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Riemann Xi function

In mathematics, the **Riemann Xi function** is a variant of the Riemann zeta function, and is defined so as to have a particularly simple functional equation. The function is named in honour of Bernhard Riemann.

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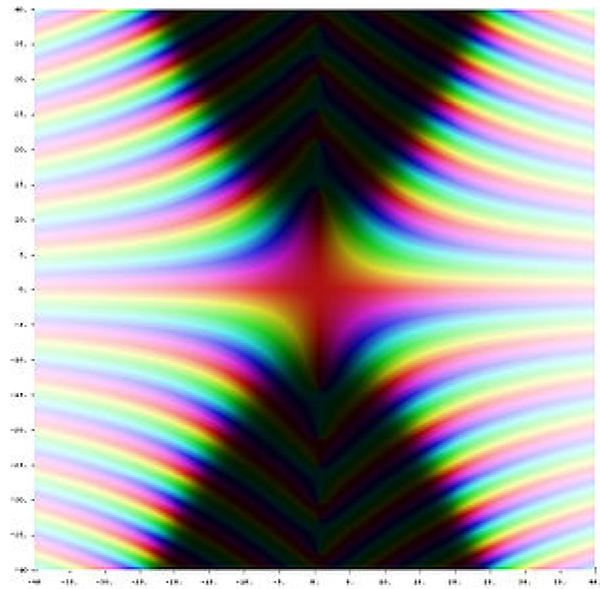

Riemann xi function $\xi(s)$ in the complex plane. The color of a point s encodes the value of the function. Darker colors denote values closer to zero and hue encodes the value's argument.

Definition

Riemann's original lower-case "xi"-function, ξ was renamed with an upper-case Ξ (Greek letter "Xi") by Edmund Landau. Landau's lower-case ξ ("xi") is defined as[1]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

for $s \in \mathbb{C}$. Here $\zeta(s)$ denotes the Riemann zeta function and $\Gamma(s)$ is the Gamma function.

The functional equation (or reflection formula) for Landau's ξ is

$$\xi(1-s) = \xi(s)\ .$$

Riemann's original function, rebaptised upper-case Ξ by Landau,[1] satisfies

$$\Xi(z) = \xi\left(\tfrac{1}{2} + zi\right),$$

and obeys the functional equation

$$\Xi(-z) = \Xi(z)\ .$$

Both functions are entire and purely real for real arguments.

Values

The general form for positive even integers is

$$\xi(2n) = (-1)^{n+1} \frac{n!}{(2n)!} B_{2n} 2^{2n-1} \pi^n (2n-1)$$

where B_n denotes the n-th Bernoulli number. For example:

$$\xi(2) = \frac{\pi}{6}$$

Series representations

The ξ function has the series expansion

$$\frac{d}{dz} \ln \xi\left(\frac{-z}{1-z}\right) = \sum_{n=0}^{\infty} \lambda_{n+1} z^n,$$

where

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[s^{n-1} \log \xi(s)\right]\Big|_{s=1} = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right],$$

where the sum extends over ρ, the non-trivial zeros of the zeta function, in order of $|\Im(\rho)|$.

This expansion plays a particularly important role in Li's criterion, which states that the Riemann hypothesis is equivalent to having $\lambda_n > 0$ for all positive n.

Hadamard product

A simple infinite product expansion is

$$\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right),$$

where ρ ranges over the roots of ξ.

To ensure convergence in the expansion, the product should be taken over "matching pairs" of zeroes, i.e., the factors for a pair of zeroes of the form ρ and 1−ρ should be grouped together.

References

1. Landau, Edmund (1974) [1909]. *Handbuch der Lehre von der Verteilung der Primzahlen* [*Handbook of the Study of Distribution of the Prime Numbers*] (Third ed.). New York: Chelsea. §70-71 and page 894.

Further references

- Weisstein, Eric W. "Xi-Function" (https://mathworld.wolfram.com/Xi-Function.html). *MathWorld*.
- Keiper, J.B. (1992). "Power series expansions of Riemann's xi function" (https://doi.org/10.1090%2FS0025-5718-1992-1122072-5). *Mathematics of Computation*. **58** (198): 765–773. Bibcode:1992MaCom..58..765K (https://ui.adsabs.harvard.edu/abs/1992MaCom..58..765K). doi:10.1090/S0025-5718-1992-1122072-5 (https://doi.org/10.1090%2FS0025-5718-1992-1122072-5).