Exploring the Riemann Hypothesis

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Introduction

If you haven't really heard of the Riemann hypothesis before, then a quick bit of research will probably make the impression on you that it is a somewhat important open problem in mathematics. You will likely take notice that it is still not proved. It has been around for over 150 years, and yet, the Riemann hypothesis still doesn't have a proof which shows with certainty that Riemann had something more than conjecture when he made his famous (or perhaps infamous, depending on how you look at it) statement.

It won't take long for you to stumble upon the connection which has been drawn between the Riemann hypothesis and its consequences on the distribution of prime numbers. Essentially, if the Riemann hypothesis is true, then we better understand how the prime numbers are distributed amongst the positive integers. If it should turn out that the Riemann hypothesis isn't true, then we might have an interesting subset of prime numbers to look at as a consequence of the places where Riemann's hypothesis breaks down. I can just imagine that set being called the non-Riemannian Primes or something like that.

Now, if you begin to wonder why the Riemann hypothesis can say something so powerful about the prime numbers and yet not be decided one way or another, then let me refer you to Euler's opinion on primes. He wrote, "Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate" [**6**, pp. 41-2]. If you look at the history of the attempts to finally prove or disprove Riemann's hypothesis, you may very well agree with Euler's claim on the limitations of the human mind. Many mathematicians and authors note that as Riemann's hypothesis is a conjecture about the zeros of the zeta function, part of the problem becomes understanding the structure of the zeta function. And, as we shall see, this is no trivial task.

Our pathway, here, to trying to get a grasp on the Riemann hypothesis will take us through some of the history of Riemann and the era in which he made his marks on mathematics. This will lay the groundwork for seeing where the zeta function came from and why Riemann's ideas on it were so original and illuminating and yet hard to truly understand. Then we will follow an analytical development of the Riemann zeta function, taking it from a simple function on a Dirichlet series to a function analytic on the entire complex plane with one pole as exception. After trying to build the zeta function from the ground up and seeing the structure we should finally be able to state the Riemann hypothesis in a way that makes sense.

Once we have the conjecture that is the Riemann hypothesis down, we will make no attempt to prove it, here, or even really to suggest the best direction to take in order to hopefully prove the zeta function behaves as Riemann believed. Rather, we will state some of the equivalences of the Riemann hypothesis and we will see some very interesting and, dare I say, important consequences of it, should it prove true. We will take a moment and consider why some mathematicians may say that the Riemann hypothesis is false and why those that believe it is true may even say it is almost false.

However, before we begin down the road I have laid out, I would like to take a moment to mention some questions I had when first looking at the Riemann hypothesis myself. And, I hope you will consider them as well as you walk down the road I have created for you.

When I first looked at the Riemann hypothesis, my first question was, “Why have I not heard of this before?” It may have been mentioned in passing during some course I have taken in my studies, but I did not find this conjecture ever really laid out in front of me. And I certainly was never taken down the winding ways to see how it could possibly be making a statement on the prime numbers. Every grade school student should have heard of prime numbers by the time he or she is getting into secondary school. This is an immediately recognizable and usually interesting concept to even those who are not mathematically inclined. And yet, a statement as powerful as Riemann’s which would answer an age old question on how the primes were laid out was never made apparent to me.

I suppose we could merely say that my own studies were lacking, and I merely failed to notice and appreciate the Riemann hypothesis when it came up. However, I was willing to suppose instead that there could be other barriers preventing students from being enriched by Riemann's ideas. And I sought to look for those whilst I was looking at Riemann's hypothesis itself.

Another question I had was, "Besides its interest to me, what really makes the Riemann hypothesis so important?" To some extent, that answer is easy.

"There are a number of great old unsolved problems in mathematics, but none of them has quite the stature of the Riemann hypothesis. This stature can be attributed to a variety of causes ranging from mathematical to cultural. As with the other old great unsolved problems, the Riemann hypothesis is clearly very difficult. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics" [**1**, p. 4].

This is to say that the difficulty in trying to find a proof for the hypothesis is part of what makes it so important and attractive to mathematicians. However, just because a problem has been tried and failed many times doesn't, to my mind at least, make it important. It is the deeper meaning of the problem that causes it to be so difficult which makes the problem of interest and importance.

Thus, as we meander down past the window through which we take a voyeuristic look at the Riemann hypothesis, we are well served to consider what the deeper meaning of the Riemann hypothesis really is. We can look at what makes the zeta function and Riemann's conclusions on its behavior so special.

As we look at the history of the zeta function, and in turn develop it, and finally draw consequences from it, we will see its tendrils sneaking across multiple mathematical disciplines. Predominantly, we will see the long shadows of number theory and analysis, but if we look even deeper or take a look at some of the modern attempts at proof of Riemann's hypothesis we may see even more disciplines and subjects in the woods. So, as we read, we are encouraged not just to read the words on the pages, but to consider the greater meaning of them. Hopefully, this is a task we are capable of completing.

Some History of Bernhard Riemann and his Hypothesis

Throughout his entire mathematical career, and indeed even before the proper start of that, Bernhard Riemann had an obsession with the perfect. He consistently worked so that everything he produced was as flawless as humanly possible. (With such a drive to only produce the immaculate it is a wonder he ever put forth the result we now know as the Riemann hypothesis, as we shall see later) [**7**, p. 61] & [**4**, p. 25].

In the early 1840s Riemann was enrolled in the Gymnasium Johanseum in Lüneberg in Hanover. The school's director, Friedrich Constantin Schmalfuß, noted young Bernhard's desire to know and create perfection. Riemann went so far as to be unwilling to submit any work he did not feel would be free of negative marks. Schmalfuß noted this and allowed Riemann the right to peruse his library. Schmalfuß did so in the hopes of stimulating the special talent for Mathematics which he observed in Bernhard. Now free to study Schmalfuß's collection of mathematical books and escape his personal peer pressures, Bernhard Riemann began to thrive [**7**, pp. 60-2].

Schmalfuß guided Riemann away from the popular direction of mathematics at the time, and so away from those books in his library. During that period of time, guided by the consequences of the French Revolution and Napoleon's subsequent reign, mathematics had taken to the direction of being studied not for its own sake, but rather, for the sake of its application and real-world consequences, including utilization in warfare. Mathematics which could not be applied to warfare or other sciences was losing its place. However, at Gymnasiums such as the one at which Riemann studied, mathematics was again beginning to be studied merely for its own sake and for an appreciation of its beauty. As such, Schmalfuß made the effort to keep Riemann away from those "mathematical texts full of formulas and rules that were aimed at feeding the demands of a growing industrial world." Instead, Schmalfuß was encouraging Riemann's studies in the direction of the classical works of Euclid, Archimedes, and Apollonius [**7**, pp. 59-61].

When Riemann was given a treatise by Descartes on analytical geometry, Riemann's distaste for such equations and formulas was apparent. Already it was clear that Bernhard had developed an appetite for what we might call pure mathematics. At one point Riemann devoured the contents of Legendre's *Théorie des Nombres* in just six days claiming "I know it by heart." Thus, we might say that Riemann's future contributions in pure mathematics can be traced back to this point in his history [**7**, pp. 61-3].

While Riemann's father clearly had the desire that Bernhard study theology at Göttingen University, Gauss and the scientific tradition at Göttingen made their mark upon the impressionable young Bernhard. Thankfully for the future of mathematics, after Bernhard went to his father with the request that he switch from studies in theology to mathematics to match his own interests, his father gave Riemann his blessing. Immediately after getting the permission the obedient youth needed from his father he began to completely immerse himself in the scientific and mathematical community at Göttingen to become a true mathematician [**4**, pp. 26-7].

With the supervision of Gauss, in the spring of 1849, Riemann began his doctorate course in mathematics at Göttingen University. For the two previous years, he had spent time studying at Berlin under the influence of Dirichlet. However, Riemann desired to be a lecturer at Göttingen which required not only a doctorate but also a second doctorate of sorts, called “habilitation,” with a thesis and trial lecture. To complete this work (his doctorate and the second doctorate) Riemann took more than five years. Upon examining Riemann’s thesis, Gauss heaped praise upon Bernhard by saying, “A substantial and valuable work, which does not merely meet the standards required for a doctoral dissertation, but far exceeds them.” His thesis could be called a masterpiece fusing together elements so new that they could hardly be called fields or subjects yet. His contributions included work in complex function theory and in function theory married with topology [**4**, pp. 119-21].

Riemann’s habilitation thesis was titled “On the representability of a function by a trigonometric series.” It is considered a landmark in mathematics, and it gave mathematics the Riemann integral. However, despite the importance of his thesis work, Riemann’s habilitation lecture is considered to have far surpassed the contributions of his thesis. He was required to select three lecture titles, of which his supervisor, Gauss, was to choose one. His choice was “On the Hypotheses that Lie at the Foundations of Geometry.” Riemann delivered this lecture to the assembled faculty of Göttingen on June 10, 1854 [**4**, p. 127].

The reading of Riemann’s paper is considered to be one of the highlights of mathematics. Riemann’s inspiration came from the work of his mentor, which is regarded as the originator of differential geometry. Gauss’s paper from 1827 was called “A General Investigation into Curved Surfaces.” Riemann expanded Gauss’s perceptions on the dimensions of space into a new way of looking. Gauss had thought of curved 2-dimensional surfaces existing in 3-dimensional space. However Riemann was able to look at 3-dimensional space from the interior of the space. It was as if Gauss was looking from the outside in and Riemann had found a way to stand in the middle of it all. The consequences of Riemann’s work even apply to such important theories as the General Theory of Relativity [**4**, pp. 127-29].

Given the work of Riemann that I have mentioned so far, it may seem surprising that his famous hypothesis is one firmly grounded in number theory. But, perhaps it shouldn’t be too surprising. Much of Riemann’s work was about looking at things in new ways. Riemann’s instinct driven perspectives gave him new insight, and this is perhaps the key to the greatness of Riemann. Riemann’s new look at 3-dimensional geometry could then be paralleled with the new look he had at something called the zeta function. It was through the tools provided by

Cauchy, with his construction of the imaginary numbers that Riemann would eventually use to gain his new perspective. [**7**, pp. 66-72]

The path to Riemann's great conjecture, his hypothesis, is not without its distractions from the consequences of life. The deaths of his fellows and respected colleagues are a series of markers along this path. It starts with the death of his mentor, Gauss in February of 1855. With his professorship vacant, Göttingen offered the post to Dirichlet, who accepted and arrived a few weeks later [**4**, pp. 132-5].

During his earlier time in Berlin, Riemann had bonded with Dirichlet and even gained some of what would be his inspiration for later looking at the prime numbers [**7**, pp. 64-5]. So the forthcoming promise of collaboration with his former mentor must have pleased Riemann.

However, yet more death was to occur around Bernhard. In 1855 he lost his father, whose powerful influence on Bernhard cannot be overstated, and one of his sisters. His remaining three sisters went to stay with his brother, Wilhelm, in Bremen. In 1857 Bernhard continued to suffer personal losses as he lost another of his sisters, Marie, and his brother. After all of that, his remaining two sisters came to live with him at Göttingen [**4**, pp. 133-4].

However it was not all loss for Riemann. He was finally appointed Assistant Professor at Göttingen, and so was finally able to earn a modest salary. Even the next death to which Riemann was forced to bear witness proved fruitful for his development [**4**, p. 134].

In the summer of 1858 his friend, mentor, and respected colleague, Dirichlet suffered a heart attack. He had been lecturing abroad and was brought back home to his wife in Göttingen. As he was himself gravely ill, lying there, dealing with his own pain, his wife suddenly died of a stroke. The following May, Dirichlet himself, breathed no more [**4**, p. 134].

Now with Dirichlet's death, Riemann was selected to be Gauss's, and indeed Dirichlet's, successor at Göttingen. Now Riemann had a guaranteed livelihood and was given the former apartments of Gauss, as a home for his two sisters and him. He was even appointed a corresponding member of the Berlin Academy, the city where he had once been nothing more than an untested student [**4**, pp. 134-5].

Finally, from all this death and emotional despair, Riemann was starting to emerge as a respected mathematician, and a true star in the mathematical sky. In recognition of his triumph at joining the Berlin Academy, Riemann wrote the paper from which comes the focus of this discussion and many mathematicians for over a century. It was titled "On the Number of Prime Numbers Less Than a Given Quantity" [**4**, pp. 135, 151].

He opened his paper with an acknowledgement of his former mentors whom had passed on to leave him the post he now occupied at Göttingen. And, by the completion of his third sentence, Riemann had introduced his zeta function [**4**, p. 135].

From his 1859 paper:

> For the consideration which the Academy has shown to me by admitting me as one of its corresponding members, I believe I can best express my thanks by availing myself at once of the privilege thereby given me to communicate an inquiry into the frequency of prime numbers; a subject which, through the interest shown in it by Gauss and Dirichlet over a long period, appears not altogether unworthy of such a communication.
>
> I take as my starting-point for this inquiry Euler's observation that the product
>
> $$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s}$$
>
> for all prime numbers *p* and all whole numbers *n*. The function of a complex variable s which both these expressions stand for, so long as they converge, I signify by $\zeta(s)$" [**4**, p. 135].

Riemann's famous hypothesis rears its head on the fourth page of his paper. Here, he finally asserts his conjecture about the zeros of his zeta function [**4**, p. 136].

For a moment, it is worth looking at how Riemann could ever have allowed himself to publish or have given lectures on something so completely obviously imperfect. For Riemann to give up on a proof, of what we now know is a very important conjecture, seems unlikely. And yet he says "One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation" [**7**, p. 83]. While it seems fair that he might set aside the proof for a moment it seems unusual for a man of Riemann's character to so cavalierly set it aside. And yet, there it lies.

However, we may rejoice in the fact that Riemann published it at all. For Riemann to submit his 10 page publication to the monthly notices of the Berlin Academy, despite his obsession with perfection, (an obsession given even more fuel from his mentor, Gauss's, insistence that he only publish perfect proofs with no gaps,) we shall remain grateful. This is even Riemann's only paper on primes. And while he did offer some hints that he had proofs of some of his results, he claimed they were unready for publication [**7**, pp. 82-3]. In that way, it seems even more remarkable that he would have published the results that he did. This is especially true if we consider that he didn't even thoroughly prove his paper's main result [**4**, pp. 151-2].

Thus, in the end, we are left valuing this paper of Riemann, not for its careful analytic clarity and well thought out proofs, but for the pure insight and originality of his ideas. It took over 30 years for any progress to be made in this field after Riemann made his mark [**4**, pp. 152-3].

Even if the Riemann hypothesis were false, and it well could be, despite being a stronger result than the Prime Number Theorem which has since been proven true, we still have the years (over a century now) of research and further originality germinated by Riemann's catalyzing paper. Many call the Riemann hypothesis, today, the greatest unsolved problem in mathematics. Karl Sabbagh even named his impressive book looking into this very topic <u>The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics</u> [**6**]. So should it prove false one day, we still cannot dismiss the quality of Riemann's ideas. He certainly created an interesting problem to ponder, and it takes quite a bit of work to wrap your head around it.

Since the Riemann hypothesis is a statement about the zeros of the zeta function, which we will explicitly state later, in order to understand the conjecture that is the Riemann hypothesis, we must first understand the zeta function itself. Shortly, we shall look at the analytic development of the Riemann zeta function, and we shall examine the properties of it due to its structure. However, we will first look into a bit of the history of the development of the zeta function which led Riemann to look at it in a relatively new manner.

We start by looking at the work of Riemann's mentors Gauss and Dirichlet. During the 1820's Dirichlet spent some years in Paris, and while there found himself drawn to and fascinated with a work of Gauss's from his youth, *Disquisitiones Arithmeticae*. With this work, Gauss had succeeded in setting off Number Theory as an independent discipline in Pure Mathematics. However, Gauss's style and the difficulty of the work had proven itself bulletproof to the attempts of many to have more than a trivial understanding of it. Dirichlet, though, was more than happy to spend the long days necessary to understand and comprehend Gauss's work. It was thus through Dirichlet that Gauss's treatise finally became accessible to the broader community and got the wide distribution and recognition his work deserved [**7**, p. 76].

Of particular interest to Dirichlet was a clock calculator of Gauss's. Specifically, he was interested in the conjecture due to a pattern in the calculations noticed by Fermat. Fermat had conjectured that if you were to input the primes into a clock calculator with *N* hours on it, then infinitely often, the clock would strike one o'clock. That is to say, if you were to take all of the prime numbers, p_i, and any natural number $N \geq 2$, then

$$1 \equiv p_i \bmod (N)$$

would be true for some infinite sequence of prime numbers [**7**, p. 76].

It was because of this interest that Dirichlet worked at the proof of this conjecture, and in 1838, at only 33 years old, Dirichlet found his proof and made his mark in the theory of numbers. The thing that made his proof seem perhaps even more special was that Dirichlet had not been as elementary and cunning as say Euclid had been with his proof that there were infinitely many primes. Instead, Dirichlet's proof mixed several ideas from various areas of mathematics. In fact, it may seem at first that the ideas that Dirichlet uses have little to do with each other [**7**, p. 76]. Dirichlet was able to mix ideas from analysis and arithmetic in order to create the beginnings of what would be called analytic number theory. While Dirichlet proved even more than the result of this clock calculator in his paper titled, (translated,) "Proof of the theorem that each unlimited arithmetic progression, whose first member and difference are whole numbers without common factor, contains infinitely many prime numbers," his ideas were grounded in Gauss's clock calculator [**4**, pp. 95-7].

To start with, Dirichlet used a function which had first appeared in Euler's time, the *zeta function*. The *zeta function* was defined on a real number, x, and was calculated by the following definition:

$$\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \ldots + \frac{1}{n^x} + \ldots$$

[**7**, p. 76].

In fact, in our modern definition of zeta functions we define a zeta function as any function which can be defined as a Dirichlet series. So, Dirichlet's mark on this subject should not be understated.

One can imagine just how interesting it would have been to compute values of his zeta function without the aid of modern computational devices. Needing to figure out the value of each ratio, one at a time, must have been maddening. The interest, though, in this sum, came long before the invention of our modern computers and calculators. The interest carries back all the way to Greek times and Pythagoras. It goes back to a connection discovered between music and the sequence of fractions $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ [**7**, pp. 76-7].

When Pythagoras hit an urn filled with water with a hammer, it produced a note. When he removed half the water and struck it again, it produced a note one octave higher. As he played subsequent notes on the urn when it was one-third full, one-quarter full, etcetera, he found the sounds he was hearing to be in harmony with the first note he played when the urn was originally full of water. Any other amounts of water seemed to produce notes in dissonance with those he had already found in harmony. Hence the name for the sum generated by adding all the fractions $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ has come to be the harmonic series [**7** pp. 77-9].

Euler realized that this was, in fact, an infinite sum which was generated by the zeta function when *x* = 1. Euler then attempted to find the value of the zeta function when *x* = 2 was input into the zeta function. Though it might seem strange when you first look at it, we know of course that with this input, the sum will no longer spiral off to infinity, but it will actually converge to a specific value. Finding the actual limit of that sum was no easy task, and Euler himself concluded at one point that they would not get much closer than the estimate of $\frac{8}{5}$ [**7**, p. 79].

However, it was Euler himself who found that the sum was actually approaching $\frac{\pi^2}{6}$. The discovery that the irrational number, which to their eyes behaved chaotically in its decimal expansion, was somehow tied to the seemingly innocent sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ hit the scientific community of the time like a wildfire [**7**, p. 80].

Another major discovery of Euler's, into which we will go more detail later, was that there was another representation for the sum which defined the zeta function. Euler found a means of expressing that infinite sum as an infinite product. Thus, we call this Euler's product formula. The beauty of his infinite product was not so much that we went from a sum to a product, but that we ended up going from a sum involving all of the positive integers to a product which was expressible using only the prime numbers [**7**, pp. 80-1].

Euler, however, never realized the full potential and significance of his discovery. It was Dirichlet who first used it in a significant way. He used the zeta function to prove that the clock calculator would strike one o'clock for an infinite number of primes input into the calculator. Thus it was Dirichlet who first used the discovery of Euler to discover something new about the primes. Proving

$$1 \equiv p_i \bmod(N)$$

occurs for an infinite number of primes was no small feat [**7**, p. 81].

Given that Dirichlet was Riemann's mentor, colleague, and friend, it seems likely that Dirichlet must have communicated to Riemann the power inherent in the infinite sums generated by the zeta function. And yet, Dirichlet and his colleagues had been content to deal with the zeta function only as a function on the real numbers. Riemann, however, was in the mindset provided by Cauchy's imaginary numbers. To Riemann, the zeta function was yet another vessel to carry the complex numbers. For him, it was of interest to look at the zeta function with complex number inputs [**7**, pp. 81-2].

So, to talk about the Euler zeta function, the one utilized by Dirichlet, is to talk about the zeta function, $\zeta(x)$, or the sum $\sum_{n=1}^{\infty} \frac{1}{n^x}$, with only real numbers as inputs for *x*. However, when we talk about the Riemann zeta function we no longer limit ourselves exclusively to the reals. We may now take inputs from all over $\mathbb{C}$, anywhere in the complex plane. So now we define the zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

[**6**, pp. 92-93].

Notice that we have gone from using *x* as our independent variable to *s*. And while *z* may be the standard variable used to represent a complex number, it is in deference to the importance of Riemann's work that we have continued to use his notation with *s* instead of *z* [**9**].

Finally, by looking at the zeta function through this new lens, Riemann had made a discovery. He felt that he could prove why Gauss's guess at the number of prime numbers would remain as accurate as it was. Riemann felt he had found the key to proving Gauss's Prime Number Conjecture. Gauss's instincts would be replaced by a firm mathematical proof. Riemann's hypothesis would force the Prime Number Theorem to come out easily as a result. And, indeed, if the Riemann hypothesis should prove to be true, the Prime Number Theorem would be one consequence of Riemann's new perspective on the zeta function. Unfortunately Riemann was unable to do anything more than lay the groundwork of proving his new discoveries. The rigorous proof we may all desire eluded him [**7**, pp. 82-3].

Building the Riemann Zeta Function

One of the early goals in looking at the Riemann hypothesis and the corresponding zeta function is to understand the structure of the Riemann zeta function. It should help us to see the impressive connection that Riemann was able to make between the prime numbers and his zeta function. We will start by building the Riemann zeta function analytically and in so doing will lead ourselves towards the functional equation which tells us something of the structure of this zeta function. We follow closely with the development laid out by Borwein, et. al [**1**, chapter 2].

The way we build the Riemann zeta function follows the course of its development through history, following Euler through to Riemann. In so doing, we shall start with a standard Dirichlet series.

We begin by defining a complex number $s = \sigma + it \ (\sigma, t \in \mathbb{R})$. Then we look at the definition of a Dirichlet series, a series of the form

$$\sum a(n)e^{-\lambda(n)z} \ ,$$

where $a(n)$ and z complex numbers and $\{\lambda(n)\}$ is a monotonic increasing sequence of real numbers. We take the specific Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots .$$

This marks our starting point as we seek to develop the Riemann zeta function. We continue by looking at this series when $s = 1$. We can clearly see that we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

which is well known as the harmonic series, which of course diverges. We notice that if the real part of s is less than or equal to 1 the Dirichlet series diverges. That is, when $\Re(s) \leq 1$, the above series is divergent.

Further, using the integral test, we can see that when $\Re(s) > 1$ the series converges. So, we use this Dirichlet series to define the Riemann zeta function for $\Re(s) > 1$ as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function has the property of being analytic in the region $\Re(s) > 1$. We would, however, like to see if we could get the zeta function to be defined on even more of the complex plane. Restricting the real part of s to those values greater than one seems far too limiting.

We start that quest by looking at the work of Euler as regards the above series. Euler was the first to evaluate the values of $\zeta(s)$with a high degree of accuracy for values of s equaling 2, 3, 4, …, 15, 16. We have, for example, the value of $\zeta(s)$for $s = 2$.

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \frac{\pi^2}{6}$$

Unfortunately, Euler did not make a comprehensive study of this series for complex values. It was Riemann who would make the first “intensive study of this series as a function of a complex variable” [**1**, Chapter 2, p. 10].

However, we use an observation of Euler that every natural number can be uniquely written as the finite product of powers of different primes. Therefore, given any $n \in \mathbb{N}$, we may take

$$n = \prod_{p_i} p_i^{e_i}$$

In the product the p_i are all primes, and the e_i are nonnegative integers; we note that we will have 0 for most of them. Of course, the e_i will vary as n varies; however, if we consider each $n \in \mathbb{N}$, we will use each possible finite combination of exponents $e_i \in \mathbb{N} \cup \{0\}$. Thus, we may take our series and represent it by

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ... \right)$$

where the infinite product is over every prime.

We do this to get the Euler product formula, the most important of Euler’s contributions to the theory of the zeta function. Euler’s product formula gives us a new way of representing our zeta function. It is for our $s = \sigma + i\,t$ where $\sigma > 1$ and is given as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

To show this holds (with some help from [**8**]) we start with the expression on the right

$$\prod_p \frac{1}{1 - \frac{1}{p^s}} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \dots .$$

As these terms are the geometric sums of the series of the form $\sum_{k=0}^{\infty} \left(\frac{1}{p^s}\right)^k$, this product is

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2^s}\right)^k \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3^s}\right)^k \cdot \sum_{k=0}^{\infty} \left(\frac{1}{5^s}\right)^k \cdot \dots$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \dots\right) \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \dots\right) \dots,$$

and if we start multiplying out the terms in the infinite product, we get the sum

$$= 1 + \sum_{1 \le i} \frac{1}{p_i^s} + \sum_{1 \le i \le j} \frac{1}{p_i^s\, p_j^s} + \sum_{1 \le i \le j \le k} \frac{1}{p_i^s\, p_j^s\, p_k^s} + \dots .$$

Here, we notice this accounts for each prime to the power of each multiple of s, so we not only have a sum involving the primes, but in fact every positive integer.

So our sum is

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Hence, we have the Euler product formula as desired.

This is called the analytic form of the fundamental theorem of arithmetic. Now, through the lens of the Euler product formula we are able to see how the Riemann zeta function is an expression about the prime number factorization of the positive integers, and we also see how the Riemann zeta function might say something on the distribution of prime numbers.

We notice that an infinite product which is convergent, such as the infinite product description we now have for the zeta function, never vanishes unless a factor vanishes. So the Euler product formula proves even more fruitful by yielding this next theorem.

Theorem 1*: For all* $s \in \mathbb{C}$ *with* $\Re(s) > 1$*, we have* $\zeta(s) \neq 0$*.*

Already, we have agreed that the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ diverges for any s with $\Re(s) \leq 1$. Of course, when $s = 1$, we have the harmonic series. So now, we finally want to head outside of our given region in the complex plane. To do this, we continue to build the zeta function, continuing to build our definitions for ways of representing it. Before we do this, we must first realize that our given definition for the zeta function, as a function of s for the given series, uniquely determines $\zeta(s)$ on the entire complex plane assuming $\zeta(s)$ is analytic on the complex plane with maybe a pole. And in fact, we will show that $\zeta(s)$ continues analytically to the entire complex plane with one point as exception, a pole at $s = 1$.

We start by reminding ourselves that an analytic continuation takes an analytic function defined on one domain to an analytic function defined on a larger domain, in a unique way, with special conditions.

Analytic continuation gives us that if there is a function, f_1, analytic on domain D_1, and another function, f_2, analytic on another domain D_2, such that $D_1 \cap D_2 \neq \varnothing$ and $f_1 = f_2$ on $D_1 \cap D_2$, then we have f_2 is the unique analytic continuation of f_1 to D_2.

Thus, we need to find a function, analytic on $\mathbb{C} \setminus \{1\}$, such that this function equals the Dirichlet series on any domain, for us $\Re(s) > 1$. If we can find such a function, we will have defined $\zeta(s)$ on all of $\mathbb{C} \setminus \{1\}$.

In his historic paper, Riemann proved that $\zeta(s)$ can be continued analytically to an analytic function on the whole of $\mathbb{C}$ minus the single point $s = 1$. Further, he showed that at $s = 1$ the zeta function has a simple pole, and it has a residue of 1.

Now, we will find the Riemann zeta function, $\zeta(s)$, as the analytic continuation of the Dirichlet series that we started with to the whole complex plane, excluding $s = 1$. We start by rewriting how we defined $\zeta(s)$ in the region we already have the zeta function analytic in, $\Re(s) > 1$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = s\sum_{n=1}^{\infty} n\int_{n}^{n+1} x^{-s-1}\,dx$$

Now, we represent x via the following method, let $x = [x] + \{x\}$, with $[x]$ as the integral part of x and $\{x\}$ as the fractional part of x. And, we notice that $[x]$ will be constant as n for any x in the interval defined by [n, n+1). Thus, we have

$$\zeta(s) = s\left(\sum_{n=1}^{\infty}\left(\int_{n}^{n+1}[x]\,x^{-s-1}\,dx\right)\right)$$

$$= s\int_{1}^{\infty}[x]x^{-s-1}\,dx$$

Now, we write $[x] = x - \{x\}$, and we get

$$\zeta(s) = s\int_{1}^{\infty} x^{-s}\,dx - s\int_{1}^{\infty}\{x\}x^{-s-1}\,dx$$

$$= \frac{s}{s-1} - s\int_{1}^{\infty}\{x\}x^{-s-1}\,dx\,,\ \sigma = \Re(s) > 1.$$

At this point we observe that because $0 \leq \{x\} < 1$, we have the improper integral,

$$\int_1^\infty [x] x^{-s-1}\, dx ,$$

from above converges when $\sigma > 0$ because the integral

$$\int_1^\infty x^{-\sigma-1}\, dx$$

converges. Thus the improper integral in our new representation for $\zeta(s)$ defines an analytic function of s in the region $\Re(s) > 0$. Thus, our new representation for $\zeta(s)$ gives it as a meromorphic function which has continued analytically $\zeta(s)$ to the new region $\Re(s) > 0$, excepting $s = 1$. Also, we can note that the $\frac{s}{s-1}$ term in the above expression gives us a simple pole of $\zeta(s)$ at $s = 1$ with a residue of 1.

The above equation for $\zeta(s)$ only gives us an extension of the Riemann zeta function to the region $\Re(s) > 0$. Riemann, however, used a similar argument with the gamma function, Γ, to continue his function analytically to the whole complex plane. And our goal has been, all along, to reach an extension of the zeta function to the entire complex plane, but continuing to except $s = 1$.

To that end, we start by defining the gamma function via Euler's integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt ,$$

and we recall that the gamma function is an extension of the factorial function to the entire complex plane excepting the nonpositive integers. Unfortunately, this integral form of $\Gamma(s)$ is only applicable for $\Re(s) > 0$. We need to use Weierstraß's form of the gamma function which applies to the whole complex plane. So we take Weierstraß's formula

$$\frac{1}{s\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

where γ is Euler's constant.

Now, Γ is analytic on the entire complex plane not including $s = 0, -1, -2, -3, \ldots$, and the residue of $\Gamma(s)$ at $s = -n$ is $\frac{(-1)^n}{n!}$. Also take note that for $s \in \mathbb{N}$ we have that $\Gamma(s) = (s-1)!$.

Using Euler's integral form, for $\sigma > 0$, we look at

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1} \, dt ,$$

and we set $t = n^2 \pi x$ to get

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x} \, dx .$$

Now, with some careful exchanging of summation and integration, for $\sigma > 1$, we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right) \mathrm{d}x$$

$$= \int_0^\infty x^{\frac{s}{2}-1}\left(\frac{\vartheta(x)-1}{2}\right) \mathrm{d}x\,,$$

where

$$\vartheta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$$

is the Jacobi theta function.

Next, we use a functional equation of $\vartheta(x)$, also from Jacobi (see [**3**, pp. 61-64] for more details,) to rewrite the zeta function. First, we have the functional equation which is valid for x > 0

$$x^{\frac{1}{2}}\vartheta(x) = \vartheta\left(x^{-1}\right)\,,$$

and using that functional equation, we obtain

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\cdot\left\{\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{s}{2}-\frac{1}{2}}\right)\cdot\left(\frac{\vartheta(x)-1}{2}\right) \mathrm{d}x\right\}.$$

Now, because of the fact that $\vartheta(x)$decays exponentially, the improper integral above converges for every $s \in \mathbb{C}$. Thus, with that integral, we have defined an entire function in $\mathbb{C}$. Finally, we have the above definition of the zeta function giving us the analytic continuation we sought on the whole complex plane, without the point $s = 1$.

The standard proof of the functional equation from Jacobi which we used above follows using Poisson summation, but we will not discuss it here.

At this point, we can state our result as the following theorem,

Theorem 2: *The function*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \left\{ \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{s}{2}-\frac{1}{2}} \right) \cdot \left(\frac{\vartheta(x)-1}{2} \right) \mathrm{d}x \right\}$$

is meromorphic with a simple pole at s = 1 with residue 1.

With this functional equation, we have finally gotten the extension of the zeta function which we were looking for. Originally we were limited to looking only at the region of the complex plane with $\Re(s) > 1$. However, we may now look at the zeta function as a meromorphic function on the entire complex plane, with a slight hiccup being the pole at $s = 1$.

As we saw earlier, there is a correlation between the zeta function and the prime numbers. In fact, if we can find the zeros of the zeta function, then we will have found information on the prime numbers. Specifically we can learn something about how the prime numbers are distributed among all the natural numbers. We will explore this more soon, but the premise is that if we could draw some conclusion about all of the zeros of the Riemann zeta function, then we would have information on all of the prime numbers. To be able to talk about all prime numbers like that certainly seems like no small feat.

First, we have to classify the zeros of the zeta function that we find. We can immediately find an infinite number of zeros of the zeta function, which we will refer to as the *trivial zeros* of $\zeta(s)$. All of these will lie outside the region $0 \leq \Re(s) \leq 1$. These zeros will prove uninteresting to us, and we will, in turn, not include them when we state the Riemann hypothesis on the zeros of the zeta function.

Before we state any conclusion on the rest of the zeros of the zeta function, the *nontrivial zeros* of $\zeta(s)$, we will need a functional equation to start. It was Riemann who noticed that our new representation for $\zeta(s)$not only gives us the analytic continuation we were seeking, but it also can be used to give us a functional equation for $\zeta(s)$. Riemann noticed that the term $\frac{1}{s(s-1)}$ from our above definition of the zeta function as well as the improper integral in the same definition are both invariant when substituting *s* for 1 – *s*. Thus, we have the following functional equation, which we shall give as a theorem.

Theorem 3: *For any s in* $\mathbb{C}$,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

For the sake of convenience and for clarity, we define the function

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Given our previous theorem on the zeta function, we can see that $\xi(s)$ is an entire function and satisfies the following simple functional equation

$$\xi(s) = \xi(1-s).$$

This gives us that $\xi(s)$ is symmetric around the point $s = \frac{1}{2}$ which of course lies on the vertical line $\Re(s) = \frac{1}{2}$ in the complex plane.

We have now gone so far as to take our original Dirichlet series representation for $\zeta(s)$ and extended it to a function which is analytic on the entirety of the complex plane minus one point, its pole at $s = 1$. From this, we have begun to understand the analytic structure of the zeta function and have even begun asking questions about the zeros of $\zeta(s)$. This includes a division of the zeros of $\zeta(s)$into two groups: the *trivial zeros* and the *nontrivial zeros*. Further, we have taken our extension of $\zeta(s)$ and found a functional equation for it. This has even shown us that there may be something special about the line $\Re(s) = \frac{1}{2}$. So now, let us make some conclusions about the behavior of $\zeta(s)$, and then we can finally state the Riemann hypothesis.

Theorem 4: *The function* $\zeta(s)$ *satisfies the following:*

1. $\zeta(s)$ *has no zero for* $\Re(s) > 1$*;*

2. the only pole of $\zeta(s)$ *is at s* = 1*; it has residue* 1 *and is simple;*

3. $\zeta(s)$ *has trivial zeros at s* = -2, -4, -6, …*;*

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ *and are symmetric about both the vertical line* $\Re(s) = \frac{1}{2}$ *and the real axis* $\Im(s) = 0$*;*

5. the zeros of $\xi(s)$ *are precisely the nontrivial zeros of* $\zeta(s)$*.*

Proof: Part 1 has already been proved by our earlier theorem on the Euler product form representation for $\zeta(s)$. We know that $\zeta(s) \neq 0$ for any *s* with $\Re(s) > 1$. Thus, it is impossible for a zero to be in this region.

For part 2, we have already agreed that there is only one pole of $\zeta(s)$ and it is at *s* = 1 with residue 1. This pole being simple follows from $(s - 1)\,\zeta(s)$ being analytic on all of $\mathbb{C}$.

To prove part 3, we look at what we know about the gamma function. We recall that $\Gamma(s)$ only has poles at *s* = 0, -1, -2, -3, … . And we also know that those poles are simple poles. Thus, it follows from our extension of $\zeta(s)$ in theorem 2, where we have $\Gamma\left(\frac{s}{2}\right)$ instead of $\Gamma(s)$, that there are zeros of the zeta function at s = -2, -4, -6, … . The pole of $\Gamma(s)$ at s = 0 is canceled by the term $\frac{1}{s(s-1)}$. Thus, we do not have a zero of $\zeta(s)$ at s = 0. We call this list of zeros of $\zeta(s)$ the *trivial zeros* of $\zeta(s)$ since they are uninteresting, as far as we are concerned with the Riemann hypothesis, and we call all other zeros of $\zeta(s)$ the *nontrivial zeros* of $\zeta(s)$.

Part 5 falls out of our definition for the function $\xi(s)$,

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

as the zeros of $\xi(s)$ are precisely the remaining zeros we have yet to find of $\zeta(s)$, namely, the *nontrivial zeros* of $\zeta(s)$.

