# 1 Allan Variance

In the characterization of sensors and measurement systems, especially those used in inertial navigation, metrology or time synchronization, it is essential to evaluate the stability and quality of the signals they produce. Measurements obtained from real sensors always contain different types of stochastic noise, such as white noise or random drift, whose identifications make it possible to understand the sensor's limitations and optimize filtering or data fusion algorithms.

In this context, the Allan variance (AVAR) and its square root, the Allan deviation (ADEV), are statistical tools specifically designed to characterize the temporal stability of noisy signals. Unlike classical variance, which is not suitable for signals with drift components or long-term correlations, the Allan variance analyzes how the average signal changes over different integration time intervals, It is thus possible to distinguish the various noise regimes present.

Allan variance or deviation analysis provides a more complete view of the dynamic behaviour of a sensor: it reveals which types of noise dominate on certain time scales and how they affect the accuracy and stability of the system. For this reason, it is an essential tool for the performance evaluation and calibration of inertial sensors, precision clocks and other measuring devices.

To begin with, consider the discrete-time model of a one-dimensional sensor:

$$d_k = p_k + b_k + v_k, \quad v_k \sim \mathcal{N}(0,R) \tag{1}$$

$$b_{k+1} = b_k + w_k, \quad w_k \sim \mathcal{N}(0,qT_s) \tag{2}$$

where:

- $d_k$: sensor measurement [u],
- $p_k$: true value [u],
- $b_k$: sensor bias [u],
- $v_k$: measurement white noise [u],
- $R$: measurement noise variance [u²],
- $w_k$: random walk of the bias [u],
- $q$: intensity of random walk [u²/s],
- $T_s$: sampling period [s].

We can form $i$ averages from groups formed by $m$ samples covering a time $\tau = mT_s$ each:

$$\bar{d}_i^{(m)} = \frac{1}{m}\sum_{k=0}^{m-1} d_{im+k} \tag{3}$$

For time intervals $\tau$, the Allan variance of averages is:

$$\sigma^2(\tau) = \frac{1}{2}E\left[\left(\bar{d}_{i+1}^{(m)} - \bar{d}_i^{(m)}\right)^2\right] \tag{4}$$

The value of $\sigma^2(\tau)$ can be derived depending on the types of noise present in the signal, such as those present in the equations (1) and (1).

## 1.1 Characterisation of measurement white noise $v_k$

If we assume $d_k = p_0 + v_k$ (ignoring the measurement bias for the moment), where $p_0$ is a known constant in the measurement and can therefore be removed from it, the average noise of each block is

$$\bar{v}_i = \frac{1}{m}\sum_{j=0}^{m-1} v_{im+j} \tag{5}$$

Since the values of $v$, being Gaussian white noise, are independent and remembering that $\mathrm{Var}(v_k) = R$,

$$\mathrm{Var}(\bar{v}_i) = \frac{1}{m^2}\sum_{j=0}^{m-1}\mathrm{Var}(v_{im+j}) = \frac{mR}{m^2} = \frac{R}{m} \tag{6}$$

Also

$$\mathrm{Var}(\bar{v}_{i+1} - \bar{v}_i) = \mathrm{Var}(\bar{v}_{i+1}) + \mathrm{Var}(\bar{v}_i) = 2\frac{R}{m} \tag{7}$$

since the noise averages in the different blocks are independent of each other, in this case the Allan variance is given by:

$$\sigma^2(\tau) = \frac{1}{2}\cdot 2\frac{R}{m} = \frac{R}{m} \tag{8}$$

Substituing $m = \tau/T_s$:

$$\sigma^2(\tau) = \frac{R}{m} = \frac{RT_s}{\tau} \tag{9}$$

Equivalently,

$$\sigma(\tau) = \sqrt{\frac{RT_s}{\tau}} \tag{10}$$

Thus, in a logarithmic double-scale graph of the Allan deviation, the Gaussian white noise region appears as a straight line of slope $-\frac{1}{2}$. From the intersection of the abscissa axis, $a_{\mathrm{wn}}$:

$$\log_{10}\sigma(\tau) = -\tfrac{1}{2}\log_{10}\tau + a_{\mathrm{wn}} \tag{11}$$

we obtain:

$$R = \frac{\left(10^{a_{\mathrm{wn}}}\right)^2}{T_s} \tag{12}$$

## 1.2 Characterization of random walk bias $b_k$

For the measurement model analysed, the bias evolves according to (2) with independent increments $w_k$ with $\mathrm{Var}(w_k) = qT_s$. That is, $w_k$ is modeled as Gaussian white noise.

We want to obtain $\sigma^2(\tau) = \frac{1}{2}E[(\overline{b}_{i+1} - \overline{b}_i)^2]$ for each average block $\overline{b}_i$ over $m$ samples.

For that, we need:

- Write $b_k$ as a cumulaitve sum: $b_k = b_0 + \sum_{j=0}^{k-1} w_j$.
- Express the average blocks $\overline{b}_i = \frac{1}{m}\sum_{n=0}^{m-1} b_{im+n}$ as a double sum of increments $w_j$ with triangular weights.

Then:

$$\overline{b}_i = \frac{1}{m}\sum_{n=0}^{m-1}\left(b_0 + \sum_{j=0}^{im+n-1} w_j\right) \tag{13}$$

We can assume, for derivation purposes that $b_0 = 0$

$$\overline{b}_i = \frac{1}{m}\sum_{n=0}^{m-1}\left(\sum_{j=0}^{im-1} w_j + \sum_{t=0}^{n-1} w_{im+t}\right) \tag{14}$$

$$\overline{b}_i = \frac{1}{m}\sum_{n=0}^{m-1}\sum_{j=0}^{im-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}\sum_{t=0}^{n-1} w_{im+t} \tag{15}$$

$$\overline{b}_i = \sum_{j=0}^{im-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{im+n} \tag{16}$$

### 1.2.1 Expression for $\bar{b}_{i+1} - \bar{b}_i$

Similarly, we compute $\bar{b}_{i+1}$

$$\bar{b}_{i+1} = \sum_{j=0}^{(i+1)m-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{(i+1)m+n} \tag{17}$$

The substract $\bar{b}_i$

$$\bar{b}_{i+1} - \bar{b}_i = \left[\sum_{j=0}^{(i+1)m-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{(i+1)m+n}\right] - \left[\sum_{j=0}^{im-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{im+n}\right] \tag{18}$$

$$\bar{b}_{i+1} - \bar{b}_i = \left[\sum_{j=0}^{im-1} w_j + \sum_{n=0}^{m-1} w_{im+n} + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{(i+1)m+n}\right] - \left[\sum_{j=0}^{im-1} w_j + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{im+n}\right] \tag{19}$$

The sum common to both $\sum_{j=0}^{im-1} w_j$ is cancelled. Arrangement of terms:

$$\bar{b}_{i+1} - \bar{b}_i = \left[\frac{1}{m}\sum_{n=0}^{m-1} m w_{im+n} + \frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{(i+1)m+n}\right] - \left[\frac{1}{m}\sum_{n=0}^{m-1}(m-n-1)w_{im+n}\right] \tag{20}$$

Then

$$\bar{b}_{i+1} - \bar{b}_i = \frac{1}{m}\sum_{n=0}^{m-1}\Big((n+1)w_{im+n} + (m-n-1)w_{(i+1)m+n}\Big) \tag{21}$$

This is a linear combination of $2m-1$ independent increments $w$ with known deterministic coefficients.

