal basis for the space V. The created vector set must have the same span[1] as the previous one to be a valid basis for the space V:

$$\text{span}(\{|v\rangle\}) = \text{span}(\{|v\rangle\}) = V$$

[1] The span of a set of vectors is all the possible linear combinations that come from those vectors.



Note that the span of the basis set is the definition of the space. In other words, every vector in V can be represented as a linear combination of the basis vectors.

To get at some intuition let's look at the case for $k = 1$:

$$|u_2\rangle = \frac{|v_2\rangle - \langle u_1|v_2\rangle\,|u_1\rangle}{\||v_2\rangle - \langle u_1|v_2\rangle\,|u_1\rangle\|}$$

We can see that the vector $|u_2\rangle$ is defined by the subtraction of the vector $|v_2\rangle$ by the projection[2] of $|v_2\rangle$ onto $|v_1\rangle$. Which is equivalent for the projection of $|v_2\rangle$ onto $|u_1\rangle$, remember that $|u_1\rangle$ is just $|v_1\rangle$ normalized. Therefore we have:

finish the graph

The proof that is an orthonormal basis is quite simple: We can see immediately that the components of $\{|u\rangle\}$ are unit vectors because they are normalized (the vectors $|v_{k+1}\rangle - \sum_{i=1}^{k} \langle u_i|v_{k+1}\rangle\,|u_i\rangle$ are divided by their norm). And we can se that they are orthogonal by checking that the inner product of non-equal vectors in the set is 0:

For $k = 1$:

$$|u_2\rangle = \frac{|v_2\rangle - \langle u_1|v_2\rangle\,|u_1\rangle}{\||v_2\rangle - \langle u_1|v_2\rangle\,|u_1\rangle\|}$$

[2] The way to project the vector $|a\rangle$ onto $|b\rangle$ is: $\langle b|a\rangle\,|b\rangle$.

Then the inner product with $|v_1\rangle$ is:

$$\begin{aligned}\langle u_1|u_2\rangle &= \langle u_1| \left(\frac{|v_2\rangle - \langle u_1|v_2\rangle |u_1\rangle}{\||v_2\rangle - \langle u_1|v_2\rangle |u_1\rangle\|} \right) \\ &= \frac{\langle u_1|v_2\rangle - \langle u_1|v_2\rangle \langle u_1|u_1\rangle}{\||v_2\rangle - \langle u_1|v_2\rangle |u_1\rangle\|} \\ &= 0\end{aligned}$$

By induction we can see that for $j \leq d$, with d being the dimension of the vector space:

$$\begin{aligned}\langle u_j|u_{n+1}\rangle &= \langle u_j| \left(\frac{|v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle |u_i\rangle}{\||v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle |u_i\rangle\|} \right) \\ &= \frac{\langle u_j|v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle \langle u_j|u_i\rangle}{\||v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle |u_i\rangle\|} \\ &= \frac{\langle u_j|v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle \delta_{ij}}{\||v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle |u_i\rangle\|} \\ &= \frac{\langle u_j|v_{n+1}\rangle - \langle u_j|v_{n+1}\rangle}{\||v_{n+1}\rangle - \sum_{i=1}^{n} \langle u_i|v_{n+1}\rangle |u_i\rangle\|} \\ &= 0\end{aligned}$$

All of this doesn't seem straightforward at first but remember that the inner product of two orthogonal vectors is zero, and the inner product between the same unit vector is one.

A.2 Dirac Notation Crash Course

On the following table there is a quick summary important mathematical concepts of linear algebra [19] expressed in Dirac Notation[3].

[3]The notation used for the complex vector spaces and complex number space are not standard Dirac Notation, but I included them in the table to explain what they mean.

