1. One of the most important **commutators** in physics is that between the coordinate, $\hat{x}$, and the momentum, $\hat{p}$.

Given that $[\hat{x},\hat{p}] = i\hbar$ deduce that (i) $[\hat{x}^2,\hat{p}] = 2i\hbar\hat{x}$;

(ii) $[\hat{x},\hat{p}^2] = 2i\hbar\hat{p}$; (iii) $[\hat{H},\hat{x}] = \frac{-i\hbar}{m}\hat{p}$;

(iv) If $g(x)$ is an arbitrary function of x, show that $[\hat{p},g] = -i\hbar\frac{dg}{dx}$ **(20 marks)**

2. Consider the operators $\hat{D},\hat{I},\hat{B},\hat{P},\hat{\Theta}$ defined below as :

$$\hat{D}\psi(x) = \frac{\partial}{\partial x}, \hat{I}\psi(x) = \psi(x), \hat{B}\psi(x) = \frac{1}{3}\psi(x), \hat{P}\psi(x) = 3\psi^2 - 2\psi, \hat{\Theta}(x) = 0$$

For each of the operators listed above,calculate the square ($\hat{D}^2 = \hat{D}\hat{D}$, for example) **(15 marks)**

3. Let $\hat{X}$ be the one-dimensional position operator , $\hat{X}\psi = x\psi$, and let $\hat{D}$ be the derivative operator: $\hat{D}\psi = \frac{\partial\psi}{\partial x}$. Calculate $\hat{D}\hat{X}$. $= 1$ **(5 marks)**

4. The **Dirac delta** function $\delta(x-a)$ may be accurately expressed as an eigenfunction of $\hat{x}$ in the coordinate representation.

(a) If a system is in a state $\psi(x) = \delta(x+2)$, what does the measurement of x give?

(b) Evaluate the following: (i) $\int_{+\infty}^{\infty} dx\,\delta(x-2)$ (ii) $\int dx(x-4)\delta(x+3)$

(iii) $\int dx(\log_{10} x)\delta(x-0.01)$ (iv) $\int dx(e^{x+2})\delta(x+2)$

(v) $\int_0^\infty dx[\cos(3x) + 2e^{ix}](\delta(x-\pi) + \delta(x))$ **(20 marks)**

$\frac{\hat{P}^2}{2m} + V$

... Answer any Four Questions

Where necessary you may assume :

$$\Delta F=\sqrt{\langle F^2\rangle-\langle F\rangle^2} \qquad \int_0^\infty x^{2n}e^{-\frac{x^2}{a}}dx=\frac{\sqrt{\pi a}}{2}\frac{(2n)!}{n!}\left(\frac{a}{4}\right)^n$$

1. (a) State the fundamental postulates of Quantum Mechanics in respect of measurement of observables. **(4 marks)**

(b) In the quantum mechanical measurement of a physical observable A, the eigenvalue equation is given by : $\hat{A}\phi=a\phi$. Identify the symbols.**(3 marks)**

(c) Consider a free particle moving in 1-D space such that the states are given by $\phi_1(x)=Ae^{-ikx}$ $\phi_1(x)=Ae^{-ikx}$ and $\phi_2(x)=B\cos kx$, where A,B and k are constants.

(i) What **momentum** is associated with the particle when in state $\phi_1(x)$?

(ii) What **energy** is associated with the particle when in state $\phi_1(x)$?

(iii) What **momentum** is associated with the particle when in state $\phi_2(x)$?

(iv) What **energy** is associated with the particle when in state $\phi_2(x)$?

(v) How do the states $\phi_1(x)$ and $\phi_2(x)$ differ with regard to measurements of **momentum** and **energy**? **(18 marks)**

The **Dirac delta** function $\delta(x-a)$ may be accurately expressed as an eigenfunction of $\hat{x}$ in the coordinate representation.

a. State any **three** properties of $\delta(x-a)$. **(3 marks)**

b. If a system is in a state $\psi(x)=\delta(x+\pi)$, what does the measurement of x give? **(2 marks)**

c. Evaluate the following: (i) $\int dx\delta(x+7)$ (ii) $\int dx(x+4)\delta(x-6)$

(iii) $\int dx(\log_{10}x)\delta(x-0.01)$ (iv) $\int dx(e^{x+2})\delta(x+2)$

(v) $\int_0^\infty dx[\cos(3x)+2e^{ix}](\delta(x-\pi)+\delta(x))$ **(20 marks)**

3. Distinguish between a **function** and an **operator** **(4 marks)**

Consider the operators $\hat{D}, \hat{I}, \hat{B}, \hat{P}, \hat{\Theta}, \hat{X}$ defined below as :

$$\hat{D}\psi(x) = \frac{\partial}{\partial x}, \hat{I}\psi(x) = \psi(x), \hat{B}\psi(x) = \frac{1}{3}\psi(x), \hat{P}\psi(x) = 3\psi^2 - 2\psi, \hat{\Theta}(x) = 0, \hat{X}\psi = x\psi$$

