Unit 3 Lagrange's Mechanics

1.0 Introduction

2.0 Aim and Objectives

- know the importance of concepts such as generalized coordinates and constrained motion.
- derive the Lagrange's equations of motion
- test the elegance and power of the Lagrange method in problem solving as being done using Newton method.

3.0 Lagrange's equations of motion

To obtain a more general form of Lagrange's equations,

Kinetic energy, $T = \frac{1}{2} m_i \dot{x}_i^2$

In generalized coordinates,

$$\frac{\partial T}{\partial \dot{x}_i} \rightarrow \frac{\partial T}{\partial \dot{q}_k}$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k}, \qquad \dot{x}_i = \frac{dx_i}{dt} = \sum \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

$$\Rightarrow \quad \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j} \quad \text{By cancellation of dots}$$

Hence,

$$\frac{\partial T}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \sum m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left\{\sum m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}\right\}$$

$$= \sum\left\{m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k} + m_i \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right)\right\}$$

$$= \left\{F_i \frac{\partial x_i}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i \dot{x}_i\right)\right\}$$

But

$$Q_k = F_i \frac{\partial x_i}{\partial q_k} \equiv \textit{Generalized force} \quad \text{and} \quad T = \frac{1}{2} m_i \dot{x}_i \equiv \textit{Kinetic energy}$$

$\therefore$

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}$$

If again, the system is a conservative one, $Q_k = -\frac{\partial V}{\partial q_k}$

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k}$$

The definition of the Lagrangian is

$$L = T - V \quad \Rightarrow \quad T = L + V$$

$$\frac{d}{dt}\left(\frac{\partial (L+V)}{\partial \dot{q}_k}\right) = \frac{\partial (T - V)}{\partial q_k} \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} + \frac{\partial V}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

The kinetic energy, T is a function of both q_k and $\dot{q}_k$ but the potential energy is a function of only position q_k, not velocity $\dot{q}_k$, therefore $\frac{\partial V}{\dot{q}_k} = 0$

and we finally have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

These are the Lagrange's equations of motion, also known as the Euler – Langrange equations.

There is one Lagrange equation for each generalized coordinate q_i. When $q_i = r_i$ (i.e. the generalized coordinates are simply the cartesian coordinates), it is straightforward to see that the Lagrange's equations reduce to Newton's second law.
The above derivation can be generalized to a system of N particles. There will be 6N generalized coordinates, related to the position coordinates by 3N transformation equations. In each of the 3N Lagrange's equations, T is the total energy of the system, and V the total potential energy.

Note:
In practice, it is easier to solve a problem using the Euler – Lagrange equations than Newton's laws. This is because not only may more appropriate generalized coordinates q_i be chosen to exploit symmetries in the system, but constraint forces are replaced with simpler equations.

4.0 Conclusion

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5.0 Summary

To set up an equation of motion:
(i) find T and V
(ii) L = T + V

Unit 4 Hamilton's Mechanics

1.0 Introduction

The Hamiltonian formulation, just like the Lagrangian, is reformulations of Newtonian mechanics and also provide simple techniques for deriving equations of motion using energy relations.

2.0 Aim and Objectives

At the end of this unit students will be able to:

- know the importance of concepts such as generalized coordinates and constrained motion.
- derive the Hamilton's equations of motion also known as 'canonical equations of Hamilton'
- test the elegance and power of the Hamilton method in problem solving as being done using Newton method.

3. 0 Hamilton's Equation of Motion

For a system of particles each having mass m_α described by a set of generalized coordinates q_α, the classical Hamiltonian function is defined by

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L\left(q_\alpha, \dot{q}_\alpha, t\right)$$

where $L\left(q_\alpha, \dot{q}_\alpha\right)$ is Lagrangian.

Now taking the total differential

$$dH = \sum_\alpha \left(p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha - \frac{\partial L}{\partial q_\alpha} dq_\alpha - \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right) - \frac{\partial L}{\partial t} dt$$

$\frac{\partial L}{\partial \dot{q}_\alpha}$ is the definition of the generalized momentum p_α and from Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \Rightarrow \frac{d}{dt} p_\alpha - \frac{\partial L}{\partial q_\alpha} = 0 \Rightarrow \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$$

So we have

$$\frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha \quad \text{and} \quad \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$$

Therefore the total differential of the classical Hamiltonian becomes

$$dH = \sum_\alpha \left(p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha - \dot{p}_\alpha dq_\alpha - p_\alpha d\dot{q}_\alpha \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\alpha \left(\dot{q}_\alpha dp_\alpha - \dot{p}_\alpha dq_\alpha \right) - \frac{\partial L}{\partial t} dt$$

From which we have the following equations

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$$

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

and $$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

These equations are called the Hamilton's equations of motion.They are also known as *canonical equations* of Hamilton

4.0 Conclusion

Hamiltoian, just like the Lagrangian, is reformulations of Newtonian mechanics and also provide simple techniques for deriving equations of motion using energy relations.

