GNSS Observables

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Pseudorange Observations

- The pseudoranges between the satellite and the receiver are derived from the difference in reception time and transmission time of an encoded satellite signal.
- The pseudorange of a reception epoch t_r can be written as

$$
\begin{aligned}
P_r^s(t_r) &= c \cdot (t_r - t^s) = c \cdot [(T_r + \delta_r) - (T^s + \delta^s)] \\
&= c \cdot \tau + c \cdot (\delta_r - \delta^s) = \rho + c \cdot (\delta_r - \delta^s)
\end{aligned}
$$

  t_r - the time given by the receiver clock at signal reception;
  t^s - the time given by satellite clock at signal transmission;
  T_r, T^s - true reception and transmission time, respectively;
  δ_r, δ^s - the receiver and satellite clock errors, respectively;
  c - travel speed; ρ - true range

- Pseudorange is also called *code observable*
- The precision of a pseudorange is about 1% of the chip length, for example
  - C/A code $\longrightarrow$ Chip length=300 m $\longrightarrow$ Precision=3 m
  - P code $\longrightarrow$ Chip length=30 m $\longrightarrow$ Precision=0.3 m

Carrier Phase Observations

- Carrier phase observations are obtained by comparing the phases between a signal transmitted by a satellite and a similar signal generated by a receiver.
- Given an initial epoch t_0, the satellite carrier wave and receiver carrier wave have initial phases of ϕ_0^s and ϕ_{0r}, respectively.
- At a signal reception time t_r given by the receiver clock, the phase of the local carrier wave (cycles) is

$$\phi_r(t_r) = f_r \cdot (t - t_0) + \phi_{0r}$$

  where f_r is the receiver clock frequency, and t is the true reception time.
- The phase of the received signal is the phase of the transmitted signal, delayed by the travel time from satellite to the receiver τ:

$$\phi^s(t_r) = f^s \cdot (t - \tau - t_0) + \phi_0^s$$

  where f^s is the satellite clock frequency.

Carrier Phase Observations

- At the reception time t_r, the phase difference is

$$\begin{aligned}\Delta\phi_r^s(t_r) &= \phi_r(t_r) - \phi^s(t_r) \\ &= f_r \cdot (t - t_0) - f^s \cdot (t - \tau - t_0) + \underbrace{(\phi_{0r} - \phi_0^s)}_{:=\Delta\phi_0 \text{ 'initial phase bias'}}\end{aligned}$$

- ϕ_{0r} and ϕ_0^s are often referred to as receiver and satellite hardware phase delays, or uncalibrated phase delays (UPDs).
- At the initial locking epoch, carrier phase (in radians) is measured modulo 2π, i.e. $\phi_r^s = mod(\Delta\phi_r^s, 2\pi)$, and the integer number of cycles N ("ambiguity") is unknown. Since after, ϕ_r^s is measured continuously
- Ideally, $f^s = f_r = f$, where f is the nominal frequency standard. In this case, the carrier phase measurement can be written as

$$\phi_r^s(t_r) = f \cdot \tau + \Delta\phi_0 + N$$

Carrier Phase Observations

- In practice, both satellite and receiver oscillators have small perturbations in the frequencies, i.e., clock errors:

$$f^s(t) = f + \delta f^s(t); \qquad f_r(t) = f + \delta f_r(t)$$

- Then the carrier phase measurement is

$$\phi_r^s(t_r) = f \cdot \tau + \Delta\phi_0 + N + \underbrace{\delta f_r(t) \cdot (t - t_0)}_{\text{"receiver clk err"} = f \cdot \delta_r} - \underbrace{\delta f^s(t) \cdot (t - t_0)}_{\text{"satellite clk err"} = f \cdot \delta^s} + \delta f^s(t) \cdot \tau$$

