NYU Computer Science Bridge to Tandon Course Winter 2021

Homework 5

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Question 3

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.1.3: Recognizing well-defined algebraic functions and their ranges.
Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

(b) $f(x) = \frac{1}{(x^2-4)}$
Solution: Not a function. If x is 2 or -2, then f(x) is not defined, not a real number. ■

(c) $f(x) = \sqrt{x^2}$
Solution: A function from $\mathbb{R}$ to $\mathbb{R}$; range of $f : \{y \mid y \geq 0\}$. ■

b) Exercise 4.1.5:Range of a function.
Express the range of each function using roster notation.

(b) Let A = {2, 3, 4, 5}. f: A $\rightarrow \mathbb{Z}$ such that $f(x) = x^2$.
Solution: {4, 9, 16, 25} ■

(d) $f : \{0, 1\}^5 \rightarrow \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x.
For example f(01101) = 3, because there are three 1's in the string "01101".
Solution: {0, 1, 2, 3, 4, 5} ■

(h)Let A = {1, 2, 3}. f: A × A $\rightarrow \mathbb{Z}\times\mathbb{Z}$, where $f(x, y) = (y, x)$.
Solution: {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} ■

(i)Let A = {1, 2, 3}. f: A × A $\rightarrow \mathbb{Z}\times\mathbb{Z}$, where $f(x, y) = (x, y + 1)$.
Solution: {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)} ■

(l)Let A = {1, 2, 3}. f: P(A) $\rightarrow$ P(A). For X $\subseteq$ A, f(X) = X - {1}.
Solution: {∅, {2}, {3}, {2, 3}} ■

Question 4

I. Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.2.2: Properties of algebraic functions.
For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(c)$h : \mathbb{Z} \to \mathbb{Z} . h(x) = x^3$
Solution: One-to-one but not onto. For example, there is no $x \in \mathbb{Z}$ such that $x^3 = 2$. ■

(g)$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x,\, y) = (x + 1,\, 2y)$
Solution: One-to-one but not onto. For example, there is no $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that $f(x, y) = (1, 3)$ ■

(k)$f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+, f(x,\, y) = 2^x + y$
Solution: Not onto. For example, there is no $x \in \mathbb{Z}^+$ such that $f(x,\, y) = 1$ because the minimum value of f is 3 where x = 1 and y = 1. Not one-to-one. For example, f(2, 1) = f(1, 3) = 5. ■

b. Exercise 4.2.4: Properties of functions on strings and power sets.
For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(b)$f : \{0,\, 1\}^3 \to \{0,\, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.
Solution: Not onto. For example, there is no $x \in \{0,\, 1\}^3$ such that $f(x) = (001)$. Not one-to-one. For example, f(100) = f(000) = (100). ■

(c)$f : \{0,\, 1\}^3 \to \{0,\, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.
Solution: Onto and one-to-one. ■

(d)$f : \{0,\, 1\}^3 \to \{0,\, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.
Solution: One-to-one and not onto. For example, there is no $x \in \{0,\, 1\}^3$ such that $f(x) = (0001)$. ■

(g)Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8} and let B = {1}. $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - B$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.
Solution: Not onto. For example, there is no $x \in X$ such that $f(x) = \{1\}$. Not one-to-one. For example,$f(\{1, 2\}) = f(\{2\}) = \{2\}$■

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto

Solution: $f : \mathbb{Z} \to \mathbb{Z}^+, f(x) = \begin{cases} 2|x|, & \text{if } x > 0 \\ 2|x| + 3, & \text{otherwise} \end{cases}$ ■

b. onto, but not one-to-one

Solution: $f : \mathbb{Z} \to \mathbb{Z}^+, f(x) = |x| + 1$■

c. one-to-one and onto

Solution: $f : \mathbb{Z} \to \mathbb{Z}^+, f(x) = \begin{cases} 2|x|, & \text{if } x > 0 \\ 2|x| + 1, & \text{otherwise} \end{cases}$ ■

d. neither one-to-one nor onto

Solution: $f : \mathbb{Z} \to \mathbb{Z}^+, f(x) = x^2 + 1$■

Question 5

Solve the following questions from the Discrete Math zyBook:

a) Exercise 4.3.2: Finding inverses of functions.
For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

(c)$f : \mathbb{R} \to \mathbb{R}. f(x) = 2x + 3$
Solution: $f^{-1}(x) = \frac{(x-3)}{2}$ ■

(d)Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. $f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For $X \subseteq A, f(X) = |X|$. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.
Solution: The function f is not one-to-one, so f^{-1} is not well-defined. ■

(g)$f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, f(011) = 110.
Solution: $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, f(011) = 110. ■

(i)$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$
Solution: $f^{-1}(x, y) = (x - 5, y + 2)$ ■

b) Exercise 4.4.8: Explicit formulas for compositions of functions.
The domain and target set of functions f, g, and h are Z. The functions are defined as:

- f(x) = 2x + 3
- g(x) = 5x + 7
- h(x) = x^2 + 1

Give an explicit formula for each function given below.

(c)f o h
Solution: f o $h(x) = 2x^2 + 5$ ■

(d)h o f
Solution: h o $f(x) = 4x^2 + 12 + 10$ ■

c) Exercise 4.4.2: Composition of functions on integers.
Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let

- $f(x) = x^2$
- $g(x) = 2^2$
- $h(x) = \lceil \frac{x}{5} \rceil$

(b)Evaluate f o h(52)
Solution: 121 ■

(c)Evaluate g o h o f(4)
Solution: 16 ■

(d)Give a mathematical expression for h o f.
Solution: h o $f(x) = \lceil \frac{x^2}{5} \rceil$ ■

d) Exercise 4.4.6: Composition of functions on sets of strings.
Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, f(001) = 101 and f(110) = 110.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, g(011) = 110.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, h(011) = 010.

(c)What is h o f(010)?
Solution: 111 ■

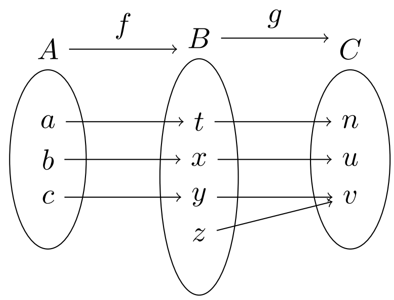
(d)What is the range of h o f?
Solution: {101, 111} ■

(e)What is the range of g o f?
Solution: {001, 011, 111, 101} ■

e) ***Extra Credit*** **Exercise 4.4.4**: Composition of onto and one-to-one functions.

(c)Is it possible that f is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.
Solution: No. If we assume that f is not one-to-one, then for some x_1 and x_2 where $x_1 \in X \,\&\, x_2 \in$ $\&\, x_1 \neq x_2$ such that $f(x_1) = f(x_2) = k$. If g o f is one-to-one, then $g(f(x_1)) = g(k) \neq g(f(x_2) = g(k)$, which is contradictory. ■

(d)Is it possible that g is not one-to-one and g o f is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.
Solution: Yes. The diagram below illustrates an example:



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