5.0 Summary

The Hamiltonian method differs from the Lagrangian method in that instead of expressing second-order differential constraints on an n-dimensional coordinate space (where n is the number of degrees of freedom of the system), it expresses first-order constraints on a $2n$-dimensional phase space.

6.0 Tutor-Marked Assignment (TMA)

Question 4.1

A particle moves in the x-y plane under the influence of a central force depending only on its distance from the origin. Set up the Hamiltonian and get the equations of motion.

Question 4.2

A particle of mass m moves in one dimension under the influence of a force

$$F(x,t) = \frac{k}{x^2} e^{-\left(\frac{t}{\tau}\right)}$$

where k and τ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and total energy, and discuss the conservation of energy for the system.

7.0 References/Further Readings

Fowles, G. R. and Cassiday, G. L. (1993) Analytical Mechanics, 5th Ed., Saunders College Publishing, New York.

Goldstein, H. (1959) Classical Mechanics, Addison-Wesley Publishing Company, Inc. New York.

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Unit 5 Between Newtonian, Lagrangian and Hamiltonian Mechanics

1.0 Introduction

Lagrangian mechanics and Hamiltonian mechanics are two important and more abstract alternative formulations of classical mechanics. They bypassed the concept of "force", instead referring to other physical quantities, such as energy, for describing mechanical systems. Lagrange's and Hamilton's equations provide a new and equivalent way of looking at classical mechanics.

2.0 Aim and Objectives

By the end of this study unit, students will be able to:

- see how concept of 'force' in Newtonian mechanics is transformed into physical quantity of energy for use in Lagrangian and Hamiltonian mechanics
- see that the Newtonian, Lagrangian and Hamiltonian mechanics provide equivalent looks into classical mechanics.

3.0 Transformation of Newton's Law from Vector to Scalar Notation

Newton's law, which is in vector notation can be transformed to scalar. The force F on a particle is

$$F = ma = \frac{dp}{dT} = m\dot{v}$$

$$F_i = \frac{d}{dt}\left(m_i \dot{x}\right) = \frac{d}{dt}\left(\frac{dT}{d\dot{x}}\right)$$

Kinetic energy

$$T = \frac{1}{2} m_i \dot{x}_i^2 \qquad \Rightarrow \qquad \frac{dT}{d\dot{x}} = m_i \dot{x}_i = p_x$$

If forces are derivable from potential energy, V then,

$$F = -\nabla V \quad = \quad -\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}\right) \quad = \quad F_i, \qquad V = V(x_i)$$

So

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) \quad = \quad -\frac{\partial V}{\partial x_i}$$

This is just Newton's law transformed from vector notation to scalar in cartesian coordinates with T = kinetic energy and V = potential energy

3.1 Between Newtonian, Lagrangian and Hamiltonian Mechanics

Classical mechanics is concerned with the set of physical laws governing and mathematically describing the motions of bodies and aggregates of bodies geometrically distributed within a certain boundary under the action of a system of forces.

The initial stage in the development of classical mechanics is often referred to as Newtonian mechanics, and is associated with the physical concepts employed by and the mathematical methods invented by Newton himself.

Lagrangian mechanics is a re-formulation of classical mechanics that combines conservation of momentum with conservation of energy.

Hamiltonian mechanics is a reformulation of classical mechanics that arose from Lagrangian mechanics, a previous reformulation of classical mechanics.

Self Assessement Exercise

1. Write down expressions for the following quantities and explained the meaning of each symbol involved: (i) generalized velocity (ii) generalized force (iii) generalized kinetic energy

2. Explain the terms 'Generalized coordinates' and 'Degrees of freedom '

3. Distinquish between Holonomic and Non-holonomic constraints.
. Give the generalized coordinates which are applicable to the motion of each of the following:
(i) A particle moving in a plane under the influence of a force directed towards the origin.
(ii) A disk rolling on the horizontal xy plane constrained to move so that the plane of the disk is always vertical.

4. Write down the Lagrange's equation of motion for a
(i) conservative system.
(ii) non-conservative system.

5. Obtain the Hamiltonian equation of motion for a 1-D harmonic oscillator, supposing that there is a damping force which is proportional to the velocity and that the system is non-conservative.

6. (a) set up the Lagragian for a simple pendulum;
(b) solve the resulting equation to find the motion of the pendulum.

7. Write three sentences on what you understand on Newtonian mechanics, Lagrangian mechanics and Hamiltonian mechanis; bringing the siimilarities and differences.

4.0 Conclusion

Each of the 3 mechanics of Newton, Lagrange and Hamilton can be preferred to the other to describe mechanical system, depending on convenience.

5.0 Summary

Hamiltonian mechanics is a reformulation of Newtonian mechanics that arose from Lagrangian mechanics, a previous reformulation of Newtonian mechanics.

6.0 Tutor-Marked Assignment (TMA)

Question 5.1

Consider a particle of mass m which moves freely in a conservative force field whose potential energy function is V. Find the Hamiltonian function and show that the canonical equations of motion reduce to Newton's equations (use rectangular coordinate)