- For $\frac{\delta f^s}{f} \approx 10^{-12}$ and $\tau \approx 0.09$ *seconds*, then the last term $\delta f^s(t) \cdot \tau < 10^{-4}$ cycles or 0.02 mm, and can be neglected.
- Thus, carrier phase measurement can be written as

$$\phi_r^s(t_r) = f \cdot \tau + \Delta\phi_0 + N + f \cdot \delta_r - f \cdot \delta^s$$

Carrier Phase Observations

- Carrier phase in unit of distance can be written as

$$\Phi^s_{r,f}(t_r) = \lambda_f \cdot \phi^s_r(t_r) = \rho + c \cdot (\delta_r - \delta^s) + \lambda_f \cdot N_f + \Delta\Phi_{0,f} + T - \frac{I}{f^2} + \varepsilon_{\phi_f}$$

  where λ_f is the wavelength of the carrier wave; $\Delta\Phi_{0,f} = \lambda_f \cdot \Delta\phi_{0,f}$ is the initial bias; T is the tropospheric delay; $I = 40.3 \cdot TEC$; ε_{ϕ_f} represents the measurement noise.

- Note the similarity to pseudorange measurement

$$P^s_{r,f}(t_r) = \rho + c \cdot (\delta_r - \delta^s) + T + \frac{I}{f^2} + \varepsilon_{P_f}$$

- The signs of ionospheric errors are opposite for carrier phase and pseudorange observations, i.e., phase advance v.s. group delay.
- The precision of phase observations is normally about 1% of the wavelength. For GPS, $\lambda_1 = 19\ cm$ and $\lambda_2 = 24\ cm$.

Summary of Basic Observables

Given frequencies f_1 and f_2.

$$\Phi^s_{r,1} = \rho + c \cdot (\delta_r - \delta^s) + \lambda_1 \cdot N_1 - \frac{I}{f_1^2} + T + \Delta\Phi_{0,1} + M_{\Phi_1} + \varepsilon_{\Phi_1}$$

$$\Phi^s_{r,2} = \rho + c \cdot (\delta_r - \delta^s) + \lambda_2 \cdot N_2 - \frac{I}{f_2^2} + T + \Delta\Phi_{0,2} + M_{\Phi_2} + \varepsilon_{\Phi_2}$$

$$\phi^s_{r,1} = \lambda_1^{-1}\rho + f_1 \cdot (\delta_r - \delta^s) + N_1 - \frac{I}{c \cdot f_1} + \lambda_1^{-1} T + \Delta\phi_{0,1} + M_{\phi_1} + \varepsilon_{\phi_1}$$

$$\phi^s_{r,2} = \lambda_2^{-1}\rho + f_2 \cdot (\delta_r - \delta^s) + N_2 - \frac{I}{c \cdot f_2} + \lambda_2^{-1} T + \Delta\phi_{0,2} + M_{\phi_2} + \varepsilon_{\phi_2}$$

$$P^s_{r,1} = \rho + c \cdot (\delta_r - \delta^s) + \frac{I}{f_1^2} + T + \Delta P_{0,1} + M_{P_1} + \varepsilon_{P_1}$$

$$P^s_{r,2} = \rho + c \cdot (\delta_r - \delta^s) + \frac{I}{f_2^2} + T + \Delta P_{0,2} + M_{P_2} + \varepsilon_{P_2}$$

where

$$\Delta\Phi_{0,f} = \Phi_{0r,f} - \Phi^s_{0,f}$$
$$\Delta P_{0,f} = P_{0r,f} - P^s_{0,f}$$

Hardware delays $\Phi_{0r,f}$ and $\Phi^s_{0,f}$ are also called uncalibrated phase delays (UPDs) of the receiver and satellites for frequency f, respectively. Similarly, $P_{0r,f}$ and $P^s_{0,f}$ are uncalibrated code delays (UCDs).