Notation	Description
z	Complex number
z^*	Complex conjugate of the a complex number z. $(a+bi)^* = (a-bi)$
$\|\psi\rangle$	Vector with label ψ. Known as *ket*
$\|\psi\rangle^T$	Transpose of vector $\|\psi\rangle$
$\|\psi\rangle^\dagger$	Hermitian conjugate of vector. $\|\psi\rangle^\dagger = (\|\psi\rangle^T)^*$
$\langle\psi\|$	Dual vector to $\|\psi\rangle$. $\|\psi\rangle = \langle\psi\|^\dagger$ and $\langle\psi\| = \|\psi\rangle^\dagger$. Known as *bra*
$\langle\varphi\|\psi\rangle$	Inner product of vectors $\langle\varphi\|$ and $\|\psi\rangle$
$\|\varphi\rangle\langle\psi\|$	Outer product of vectors $\langle\varphi\|$ and $\|\psi\rangle$
$\|\psi\rangle \otimes \|\varphi\rangle$	Tensor product of vectors $\|\varphi\rangle$ and $\|\psi\rangle$
0	Zero vector and zero operator
$\mathbb{I}_n$	Identity matrix of dimension n
$\mathbb{C}_n$	Complex vector space of dimension n
$\mathbb{C}_1$ or $\mathbb{C}$	Complex number space

A.3 More on the Partial Trace

Apèndix B

Quantum Computation vs Quantum Mechanics

On the introduction I mentioned that one of the reasons for with I started learning and researching quantum computing is because it is easy, on this appendix I am going to present why this is the case with a practical example.

On quantum mechanics the more general way to represent quantum state, like the orbitals of an atom of hydrogen, are wavefunctions, not statevectors. Wavefunctions are extremely useful, however, working with them adds a hole new level of complexity because they are continuous functions that depend on time. Compare them with vectors, which are discrete packets of information that evolve through discrete amounts of time.

B.1 Normalizing

Because the probabilistic interpretation of both statevectors and wavefunctions, these two objects have to be normalize upon measurement, thus making the total sum of probabilities 1. How to normalize these objects is a perfect example to illustrate the difference in complexity that I see between quantum mechanics and quantum computation.

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

Figura B.1: **Schrödinger's Equation**. Where $\hbar$ is $h/2\pi$, and V is the potential energy function.

For a wavefunction Ψ that represents a particle, the probability of finding the particle in a point x is $|\Psi(x,t)|^2$. Then, the wavefunction has to be normalized like the following:

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx \ = 1 \tag{B.1}$$

Since the wavefunctions evolves over time with the Schrödinger equation, see Fig.B.1, any solution this equation has to be normalized. Luckily, if the wavefunction is normalized at time $t = 0$ it stays this way, in other words, the Schrödinger equations preserves the normalization of the wavefunction [20].

We can prove that the equation, preserves B.1, starting from the trivial equality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx \ = \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx$$

By the product rule we have that[1]:

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi\Psi^*) = \Psi^*\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}\Psi$$

Now the Schrödinger equation says that

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi$$

then by taking the complex conjugate we have that

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*$$

so

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]$$

finally we can evaluate the integral from the start:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx \ = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Big|_{-\infty}^{+\infty}$$

[1] From now on I am going to write $\Psi(x,t)$ as simply Ψ to not make the equations to complicated.

Because $\Psi(x,t)$ has to go to zero when x goes to either infinity, is true that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2\, dx \; = 0$$

Thus the integral is constant and went Ψ is normalized at $t = 0$, it stays that way for any t (positive of course).

falta la prueba para la normalización de un statevector

Apèndix C

Polarization of a photon

On equation 2.1 I have excluded the concept of phase, which determinates the type of polarization that the photon has. There are three types of polarization:

1. **Linear:** A photon is linear polarized when the phase angles α_x, α_y of both basis $|x\rangle, |y\rangle$ states are equal:

$$\begin{aligned} |\nearrow\rangle &= \cos(\theta)e^{i\alpha_x}|x\rangle + \cos(\theta)e^{i\alpha_y}|y\rangle \\ &= [\cos(\theta)|x\rangle + \sin(\theta)|y\rangle]e^{i\alpha} \end{aligned}$$

   Where $\alpha = \alpha_x = \alpha_y$.

