Analysis of Algorithms

Outline

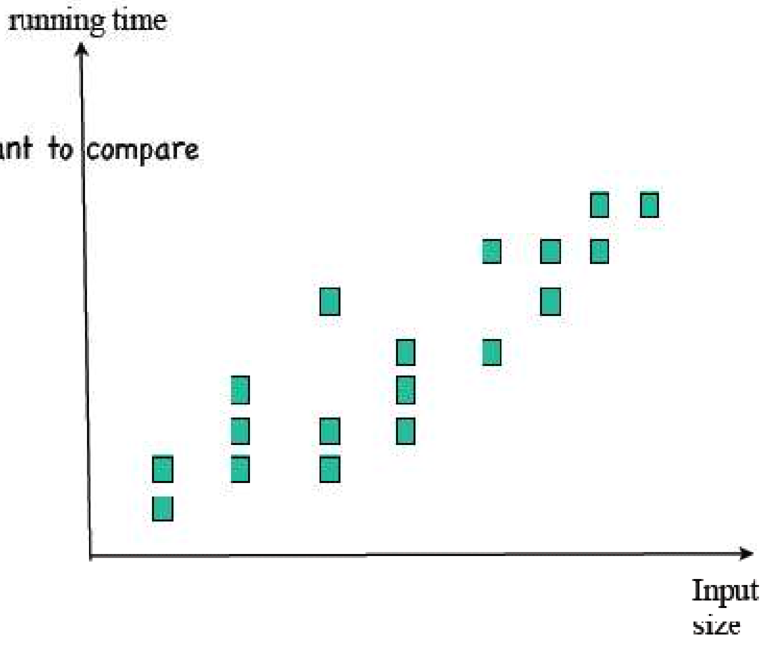
- analysis of algorithms
- asymptotic analysis
- big-O
- big-Omega Ω-notation
- big-theta Θ-notation
- asymptotic notation
- commonly used functions
- discrete math refresher

 READING:

- GT textbook chapter 4

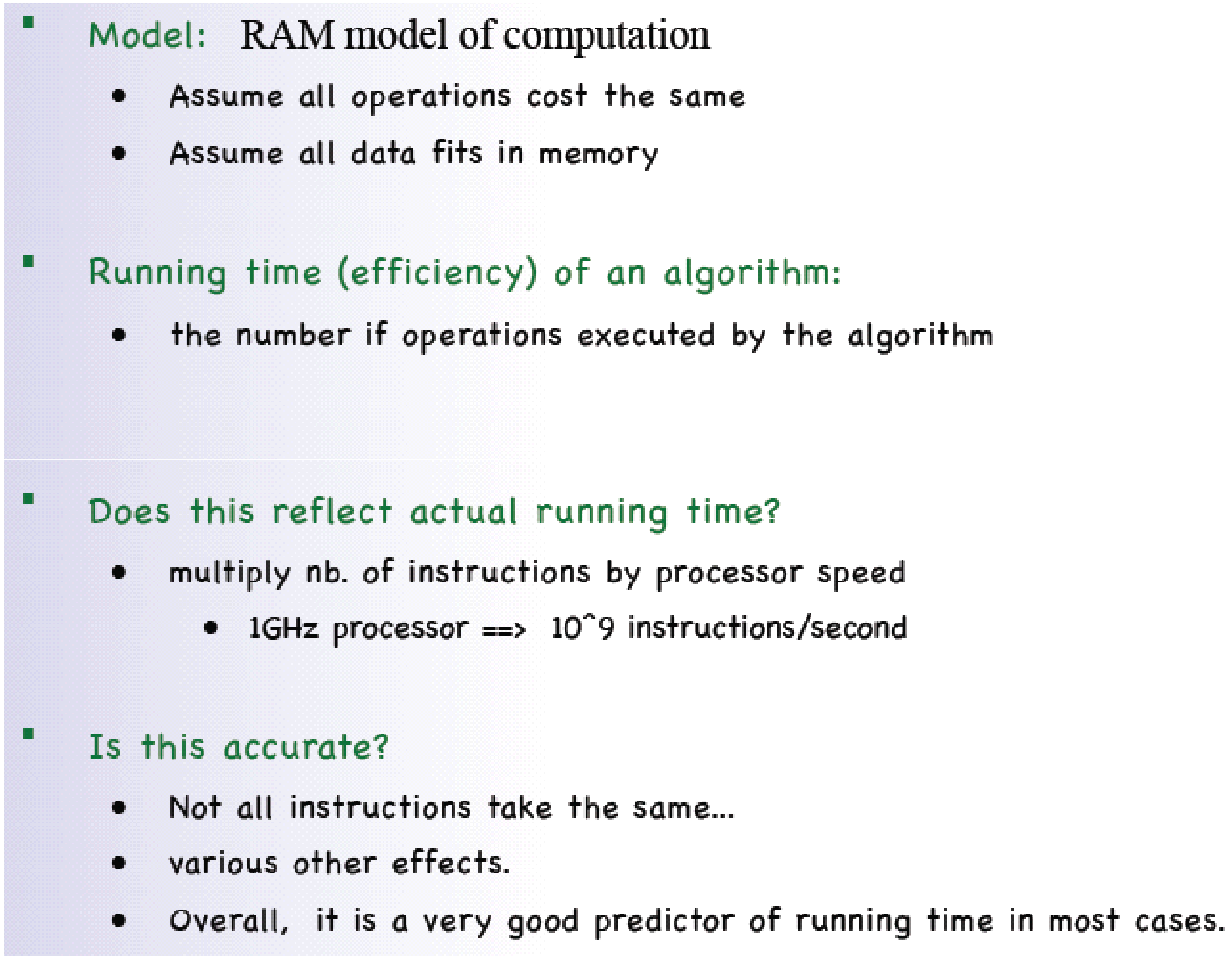
Analysis of algorithms

- Limitations
  - need to implement the algorithm
    - need to implement all algorithms that we want to compare
  - need many experiments
  - try several platforms
- Advantages
  - find the best algorithm in practice



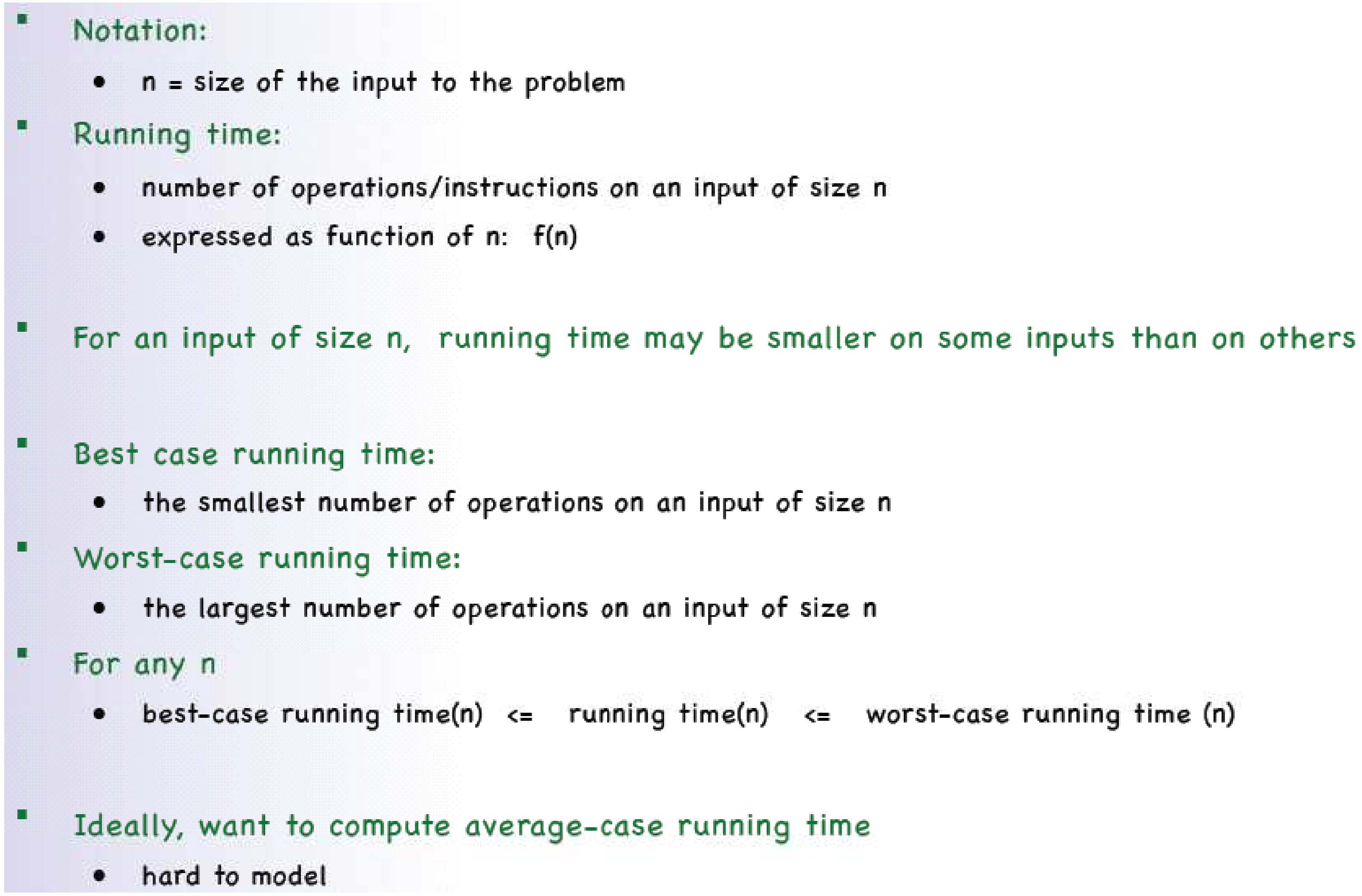
- We would like to analyze algorithms without having to implement them
- Basically, we would like to be able to look at two algorithms flowcharts and decide which one is better

Theoretical analysis



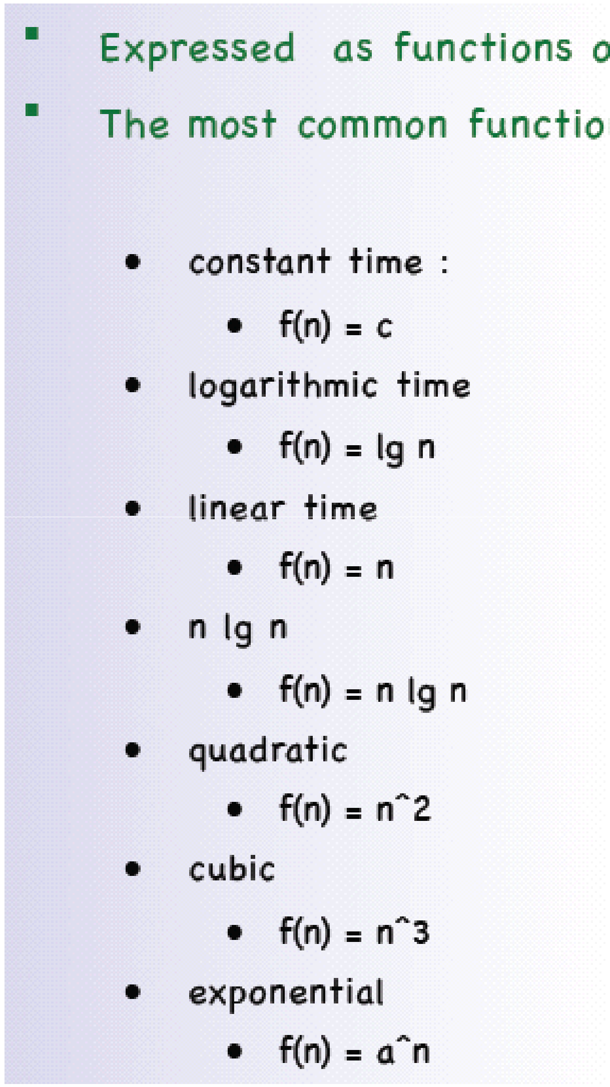
- Model: RAM model of computation
  - Assume all operations cost the same
  - Assume all data fits in memory
- Running time (efficiency) of an algorithm:
  - the number if operations executed by the algorithm
- Does this reflect actual running time?
  - multiply nb. of instructions by processor speed
    - 1GHz processor ==> 10ˆ9 instructions/second
- Is this accurate?
  - Not all instructions take the same...
  - various other effects.
  - Overall, it is a very good predictor of running time in most cases.

Notations



- Notation:
  - n = size of the input to the problem
- Running time:
  - number of operations/instructions on an input of size n
  - expressed as function of n: f(n)
- For an input of size n, running time may be smaller on some inputs than on others
- Best case running time:
  - the smallest number of operations on an input of size n
- Worst-case running time:
  - the largest number of operations on an input of size n
- For any n
  - best-case running time(n) <= running time(n) <= worst-case running time (n)
- Ideally, want to compute average-case running time
  - hard to model

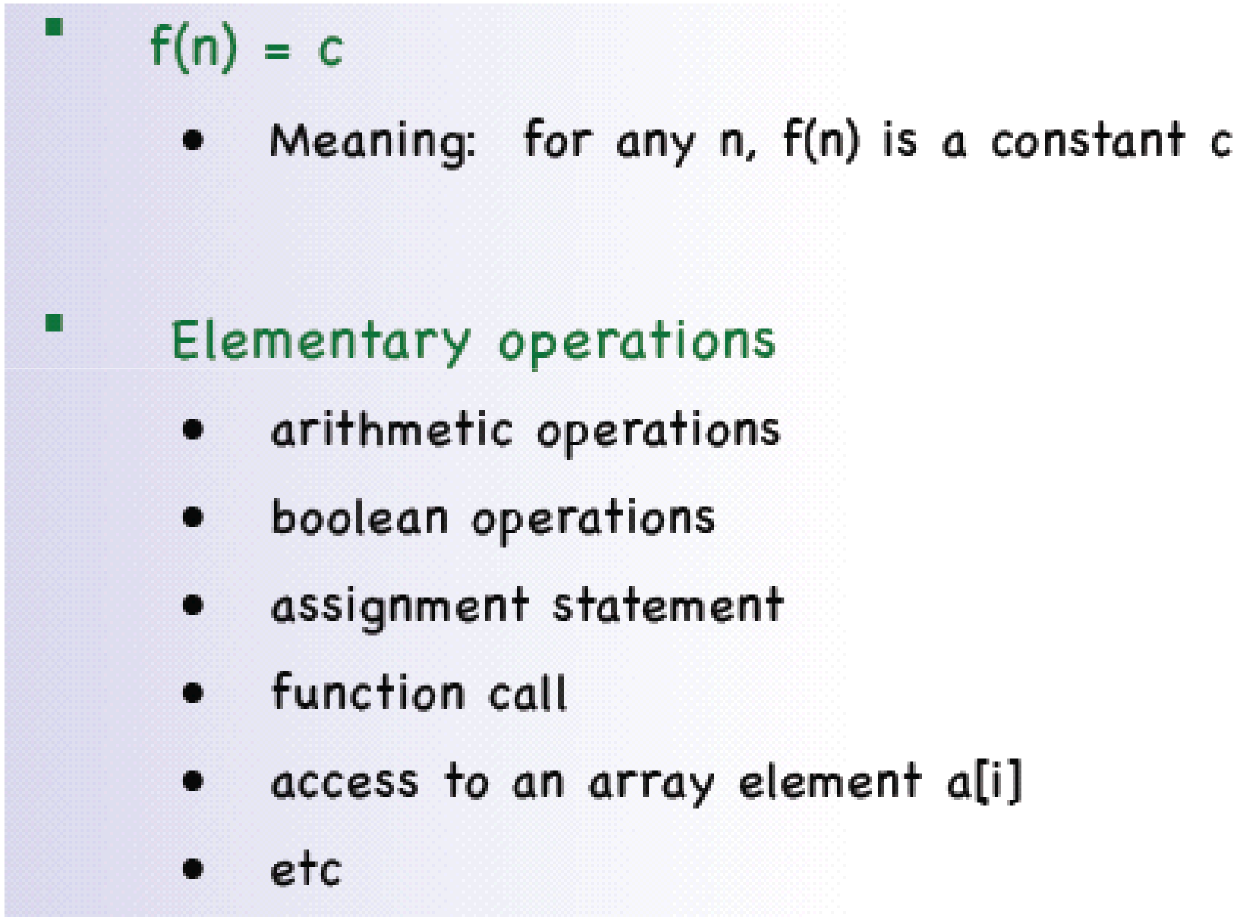
Running time



- Expressed as functions of n: f(n)
- The most common functions for running times are the following:

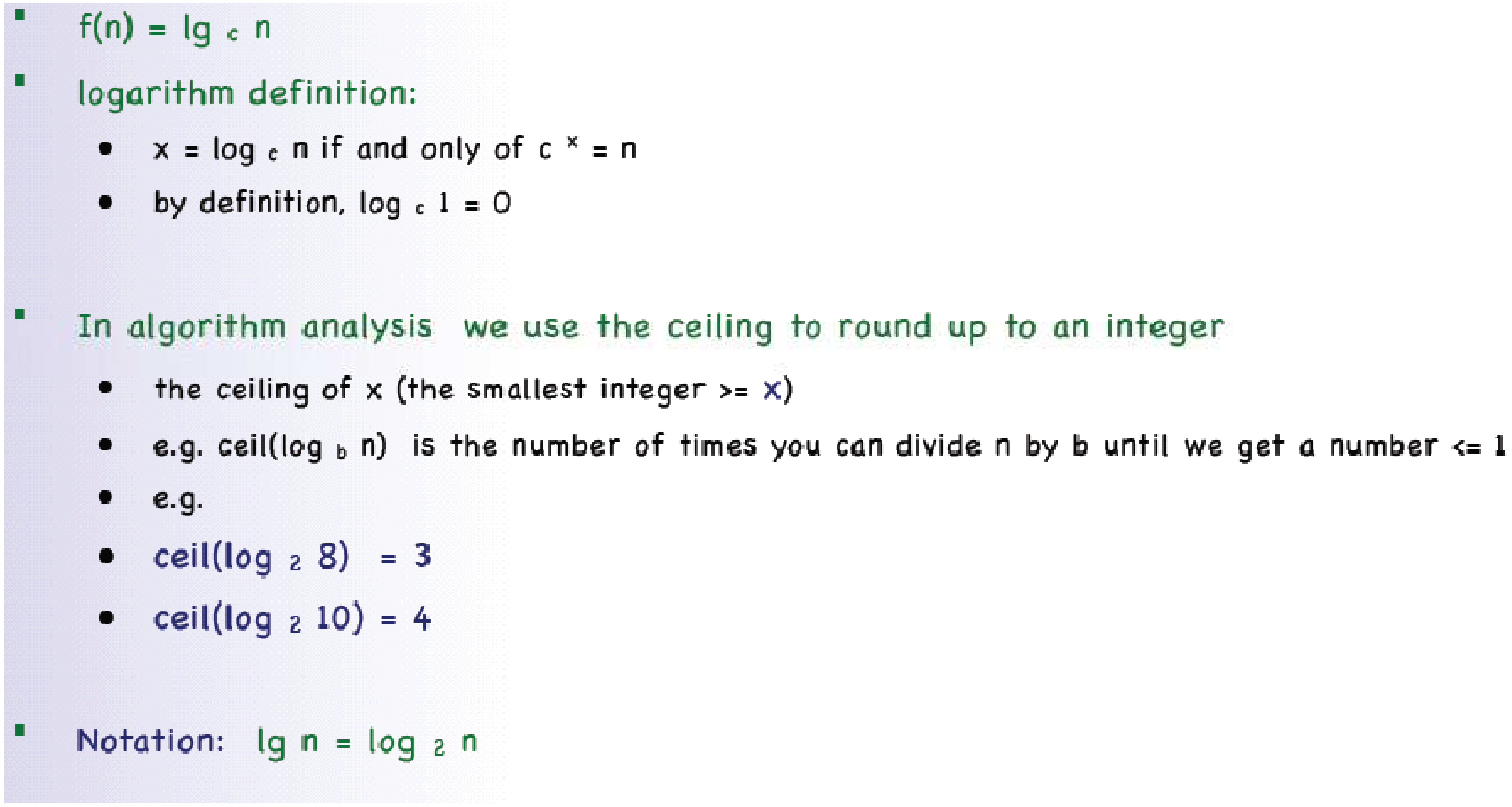
  - constant time :
    - f(n) = c
  - logarithmic time
    - f(n) = lg n
  - linear time
    - f(n) = n
  - n lg n
    - f(n) = n lg n
  - quadratic
    - f(n) = n^2
  - cubic
    - f(n) = n^3
  - exponential
    - f(n) = a^n

Constant running time O(1)



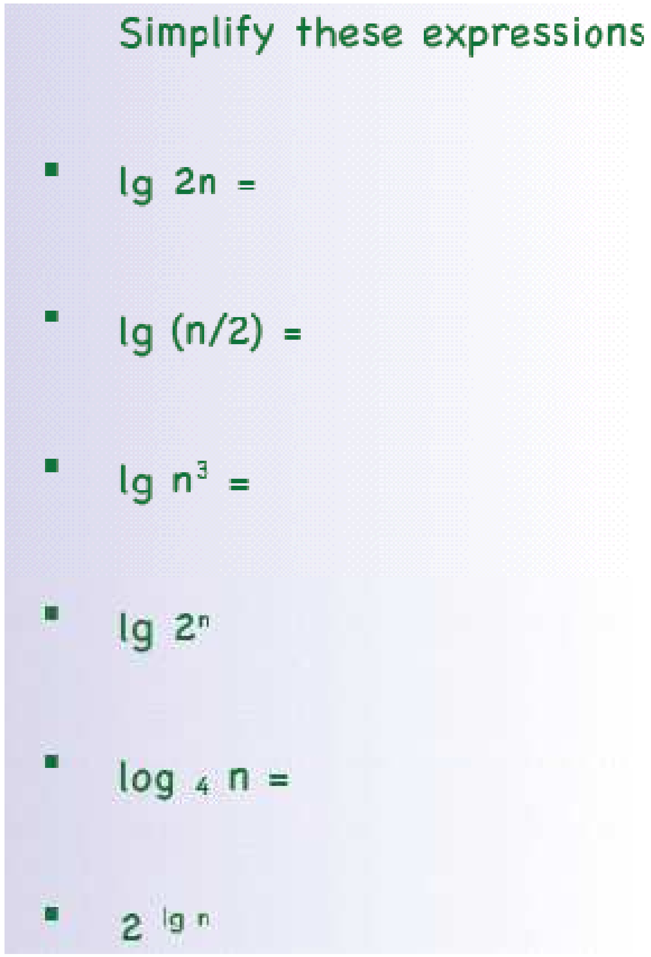
- $f(n) = c$
  - Meaning: for any n, f(n) is a constant c
- Elementary operations
  - arithmetic operations
  - boolean operations
  - assignment statement
  - function call
  - access to an array element a[i]
  - etc

Logarithmic running time



- $f(n) = \lg_c n$
- logarithm definition:
  - $x = \log_c n$ if and only of $c^x = n$
  - by definition, $\log_c 1 = 0$
- In algorithm analysis we use the ceiling to round up to an integer
  - the ceiling of x (the smallest integer >= x)
  - e.g. $\text{ceil}(\log_b n)$ is the number of times you can divide n by b until we get a number <= 1
  - e.g.
  - $\text{ceil}(\log_2 8) = 3$
  - $\text{ceil}(\log_2 10) = 4$
- Notation: $\lg n = \log_2 n$

exercise



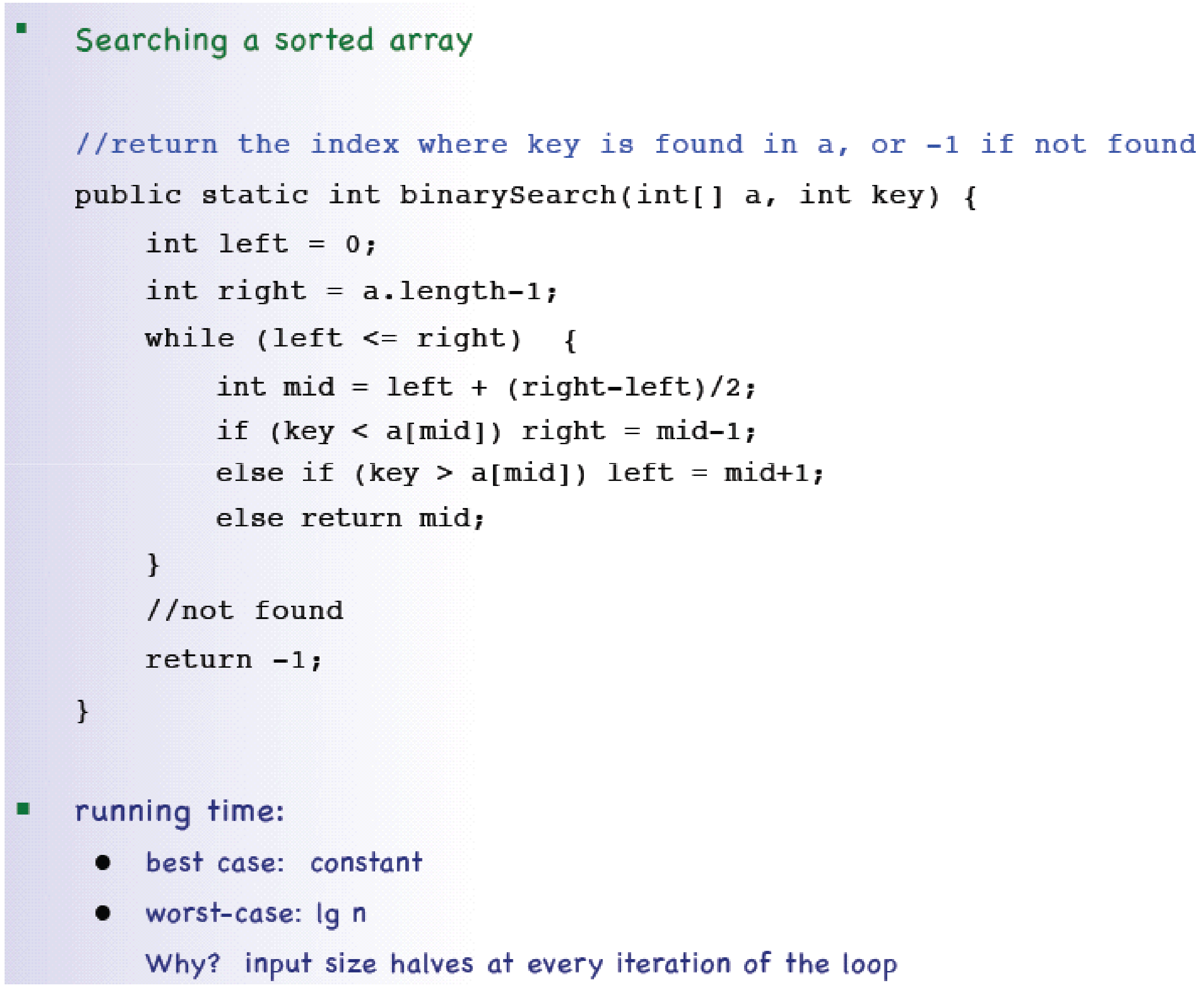
Simplify these expressions

- $\lg 2n =$
- $\lg (n/2) =$
- $\lg n^3 =$
- $\lg 2^n$
- $\log_4 n =$
- $2^{\lg n}$

Answer:

- $\lg 2n = \lg n + 1$
- $\lg (n/2) = \lg n - 1$
- $\lg n^3 = 3 \lg n$
- $\lg 2^n = n$
- $\lg_4 n = \lg n / \lg 4 = (\lg n)/2$
- $2^{\lg n} = n$

Binary Search



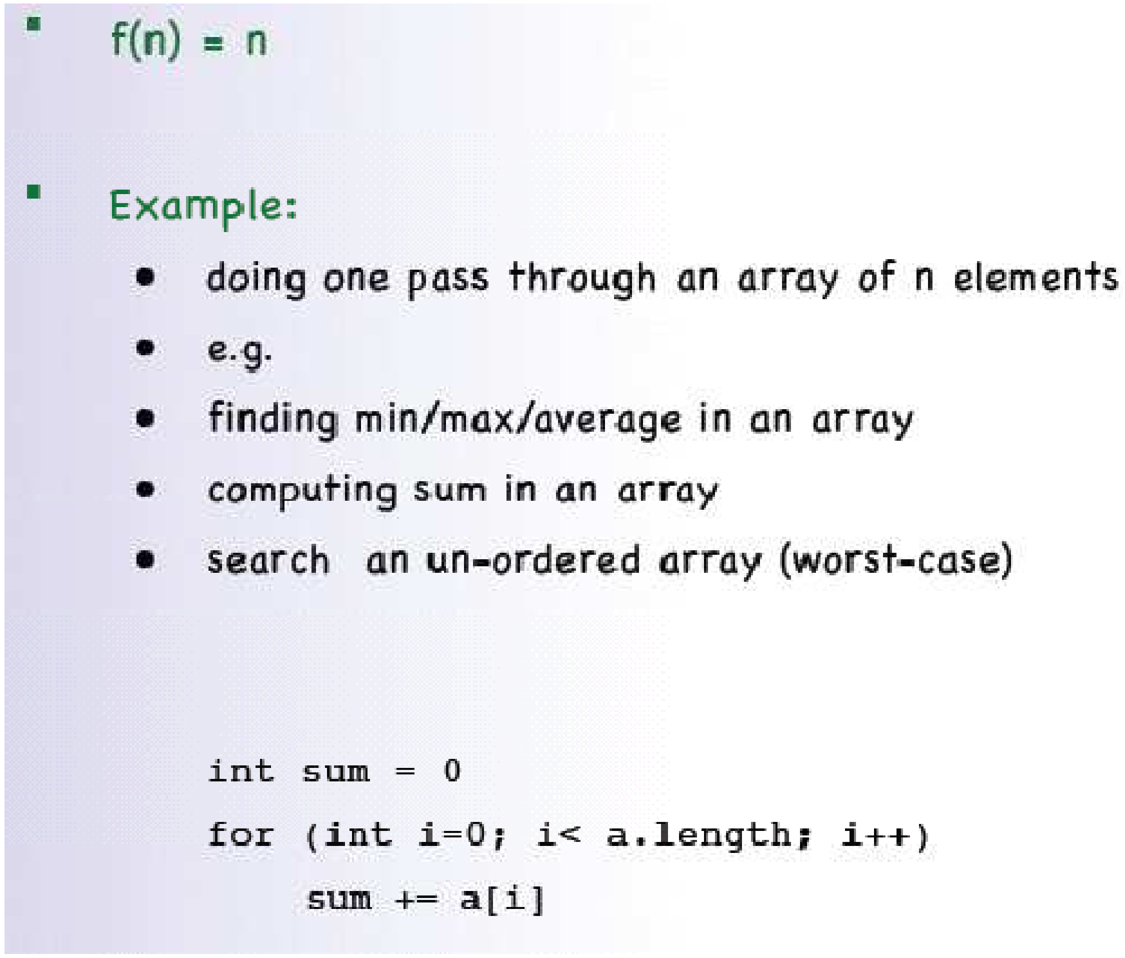
- Searching a sorted array

```
//return the index where key is found in a, or -1 if not found
public static int binarySearch(int[] a, int key) {
    int left = 0;
    int right = a.length-1;
    while (left <= right)  {
        int mid = left + (right-left)/2;
        if (key < a[mid]) right = mid-1;
        else if (key > a[mid]) left = mid+1;
        else return mid;
    }
    //not found
    return -1;
}
```

- running time:
  - best case: constant
  - worst-case: lg n

    Why? input size halves at every iteration of the loop

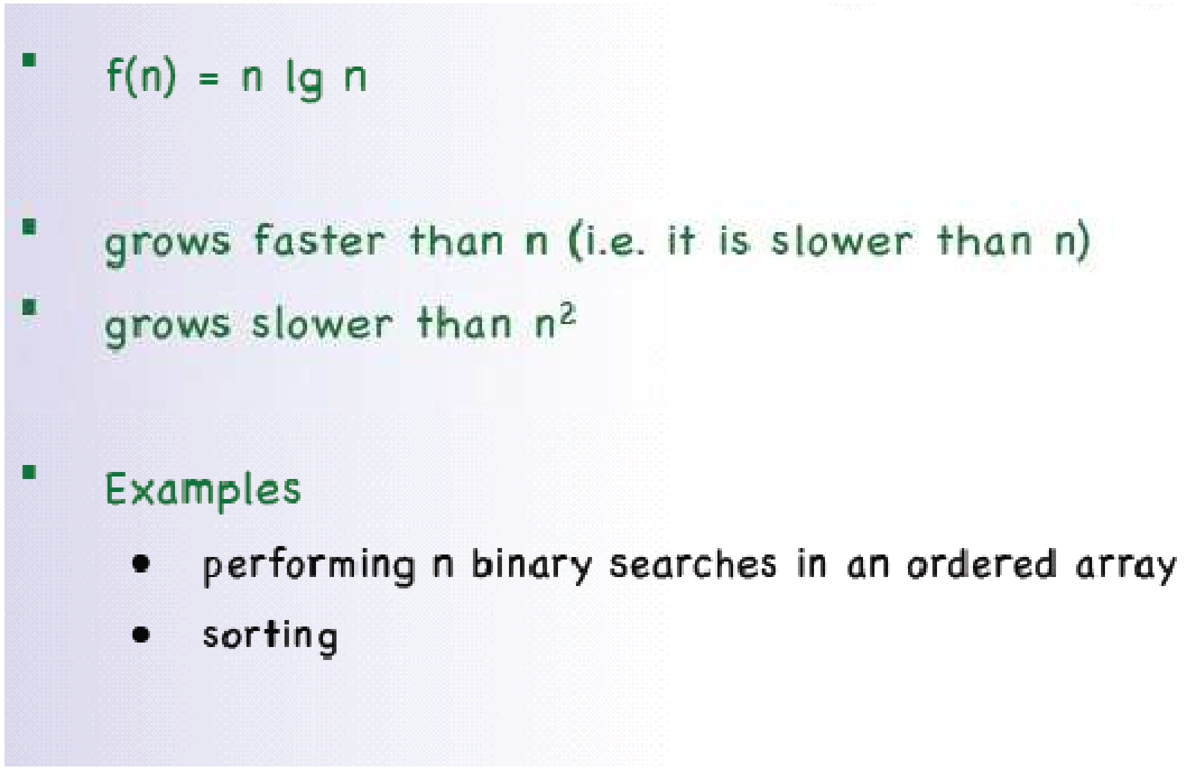
Linear running time example



- f(n) = n
- Example:
  - doing one pass through an array of n elements
  - e.g.
  - finding min/max/average in an array
  - computing sum in an array
  - search an un-ordered array (worst-case)

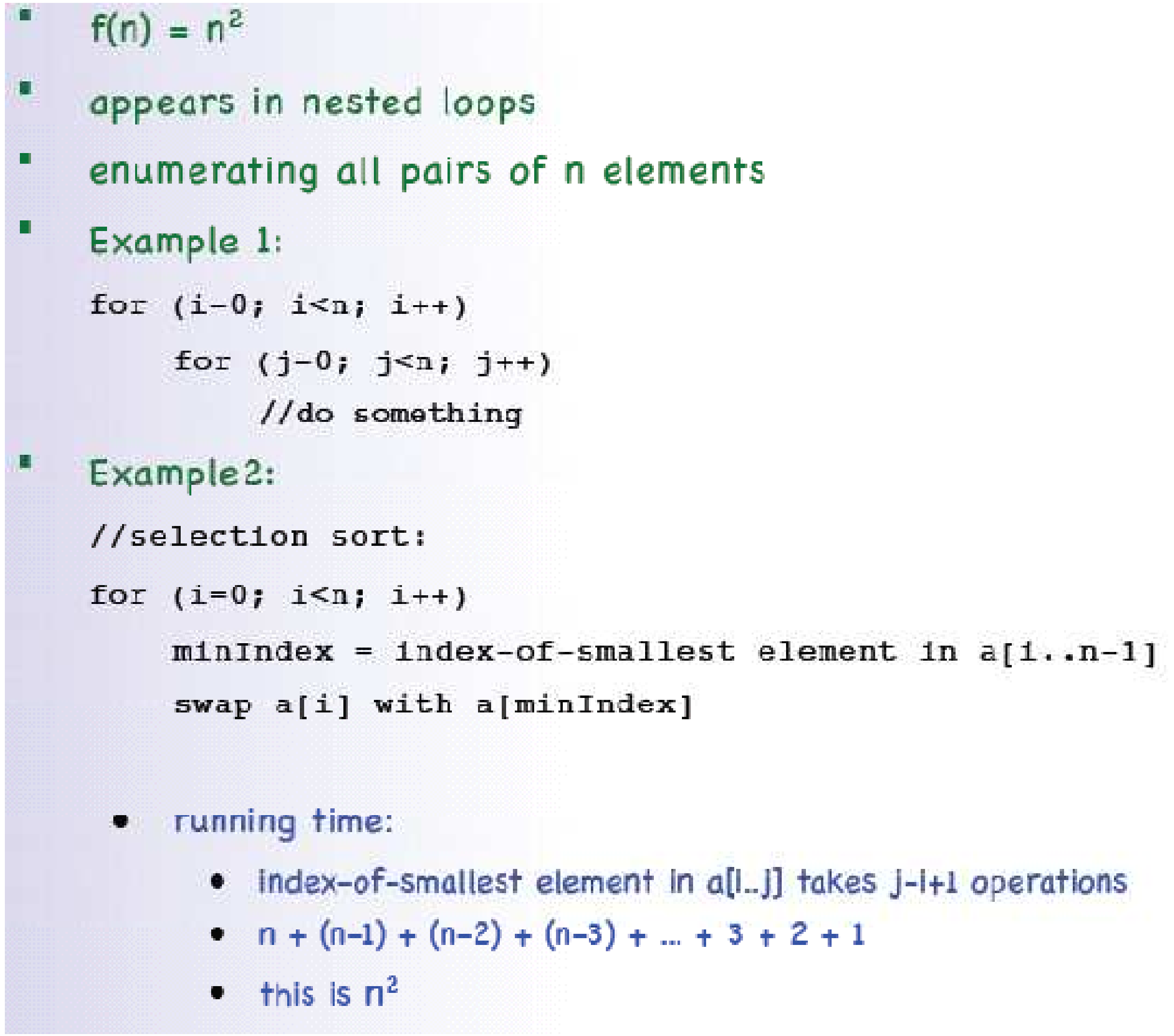
```
int sum = 0
for (int i=0; i< a.length; i++)
    sum += a[i]
```

O(n lg n) running time



- $f(n) = n \lg n$
- grows faster than n (i.e. it is slower than n)
- grows slower than n^2
- Examples
  - performing n binary searches in an ordered array
  - sorting

Quadratic running time - O(n^2)



- $f(n) = n^2$
- appears in nested loops
- enumerating all pairs of n elements
- Example 1:

```
for (i-0; i<n; i++)
    for (j-0; j<n; j++)
        //do something
```

- Example2:

```
//selection sort:
for (i=0; i<n; i++)
    minIndex = index-of-smallest element in a[i..n-1]
    swap a[i] with a[minIndex]
```

  - running time:
    - index-of-smallest element in a[i..j] takes j-i+1 operations
    - $n + (n-1) + (n-2) + (n-3) + \ldots + 3 + 2 + 1$
    - this is n^2

Useful formula

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{1}{2} n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{1}{4} \left[n(n+1) \right]^2$$

Useful formula

$$\sum_{i=1}^{n} i^k = 1^k + 2^k + \ldots + n^k = \Theta(n^{k+1})$$

Cubic running time $O(n^3)$

- Cubic running time: $f(n) = n^3$
- In general, a polynomial running time is:

  $f(n) = n^d, d>0$

- Examples:
  - nested loops

Exponential running time $O(2^n)$

- Exponential running time: $f(n) = a^n$, $a > 1$
- Examples:

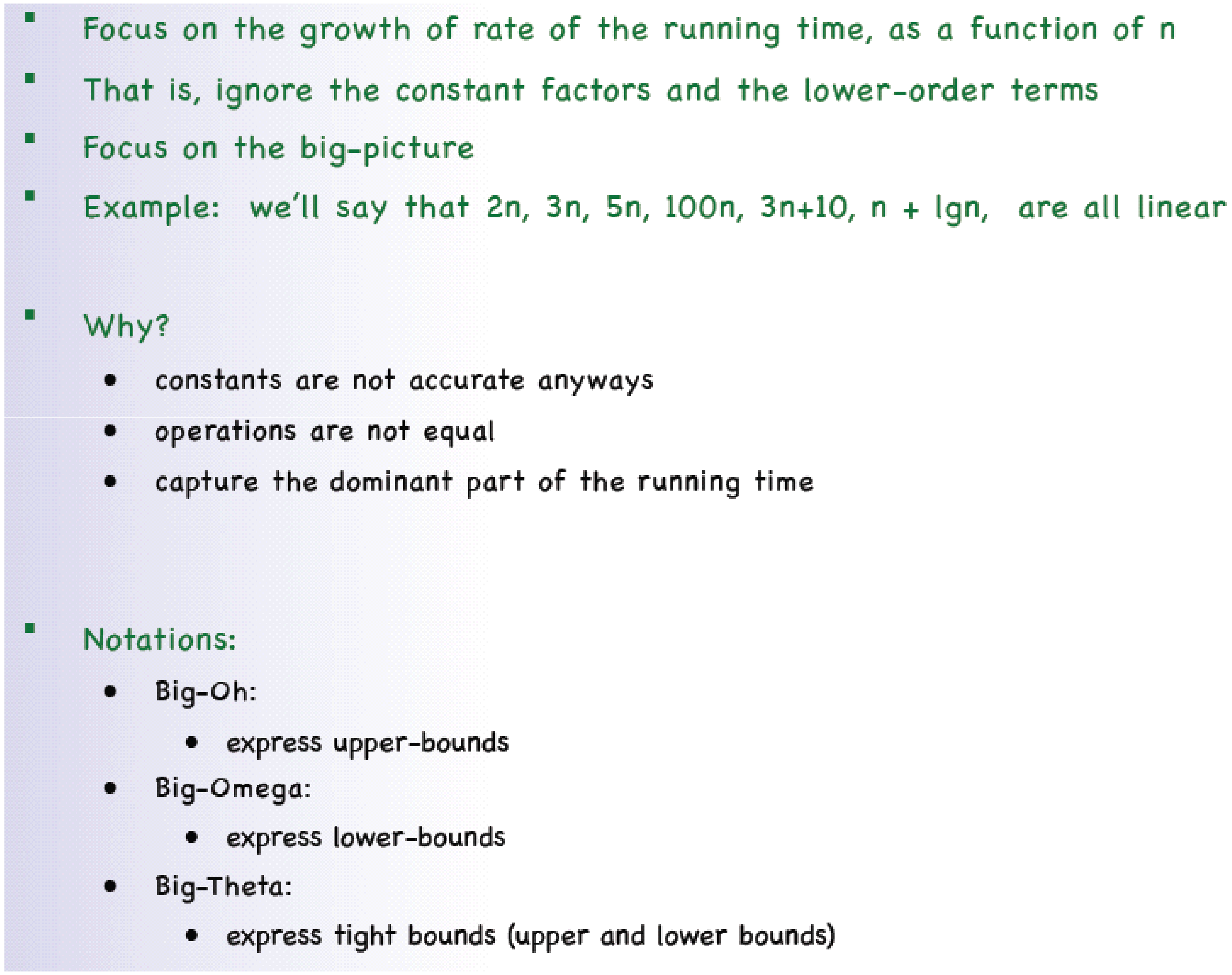
  running time of Tower of Hanoi

  moving n disks from A to B requires at least 2^n moves; which means it requires at least this much time