Geometry-free Combination (Ionospheric functions)

$$P_I = P_1 - P_2 = (\frac{1}{f_1^2} - \frac{1}{f_2^2}) \cdot I + \Delta P_{0,I} + \varepsilon_{P_I}$$

$$\Phi_I = \Phi_2 - \Phi_1 = (\frac{1}{f_1^2} - \frac{1}{f_2^2}) \cdot I + (\lambda_2 N_2 - \lambda_1 N_1) + \Delta\Phi_{0,I} + \varepsilon_{\Phi_I}$$

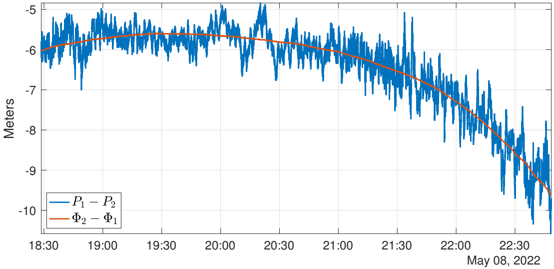


Figure 1: Geometry-free combination (Pseudorange and Phase for PRN06). Pseudoranges are noisy but unambiguous; Carrier phases are precise but ambiguous.

Geometry-free Combination (Ionospheric functions)

- Geometry-free combination
  1. cancels all frequency-independent effects (i.e., geometry & troposphere)
  2. leaves all frequency-dependent effects (i.e., ionosphere, hardware delays & multipath)
  3. often used to estimate ionospheric electron content
  4. also used to detect cycle slips in carrier phase measurements.

Ionosphere-free Combination

- The ionosphere-free combination of carrier phase is

$$\begin{aligned}\Phi^s_{r,c} &= \frac{f_1^2 \cdot \Phi^s_{r,1} - f_2^2 \cdot \Phi^s_{r,2}}{f_1^2 - f_2^2} \\ &= \rho + c \cdot (\delta_r - \delta^s) + c \cdot \frac{f_1 - f_2}{f_1^2 - f_2^2} \cdot N_1 + c \cdot \frac{f_2}{f_1^2 - f_2^2} \cdot \underbrace{(N_1 - N_2)}_{N_{12}} + T + \Delta\Phi_{0,c} + \varepsilon_{\phi_c}\end{aligned}$$

  - For GPS L1 and L2 frequencies

$$\Phi^s_{r,c} = \rho + c \cdot (\delta_r - \delta^s) + \underbrace{\frac{2c \cdot f_0}{f_1^2 - f_2^2}}_{:=\lambda_c} \cdot \underbrace{(17N_1 + 60N_{12})}_{:=N_c} + T + \Delta\Phi_{0,c} + \varepsilon_{\phi_c}$$

- Similarly, the ionosphere-free combination of pseudorange is

$$P^s_{r,c} = \frac{f_1^2 \cdot P^s_{r,1} - f_2^2 \cdot P^s_{r,2}}{f_1^2 - f_2^2} = \rho + c \cdot (\delta_r - \delta^s) + T + \Delta P_{0,c} + \varepsilon_{P_c}$$

Ionosphere-free Combination

- Advantage: removes the 1st order (99.9%) ionospheric errors.
- Drawbacks:
  1. The integer nature of ambiguities is not preserved.
  2. The level of noise will be amplified. Take GPS for example,
     $f_1 = 154 f_0, f_2 = 120 f_0 \Rightarrow \Phi^s_{r,c} = 2.55\Phi^2_{r,1} - 1.55\Phi^2_{r,2}$;
     if $\sigma_\Phi = 5\ mm\ (i.e., 0.025\ cycle) \Rightarrow \sigma_{\Phi_c} = 2.98\sigma_\Phi$

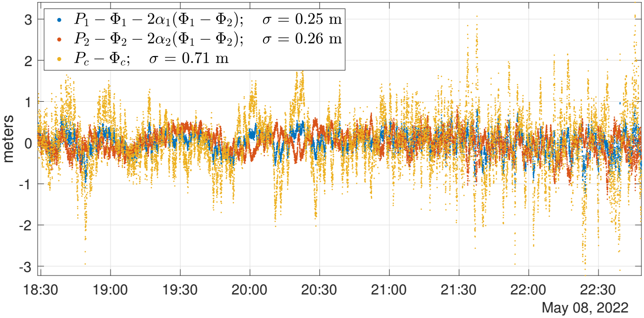


Figure 2: Comparison of pseudorange noise for GPS P_1 and P_2, and their Iono-free combination (PRN06).

