Question 5:

Exercise 1.12.2: Proving arguments are valid using rules of inference.

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

(b)

$$\begin{array}{l} \boldsymbol{p \to (q \land r)} \\ \boldsymbol{\neg q} \\ \hline \therefore \boldsymbol{\neg p} \end{array}$$

Solution.

1.	$p \to (q \land r)$	Hypothesis
2.	$\neg q$	Hypothesis
3.	$\cancel{\neg(q \land r)}$	~~Domination laws, 2.~~
4.	$\neg p$	Modus tollens, 1, 3.

■

1. negation q: Hypothesis
2. negation q or negation r: Addition, 1.
3. negation(q and r): De Morgan's law, 2.
4. p so that (q and r): Hypothesis
5. negation p: Modus Tollens, 3, 4.

(e)

$$\begin{array}{l} \boldsymbol{p \lor q} \\ \boldsymbol{\neg p \lor r} \\ \boldsymbol{\neg q} \\ \hline \therefore \text{r} \end{array}$$

Solution.

1.	$p \lor q$	Hypothesis
2.	$\neg p \lor r$	Hypothesis
3.	$q \lor r$	Resolution, 1, 2.
4.	$\neg q$	Hypothesis
5.	r	Disjunctive Syllogism, 3, 4.

■

Exercise 1.12.3: Proving the rules of inference using other rules.

Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

One of the rules of inference is Disjunctive syllogism :

$$\begin{array}{l} \boldsymbol{p \lor q} \\ \boldsymbol{\neg p} \\ \hline \therefore \textbf{q} \end{array}$$

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive syllogism. (Hint: you will need one of the conditional identities from the laws of propositional logic).

Solution.

1.	$p \vee q$	Hypothesis
2.	$\neg(\neg p) \vee q$	Double negation, 1.
3.	$\neg p \rightarrow q$	Conditional identities, 1.
4.	$\neg p$	Hypothesis
5.	q	Modus Ponens, 2, 3.

■

Exercise 1.12.5: Proving arguments in English are valid or invalid.
Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

(c)

$$I\ will\ buy\ a\ new\ car\ and\ a\ new\ house\ only\ if\ I\ get\ a\ job.$$

$$I\ am\ not\ going\ to\ ge\ a\ job.$$

$$\therefore \text{ I will not buy a new car.}$$

Solution.

j: I will get a job.
c: I will buy a new car
h: I will buy a new house

The form of the argument is

$$(\boldsymbol{c} \wedge \boldsymbol{h}) \rightarrow \boldsymbol{j}$$

$$\neg \boldsymbol{j}$$

$$\therefore \neg \boldsymbol{c}$$

The argument is not valid. When j = h = F, and c = T, the hypotheses are both true and the conclusion $\neg \boldsymbol{c}$ is false. ■

(d)

$$I\ will\ buy\ a\ new\ car\ and\ a\ new\ house\ only\ if\ I\ get\ a\ job.$$

$$I\ am\ not\ going\ to\ ge\ a\ job.$$

$$I\ will\ buy\ a\ new\ house.$$

$$\therefore \text{ I will not buy a new car.}$$

Solution.

j: I will get a job.
c: I will buy a new car.
h: I will buy a new house.

The form of the argument is

$$(\boldsymbol{c} \wedge \boldsymbol{h}) \to \boldsymbol{j}$$

$$\neg \boldsymbol{j}$$

$$\boldsymbol{h}$$

$$\therefore \neg \boldsymbol{c}$$

The argument is valid.

1.	$(c \wedge h) \to j$	Hypothesis
2.	$\neg j$	Hypothesis
3.	$\neg(c \wedge h)$	Modus tollens, 1, 2.
4.	$\neg c \vee \neg h$	De Morgan's laws, 3.
5.	h	Hypothesis
6.	$\neg c$	Disjunctive Syllogism, 4, 5.

■

Exercise 1.13.3: Show an argument with quantified statements is invalid.
Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}.

(b)

$$\exists \boldsymbol{x}(\boldsymbol{P}(\boldsymbol{x}) \vee \boldsymbol{Q}(\boldsymbol{x}))$$

$$\exists \boldsymbol{x} \neg \boldsymbol{Q}(\boldsymbol{x})$$

$$\therefore \exists \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x})$$

Solution.