### 1.2.2 Variance of the difference

Since the values of $w$ are independent, the variance of the linear combination is equal to $Q$ multiplied by the sum of the square coefficients:

$$\mathrm{Var}(\bar{b}_{i+1}-\bar{b}_i) = \frac{Q}{m^2}\sum_{n=0}^{m-1}\left((n+1)^2+(m-n-1)^2\right) \tag{22}$$

$$= \frac{Q}{m^2}\left[\sum_{n=0}^{m-1}(n+1)^2+\sum_{n=0}^{m-1}(m-n-1)^2\right] \tag{23}$$

$$= \frac{Q}{m^2}\left[\sum_{n=1}^{m}n^2+\sum_{n=1}^{m}(m-n)^2\right] \tag{24}$$

$$= \frac{Q}{m^2}\left[\sum_{n=1}^{m}n^2+\sum_{n=1}^{m}m^2-2\sum_{n=1}^{m}mn+\sum_{n=1}^{m}n^2\right] \tag{25}$$

$$= \frac{Q}{m^2}\left[2\sum_{n=1}^{m}n^2+m^3-2m\sum_{n=1}^{m}n\right] \tag{26}$$

To evaluate the sum, we use the formulas:

$$\sum_{k=1}^{m}k^2=\frac{m(m+1)(2m+1)}{6} \qquad \sum_{k=1}^{m}k=\frac{m(m+1)}{2} \tag{27}$$

Then:

$$S = 2\sum_{n=1}^{m}n^2+m^3-2m\sum_{n=1}^{m}n \tag{28}$$

$$= 2\frac{m(m+1)(2m+1)}{6}+m^3-2m\frac{m(m+1)}{2} \tag{29}$$

$$= \frac{(m^2+m)(2m+1)}{3}+m^3-m^3-m^2 \tag{30}$$

$$= \frac{2m^3+3m^2+m}{3}-m^2 \tag{31}$$

$$= \frac{2m^3+m}{3} \tag{32}$$

$$= \frac{m(2m^2+1)}{3} \tag{33}$$

Hence:

$$\mathrm{Var}(\bar{b}_{i+1}-\bar{b}_i) = \frac{Q}{m^2}\cdot\frac{m(2m^2+1)}{3} \tag{34}$$

$$= Q\cdot\frac{2m^2+1}{3m} \tag{35}$$

### 1.2.3 Allan Variance

Recalling that the Allan variance is half of the expected value of the square of the difference:

$$\sigma^2(\tau) = \tfrac{1}{2}\mathrm{Var}(\bar{b}_{i+1} - \bar{b}_i) \tag{36}$$
$$= \frac{Q}{2} \cdot \frac{2m^2+1}{3m} \tag{37}$$
$$= Q \cdot \frac{2m^2+1}{6m} \tag{38}$$

We replace $Q = qT_s$ and $m = \tau/T_s$ to express in terms of $\tau$ and $T_s$.

Then we expand to isolate the dominant term and the correction term:

$$\sigma^2(\tau) = \frac{q}{3}\tau + \frac{qT_s^2}{6\tau} \tag{39}$$

### 1.2.4 Approximation for values of $m$ large

For $m \gg 1$ ($\tau \gg T_s$), the correction term is negligible; then:

$$\sigma^2(\tau) \approx \frac{q}{3}\tau \qquad \Longrightarrow \qquad \sigma(\tau) \approx \sqrt{\frac{q}{3}}\sqrt{\tau} \tag{40}$$

n this way, on a logarithmic double-scale graph of the Allan deviation, the random walk noise region of the bias appears as a straight line slope $+\frac{1}{2}$. From the intersection of the abscissa axis $a_{\mathrm{rw}}$:

$$\log_{10}\sigma(\tau) = \tfrac{1}{2}\log_{10}\tau + a_{\mathrm{rw}} \tag{41}$$

in this way, we obtain (disregarding correction by finite samples):

$$q \approx 3 \cdot \left(10^{a_{\mathrm{rw}}}\right)^2 \tag{42}$$

## 1.3 Summary of the method

- **Gaussian white noise region** $\sigma(\tau) = \sqrt{\frac{RT_s}{\tau}}$
- **Random walk region of bias** $\sigma(\tau) = \sqrt{\frac{q}{3}}\sqrt{\tau}$

These relationships allow the estimation of $R$ and $q$ directly from the collected data.