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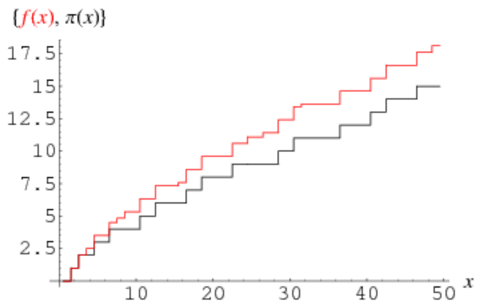
Riemann Prime Counting Function



Riemann defined the function $f(x)$ by

$$f(x) \equiv \sum_{\substack{p^\nu \le x \\ p \text{ prime}}} \frac{1}{\nu} \tag{1}$$

$$= \sum_{n=1}^{\lfloor \lg x \rfloor} \frac{\pi\left(x^{1/n}\right)}{n} \tag{2}$$

$$= \pi(x) + \tfrac{1}{2}\pi\left(x^{1/2}\right) + \tfrac{1}{3}\pi\left(x^{1/3}\right) + \ldots \tag{3}$$

(Hardy 1999, p. 30; Borwein *et al.* 2000; Havil 2003, pp. 189-191 and 196-197; Derbyshire 2004, p. 299), sometimes denoted $\pi^*(x)$, $J(x)$ (Edwards 2001, pp. 22 and 33; Derbyshire 2004, p. 298), or $\Pi(x)$ (Havil 2003, p. 189). Note that this is not an infinite series since the terms become zero starting at $n = \lfloor \lg x \rfloor$, and where $\lfloor x \rfloor$ is the floor function and $\lg x$ is the base-2 logarithm. For $x = 1$, 2, ..., the first few values are 0, 1, 2, 5/2, 7/2, 7/2, 9/2, 29/6, 16/3, 16/3, ... (OEIS A096624 and A096625). As can be seen, when x is a prime, $f(x)$ jumps by 1; when it is the square of a prime, it jumps by 1/2; when it is a cube of a prime, it jumps by 1/3; and so on (Derbyshire 2004, pp. 300-301), as illustrated above.

Amazingly, the prime counting function $\pi(x)$ is related to $f(x)$ by the Möbius transform

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f\left(x^{1/n}\right), \tag{4}$$

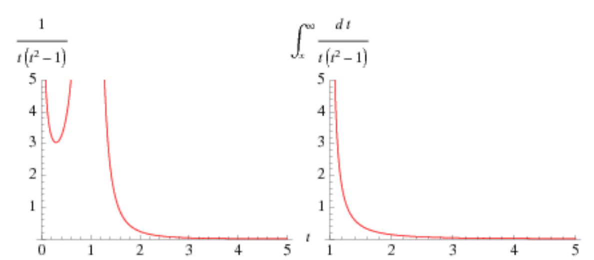
where $\mu(n)$ is the Möbius function (Riesel 1994, p. 49; Havil 2003, p. 196; Derbyshire 2004, p. 302). More amazingly still, $f(x)$ is connected with the Riemann zeta function $\zeta(s)$ by

$$\frac{\ln[\zeta(s)]}{s} = \int_0^\infty f(x)\, x^{-s-1}\, d x \tag{5}$$

(Riesel 1994, p. 47; Edwards 2001, p. 23; Derbyshire 2004, p. 309). $f(x)$ is also given by

$$f(x) = \lim_{t \to \infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{s} \ln \zeta(s)\, d s, \tag{6}$$

where $\zeta(z)$ is the Riemann zeta function, and (5) and (6) form a Mellin transform pair.



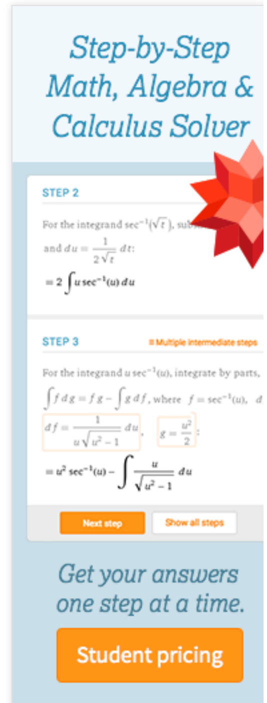
Riemann (1859) proposed that

$$f(x) = \operatorname{li}(x) - \sum_\rho \operatorname{li}\left(x^\rho\right) - \ln 2 + \int_x^\infty \frac{d t}{t\left(t^2-1\right)\ln t}, \tag{7}$$

where $\operatorname{li}(x)$ is the logarithmic integral and the sum is over all nontrivial zeros ρ of the Riemann zeta function $\zeta(z)$ (Mathews 1961, Ch. 10; Landau 1974, Ch. 19; Ingham 1990, Ch. 4; Hardy 1999, p. 40; Borwein *et al.* 2000; Edwards 2001, pp. 33-34; Havil 2003, p. 196; Derbyshire 2004, p. 328). Actually, since the sum of roots is only conditionally convergent, it must be summed in order of increasing $I[\rho]$ even when pairing terms ρ with their "twins" $1-\rho$, so

$$\sum_\rho \operatorname{li}\left(x^\rho\right) = \sum_{I[\rho]>0} \left[\operatorname{Li}\left(x^\rho\right) + \operatorname{Li}\left(x^{1-\rho}\right)\right] \tag{8}$$

