NYU Computer Science Bridge to Tandon Course	Winter 2021

Homework 7

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Question 3

(a) **Exercise 8.2.2:** Proving the growth rate for polynomials.

(b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

solution:
We will prove that $f = O(n^3)$ and $f = \Omega(n^3)$ to satisfy the definition of $\Theta(n^3)$.

First, we select $c = 8$ and $n_0 = 1$; because for $n \geq 1$, $3n^3 \geq 3n^2$ and $4n^3 \geq 4$, $n^3 + 3n^3 + 4n^3 = 8n^3 \geq n^3 + 3n^2 + 4$. Thus, $f = O(n^3)$

Second, we select $c = 1$ and $n_0 = 1$; because for $n \geq 1$, $3n^3 > 0$ and $4 > 0$, $n^3 + 3n^2 + 4 > n^3$. Thus, $f = \Omega(n^3)$

Therefore, $f = \Theta(n^3)$. ■

(b) **Exercise 8.3.5:** Worst-case time complexity - mystery algorithm.
The algorithm below makes some changes to an input sequence of numbers.

```
MysteryAlgorithm

Input: a_1, a_2,...,a_n
        n, the length of the sequence.
        p, a number.
Output: ??

i := 1
j := n

While (i < j)
    While (i < j and a_i < p)
        i := i + 1
    End-while
    While (i < j and a_j ≥ p)
        j := j - 1
    End-while
    If (i < j), swap a_i and a_j
End-while

Return( a_1, a_2,...,a_n )
```

(a)Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p = 0)

The algorithm sorts the sequence so that all the numbers that are less than the input value p appear before all the numbers that are greater than or equal to p. ■

(b)What is the total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

The number of times i is incremented or j is decremented is exactly n-1, regardless of the values in the input sequence. ■

(c)What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

Number of swaps does depend on input. Number of swaps is at minimum 0 and at maximum n/2. ■

(d)Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

The number of times i is incremented or j is decremented is exactly n, so the time complexity of the algorithm is $\Omega(n)$. ■

(e)Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

O(n) because the maximum number of times i is incremented or j is decremented will not exceed n. ■

Question 4

(a) **Exercise 5.1.1:** Counting passwords made up of letters, digits, and special characters.

Consider the following definitions for sets of characters:

Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

(b)Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.
Let D be the set of digits, L the set of letters, and S the set of special characters.

solution:
The three sets are mutually disjoint, so the total number of characters is:
$|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$

Each of the 7, 8, 9 characters in the string can be any of the 40 characters, so there are a total of 40^7, 40^8, 40^9 for strings of length 7, 8, 9 respectively.

So the final answer is $40^7 + 40^8 + 40^9$ ■

(c)Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

solution:
Let D be the set of digits and S the set of special characters.
The two sets are mutually disjoint, so the total number of characters in the two sets is:
$|D \cup S| = |D| + |S| = 10 + 4 = 14$

The first character in the string can only be any of the 14 characters, and the rest of the characters in the string an be any of the 40 characters. Thus, there are 14×40^6 for string of length 7, there are 14×40^7 for string of length 8, and there are 14×40^8 for string of length 9.

So the final answer is $14(40^6 + 40^7 + 40^8)$ ■

(b) **Exercise 5.3.2:** Strings with no repetitions.
(a)How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

solution:
$3*2^9$. The first character in the string has three possible choices and the following characters have two possible choices as two consecutive characters are not allowed. ■

(c) **Exercise 5.3.3:** Counting license plate numbers.
License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

(b)How many license plate numbers are possible if no digit appears more than once?

solution:
$10 * 9 * 8 * 26^4$. Using the product rule; there are 26 possible choices for the letter locations and 10 possible choices for the first digit location, 9 for the second, and 8 for the third. ■

(c)How many license plate numbers are possible if no digit or letter appears more than once?

solution:
I am interpreting the question as "How many license plate numbers are possible if **no digit and no letter** appears more than once?"

$10 * 9 * 8 * 26 * 25 * 24 * 23$. Using the product rule; there are 10 possible choices for the first digit location, 9 for the second, and 8 for the third. The same logic applies to the letter locations; there are 26 possible choices for the first letter location, 25 for the second, so on so forth.■

(d) **Exercise 5.2.3:** Using the bijection rule to count binary strings with even parity.
Let B = {0, 1}. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a)Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

solution:
Define the function $f : B^9 \rightarrow E_{10}$ such that if $x \in B^9$, then f(x) is obtained by adding 1 to the last bit of x if x has an odd number of 1's, 0 to the last bit if x has an even number of 1's. For example, $f(111111111) = 1111111111$ and $f(111111110) = 11111111100$. f is onto, because for any $y \in E_{10}$, y can be obtained by adding 1 or 0 to the last bit of any binary string with the length of 9. To satisfy the condition of E, either 1 or 0 can be added to the last bit. Every string with the length of 9 is covered in B^9 and the function specify that

$f(x)$ is obtained by adding 1 or 0 according the the condition of E. f is one-to-one because f only extends the string by adding a single bit (1 or 0) to the last bit without causing any possibility for repetition. ■

(b)What is $|E_{10}|$?

solution:
Since there is a bijection from E_{10} to B^9, $|E_{10}| = |B^9| = 2^9$. ■

Question 5

(a) **Exercise 5.4.2:** Counting telephone numbers.
At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a)How many different phone numbers are possible?

solution:
There are 2 way for the beginning of the phone numbers and 10 possible digits for each of the following number. 2×10^4 ■

(b)How many different phone numbers are there in which the last four digits are all different?

solution:
There are 2 way for the beginning of the phone numbers and 10 possible digits for the next number, then 9 for the next, and so on so forth. $2 \times 10 \times 9 \times 8 \times 7$ ■

(b) **Exercise 5.5.3:** Counting bit strings.
How many 10-bit strings are there subject to each of the following restrictions?

(a)No restrictions.
solution: 2^{10} ■

(b)The string starts with 001.
solution: 2^7 ■

(c)The string starts with 001 or 10.
solution: $2^7 + 2^8$ ■

(d)The first two bits are the same as the last two bits.
solution: $2^2 \times 2^6$ ■

(e)The string has exactly six 0's.
solution: $\binom{10}{6}$ ■

(f)The string has exactly six 0's and the first bit is 1.
solution: $\binom{9}{6}$ ■

(g)There is exactly one 1 in the first half and exactly three 1's in the second half.
solution: $\binom{5}{1} \times \binom{5}{3}$ ■

(c) **Exercise 5.5.5:** Choosing a chorus.
There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

solution: There are 30 boys to be chosen from for the 10-boy part of the chorus and 35 girls to be chosen from for the 10-girl part of the chorus: $\binom{30}{10} \times \binom{35}{10}$ ■

(d) **Exercise 5.5.8:** Counting five-card poker hands.
This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(c)How many five-card hands are made entirely of hearts and diamonds?
solution: $\binom{26}{5}$ ■

(d)How many five-card hands have four cards of the same rank?
solution: 13×48 ■

(e)A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?
solution: $13 \times 12 \times \binom{4}{2} \times \binom{4}{3}$ ■

(f)How many five-card hands do not have any two cards of the same rank?
solution: $\binom{13}{5} \times 4^5$ ■

(e) **Exercise 5.6.6:** Counting five-card poker hands.
A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a)How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?
solution: $\binom{44}{5} \times \binom{56}{5}$ ■

(b)Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

solution:
I am assuming that a speaker cannot be a vice speaker simultaneously. $44 \times 43 \times 56 \times 55$ ■

Question 6

(a) **Exercise 5.7.2:** Counting 5-card hands from a deck of standard playing cards.
A 5-card hand is drawn from a deck of standard playing cards.

(a)How many 5-card hands have at least one club?

solution:
First we compute the complement cases; there are 13 clubs in the card. So we first select a card from 39 non-club cards, then select a card from 38 non-club cards, and so on so forth. Since the order of the 5-card does not matter, we divide the combination by the factorial of 5.

Then we subtract the number of complement cases from the number of every possible 5-card hands, which is $\binom{52}{5}$.

So the final answer is $\binom{52}{5} - \binom{39}{5}$. ■

(b)How many 5-card hands have at least two cards with the same rank?

solution:
We get the answer by subtracting all the possible combination with the scenario where every cards are different ranked. There are 13 ranks to chosen from, and for each rank, there are 4 colors to be chosen from. $\binom{52}{5} - \binom{13}{5}4^5$ ■

(b) **Exercise 5.8.4:**Distributing comic books.
20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

For every comic book, there are 5 kids to be chosen from. Thus, 5^{20}. ■

(b)How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

For the first kid, there are 20 books to be chosen from, 16 for the second kid, and so on so forth.
$\binom{20}{4} \times \binom{16}{4} \times \binom{12}{4} \times \binom{8}{4}$ ■

Question 7

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

(a) 4
solution: 0. ■

(b) 5
solution: $5! = 120$ ■

(c) 6
solution: $\mathrm{P}(6, 5) = 720$. ■

(d) 7
solution: $\mathrm{P}(7, 5) = 2520$ ■