Ionosphere-free Combination

Figure 3 doesn't show the raw pseudoranges, but the following combinations that remove geometry and dispersive delays, and thus enable the better demonstration of pseudorange noise:

$$P_1 - \Phi_1 - 2\alpha_1(\Phi_1 - \Phi_2) = \varepsilon_{P_1} - \varepsilon_{\Phi_1} + \Delta P_{0,1} - (\lambda_1 N_1 + \Delta\Phi_{0,1}) - 2\alpha_1[(\lambda_1 N_1 + \Delta\Phi_{0,1}) - (\lambda_2 N_2 + \Delta\Phi_{0,2})]$$

$$P_2 - \Phi_2 - 2\alpha_2(\Phi_1 - \Phi_2) = \varepsilon_{P_2} - \varepsilon_{\Phi_2} + \Delta P_{0,2} - (\lambda_2 N_2 + \Delta\Phi_{0,2}) - 2\alpha_2[(\lambda_1 N_1 + \Delta\Phi_{0,1}) - (\lambda_2 N_2 + \Delta\Phi_{0,2})]$$

$$P_c - \Phi_c = \varepsilon_{P_c} - \varepsilon_{\Phi_c} + \Delta P_{0,c} - (\lambda_c N_c + \Delta\Phi_{0,c})$$

where

$$\alpha_1 = \frac{f_2^2}{f_1^2 - f_2^2}; \quad \alpha_2 = \frac{f_1^2}{f_1^2 - f_2^2}$$

Wide-lane (WL) Combination

- Wide-lane combination

$$\phi_w = \phi_1 - \phi_2 = \underbrace{(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})}_{\lambda_w^{-1}}\rho + \underbrace{(f_1 - f_2)}_{f_w}(\delta_r - \delta^s) + \underbrace{(N_1 - N_2)}_{N_w} + \lambda_w^{-1}\frac{I}{f_1 \cdot f_2} + \lambda_w^{-1}T + \Delta\phi_{0,w} + \varepsilon_{\phi_w}$$

$$\Phi_w = \lambda_w \cdot \phi_w = \frac{f_1\Phi_1 - f_2\Phi_2}{f_1 - f_2} = \rho + c \cdot (\delta_r - \delta^s) + \lambda_w \cdot (N_1 - N_2) + \frac{I}{f_1 \cdot f_2} + T + \Delta\Phi_{0,w} + \varepsilon_{\Phi_w}$$

$$P_w = \frac{f_1 P_1 - f_2 P_2}{f_1 - f_2} = \rho + c \cdot (\delta_r - \delta^s) - \frac{I}{f_1 \cdot f_2} + T + \Delta P_{0,w} + \varepsilon_{P_w}$$

- Create a signal with a significant wide wavelength, and thus useful for cycle-slip detection and ambiguity fixing.
    - Take GPS f_1 and f_2 frequencies for example:
      $f_w = f_1 - f_2 = 347.82\ MHz; \quad \lambda_w = \frac{c}{f_w} = 0.862\ m$
- The wide-lane measurements are significantly noisier than the Φ_1 and Φ_2 measurements.
    - Take GPS f_1 and f_2 frequencies for example. $\Phi_w = \frac{154}{34}\Phi_1 - \frac{120}{34}\Phi_2$,
      $\Rightarrow \quad \sigma_{\Phi_w} = \sqrt{(\frac{154}{34})^2 + (\frac{120}{34})^2} \cdot \sigma_\Phi = 5.7\sigma_\Phi$
      The noise in Φ_w is amplified to about 6 times !