Comparing Growth-Rates

$1 < \lg n < n < n \lg n < n^2 < n^3 < a^n$

Asymptotic analysis

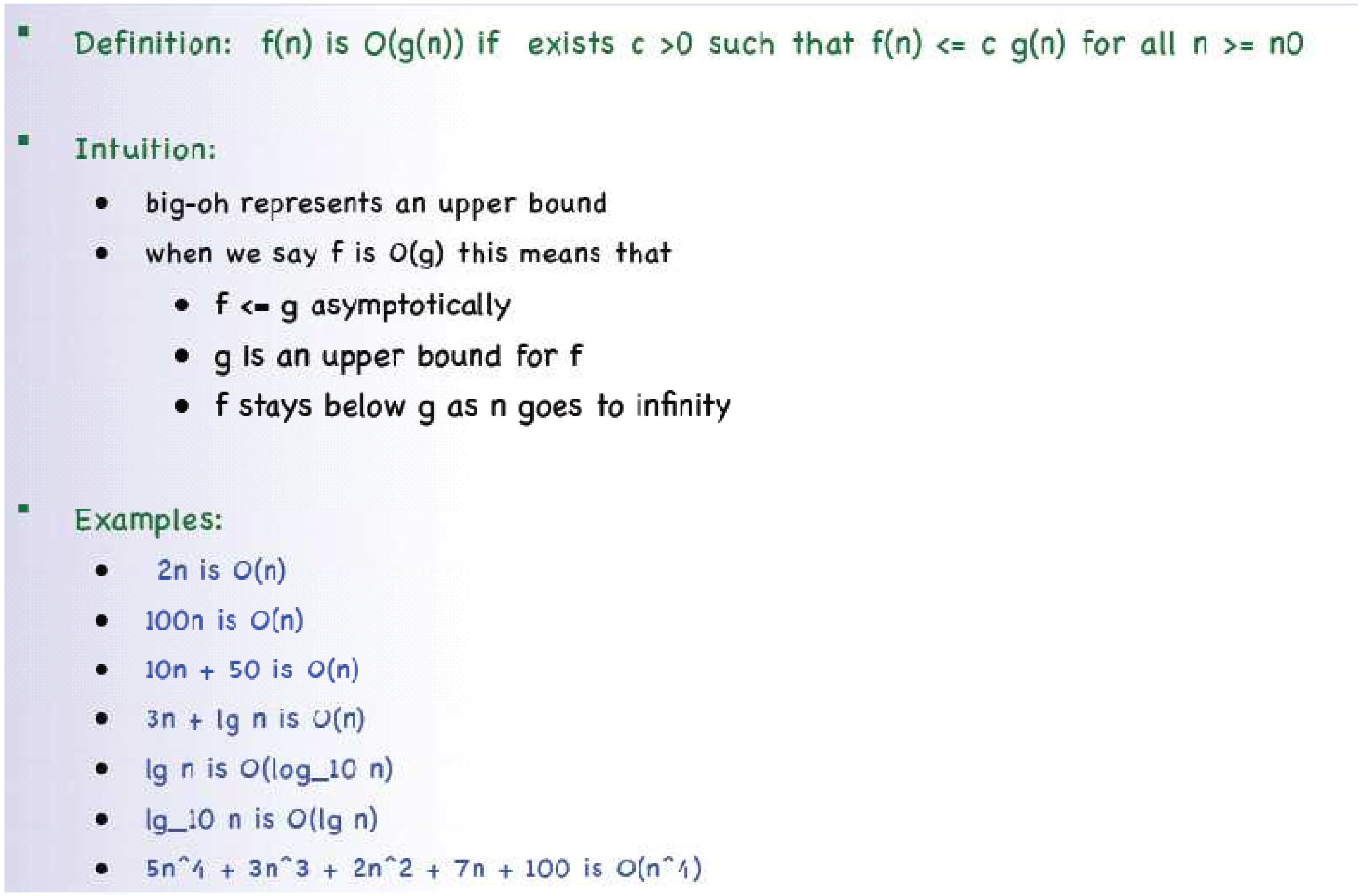


- Focus on the growth of rate of the running time, as a function of n
- That is, ignore the constant factors and the lower-order terms
- Focus on the big-picture
- Example: we'll say that 2n, 3n, 5n, 100n, 3n+10, n + lgn, are all linear

- Why?
  - constants are not accurate anyways
  - operations are not equal
  - capture the dominant part of the running time

- Notations:
  - Big-Oh:
    - express upper-bounds
  - Big-Omega:
    - express lower-bounds
  - Big-Theta:
    - express tight bounds (upper and lower bounds)

Big-oh O-notation

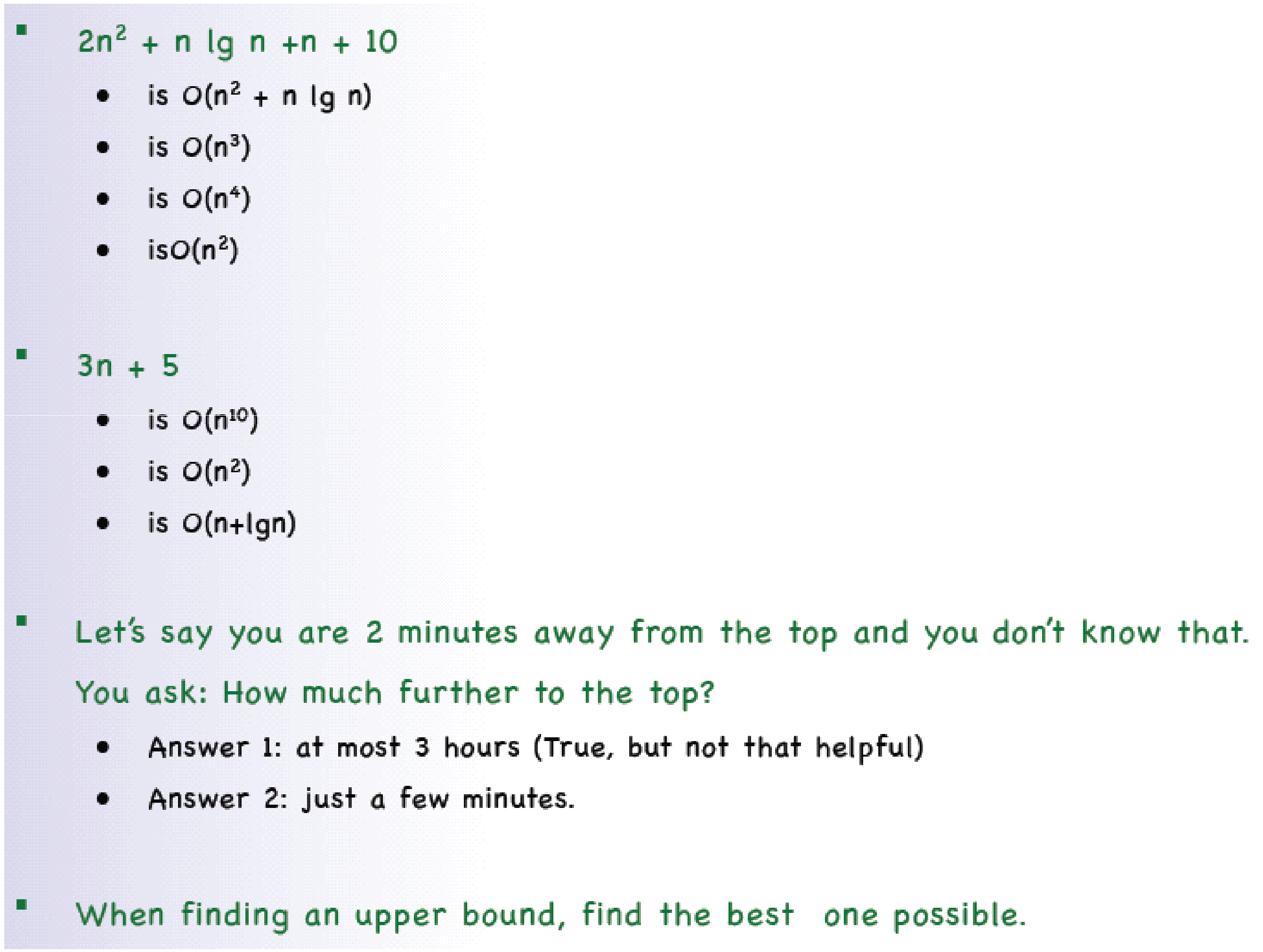


- Definition: f(n) is O(g(n)) if exists c >0 such that f(n) <= c g(n) for all n >= n0

- Intuition:
  - big-oh represents an upper bound
  - when we say f is O(g) this means that
    - f <= g asymptotically
    - g is an upper bound for f
    - f stays below g as n goes to infinity

- Examples:
  - 2n is O(n)
  - 100n is O(n)
  - 10n + 50 is O(n)
  - 3n + lg n is O(n)
  - lg n is O(log_10 n)
  - lg_10 n is O(lg n)
  - 5n^4 + 3n^3 + 2n^2 + 7n + 100 is O(n^4)

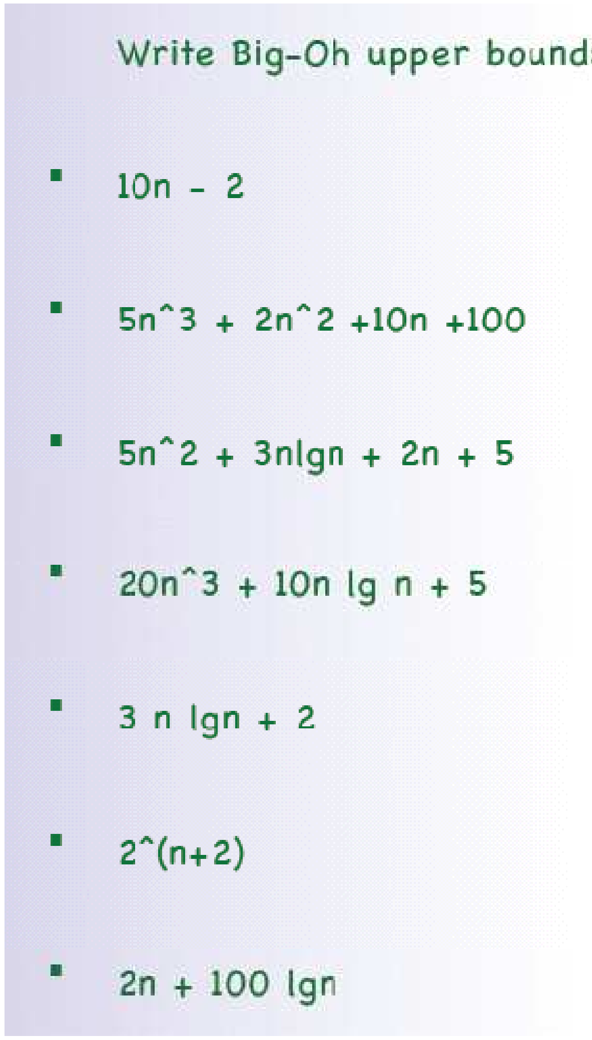
More examples



- $2n^2 + n \lg n + n + 10$
  - is $O(n^2 + n \lg n)$
  - is $O(n^3)$
  - is $O(n^4)$
  - is$O(n^2)$
- $3n + 5$
  - is $O(n^{10})$
  - is $O(n^2)$
  - is $O(n + \lg n)$
- Let's say you are 2 minutes away from the top and you don't know that.

  You ask: How much further to the top?
  - Answer 1: at most 3 hours (True, but not that helpful)
  - Answer 2: just a few minutes.
- When finding an upper bound, find the best one possible.