Calculate the following: (a) $\hat{D}\hat{X}$, (b) $\hat{B}\hat{P}$, (c) $\hat{I}\hat{P}$, (d) $\hat{D}\hat{P}$, (e) $\hat{I}\hat{B}$, (f) $\hat{P}\hat{B}$ **(21 marks)**

4. Consider A,B, and C as three operators in Quantum Physics.

(a) What is meant by **Commutator** of A and B ? **(4 marks)**

(b) If B and C are **Compatible.** What does this mean? **(5 marks)**

(c) Prove that for the operators A,B and C, the following identities are valid :

(i) $[A, B+C] = [A,B] + [A,C]$ (ii) $[AB, C] = A[B,C] + [A,C]B$

(iii) $[A+B, C] = [A,C] + [B,C]$ (iv) $[A, BC] = [A,B]C + B[A,C]$ **(16 marks)**

5. The time-dependent state $\psi(x,t)$ of a 1-D system is given by:

$$\psi(x,t) = e^{i\beta t}(A\sin\alpha x + Bi\cos\alpha x) .$$

If the potential energy is given by V_0,

(a) determine whether $\psi(x,t)$ is an energy eigenfunction. **(5 marks)**

(b) If so, calculate the measurable energy value in terms of α .**(10 marks)**

(c) What is the measurable energy value in terms of β ? **(10 marks)**

6. One of the most important **commutators** in physics is that between the coordinate, $\hat{x}$, and the momentum, $\hat{p}$.

Given that $[\hat{x}, \hat{p}] = i\hbar$ deduce that (i) $[\hat{x}^2, \hat{p}] = 2i\hbar\hat{x}$;

(ii) $[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$; (iii) $[\hat{H}, \hat{x}] = \frac{-i\hbar}{m}\hat{p}$;

(iv) If $g(x)$ is an arbitrary function of x, show that $[\hat{p}, g] = -i\hbar\frac{dg}{dx}$;

(v) Calculate $[2\hat{x}, 3\hat{p}]$ **(25 marks)**

Where necessary you may assume :

$$\Delta F = \sqrt{\langle F^2\rangle - \langle F\rangle^2} \qquad \int_0^{\infty} x^{2n} e^{-\frac{x^2}{a}}\, dx = \frac{\sqrt{\pi a}}{2}\frac{(2n)!}{n!}\left(\frac{a}{4}\right)^n$$

1. (a) State the fundamental postulates of Quantum Mechanics in respect of measurement of observables. **(4 marks)**

(b) In the quantum mechanical measurement of a physical observable A, the eigenvalue equation is given by : $\hat{A}\phi = a\phi$. Identify the symbols. **(3 marks)**

(c) Consider a free particle moving in 1-D space such that the states are given by $\phi_1(x) = Ae^{-ikx}$ $\phi_1(x) = Ae^{-ikx}$ and $\phi_2(x) = B\cos kx$, where A,B and k are constants. e^{-ikx}

(i) What **momentum** is associated with the particle when in state $\phi_1(x)$? $P_z = -i\hbar \nabla$

(ii) What **energy** is associated with the particle when in state $\phi_1(x)$?

(iii) What **momentum** is associated with the particle when in state $\phi_2(x)$?

(iv) What **energy** is associated with the particle when in state $\phi_2(x)$?

(v) How do the states $\phi_1(x)$ and $\phi_2(x)$ differ with regard to measurements of **momentum** and **energy**? **(18 marks)** $K.E = \frac{p^2}{2m}$

The **Dirac delta** function $\delta(x-a)$ may be accurately expressed as an eigenfunction of $\hat{x}$ in the coordinate representation.

a. State any **three** properties of $\delta(x-a)$. **(3 marks)**

b. If a system is in a state $\psi(x) = \delta(x+\pi)$, what does the measurement of x give? **(1 mark)**

c. Evaluate the following: (i) $\int dx\,\delta(x+7)$ (ii) $\int dx(x+4)\delta(x-6)$

(iii) $\int dx(\log_{10} x)\delta(x-0.01)$ (iv) $\int dx(e^{x+2})\delta(x+2)$ **(12 marks)**

d. (i) State whether or not it is possible to measure the momentum and get a unique answer when an electron is associated with the state function

$$\Phi = A\exp(-i\beta t)\sin(2\pi x/\lambda)$$

(ii) How does the value $\Phi\,\Phi^*$ vary with position in this case?

(iii) If $\beta = 0$, what would be the energy associated with Φ ? **(9 marks)**

Annihilation and creation operators $\hat{a}$ and $\hat{a}^+$ are defined in the Harmonic Oscillator problem respectively as : $\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$, $\hat{a}^+ = \frac{\beta}{\sqrt{2}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$

where $\beta^2 = \frac{m\omega}{\hbar}$ and other symbols have their usual meanings.