Narrow-lane (NL) Combination

- Narrow-lane combination

$$\phi_n = \phi_1 + \phi_2 = \underbrace{(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})}_{\lambda_n^{-1}} \rho + \underbrace{(f_1 + f_2)}_{f_n}(\delta_r - \delta^s) + \underbrace{(N_1 + N_2)}_{N_n} - \lambda_n^{-1} \frac{I}{f_1 \cdot f_2} + \lambda_n^{-1} T + \Delta\phi_{0,n} + \varepsilon_{\phi_n}$$

$$\Phi_n = \lambda_n \cdot \phi_n = \frac{f_1 \Phi_1 + f_2 \Phi_2}{f_1 + f_2} = \rho + c \cdot (\delta_r - \delta^s) + \lambda_n \cdot (N_1 + N_2) - \frac{I}{f_1 \cdot f_2} + T + \Delta\Phi_{0,n} + \varepsilon_{\Phi_n}$$

$$P_n = \frac{f_1 P_1 + f_2 P_2}{f_1 + f_2} = \rho + c \cdot (\delta_r - \delta^s) + \frac{I}{f_1 \cdot f_2} + T + \Delta P_{0,n} + \varepsilon_{P_n}$$

- Narrow-lane measurements have shorter wavelength, which makes it harder to resolve the corresponding integer ambiguity.
  - Take GPS f_1 and f_2 for example:
    $f_n = f_1 + f_2 = 2803.02\ MHz; \quad \lambda_n = \frac{c}{f_n} = 10.7\ cm$
- However, the narrow-lane measurements are less noisy than those at f_1 and f_2, and make the position estimates more precise than those from the wide-lane measurements.
  - Take GPS f_1 and f_2 frequencies for example. $\Phi_n = \frac{154}{274}\Phi_1 + \frac{120}{274}\Phi_2$,
    $\Rightarrow \quad \sigma_{\Phi_n} = \sqrt{(\frac{154}{274})^2 + (\frac{120}{274})^2} \cdot \sigma_\Phi = 0.7\sigma_\Phi$

Melbourne-Wubbena (MW) Wide-lane Combination

- $$N_{MW} = \frac{\Phi_w - P_n}{\lambda_w} = \underbrace{(N_1 - N_2)}_{N_w} + \frac{\Delta\Phi_{0,w} - \Delta P_{0,n}}{\lambda_w}$$
- MW combination provides a noisy estimation of the wide-land ambiguity N_w.

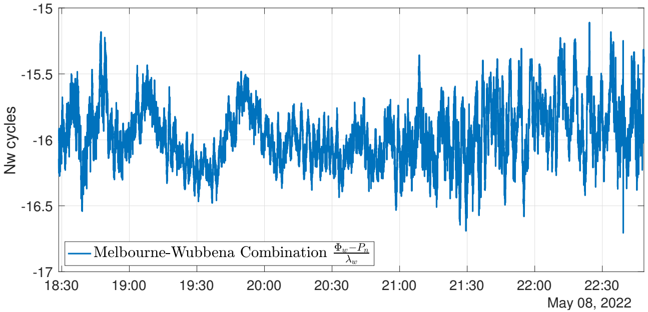
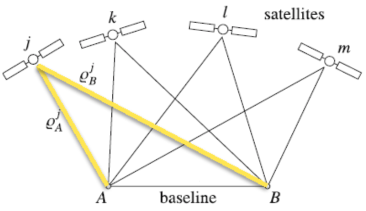


Figure 3: Melbourne-Wubbena combination (raw measurements without smoothing) (PRN06).