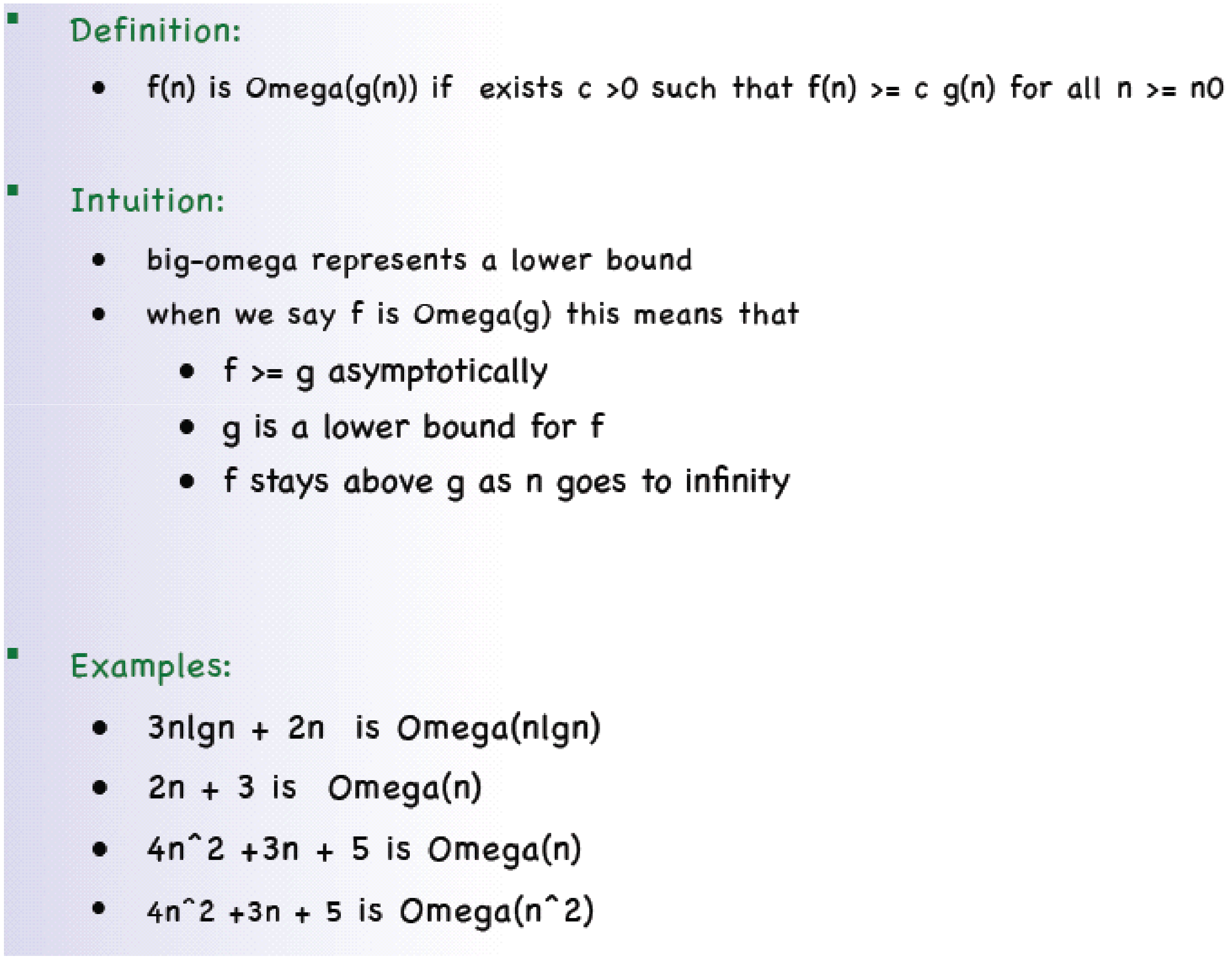
Want more exercises/examples?



Write Big-Oh upper bounds for each of the following.

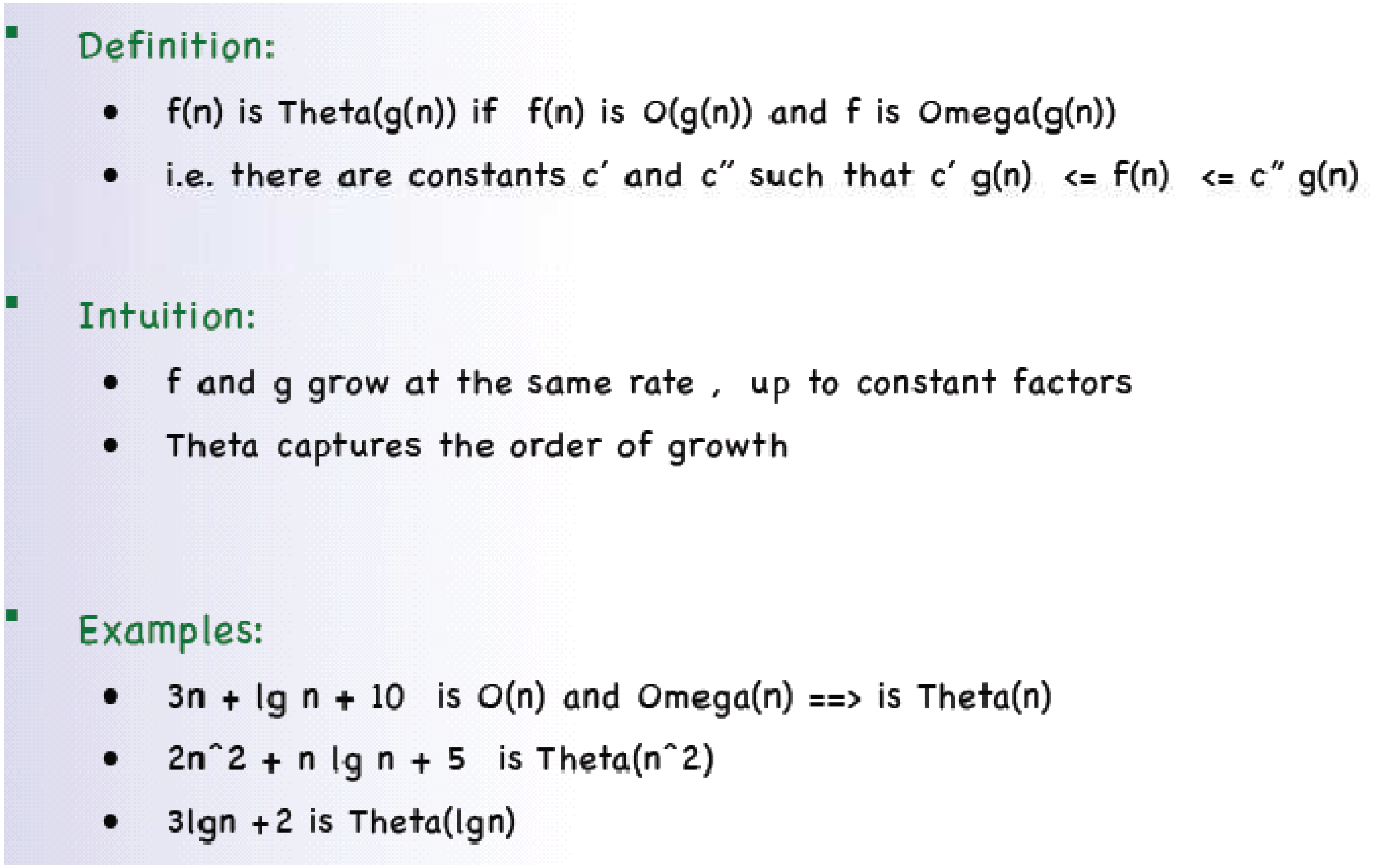
- 10n - 2
- 5n^3 + 2n^2 +10n +100
- 5n^2 + 3nlgn + 2n + 5
- 20n^3 + 10n lg n + 5
- 3 n lgn + 2
- 2^(n+2)
- 2n + 100 lgn

Big-Omega **Ω-notation**



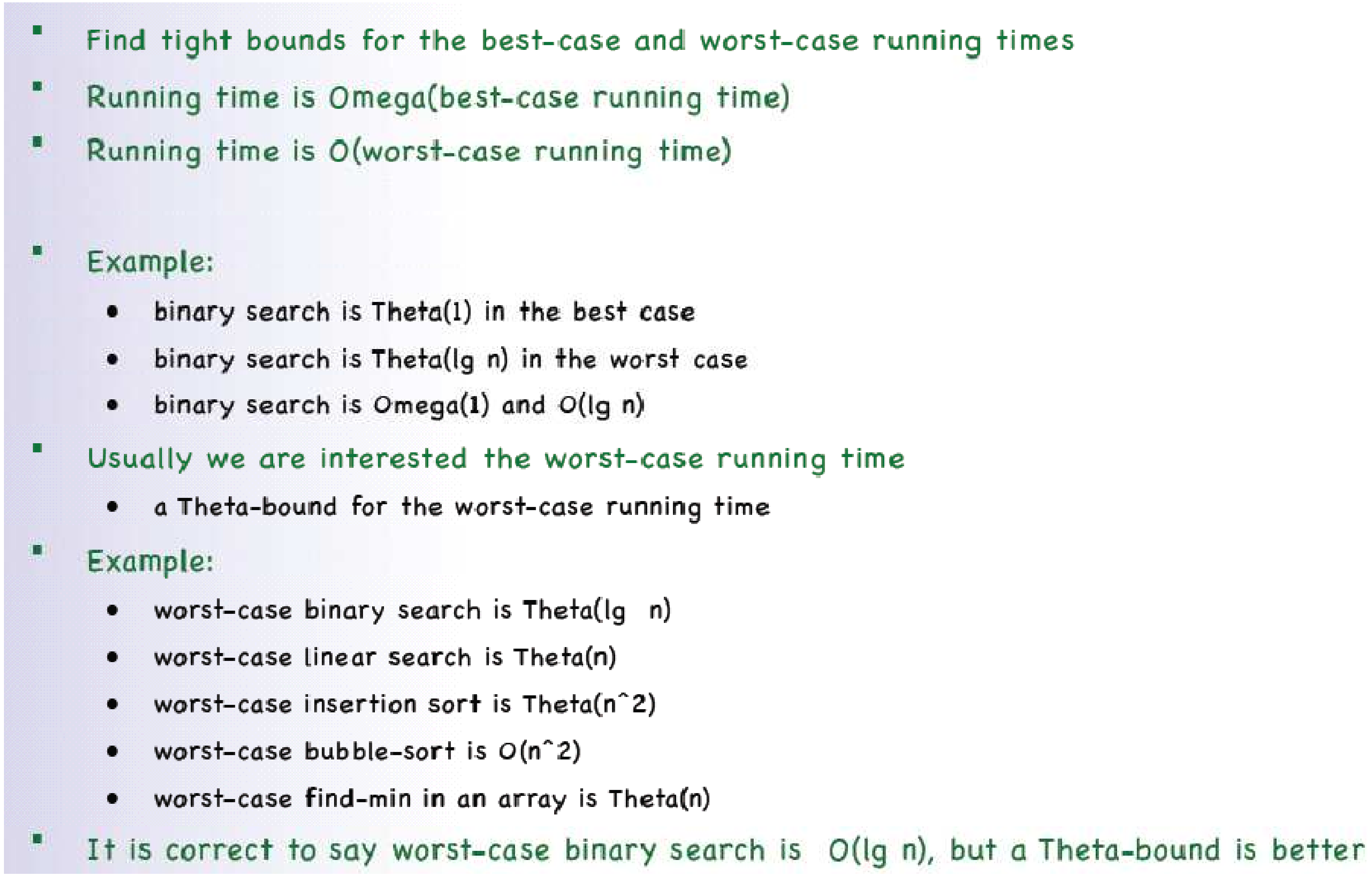
- Definition:
  - f(n) is Omega(g(n)) if exists c >0 such that f(n) >= c g(n) for all n >= n0
- Intuition:
  - big-omega represents a lower bound
  - when we say f is Omega(g) this means that
    - f >= g asymptotically
    - g is a lower bound for f
    - f stays above g as n goes to infinity
- Examples:
  - 3nlgn + 2n is Omega(nlgn)
  - 2n + 3 is Omega(n)
  - 4n^2 +3n + 5 is Omega(n)
  - 4n^2 +3n + 5 is Omega(n^2)

Θ-notation



- **Definition:**
  - f(n) is Theta(g(n)) if f(n) is O(g(n)) and f is Omega(g(n))
  - i.e. there are constants c′ and c″ such that c′ g(n) <= f(n) <= c″ g(n)
- **Intuition:**
  - f and g grow at the same rate , up to constant factors
  - Theta captures the order of growth
- **Examples:**
  - 3n + lg n + 10 is O(n) and Omega(n) ==> is Theta(n)
  - 2n^2 + n lg n + 5 is Theta(n^2)
  - 3lgn +2 is Theta(lgn)

