Recurrence Relations

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n is terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with
$n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if it terms satisfy the recurrence relation.

Recurrence Relations

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relations

Example:

Consider the recurrence relation
$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

Is the sequence $\{a_n\}$ with $a_n=3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that
$2a_{n-1} - a_{n-2} = 2(3(n - 1)) - 3(n - 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n=3n$ is a solution of the recurrence relation.

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Modeling with Recurrence Relations

Example:

Someone deposits $10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

Modeling with Recurrence Relations

We can derive the following **recurrence relation**:

$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.

The initial condition is $P_0 = 10,000$.

Then we have:

$P_1 = 1.05P_0$

$P_2 = 1.05P_1 = (1.05)^2P_0$

$P_3 = 1.05P_2 = (1.05)^3P_0$

…

$P_n = 1.05P_{n-1} = (1.05)^nP_0$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

$P_{30} = (1.05)^{30} \cdot 10{,}000 = 43{,}219.42$

After 30 years, the account contains \$43,219.42.

Modeling with Recurrence Relations

Another example:

Let a_n denote the number of bit strings of length n that do not have two consecutive 0s ("valid strings"). Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$.

Solution:

Idea: The number of valid strings equals the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

Modeling with Recurrence Relations

- Let us assume that $n \geq 3$, so that the string contains at least 3 bits.
- Let us further assume that we know the number a_{n-1} of valid strings of length $(n – 1)$.
- Then how many valid strings of length n are there, if the string ends with a 1?
- There are a_{n-1} such strings, namely the set of valid strings of length $(n – 1)$ with a 1 appended to them.
- **Note:** Whenever we append a 1 to a valid string, that string remains valid.

Modeling with Recurrence Relations

Now we need to know: How many valid strings of length n are there, if the string ends with a **0**?

Valid strings of length n ending with a 0 **must have a 1 as their (n – 1)st bit** (otherwise they would end with 00 and would not be valid).

And what is the number of valid strings of length (n – 1) that end with a 1?

We already know that there are a_{n-1} strings of length n that end with a 1.

Therefore, there are a_{n-2} strings of length (n – 1) that end with a 1.

Modeling with Recurrence Relations

So there are a_{n-2} valid strings of length n that end with a 0 (all valid strings of length (n – 2) with 10 appended to them).

As we said before, the number of valid strings is the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

That gives us the following **recurrence relation**:

$a_n = a_{n-1} + a_{n-2}$

Modeling with Recurrence Relations

What are the **initial conditions**?

$a_1 = 2$ (0 and 1)

$a_2 = 3$ (01, 10, and 11)

$a_3 = a_2 + a_1 = 3 + 2 = 5$

$a_4 = a_3 + a_2 = 5 + 3 = 8$

$a_5 = a_4 + a_3 = 8 + 5 = 13$

…

This sequence satisfies the same recurrence relation as the **Fibonacci sequence**.

Since $a_1 = f_3$ and $a_2 = f_4$, we have $a_n = f_{n+2}$.

Solving Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$,
Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$.

Solving Recurrence Relations

Examples:

The recurrence relation $P_n = (1.05)P_{n-1}$
is a linear homogeneous recurrence relation of **degree one**.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$
is a linear homogeneous recurrence relation of **degree two**.

The recurrence relation $a_n = a_{n-5}$
is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Basically, when solving such recurrence relations, we try to find solutions of the form **$a_n = r^n$**, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if
$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Let us consider linear homogeneous recurrence relations of **degree two**.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.
Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

(See Text pp. 462 and 463 for the proof of Theorem 1.)

Solving Recurrence Relations

Example: What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$a_n = \alpha_1 2^n + \alpha_2(-1)^n$ for some constants α_1 and α_2.

Solving Recurrence Relations

Given the equation $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$

Solving these two equations gives us

$\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$a_n = 3 \cdot 2^n - (-1)^n$.

Solving Recurrence Relations

$a_n = r^n$ is a solution of the linear homogeneous recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$

if and only if

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

Example: Give an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

The characteristic equation is $r^2 - r - 1 = 0$.

Its roots are

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2.

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$$

Solving Recurrence Relations

The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}\ ,\ \ \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations

But what happens if the characteristic equation has only one root?

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.
A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if
$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for n = 0, 1, 2, …, where α_1 and α_2 are constants.

Solving Recurrence Relations

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$a_n = 3^n + n3^n$.

Divide and Conquer Algorithm

An algorithm that solves a problem by first divides the problem into one or more instance of the same problem of smaller size, then solves each of the smaller subproblem and obtains the solution to the original problem from those solutions of the smaller subproblems.

Some well-known divide-and-conquer algorithms :

MergeSort, Quicksort , Binary Search

Divide and Conquer Recurrence Relation

A recursive algorithm divides a problem of size n into a subproblems, where each subproblem of of size n/b (for simplicity, let us assume that n is some power of b.)

Also, suppose that a total of f(n) extra operations are required in the conquer step to combine the solutions of the subproblems into a solution of the original problem.

$$T(n) = a\,T(n/b) + f(n)$$

Master Theorem

Let f be an increasing function that satisfies the recurrence relation

$$T(n) = a\,T(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, $b>1$ is an integer, and c and d are real numbers with $c > 0$ and $d \geq 0$. Then $T(n)$ is

a)	$O(n^d)$	***if*** $a < b^d$
b)	$O(n^d \log n)$	***if*** $a = b^d$
c)	$O(n^{\log_b a})$	***if*** $a > b^d$

Master Theorem in more general form

Let $a \geq 1$ and $b > 1$ be constants, let f(n) be a function , and let T(n) be defined on the nonnegative integers by the recurrence

T(n) = aT(n/b) + f(n), then T(n) =

$\Theta(n^{\log_b a})$ if f(n) = $O(n^{\log_b a-\varepsilon})$ for some constant $\varepsilon > 0$.

$\Theta(n^{\log_b a} \lg n)$ if f(n) = $\Theta(n^{\log_b a})$

$\Theta(f(n))$ if f(n) = $\Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon > 0$,

and if $af(n/b) \leq cf(n)$ for some constant $c < 1$and all sufficiently large n

Example: Merge-Sort

- The recurrence equation is:

$$T(n) = 2T(n/2) + n$$

Here, $a = 2$, $b = 2$, and $f(n) = n$ and $c = 1$

Case 2 applies

$$T(n) = \Theta(n^{\log_2 2} \lg n)$$

Conclusion:

$$T(n) = \Theta(n \lg n)$$

Useful math to know…

Logarithms

$\log_c(ab) = \log_c a + \log_c b$

$\log_b a^n = n\log_b a$

$\log_b a^n = n\log_b a$

$\log_b(1/a) = -\log_b a$

$\log_b a = 1 / \log_a b$

$$a = b^{\log_b a}$$

$$a^{\log_b c} = c^{\log_b a}$$

Geometric series: $\displaystyle\sum_{i=0}^{k} c^i = \frac{c^{k+1}-1}{c-1}$

Recurrence equations to remember

$T(n) = T(n-1) + O(1) \rightarrow O(n)$

$T(n) = T(n-1) + O(n) \rightarrow O(n^2)$

$T(n) = 2T(n-1) + O(1) \rightarrow O(2^n)$

$T(n) = T(n/2) + O(1) \rightarrow O(\lg n)$

$T(n) = 2T(n/2) + O(1) \rightarrow O(n)$

$T(n) = 2T(n/2) + O(n) \rightarrow O(n \lg n)$