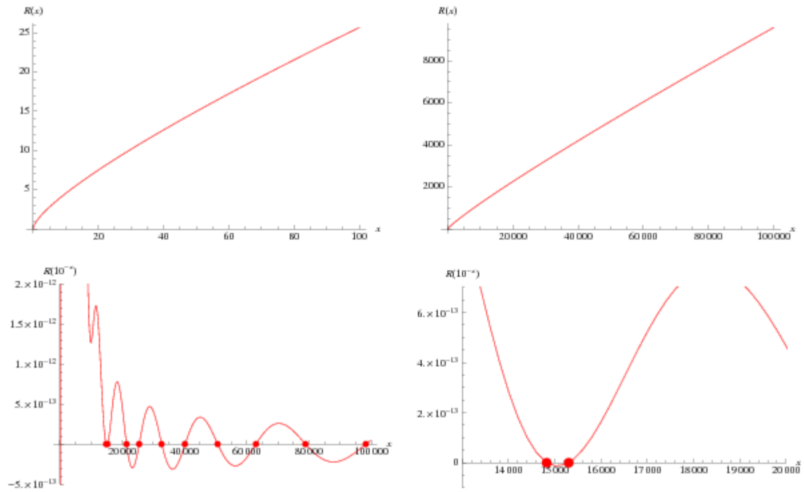
(Edwards 2001, pp. 30 and 33).



This formula was subsequently proved by Mangoldt (1895; Riesel 1994, p. 47; Edwards 2001, pp. 48 and 62-65). The integral on the right-hand side converges only for $x > 1$, but since there are no primes less than 2, the only values of interest are for $x \geq 2$. Since it is monotonic decreasing, the maximum therefore occurs at $x = 2$, which has value

$$\int_2^\infty \frac{d\,t}{t \ln t \left(t^2 - 1\right)} = 0.14001010114328692668 \ldots \qquad (9)$$

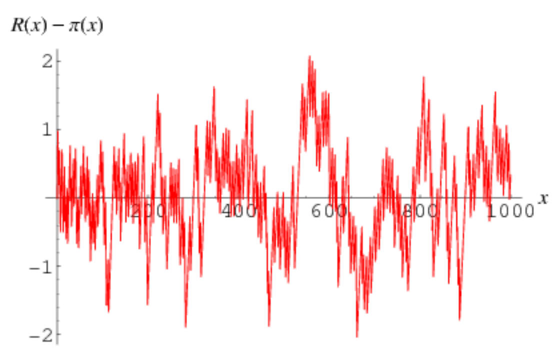
(OEIS A096623; Derbyshire 2004, p. 329).



Riemann also considered the function

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}\left(x^{1/n}\right), \qquad (10)$$

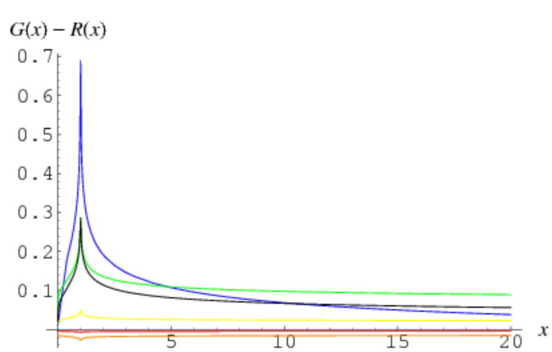
sometimes also denoted $\operatorname{Ri}(x)$ (Borwein *et al.* 2000), obtained by replacing $f\left(x^{1/n}\right)$ in the Riemann function with the logarithmic integral $\operatorname{li}\left(x^{1/n}\right)$, where $\zeta(z)$ is the Riemann zeta function and $\mu(n)$ is the Möbius function (Hardy 1999, pp. 16 and 23; Borwein *et al.* 2000; Havil 2003, p. 198). $R(x)$ is plotted above, including on a semilogarithmic scale (bottom two plots), which illustrate the fact that $R(x)$ has a series of zeros near the origin. These occur at 10^{-x} for $x = 14\,827.7$ (OEIS A143530), 15300.7, 21381.5, 25461.7, 32711.9, 40219.6, 50689.8, 62979.8, 78890.2, 98357.8, ..., corresponding to $x = 1.829 \times 10^{-14\,828}$ (OEIS A143531), $2.040 \times 10^{-15\,301}$, $3.289 \times 10^{-21\,382}$, $2.001 \times 10^{-25\,462}$, $1.374 \times 10^{-32\,712}$, $2.378 \times 10^{-40\,220}$, $1.420 \times 10^{-50\,690}$, $1.619 \times 10^{-62\,980}$, $6.835 \times 10^{-78\,891}$, $1.588 \times 10^{-98\,358}$, ...