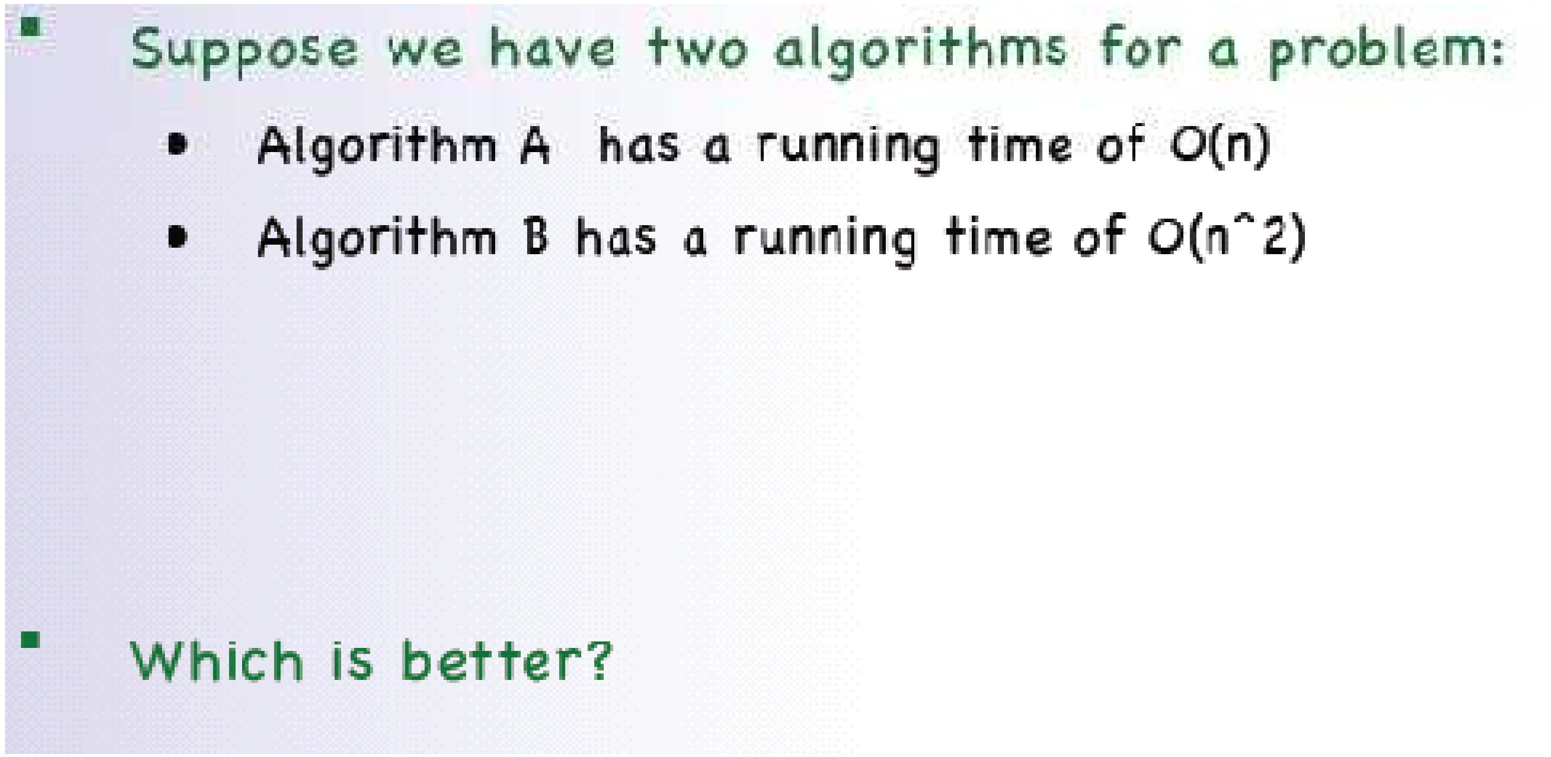
Asymptotic Analysis



- Find tight bounds for the best-case and worst-case running times
- Running time is Omega(best-case running time)
- Running time is O(worst-case running time)

- Example:
  - binary search is Theta(1) in the best case
  - binary search is Theta(lg n) in the worst case
  - binary search is Omega(1) and O(lg n)
- Usually we are interested the worst-case running time
  - a Theta-bound for the worst-case running time
- Example:
  - worst-case binary search is Theta(lg n)
  - worst-case linear search is Theta(n)
  - worst-case insertion sort is Theta(n^2)
  - worst-case bubble-sort is O(n^2)
  - worst-case find-min in an array is Theta(n)
- It is correct to say worst-case binary search is O(lg n), but a Theta-bound is better

Asymptotic Analysis



- Suppose we have two algorithms for a problem:
  - Algorithm A has a running time of O(n)
  - Algorithm B has a running time of O(n^2)
- Which is better?

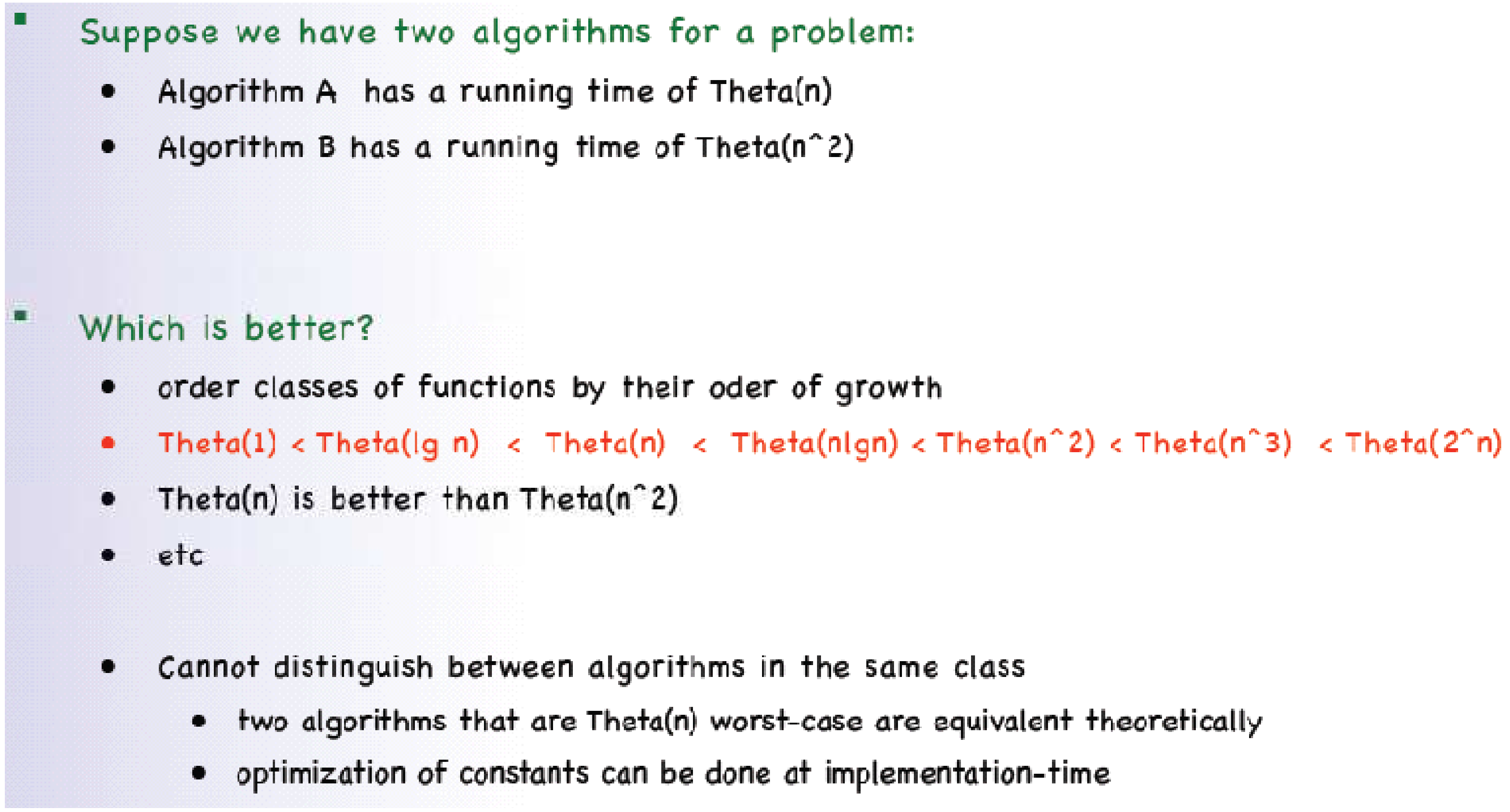
Asymptotic Analysis

Actually, we can’t tell.

O(n) and O(2^n) only give the upper bounds.

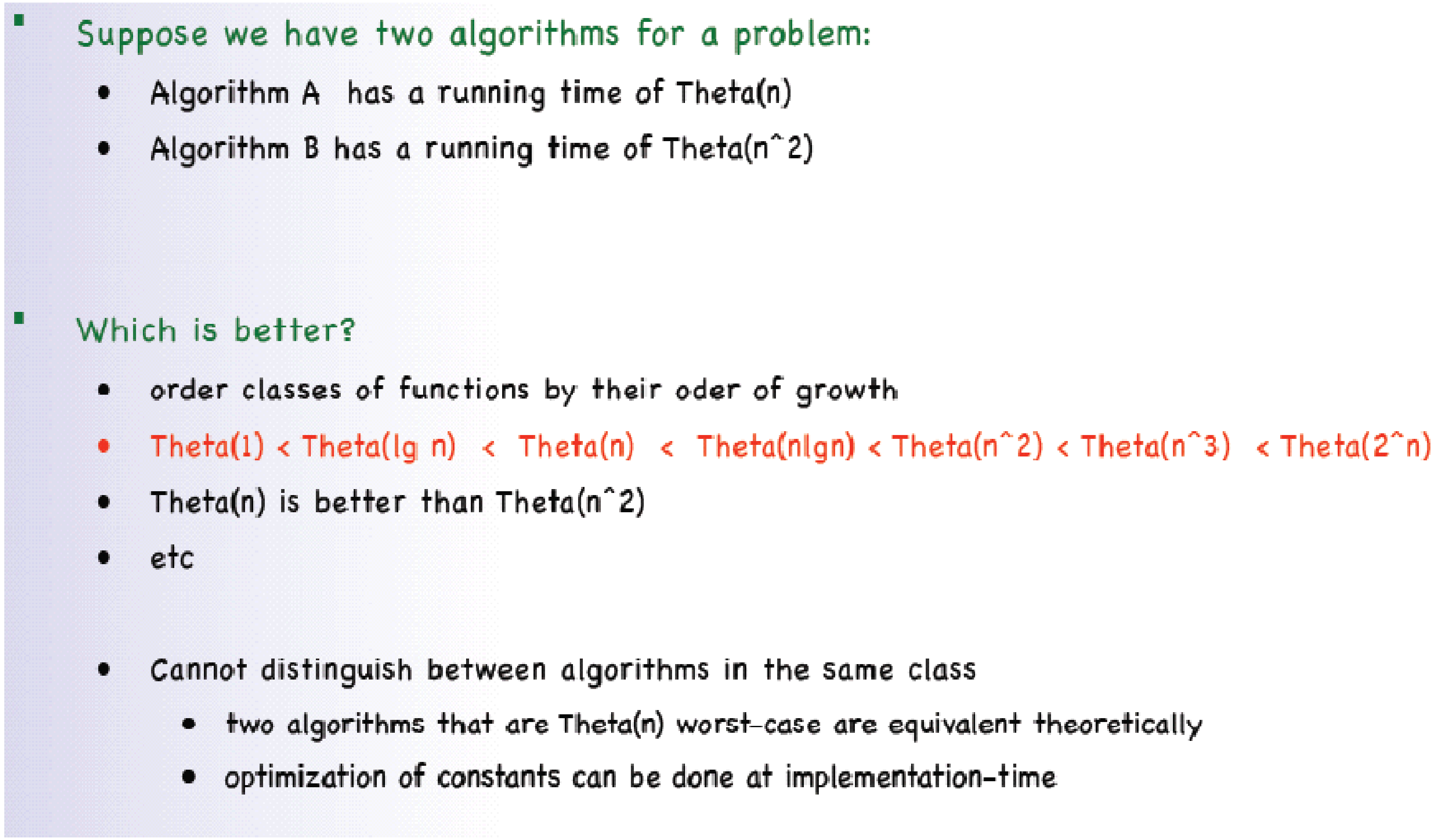
Usually, we tends to say O(n) algorithm is better (runs faster) than O(2^n) algorithm.