Single Difference

Across-Receiver Single Difference

It is formed by differencing the observations of two receivers (i.e., A, B) and the same satellite (i.e, j)"



$$\Phi_{AB}^j = \rho_{AB}^j + c \cdot \delta_{AB} + \lambda \cdot N_{AB}^j + T_{AB}^j - I_{AB}^j + \Phi_{0AB} + \varepsilon_{AB,\phi}^j \tag{1}$$

where $\Phi_{0AB} = \Phi_{0A} - \Phi_{0B}$ is the difference between the initial (receiver carrier wave) phases of the two receivers (or hardware delays of the two receivers)

Single Difference

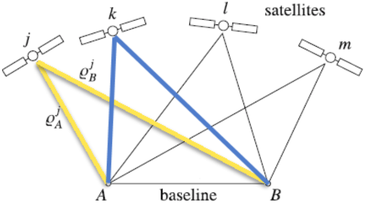
- To obtain position estimation with cm-level accuracy, the error terms in Equation (1) must be reduced to cm level
- To fix integer ambiguities requires the measurement error is less than $\frac{1}{4}$ cycles.
- "Short baseline": the residual T^j_{AB} and I^j_{AB} would be smaller than the typical errors due to receiver noise and multipath.
- For short baseline, the signal-difference measurement can be simplified as

$$\Phi^j_{AB} = \rho^j_{AB} + c \cdot \delta_{AB} + \lambda \cdot N^j_{AB} + \Phi_{0AB} + \varepsilon^j_{AB,\phi}$$

Note: tropospheric delay is usually corrected for the measurements at each receiver, separately.

Double Difference

- A double difference can be formed when two receivers observe two satellites *simultaneously*. It can be formed by differencing two across-receiver single differences or two across-satellite single differences



$$\Phi_{AB}^{jk} = \Phi_{AB}^{j} - \Phi_{AB}^{k} = \rho_{AB}^{jk} + \lambda \cdot N_{AB}^{jk} + T_{AB}^{jk} - I_{AB}^{jk} + \varepsilon_{AB,\phi}^{jk}$$

- The errors cancelled in double difference include: receiver and satellite clock errors, receiver and satellite hardware delays.
- For short baseline, the ionospheric and tropospheric delays can be largely cancelled.

Double Difference

- For short baseline, the double difference can be simplified as:

$$P_{AB}^{jk} = \rho_{AB}^{jk} + \varepsilon_{AB,p}^{jk}$$
$$\Phi_{AB}^{jk} = \rho_{AB}^{jk} + \lambda \cdot N_{AB}^{jk} + \varepsilon_{AB,\Phi}^{jk}$$

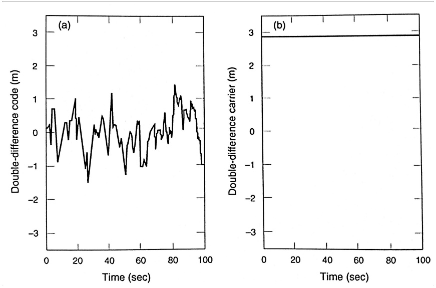


Figure 4: Code and carrier phase double differences from a 150-meter baseline.

- The difference between the above two plots is entirely due to integer

Triple Difference

- The triple difference is the difference of two double differences over time

$$\begin{aligned}\Delta\Phi_{AB}^{jk} &= \Phi_{AB}^{jk}(t_{i+1}) - \Phi_{AB}^{jk}(t_i) \\ &= \Delta\rho_{AB}^{jk} + \Delta T_{AB}^{jk} - \Delta I_{AB}^{jk} + \Delta\varepsilon_{AB,\Phi}^{jk}\end{aligned}$$

- All nuisance parameters are gone. The triple-difference observable is probably the easiest observable to process.
- The tripe-difference solution, which is less precise than double-difference solution, serves as an initial position estimation
- The triple differences are useful in identifying discontinuities in carrier tracking resulting in loss of cycle count or cycle slip.