2. **Circular:** Where the angles α_x, α_y are apart exactly $\frac{\pi}{2}$ and the amplitude of both basis states is the same:

$$\begin{aligned} |\nearrow\rangle &= \frac{1}{\sqrt{2}}\cos(\theta)e^{i\alpha_x}|x\rangle \pm i\frac{1}{\sqrt{2}}\sin(\theta)e^{i\alpha_y}|y\rangle \\ &= [\cos(\theta)e^{i\alpha_x}|x\rangle \pm i\sin(\theta)e^{i\alpha_y}|y\rangle]\frac{1}{\sqrt{2}} \end{aligned}$$

   Where the $\pm$ sign indicates the difference between right and left circular polarization, $+$ and $-$, respectively.

3. **Elliptical:** Where just the phase angles α_x, α_y differ in some amount:

$$|\nearrow\rangle = \cos(\theta)e^{i\alpha_x}|x\rangle + \sin(\theta)e^{i\alpha_y}|y\rangle$$

   This is the more general case.

Bibliografia

[1] Maria Schuld, Ilya Sinayskiy, and Francesco Petruccione. The quest for a quantum neural network. *Quantum Information Processing*, 13:2567–2586, 2014. [Online; accedit en 19/09/2021: Link].

[2] Michael Broughton, Guillaume Verdon, Trevor McCourt, Antonio J. Martinez, Jae Hyeon Yoo, Sergei V. Isakov, Philip Massey, Ramin Halavati, Murphy Yuezhen Niu, Alexander Zlokapa, Evan Peters, Owen Lockwood, Andrea Skolik, Sofiene Jerbi, Vedran Dunjko, Martin Leib, Michael Streif, David Von Dollen, Hongxiang Chen, Shuxiang Cao, Roeland Wiersema, Hsin-Yuan Huang, Jarrod R. McClean, Ryan Babbush, Sergio Boixo, Dave Bacon, Alan K. Ho, Hartmut Neven, and Masoud Mohseni. Tensorflow quantum: A software framework for quantum machine learning. 2021. [Online; accedit en 19/09/2021: Link].

[3] IBM. Ibm quantum experience, 2021. [Online; accedit en 19/09/2021: Link].

[4] Ian J. Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial networks. 2014. [Online; accedit en 19/09/2021: Link].

[5] He-Liang Huang, Yuxuan Du, Ming Gong, Youwei Zhao, Yulin Wu, Chaoyue Wang, Shaowei Li, Futian Liang, Jin Lin, Yu Xu, and et al. Experimental quantum generative adversarial networks for image generation. *Physical Review Applied*, 16(2), 2021. [Online; accedit en 19/09/2021: Link].

[6] Gilbert Strang. Linear algebra mit ocw, 2011. [Online; accedit en 19/09/2021: Link].

[7] Gilbert Strang. Matrix methods in data analysis, signal processing, and machine learning mit ocw, 2018. [Online; accedit en 19/09/2021: Link].

[8] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambrindge University Press, 10 edition, 2010.

[9] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, chapter 2, page 66. Cambrindge University Press, 10 edition, 2010.

[10] Sheldon Axler. *Linear Algebra Done Right*, chapter 3, pages 37–58. Springer, 2 edition, 1997.

[11] Sheldon Axler. *Linear Algebra Done Right*, chapter 3, pages 48–52. Springer, 2 edition, 1997.

[12] Wolfram MathWorld. L2-norm, 2021. [Online; accedit en 19/09/2021: Link].

[13] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, chapter 2, page 74. Cambrindge University Press, 10 edition, 2010.

[14] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, chapter 2, page 73. Cambrindge University Press, 10 edition, 2010.

[15] Wikipedia. Tensor product, 2021. [Online; accedit en 19/09/2021: Link].

[16] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, chapter 2, pages 80–96. Cambrindge University Press, 10 edition, 2010.

[17] Asher Peres. *Quantum Theory: Concepts and Methods*, chapter 2, pages 24–29. Kluwer Academic Publishers, 2002.

[18] Eleanor Rieffel and Wolfgang Polak. *Quantum Computing: A Gentle Introduction*, chapter 2, pages 12–13. MIT Press, 1 edition, 2011.

[19] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*, chapter 2, page 62. Cambrindge University Press, 10 edition, 2010.

[20] David J. Griffiths. *Introduction to Quantum Mechanics*, chapter 1, pages 11–12. Prentice Hall, 1995.