Asymptotic Analysis



- Suppose we have two algorithms for a problem:
  - Algorithm A has a running time of Theta(n)
  - Algorithm B has a running time of Theta(n^2)
- Which is better?
  - order classes of functions by their oder of growth
  - Theta(1) < Theta(lg n) < Theta(n) < Theta(nlgn) < Theta(n^2) < Theta(n^3) < Theta(2^n)
  - Theta(n) is better than Theta(n^2)
  - etc
  - Cannot distinguish between algorithms in the same class
    - two algorithms that are Theta(n) worst-case are equivalent theoretically
    - optimization of constants can be done at implementation-time

Asymptotic Analysis



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  - Theta(1) < Theta(lg n) < Theta(n) < Theta(nlgn) < Theta(n^2) < Theta(n^3) < Theta(2^n)
  - Theta(n) is better than Theta(n^2)
  - etc

  - Cannot distinguish between algorithms in the same class
    - two algorithms that are Theta(n) worst-case are equivalent theoretically
    - optimization of constants can be done at implementation-time

Order of growth matters

Example:

- Say $n = 10^9$ (1 billion elements)
  - 10 MHz computer ==> 1 instruction takes 10^{-7}sec seconds
    - Binary search would take

      $\Theta(\lg n) = \lg 10^9 \times 10^{-7}$sec $= 30 \times 10^{-7}$sec = 3 μsec
  - Sequential search would take

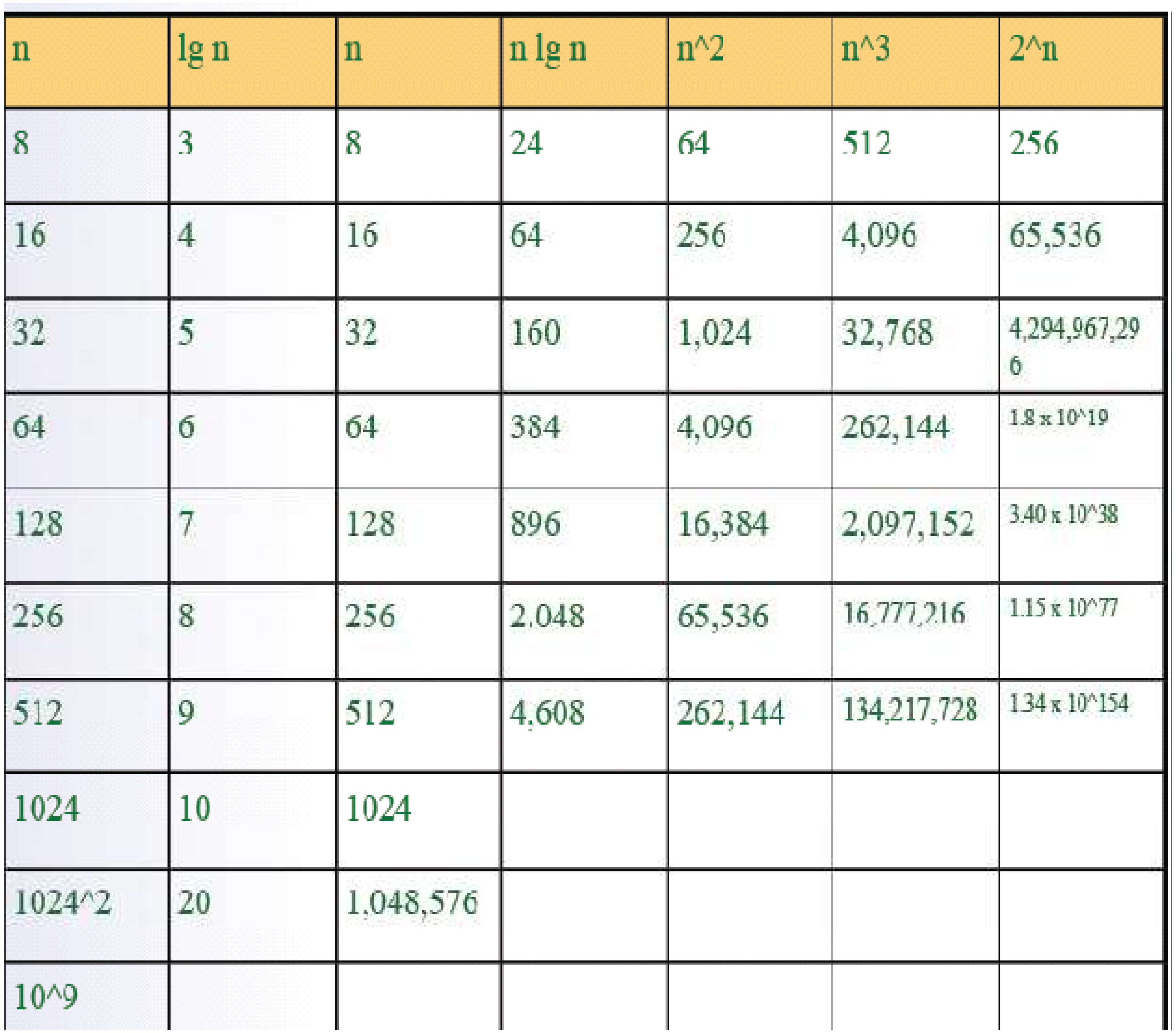
    $\Theta(n) = 10^9 \times 10^{-7}$sec = 100 seconds
  - Finding all pairs of elements would take

    $\Theta(n^2) = (10^9)^2 \times 10^{-7}$sec $= 10^{11}$ seconds = 3170 years

Imagine how much time it would take for an $\Theta(n^3)$-

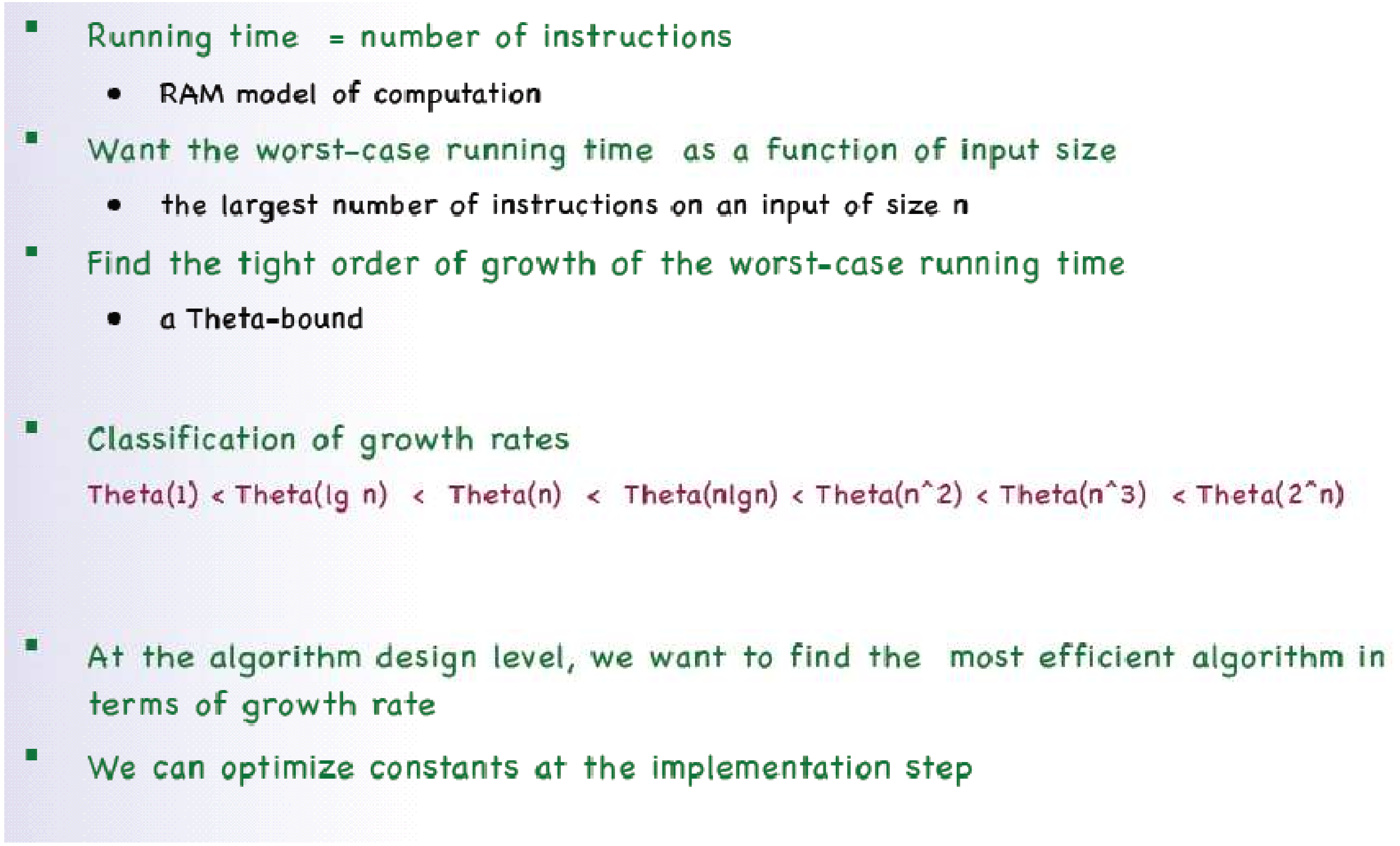
or $\Theta(2^n)$- running time algorithm.

Order of growth matters



n	lg n	n	n lg n	n^2	n^3	2^n
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	1.8 x 10^19
128	7	128	896	16,384	2,097,152	3.40 x 10^38
256	8	256	2.048	65,536	16,777,216	1.15 x 10^77
512	9	512	4.608	262,144	134,217,728	1.34 x 10^154
1024	10	1024				
1024^2	20	1,048,576				
10^9						

Conclusion



- Running time = number of instructions
  - RAM model of computation
- Want the worst-case running time as a function of input size
  - the largest number of instructions on an input of size n
- Find the tight order of growth of the worst-case running time
  - a Theta-bound

- Classification of growth rates

  Theta(1) < Theta(lg n) < Theta(n) < Theta(nlgn) < Theta(n^2) < Theta(n^3) < Theta(2^n)

- At the algorithm design level, we want to find the most efficient algorithm in terms of growth rate
- We can optimize constants at the implementation step