	P	Q
a	F	T
b	F	F

$\exists \boldsymbol{x}(\boldsymbol{P}(\boldsymbol{x}) \vee \boldsymbol{Q}(\boldsymbol{x}))$is true because $\boldsymbol{Q}(\boldsymbol{x})$ is true for inputs a. $\exists \boldsymbol{x} \neg \boldsymbol{Q}(\boldsymbol{x})$ is true since b is the example. However, since $\boldsymbol{P}(\boldsymbol{b}) = \boldsymbol{P}(\boldsymbol{a}) = \boldsymbol{F}$, $\exists \boldsymbol{x} \boldsymbol{P}(\boldsymbol{x})$ is false. Therefore both hypotheses are true and the conclusion is false. ■

Exercise 1.13.5: Determine and prove whether an argument in English is valid or invalid.
Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid. The domain for each problem is the set of students in a class.

(d)

Every student who missed class got a detention.

Penelope is a student in the class.

Penelope did not miss class.

∴ Penelope did not get a detention.

Solution.

C: x missed class.
D: x got a detention.
The form of the argument is

$$\forall x(C(x) \to D(x))$$

$$\textit{Penelope is a student in the class.}$$

$$\neg C(Penelope)$$

$$\therefore \neg D(Penelope)$$

The argument is invalid. Consider the case that $\forall x C(x)$ is false and $\forall x D(x)$ is true, then $\forall x(C(x) \to D(x))$ is true, and $\neg C(Penelope)$ is true. However, $\neg D(Penelope)$ is false, rendering all the hypothesis to be true but the conclusion to be false. Therefore, the argument is invalid. ■

(e)

$$\textit{Every student who missed class or got a detention did not get an A.}$$

$$\textit{Penelope is a student in the class.}$$

$$\textit{Penelope got an A.}$$

$\therefore$ Penelope did not get a detention.

Solution.

C: x missed class.
D: x got a detention.
A: x got an A.
The form of the argument is

$$\forall x((C(x) \lor D(x)) \to \neg A(x))$$

$$\textit{Penelope is a student in the class.}$$

$$A(Penelope)$$

$$\therefore \neg D(Penelope)$$

The argument is valid.

1.	$\forall x((C(x) \lor D(x)) \to \neg A(x))$	Hypothesis
2.	$\forall x(\neg(C(x) \lor D(x)) \lor \neg A(x))$	Conditional identities, 1.
3.	*Penelope is a student in the class.*	Hypothesis
4.	$\neg(C(Penelope) \lor D(Penelope)) \lor \neg A(Penelope)$	Universal instantiation, 3.
5.	$A(Penelope)$	Hypothesis
6.	$\neg(C(Penelope) \lor D(Penelope))$	Disjunctive syllogism, 4, 5.
7.	$\neg C(Penelope) \land \neg D(Penelope)$	De Morgan's laws, 6.
8.	$\neg D(Penelope)$	Simplification, 7.

■

Question 6:

Exercise 2.2.1: Proving conditional statements with direct proofs.
Prove each of the following statements using a direct proof.

(c)If x is a real number and x ≤ 3, then $12 - 7x + x^2 \geq 0$..

Proof.
Direct proof. Assume that x is a real number and x ≤ 3. We will show that $12 - 7x + x^2 \geq 0$.
Since is a real number, $12 - 7x + x^2 = x^2 - 7x + 12 = (x - 3)(x - 4)$.
Additionally, because x ≤ 3, $(x - 3) \leq 0$ and $(x - 4) \leq 0$.
Since $a \leq 0$, b ≤ 0, and ab ≥ 0, where $a = (x - 3)$ and $b = (y - 4)$, $x^2 - 7x + 12 = (x - 3)(x - 4) \geq 0$.
Therefore $x^2 - 7x + 12 \geq 0$. ■

(d)The product of two odd integers is an odd integer.

Proof.
Direct proof. Assume a = 2k+1 and b = 2t+1 are 2 arbitrary odd integers, where k and t are 2 arbitrary integers. Plug the expression for the argument.

