NYU Computer Science Bridge to Tandon Course Winter 2021

Homework 8

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Question 7

(a) **Exercise 6.1.5:** The probability of an event under the uniform distribution - 5-card hands. A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, $\{4\heartsuit, 4\diamondsuit, 4\clubsuit, J\spadesuit, 8\heartsuit\}$ is a three of a kind.

solution:
There are 13 ranks to be chosen from and 3 out of 4 suits to be chosen from for the 3-kind cards, combining with the remaining cards, which has 2 different ranks from the remaining 12 ranks, and 4 suits to be chosen from for each of the remaining cards.

$$\frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1}^2}{\binom{52}{5}} \approx 0.21128 \blacksquare$$

(c)What is the probability that all 5 cards have the same suit?

solution:
There are 4 suits to be chosen from to satisfy the condition, and there are 5 out of 13 ranks in one suit to be chosen from for the chosen cards.

$$\frac{\binom{4}{1} \times \binom{13}{5}}{\binom{52}{5}} \approx 0.00198 \blacksquare$$

(d)What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, $\{4\spadesuit, 4\diamondsuit, J\spadesuit, K\clubsuit, 8\heartsuit\}$ is a two of a kind.

solution:
There are 13 ranks to be chosen from and 2 out of 4 suits to be chosen from for the 2-pair cards, combining with the remaining cards, which has 3 different ranks from the remaining 12 ranks, and 4 suits to be chosen from for each of the remaining cards.

$$\frac{\binom{13}{1} \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1}^3}{\binom{52}{5}} \approx 0.4225 \blacksquare$$

(b) **Exercise 6.2.4:** The probability of events - 5-card hands.
A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a)The hand has at least one club.
solution:

$$1 - \frac{\binom{39}{5}}{\binom{52}{5}} \approx 0.77846 \blacksquare$$

(b)The hand has at least two cards with the same rank.
solution:

$$1 - \frac{\binom{13}{5} \times \binom{4}{1}^5}{\binom{52}{5}} \approx 0.49291716687 \blacksquare$$

(c)The hand has exactly one club or exactly one spade.
solution:
The probability is calculated by adding the probability of exactly one club to the probability of exactly one spade, subtracted by the probability of exactly one spade and one club

$$\frac{\binom{13}{1} \times \binom{39}{4}}{\binom{52}{5}} \times 2 - \frac{\binom{13}{1}^2 \times \binom{26}{3}}{\binom{52}{5}} = \frac{1069263 \times 2 - 439400}{\binom{52}{5}} = \frac{1699126}{2598960} \approx 0.6537715086 \blacksquare$$

(d)The hand has at least one club or at least one spade.
solution:
$1 - \binom{26}{5}/\binom{52}{5} \approx 0.97469 \blacksquare$

Question 8

(a) **Exercise 6.3.2:** Calculating conditional probabilities - random permutations.

The letters a, b, c, d, e, f, g are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)
B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.
C: The letters "def" occur together in that order (e.g. "gdefbca")

(a)Calculate the probability of each individual event. That is, calculate p(A), p(B), and p(C).
solution:

$p(A) = \frac{6!}{7!} = \frac{1}{7}$

$p(B) = \frac{6!+5!\times5+5!\times4+5!\times3+5!\times2+5!\times1}{7!} = \frac{1}{2}$

$p(C) = \frac{5!}{7!} = \frac{1}{42}$ ■

(b)What is p(A|C)?
solution:

$\frac{p(A\cap C)}{p(C)} = \frac{3!\times2}{5!} = \frac{1}{10}$ ■

(c)What is p(B|C)?
solution:

$\frac{p(B\cap C)}{p(C)} = \frac{4!+3!\times3+3!\times2+4!\times1}{5!} = \frac{1}{2}$ ■

(d)What is p(A|B)?
solution:

$\frac{p(A\cap B)}{p(B)} = \frac{5!\times3}{6!+5!\times5+5!\times4+5!\times3+5!\times2+5!\times1} = \frac{1}{7}$ ■

(e)Which pairs of events among A, B, and C are independent?
solution: (B, C) and (A, B) ■

(b) **Exercise 6.3.6:** Calculating probabilities of independent events.
A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is 1/3 and the probability of tails is 2/3. The outcomes of the coin flips are mutually independent.

What is the probability of each event?