The quantity $R(x) - \pi(x)$ is plotted above.

This function is implemented in the Wolfram Language as `RiemannR`[*x*].

Ramanujan independently derived the formula for $R(n)$, but nonrigorously (Berndt 1994, p. 123; Hardy 1999, p. 23). The following table compares $\pi(10^n)$ and $R(10^n)$ for small n. Riemann conjectured that $R(n) = \pi(n)$ (Knuth 1998, p. 382), but this was disproved by Littlewood in 1914 (Hardy and Littlewood 1918).



The Riemann prime counting function is identical to the Gram series

$$G(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k\,k!\,\zeta(k+1)}, \qquad (11)$$

where $\zeta(z)$ is the Riemann zeta function (Hardy 1999, pp. 24-25), but the Gram series is much more tractable for numeric computations. For example, the plots above show the difference $G(x) - R(x)$ where $R(x)$ is computed using the Wolfram Language's built-in `NSum` command (black) and approximated using the first 10^1 (blue), 10^2 (green), 10^3 (yellow), 10^4 (orange), and 10^5 (red) points.

In the table, $[x]$ denotes the nearest integer function. Note that the values given by Hardy (1999, p. 26) for $x = 10^9$ are incorrect.

n	$\operatorname{nint}(R(10^n))$	$\operatorname{nint}(R(10^n) - \pi(10^n))$
Sloane	A057793	A057794
1	5	1
2	26	1
3	168	0
4	1227	-2
5	9587	-5
6	78527	29
7	664667	88
8	5761552	97
9	50847455	-79
10	455050683	-1828
11	4118052495	-2318
12	37607910542	-1476

Riemann's function is related to the prime counting function by

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}), \tag{12}$$

where the sum is over all complex (nontrivial) zeros ρ of $\zeta(s)$ (Ribenboim 1996), i.e., those in the critical strip so $0 < \mathbb{R}[\rho] < 1$, interpreted to mean

$$\sum_{\rho} R(x^{\rho}) = \lim_{t \to \infty} \sum_{|\mathbb{I}(\rho)|<t} R(x^{\rho}). \tag{13}$$

However, no proof of the equality of (12) appears to exist in the literature (Borwein *et al.* 2000).

SEE ALSO:
Gram Series, Prime Counting Function, Prime Number Theorem, Riemann Function, Riemann Hypothesis, Soldner's Constant

REFERENCES:

Borwein, J. M.; Bradley, D. M.; and Crandall, R. E. "Computational Strategies for the Riemann Zeta Function." *J. Comput. Appl. Math.* **121**, 247-296, 2000.

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, 1994.

Derbyshire, J. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics.* New York: Penguin, 2004.

Edwards, H. M. *Riemann's Zeta Function.* New York: Dover, 2001.

Hardy, G. H. and Littlewood, J. E. *Acta Math.* **41**, 119-196, 1918.

Hardy, G. H. "The Series $R(x)$." §2.3 in *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, 1999.

Havil, J. *Gamma: Exploring Euler's Constant.* Princeton, NJ: Princeton University Press, p. 42, 2003.

Ingham, A. E. *The Distribution of Prime Numbers.* London: Cambridge University Press, p. 83, 1990.

Knuth, D. E. *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms, 3rd ed.* Reading, MA: Addison-Wesley, 1998.

Landau, E. *Handbuch der Lehre von der Verteilung der Primzahlen, 3rd ed.* New York: Chelsea, 1974.

Mangoldt, H. von. "Zu Riemann's Abhandlung 'Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.' " *J. reine angew. Math.* **114**, 255-305, 1895.

Mathews, G. B. Ch. 10 in *Theory of Numbers.* New York: Chelsea, 1961.

Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 224-225, 1996.

Riemann, G. F. B. "Über die Anzahl der Primzahlen unter einer gegebenen Grösse." *Monatsber. Königl. Preuss. Akad. Wiss. Berlin*, 671-680, Nov. 1859. Reprinted in *Das Kontinuum und Andere Monographen* (Ed. H. Weyl). New York: Chelsea, 1972. Also reprinted in English translation in Edwards, H. M. Appendix. *Riemann's Zeta Function.* New York: Dover, pp. 299-305, 2001.

Riemann, B. "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe." Reprinted in *Gesammelte math. Abhandlungen.* New York: Dover, pp. 227-264, 1957.

Riesel, H. and Göhl, G. "Some Calculations Related to Riemann's Prime Number Formula." *Math. Comput.* **24**, 969-983, 1970.

Riesel, H. "The Riemann Prime Number Formula." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, pp. 50-52, 1994.

Sloane, N. J. A. Sequences A057793, A057794, A096623, A096624, A096625, A143530, A143531 in "The On-Line Encyclopedia of Integer Sequences."

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 28-29 and 362-372, 1991.

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