(a). If the non-dimensional displacement ζ is defined as : $\zeta^2 \equiv \beta^2 x^2 \equiv \frac{m\omega}{\hbar}x^2$,

show that $\hat{a}$ and $\hat{a}^+$ transform as:

(i) $\hat{a} = \frac{1}{\sqrt{2}}\left(\zeta + \frac{\partial}{\partial\zeta}\right)$, (ii) $\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\zeta - \frac{\partial}{\partial\zeta}\right)$, (iii) $\hat{a}^+\hat{a} = \frac{1}{2}\left(\zeta^2 - \frac{\partial^2}{\partial\zeta^2} - 1\right)$

(9 marks)

(b) Given that $[\hat{x}, \hat{p}] = i\hbar$,show that $[\hat{a}, \hat{a}^+] = 1$ **(4 marks)**

(c) Show that the Hamiltonian $\hat{H}$ of the 1-D harmonic oscillator given by

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ is expressible as $\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$; **(6 marks)** $\hat{a}^+\hat{a}$

(d) Hence, deduce that the ground state energy :

$E_0 = \frac{1}{2}\hbar\omega$. **(6 marks)**

4. The time-dependent state $\psi(x,t)$ of a 1-D system is given by:

$\psi(x,t) = e^{i\beta t}(A\sin\alpha x + Bi\cos\alpha x)$

If the potential energy is given by V_0 ,

(a) determine whether $\psi(x,t)$ is an energy eigenfunction. **(5 marks)**

(b) If so, calculate the measurable energy value in terms of α .**(10 marks)**

(c) What is the measurable energy value in terms of β ? **(10 marks)**

5. (a) Show that the linear momentum operator is **Hermitian.** **(8 marks)**

(b) Show that the eigenvalues of a Hermitian operator are **real.** **(8 marks)**

(c) Spin Matrices are special matrices that occur in Quantum mechanics.

In 2-D, they are namely : $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Show that each of the matrices is **Hermitian.** **(9 marks)**

6. (a) Show that the commutator of the momentum operator with a function of the position operator is given by :

$\left[f(x), \hat{P}\right] = i\hbar\,\partial f/\partial x$ **(9 marks)**

(b) Hence, deduce that the commutator $\left[f(x), \hat{P}\right]$ is approximately equal to $-5i\hbar$ when x= 1 unit, given that :

$f(x) = \pi\cos\frac{\pi}{2}x + \pi\sin\frac{\pi}{2}x$ **(6 marks)**.

(c) Show that: (i) $\left[\hat{N}, \hat{a}\right] = -\hat{a}$; (ii) $\left[\hat{N}, \hat{a}^+\right] = \hat{a}^+$ **(10 marks)**

$\hat{a}^+\hat{a}$ $\hat{a}^+\hat{a}$

HeB

1. Consider the operators $\hat{D}, \hat{I}, \hat{B}, \hat{P}, \hat{\Theta}$ defined below as :

$$\hat{D}\psi(x) = \frac{\partial}{\partial x}, \hat{I}\psi(x) = \psi(x), \hat{B}\psi(x) = \frac{1}{3}\psi(x), \hat{P}\psi(x) = 3\psi^2 - 2\psi, \hat{\Theta}(x) = 0$$

For each of the operators listed above, construct the square, that is $\hat{D}^2$.

2. Let X be the one-dimensional position operator, $X\psi = x\psi$, and let D be the derivative operator: $D\psi = d\psi/dx$. Calculate DX

3. The state $\psi(x)$ of a system of electrons of mass m in 1-D is given by $\psi(x) = Ae^{-2\pi x^2}$.
   (a) Normalize to determine the value of A.
   (b) What is the normalized state function?
   (c) Calculate the average energy of the electrons in this normalized state.

4. The **Dirac delta** function $\delta(x-a)$ may be accurately expressed as an eigenfunction of $\hat{x}$ in the coordinate representation.
   Evaluate the following: (i) $\int dx\,\delta(x-2)$ (ii) $\int dx(x-4)\delta(x+3)$
   (iii) $\int dx(\log_{10} x)\delta(x-0.01)$ (iv) $\int dx(e^{x+2})\delta(x+2)$
   (v) $\int_0^{\infty} dx[\cos(3x) + 2e^{ix}](\delta(x-\pi) + \delta(x))$

5. Prove that for the operators A,B and C, the following identities are valid :
   (i) $[A+B,C] = [A,C]+[B,C]$ (ii) $[A,BC] = [A,B]C + B[A,C]$
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6. Consider a free particle moving in 1-D space such that the states are given by $\phi_1(x) = Ae^{-ikx}$ and $\phi_2(x) = B\cos kx$, where A,B and k are constants.
   (a) What **momentum** is associated with the particle when in state $\phi_1(x)$?
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   (c) What **momentum** is associated with the particle when in state $\phi_2(x)$?
   (d) What **energy** is associated with the particle when in state $\phi_2(x)$?
   (e) How do the states $\phi_1(x)$ and $\phi_2(x)$ differ with regard to measurements of **momentum** and **energy**?