(b)The first 5 flips come up heads. The last 5 flips come up tails.
solution: $(\frac{1}{3})^5 \times (\frac{2}{3})^5$ ■

(c)The first flip comes up heads. The rest of the flips come up tails.
solution: $(\frac{1}{3}) \times (\frac{2}{3})^9$ ■

(c) **Exercise 6.4.2:** Bayes' Theorem - detecting a loaded die.
(a)Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

solution:
The dice is chosen at random. Thus, $p(F) = p(\bar{F}) = \frac{1}{2}$ as the F to be the event that the fair die is chosen. Let Y be the event that the outcome is 4, 3, 6, 6, 5, 5.
Using Bayes' Theorem:

$$\begin{aligned} p(F \mid Y) &= \frac{p(Y \mid F)p(F)}{(Y \mid F)p(F) + p(Y \mid \bar{F})p(\bar{F})} \\ &= \frac{(\frac{1}{6})^6(\frac{1}{2})}{(\frac{1}{6})^6(\frac{1}{2}) + (0.15)^4(0.25)^2(\frac{1}{2})} \\ &\approx 0.404 \ ■ \end{aligned}$$

Question 9

(a) **Exercise 6.5.2:** Distribution over a random variable - aces in a 5-card hand.
A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a)What is the range of A?
solution: {0, 1, 2, 3, 4} ■

(b)Give the distribution over the random variable A.
solution: $(0, \binom{48}{5}/\binom{52}{5})$, $(1, 4\binom{48}{4}/\binom{52}{5})$, $(2, \binom{4}{2}\binom{48}{3}/\binom{52}{5})$, $(3, 4\binom{48}{2}/\binom{52}{5})$, $(4, 48/\binom{52}{5})$ ■

(b) **Exercise 6.6.1:** Random variable expectations - expected number of girls chosen in a group.
(a)Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

solution:

$$
\begin{aligned}
E[G] &= 1 \times \frac{7 \times 3}{\binom{10}{2}} + 2 \times \frac{\binom{7}{2}}{\binom{10}{2}} \\
&= \frac{21 + 2 \times 21}{45} \\
&= 1.4 \blacksquare
\end{aligned}
$$

(c) **Exercise 6.6.4:** Expected values of squares.
(a)A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?

solution:

$$
\begin{aligned}
E[X] &= 1(\frac{1}{6}) + 4(\frac{1}{6}) + 9(\frac{1}{6}) + 16(\frac{1}{6}) + 25(\frac{1}{6}) + 36(\frac{1}{6}) \\
&= \frac{91}{6} \blacksquare
\end{aligned}
$$

(b)A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and Y = 4. What is E[Y]?

solution:

$$
\begin{aligned}
E[Y] &= 1(\frac{3}{8}) + 4(\frac{3}{8}) + 9(\frac{1}{8}) \\
&= \frac{24}{8} = 3
\end{aligned}
$$

(d) **Exercise 6.7.4:** Expected values - matching coats.

(a)A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

solution:

Let P_i be a random variable that is equal to 1 if the i th child who get his or her coat. If P is the number of children who have the same birthday, then $P = P_1 + \ldots + P_{10}$.

The probability that any child get his or her coat is 1/10. Thus, $E[P_i] = 1/10$.
We have $E[P] = 1/10 \times 10 = 1$ ■

Question 10

(a) **Exercise 6.8.1:** Probability of manufacturing defects.
The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a)What is the probability that out of 100 circuit boards made exactly 2 have defects?

solution:
$b(2; 100, 0.01) = \binom{100}{2} \cdot (0.01)^2 \cdot (0.99)^{98}$ ■

(b)What is the probability that out of 100 circuit boards made at least 2 have defects?

solution:

$$\begin{aligned} 1 - b(0; 100, 0.01) - b(1; 100, 0.01) &= 1 - \binom{100}{0} \cdot (0.01)^0 \cdot (0.99)^{100} - \binom{100}{1} \cdot (0.01)^1 \cdot (0.99)^{99} \\ &= 1 - (0.99)^{100} - (0.99)^{99} \ \blacksquare \end{aligned}$$

(c)What is the expected number of circuit boards with defects out of the 100 made?

solution:
Let X be the random variable that denotes the number of circuit boards with defects out of the 100. Then $E[X] = 100 \cdot 0.01 = 1$ ■

(d)Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

solution:
$1 - b(0; 50, 0.01) = 1 - \binom{50}{0} \cdot (0.01)^0 \cdot (0.01)^{50}$

Let Y be the variable that denotes the number of batches with defects out of the 50 batches. $E[Y] = 50 \cdot 0.01 = 0.5$. Becasue X = 2Y, $E[X] = E[2Y] = 2E[Y] = 1$. The situation where the circuit board is made in batches of two shares the same expected value of individual circuit board defects with the situation where circuit board is made separately. In essence, two situations entail that the possibility of a circuit board defect is equally likely. However, the minimum unit of the batches situation is 2 circuit boards, resulting in a different binomial distribution. ■

(b) **Exercise 6.8.3:** Detecting a biased coin

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

(b)What is the probability that you reach an incorrect conclusion if the coin is biased?

solution:

The incorrect conclusion is reached if there are at least four heads. The probability that there are 4, 5, 6, 7, 8, 9, or 10 heads when using a biased coin is:

$$1 - \binom{10}{0} \cdot (0.3)^0 \cdot (0.7)^{10} - \binom{10}{1} \cdot (0.3)^1 \cdot (0.7)^9 - \binom{10}{2} \cdot (0.3)^2 \cdot (0.7)^8 - \binom{10}{3} \cdot (0.3)^3 \cdot (0.7)^7 \blacksquare$$