

Homework 5

Name: Lin, Kuan-You

Question 3

Solve the following questions from the Discrete Math zyBook:

a) **Exercise 4.1.3:** Recognizing well-defined algebraic functions and their ranges. Which of the following are functions from \mathbb{R} to \mathbb{R} ? If f is a function, give its range.

(b) $f(x) = 1/(x^2 - 4)$

Solution: Not a function. If x is 2 or -2, then $f(x)$ is not defined, not a real number. ■

(c) $f(x) = \sqrt{x^2}$

Solution: A function from \mathbb{R} to \mathbb{R} ; range of $f : \{y \mid y \geq 0\}$. ■

b) **Exercise 4.1.5:** Range of a function.

Express the range of each function using roster notation.

(b) Let $A = \{2, 3, 4, 5\}$. $f: A \rightarrow \mathbb{Z}$ such that $f(x) = x^2$.

Solution: $\{4, 9, 16, 25\}$ ■

(d) $f: \{0, 1\}^5 \rightarrow \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x . For example $f(01101) = 3$, because there are three 1's in the string "01101".

Solution: $\{0, 1, 2, 3, 4, 5\}$ ■

(h) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

Solution: $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ ■

(i) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

Solution: $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$ ■

(l) Let $A = \{1, 2, 3\}$. $f: P(A) \rightarrow P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

Solution: $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$ ■

Question 4

I. Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.2.2: Properties of algebraic functions.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(c) $h : \mathbb{Z} \rightarrow \mathbb{Z}, h(x) = x^3$

Solution: One-to-one but not onto. For example, there is no $x \in \mathbb{Z}$ such that $x^3 = 2$. ■

(g) $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 1, 2y)$

Solution: One-to-one but not onto. For example, there is no $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that $f(x, y) = (1, 3)$ ■

(k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+, f(x, y) = 2^x + y$

Solution: Not onto. For example, there is no $x \in \mathbb{Z}^+$ such that $f(x, y) = 1$ because the minimum value of f is 3 where $x = 1$ and $y = 1$. Not one-to-one. For example, $f(2, 1) = f(1, 3) = 5$. ■

b. Exercise 4.2.4: Properties of functions on strings and power sets.

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(b) $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

Solution: Not onto. For example, there is no $x \in \{0, 1\}^3$ such that $f(x) = (001)$. Not one-to-one. For example, $f(100) = f(000) = (100)$. ■

(c) $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

Solution: Onto and one-to-one. ■

(d) $f : \{0, 1\}^3 \rightarrow \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.

Solution: One-to-one and not onto. For example, there is no $x \in \{0, 1\}^3$ such that $f(x) = (0001)$. ■

(g) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f : P(A) \rightarrow P(A)$. For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A , $P(A)$ denotes the power set of A which is the set of all subsets of A .

Solution: Not onto. For example, there is no $x \in X$ such that $f(x) = \emptyset$. Not one-to-one. For example, $f(\{1, 2\}) = f(\{2\}) = \{2\}$ ■

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto

Solution: $f : \mathbb{Z} \rightarrow \mathbb{Z}^+, f(x) = \begin{cases} 2|x|, & \text{if } x > 0 \\ 2|x| + 3, & \text{otherwise} \end{cases}$ ■

b. onto, but not one-to-one

Solution: $f : \mathbb{Z} \rightarrow \mathbb{Z}^+, f(x) = |x|$ ■

c. one-to-one and onto

Solution: $f : \mathbb{Z} \rightarrow \mathbb{Z}^+, f(x) = \begin{cases} 2|x|, & \text{if } x > 0 \\ 2|x| + 1, & \text{otherwise} \end{cases}$ ■

d. neither one-to-one nor onto

Solution: $f : \mathbb{Z} \rightarrow \mathbb{Z}^+, f(x) = x^2$ ■

Question 5

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2: Finding inverses of functions.

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1} .

(c) $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3$

Solution: $f^{-1}(x) = 1/2(x - 3)$ ■

(d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $f : P(A) \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A , $P(A)$ denotes the power set of A which is the set of all subsets of A .

Solution: The function f is not one-to-one, so f^{-1} is not well-defined. ■

(g) $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

Solution: $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$. ■

(i) $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

Solution: $f^{-1}(x, y) = (x - 5, y + 2)$ ■

b) Exercise 4.4.8: Explicit formulas for compositions of functions.

The domain and target set of functions f , g , and h are \mathbb{Z} . The functions are defined as:

- $f(x) = 2x + 3$

- $g(x) = 5x + 7$

- $h(x) = x^2 + 1$

Give an explicit formula for each function given below.

(c) $f \circ h$

Solution: $f \circ h(x) = 2x^2 + 5$ ■

(d) $h \circ f$

Solution: $h \circ f(x) = 4x^2 + 12 + 10$ ■

c) Exercise 4.4.2: Composition of functions on integers.

Consider three functions f , g , and h , whose domain and target are \mathbb{Z} . Let

- $f(x) = x^2$
- $g(x) = 2^x$
- $h(x) = \lceil x/5 \rceil$

(b) Evaluate $f \circ h(52)$

Solution: 121 ■

(c) Evaluate $g \circ h \circ f(4)$

Solution: 16 ■

(d) Give a mathematical expression for $h \circ f$.

Solution: $h \circ f(x) = \lceil x^2/5 \rceil$ ■

d) Exercise 4.4.6: Composition of functions on sets of strings.

Define the following functions f , g , and h :

- $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of h is obtained by taking the input string x , and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

(c) What is $h \circ f(010)$?

Solution: 111 ■

(d) What is the range of $h \circ f$?

Solution: $\{101, 111\}$ ■

(e) What is the range of $g \circ f$?

Solution: $\{001, 011, 111, 101\}$ ■

e) **Extra Credit Exercise 4.4.4:** Composition of onto and one-to-one functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g .

Solution: No. If we assume that f is not one-to-one, then for some x_1 and x_2 where $x_1 \in X$ & $x_2 \in X$ & $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = k$. If $g \circ f$ is one-to-one, then $g(f(x_1)) = g(k) \neq g(f(x_2)) = g(k)$, which is contradictory. ■

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g .

Solution: Yes. The diagram below illustrates an example:

