# The Zeta Function

In his great 1859 paper, “Über die Anzahl der Primzahlen unter eine gegebene Grösse,” Riemann gave two proofs of the analytic continuation and functional equation of the zeta function:

**Theorem.** *Let* $\Lambda(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\zeta(s)$*. Then* $\Lambda$ *has meromorphic continuation to all* $s$*, analytic except at simple poles at* $s = 0$ *and* $1$*, and satisfies* $\Lambda(s) = \Lambda(1-s)$*.*

The first proof is described in Ahlfors. *Both* proofs are important: the first proof gives the values of the zeta functions at negative odd integers (or, using the functional equation, at positive even integers) while the second proof expresses the zeto function as the Mellin tranform of an automorphic form, and leads to the result that $s(s-1)\,\Lambda(s)$ is an entire function of order one. I’ll describe the second proof here.

Let $f$ be an $L^1$ function on $\mathbb{R}$. Then the *Fourier transform* of $f$ is defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)\, e^{2\pi i x y}\, dy.$$

**Proposition 1.** *If* $f(x) = e^{-\pi x^2}$ *then* $f = \hat{f}$*.*

Proof. Completing the square,

$$\hat{f}(x) = e^{-\pi y^2} \int_{-\infty}^{\infty} e^{-\pi (x+iy)^2}\, dy.$$

It is easy to justify moving the line of integration by Cauchy’s theorem, and

$$\int_{-\infty}^{\infty} e^{-\pi (y+ix)^2}\, dx = \int_{-\infty}^{\infty} e^{-\pi y^2}\, dy.$$

This integral equals one, since it’s square equals

$$\int_{\mathbb{R}^2} e^{-\pi (x^2+y^2)}\, dx\, dy,$$

which is easily evaluated by switching to polar coordinates. $\square$

**Proposition 2 (Poisson summation formula).** *Suppose that $f$ is a smooth function such that $(1+x^2)^N f(x)$ is bounded for all $N$. Then*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n), \tag{1}$$

Proof. We introduce the auxiliary function

$$F(x) = \sum_{-\infty}^{\infty} f(x+n).$$

This is absolutely and uniformly convergent, and is a smooth function with period 1. It therefore has a Fourier expansion:

$$F(x) = \sum_{-\infty}^{\infty} a_m \, e^{-2\pi i m x}.$$

We evaluate the coefficients

$$a_m = \int_0^1 F(x)\, e^{2\pi i m x}\, dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n)\, e^{2\pi i m x}\, dx.$$

Since $e^{2\pi i m x} = e^{2\pi i m(x+n)}$, we may collapse the summation and the integration, and write

$$a_m = \int_{-\infty}^{\infty} f(x)\, e^{2\pi i m x}\, dx = \hat{f}(m).$$

Now

$$F(x) = \sum_{-\infty}^{\infty} \hat{f}(m)\, e^{2\pi i m x}.$$

Putting $x = 0$ we obtain (1). $\square$

**Proposition 3.** *Let $t > 0$, and let*

$$\theta(t) = \sum_{-\infty}^{\infty} e^{-\pi t n^2}.$$

*Then*

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right). \tag{2}$$

PROOF. Proposition 1 has the following extension: let $f_t = e^{-\pi t x^2}$. Then

$$\hat{f}_t = \frac{1}{\sqrt{t}} f_{1/t}.$$

This is easily deduced from the Lemma by a change of variables. Since

$$\theta(t) = \sum f_t(n),$$

this result follows by Poisson summation. $\square$

We write $\theta(t) = 1 + 2\psi(t)$, where

$$\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}. \tag{3}$$

**Proposition 4.** *If* $\mathrm{re}(s) > 1$

$$\Lambda(s) = 2 \int_0^{\infty} \psi(t)\, t^{s/2}\, dt. \tag{4}$$

PROOF. We have

$$\int_0^{\infty} \psi(t)\, t^{s/2} \frac{dt}{t} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t}\, t^{s/2} \frac{dt}{t}.$$

Making a change of variables $t \to n^{-1/2}$ we have

$$\int_0^{\infty} e^{-\pi n^2 t}\, t^{s/2} \frac{dt}{t} = n^{-s} \int_0^{\infty} e^{-\pi t}\, t^{s/2} \frac{dt}{t} = n^{-s}\, \Gamma\left(\frac{s}{2}\right),$$

and summing over $n$, we obtain $\Lambda(s)$. $\square$

We may now give Riemann's second proof of the functional equation. By Proposition 4, $\Lambda(s)$ is the Mellin transform of $\psi$, that is, essentially of $\theta$, and the functional equation of $\Lambda(s)$ is a reflection of the functional equation of $\theta$. Assume $\mathrm{re}(s) > 1$. Then

$$\begin{aligned}\Lambda(s) &= \int_1^{\infty} \psi(t)\, t^{s/2} \frac{dt}{t} + \tfrac{1}{2} \int_0^1 \theta(t)\, t^{s/2} \frac{dt}{t} - \tfrac{1}{2} \int_0^1 t^{s/2} \frac{dt}{t} \\ &= \int_1^{\infty} \psi(t)\, t^{s/2} \frac{dt}{t} + \tfrac{1}{2} \int_0^1 \theta(t)\, t^{s/2} \frac{dt}{t} - \frac{1}{s}.\end{aligned}$$

Now we make the substitution $t \to 1/t$ and make use of the functional equation (2) of theta, to write

$$\begin{aligned}
\tfrac{1}{2}\int_0^1 \theta(t)\, t^{s/2}\,\frac{dt}{t} &= \tfrac{1}{2}\int_1^\infty \theta\left(\frac{1}{t}\right)\, t^{-s/2}\,\frac{dt}{t} \\
&= \tfrac{1}{2}\int_1^\infty \theta(t)\, t^{(1-s)/2}\,\frac{dt}{t} = \\
&= \int_1^\infty \psi(t)\, t^{(1-s)/2}\,\frac{dt}{t} + \int_1^\infty t^{(1-s)/2}\,\frac{dt}{t} = \\
&= \int_1^\infty \psi(t)\, t^{(1-s)/2}\,\frac{dt}{t} - \frac{1}{1-s}.
\end{aligned}$$

Therefore

$$\Lambda(s) = \int_1^\infty \psi(t)\,\left[t^{s/2} + t^{(1-s)/2}\right]\,\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.$$

We have proved this if $\mathrm{re}(s) > 1$. However the expression on the right is valid for all $s$ due to the rapid decay of $\psi(t)$ as $t \to \infty$, which is clear from (3). This gives the analytic continuation of $\Lambda(s)$ to all $s$, and since this expression is symmetric under $s \to 1-s$, it also gives us the functional equation. ■