$a * b = (2k + 1)(2t + 1) = 4tk + 2t + 2k + 1 = 2(2tk + t + k) + 1.$

Since t and k are integers, then 2tk+t+k is also an integer. Since $(2k + 1)(2t + 1) = 2n + 1$, where n = 2tk+t+k is an integer, then $(2k + 1)(2t + 1)$ is an odd integer. Because k and t are 2 arbitrary integers, and a and b are 2 odd integers, the product of two odd integers is an odd integer. ■

Question 7:

Exercise 2.3.1: Proving conditional statements by contrapositive.
Prove each statement by contrapositive

(d)For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof.
Proof by contrapositive. We assume that n is an even integer and show that $n^2 - 2n + 7$ is an odd integer.
If n is an even integer, then n=2k for some integer k. Plugging in the expression 2k for n in $n^2 - 2n + 7$ gives

$$(2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2(2k^2 - 2k + 3) + 1$$

Since k is an integer, $2k^2 - 2k + 3$ is also an integer. $n^2 - 2n + 7$ can be expressed as 2 times an integer plus 1. Therefore, $n^2 - 2n + 7$ is odd. ■

(f)For every non-zero real number x, if $1/x$ is irrational, then 1x is also irrational.

Proof.
Proof by contrapositive. We assume that for every non-zero real number x that $1/x$ is rational, and show that x is also rational. By the definition of a rational number, $\frac{1}{x} = \frac{a}{b}$, where a and b are integers and $b \neq 0$. Since x is a non-zero real number, $\frac{1}{x}$ is also a non-zero real number, and $a \neq 0$. Therefore

$x = \frac{b}{a}$

Because b and a are integers and the denominator $a \neq 0$, x is rational. ■

(g)For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Proof.
Proof by contrapositive. We assume that for every pair of real number x and y, that $x > y$, and show that $x^3 + xy^2 > x^2y + y^3$. Because x and y are real numbers and $x > y$, that either x or y is not equal to zero, $x^2 + y^2 > 0$. We can multiple both sides of the formula with $x^2 + y^2$ and not alter the direction of the inequality.:

$$x * (x^2 + y^2) > y * (x^2 + y^2)$$

Since $x(x^2 + y^2) > y(x^2 + y^2)$ is true, and $x(x^2 + y^2) > y(x^2 + y^2) = x^3 + xy^2 > x^2y + y^3$, $x^3 + xy^2 > x^2y + y^3$ is true. ■

(l)For every pair of real numbers x and y, if x+y>20, then x>10 or y>10.

Proof.
Proof by contrapositive. We assume that for every pair of real number x and y, that $x > 10 \; or \; y > 10$ is not true, and show that $x + y \leq 20$. Since that $x > 10 \; or \; y > 10$ is not true, then $x \leq 10 \; and \; y \leq 10$. Since x and y are real numbers, if we plug the minimum value 10 of the x and y into the equation $x + y$, we get the minimum value of the sum of x and y is 20. Therefore, $x + y \leq 20$. ■

Question 8:

Exercise 2.4.1: Proofs by contradiction.

Give a proof for each statement.

(c)The average of three real numbers is greater than or equal to at least one of the numbers.

Proof.

Proof by contradiction. Suppose that the average of three real numbers is smaller than all of the numbers. If we assume a, b, and c to be arbitrary real numbers, the argument can be expressed as:

$$\frac{a+b+c}{3} < a$$
$$\frac{a+b+c}{3} < b$$
$$\frac{a+b+c}{3} < c$$

Then, we multiple 3 for both sides for 3 formula, and the sum of them will be:

$$3(a+b+c) < 3a+3b+3c$$

We get $a+b+c < a+b+c$, which is contradictory. ■

(e)There is no smallest integer.

Proof.

Proof by contradiction. Suppose there is a smallest integer, and we assume x to be this smallest particular integer. Since x is an integer, $x-1$ is also an integer, and $x-1 < x$, which contradicts the fact that x is the smallest integer. ■

Question 9:

Exercise 2.5.1: Proofs by cases.

Prove each statement.

(c)If integers x and y have the same parity, then x + y is even.

The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

Proof.

We consider two cases: x and y are even, and x and y are odd.

Case 1: x and y are even. If x and y are even, then $x = 2k$ and $y = 2t$ for arbitrary integers k and t.

$$x + y = 2k + 2t = 2(k + t)$$

Since k and t are integers, $(k + t)$ is also an integer. Therefore, $x + y$ is equal to 2 times an integer and $x + y$ is even.

Case 2: x and y are odd. If x and y are odd, then $x = 2k + 1$ and $y = 2t + 1$ for arbitrary integers k and t.

$$x + y = (2k + 1) + (2t + 1) = 2(k + t + 1)$$

Since k and t are integers, $(k + t + 1)$ is also an integer. Therefore, $x + y$ is equal to 2 times an integer and $x + y$ is even.

■