NYU Computer Science Bridge to Tandon Course Winter 2021

Homework 11

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Question 5

(a) **Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder)**

Proof
By induction on n.

Base case: n = 1.
$1^3 + 2 \cdot 1 = 3$; 3 is divided by 3.

Inductive step: We will show that for any integer $n \geq 1$, if $k^3 + 2k$ is divided by 3, then $(k+1)^3 + 2(k+1)$ is divided by 3

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= 1^3 + 3 \cdot 1^2 \cdot k + 3 \cdot 1 \cdot k^2 + k^3 + 2k + 1 && \text{by algebra} \\
&= k^3 + 3k^2 + 5k + 2 && \text{by algebra} \\
&= (k^3 + 2k) + 3(k^2 + k + 1) && \text{by algebra}
\end{aligned}$$

$(k^3 + 2k)$ is divided by 3 by the inductive hypothesis. Because k is an integer, $(k^2 + k + 1)$ is also an integer, and $3(k^2 + k + 1)$ is divided by 3.

Therefore, $(k+1)^3 + 2(k+1)$ is divided by 3. ■

(b) **Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.**

Proof
By induction on n.

Base case: n = 2.
Since 2 is a prime number, it already is a product of one prime number: 2

Inductive step: Assume that for $k \geq 2$, any integer j in the range from 2 through k can be expressed as a product of prime numbers. We will show that k + 1 can be expressed as a product of prime numbers.

If k + 1 is prime, then it is a product of one prime number, k + 1. If k + 1 is not prime, k + 1 is composite and can be expressed as the product of two integers, a and b, that are each at least 2.

Since $k + 1 = a \cdot b$, then $a = (k + 1)/b$. Furthermore, since $b \geq 2$, then $a = (k + 1)/b < k + 1$. If a is an integer which is strictly less than k + 1, then $a \leq k$. The symmetric argument can be used to show that $b = (k + 1)/a \leq k$. Thus a and b both fall in the range from 2 through k which means that the inductive hypothesis can then be applied and they can each be expressed as a product of primes. Therefore, (k+1) can be expressed as $a \cdot b$, which is a product of primes. ■

Question 6

(a) **Exercise 7.4.1: Components of an inductive proof.**
Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that P(3) is true.

$$\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\frac{n(n+1)(2n+1)}{6} = \frac{(3 \cdot 4 \cdot 7)}{6} = 14$$

The right-hand side yield the same result as the left-hand side, thus the function holds for $n = 3.$■

(b) Express P(k).

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6} = \frac{2k^3 + 3k^2 + 1k}{6}■$$

(c) Express P(k + 1).

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}■$$

(d) In an inductive proof that for every positive integer n,
$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
what must be proven in the base case?

n = 1 must be proven in the base case. ■

(e) In an inductive proof that for every positive integer n,
$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
what must be proven in the inductive step?

For $k \geq 1, P(k+1)$ is true must be proven in the inductive step. ■

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

The inductive hypothesis is that for $k \geq 1, P(k)$ is true. ■

(g) Prove by induction that for any positive integer n, $\sum_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof
By induction on n.

Base case: n = 1.

$$P(1) = \sum_{j=1}^{1} j^2 = \frac{(1)(2)(3)}{6} = 1$$

Inductive step: We will show that under the hypothesis that $P(k)$ is true for $k \geq 1$, then $P(k+1)$ is true for $k \geq 1$.

$$\begin{aligned}
P(k+1) &= \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6} && \text{by definition} \\
&= \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{2k^3 + 9k^2 + 13k + 6}{6} && \text{by algebra} \\
&= \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{2k^3 + 3k^2 + 1k}{6} + \frac{6k^2 + 12k + 6}{6} && \text{by algebra}
\end{aligned}$$

Because $\sum_{j=1}^{k} j^2 = \dfrac{k(k+1)(2k+1)}{6}$ is true by the hypothesis, we only need to prove that $(k+1)^2 = \dfrac{6k^2 + 12k + 6}{6}$.

$$\begin{aligned}
(k+1)^2 = k^2 + 2k + 1 &= \frac{6k^2 + 12k + 6}{6} \\
&= \frac{6 \cdot (k^2 + 2k + 1)}{6} \\
&= k^2 + 2k + 1
\end{aligned}$$

Therefore, P(k+1) is true. ■

(b) **Exercise 7.4.3: Proving inequalities by induction.**
Prove that for $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Proof
By induction on n.

Base case: n = 1.

$$\sum_{j=1}^{1} \frac{1}{j^2} = 1 \leq 2 - \frac{1}{1} = 1$$

Inductive step: Suppose that $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$ is true for $k \geq 1$, we will show that $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ is true for $k \geq 1$.

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{k+1} && \text{by algebra} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k+1} && \text{by hypothesis and algebra} \\
&= 2 - \frac{1}{k(k+1)} && \text{by algebra} \\
&\leq 2 - \frac{1}{k+1} && \text{for } k+1 \geq k(k+1) \text{ when } k \geq 1
\end{aligned}$$

Therefore, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ for $k \geq 1$. ■

(c) **Exercise 7.5.1: Proving divisibility results by induction.**
Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Proof
By induction on n.

Base case: n = 1.
$3^{2 \cdot 1} - 1 = 8$. 4 evenly divides 8

Inductive step: Suppose that for any positive integer k, 4 evenly divides $3^{2k} - 1$, we will show that 4 evenly divides $3^{2(k+1)} - 1$.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^2(3^{2k}) - 1 && \text{by algebra} \\ &= 9(3^{2k} - 1) + 8 && \text{by algebra} \end{aligned}$$

Since 4 evenly divides $3^{2k} - 1$ by hypothesis, it can be express as $4 \cdot t$ where t is a positive integer.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 9(4 \cdot t) + 8 && \text{by hypothesis} \\ &= 4(9 \cdot t + 2) && \text{by algebra} \end{aligned}$$

Therefore, 4 evenly divides $3^{2(k+1)} - 1$. ■