Thus, our statement in Part 4 that the *nontrivial zeros* of $\zeta(s)$ are symmetric about $\Re(s) = \frac{1}{2}$, is a consequence of the functional equation for $\xi(s)$. We know from part 1 that $\zeta(s)$ has no zeros in the region $\Re(s) > 1$. So it must be that by the symmetry of $\xi(s)$ about $s = \frac{1}{2}$ that there are no zeros in the region $\Re(s) < 0$. That is, $\zeta(s)$ is symmetric about $\Re(s) = \frac{1}{2}$. Therefore, all of the *nontrivial zeros* of $\zeta(s)$ must lie inside the region $0 \leq \Re(s) \leq 1$.

That the *nontrivial zeros* are also symmetric about the real axis, where t = 0, follows from our analytic continuation of $\zeta(s)$.

As we can see, there are a line and a region which bear significant consequence on our knowledge about the zeros of the zeta function. Thus, we assign special names to these two objects. The vertical line $\Re(s) = \frac{1}{2}$ is called the *critical line*. And, the region $0 \leq \Re(s) \leq 1$ is called the *critical strip*.

All this has led us on our unending hunt for the zeros of the zeta function. Specifically, we are concerned with the *nontrivial zeros* of the zeta function. Now that we have some understanding of the structure and analytic development of $\zeta(s)$, we are finally ready to state the Riemann hypothesis.

As we have already stated, Riemann made his famous conjecture about the zeros of the Riemann zeta function on page four of his manuscript. And now, we may finally see it ourselves.

Conjecture (The Riemann Hypothesis): *All nontrivial zeros of* $\zeta(s)$ *lie on the critical line* $\Re(s) = \frac{1}{2}$.

Thus, Riemann made the bold conjecture that all zeros of $\zeta(s)$, excluding those found from the poles of $\Gamma(s)$, must have real part equal to one half.

The Likelihood that the Riemann Hypothesis is True and the PNT

While a proof of the Riemann hypothesis may elude us, we have anything but a short supply of its consequences. One of the most immediately available consequences of the Riemann hypothesis is the Prime Number Theorem (PNT.) A simple reformulation of the Riemann hypothesis shows that if it is true, then the reformulation of the PNT must also be true. The Prime Number Theorem thus becomes a trivial consequence of the Riemann hypothesis. Riemann, to an extent, saw that his hypothesis had the potential to take care of Gauss's missing proof for the PNT. Gauss had made his prediction for the prime counting function, but he had been unable to supply a proof himself. However, Riemann was also unable to find the rigorous proof necessary [**7**, pp. 82-3].

With that said, many mathematicians have made various reformulations of the Riemann hypothesis and they have all been unable to rigorously prove the conjecture as well. For emphasis, if one were to look at Borwein et al. [**1**, chapter 5] they have taken a whole chapter dedicated to the various equivalent statements of the Riemann hypothesis. There are 24 equivalences given in that chapter alone.

There are even extensions which take the Riemann hypothesis further. They have names such as *The Generalized Riemann Hypothesis*, *The Extended Riemann Hypothesis*, and *The Grand Riemann Hypothesis* [**1**, chapter 6]. While we won't take the time to look at these extensions in our endeavors here we can certainly see that the consequences of the Riemann hypothesis are important to number theory and the Prime Number Theorem in particular.

For our interest, it makes sense to start by looking at the consequences of the Riemann hypothesis on the Prime Number Theorem. Thus, we will start by looking at an equivalent statement for each, and hopefully, we can see why the PNT is a direct consequence of Riemann's hypothesis.

We will start with the definition of the Liouville function. However, before we go further down that rabbit hole, let us take a minute to state and look at the Prime Number Theorem.

Theorem 5 (The Prime Number Theorem): *Let $\pi(n)$ denote the number of primes less than or equal to n. Then*

$$\lim_{n \to \infty} \frac{\pi(n)\log(n)}{n} = 1$$

We call $\pi(n)$ the prime counting function.

So we could say that this theorem states that the number of prime numbers less than or equal to any number n is approximately $\frac{n}{\log(n)}$. As was already mentioned, Gauss was the one to conjecture this first, in 1792, and his conjecture was based on a substantial amount of computation and natural insight on his part [**1**, chapter 1].

Gauss was unable to supply a proof of this conjecture, and while Riemann felt he was making headway towards finding the missing proof of his mentor's hypothesis he was unable to prove his own hypothesis which he had hoped would be used to answer this issue of Gauss. It was actually in 1896, thirty-seven years after Riemann made his attempts that Jacques Hadamard and Charles de la Vallée-Poussin each independently worked out proofs of the PNT. However they could not use Riemann's hypothesis since it still was unproved [**6**, p. 28]. In fact, they merely proved something far less than the Riemann hypothesis, that the Riemann zeta function had no zeros with real part equal to 1, in order to get the PNT as a consequence.

It is worth noting that the PNT as we have just stated it has a slight problem. The problem with Gauss's Prime Number Theorem is that this ratio holds in the limit, but if we were to take a specific value for n then $\pi(n)$ will always be off from $\frac{n}{\log(n)}$ by a certain amount. Thus we might rephrase the Prime Number Theorem to say that the number of primes less than n is equal to $\frac{n}{\log(n)}$ with at least certain percentage of error, or it is off by some number which is incredibly small by comparison. So we could actually say that for a given n, $\pi(n) = \frac{n}{\log(n)} + E$, with E fairly small but still not insignificant [**6**, p. 29]. This is to say that the difference between the prime counting function and $\frac{n}{\log(n)}$ is a value, we called it E, that is not quite as close to zero as we might like.

For example, if we take the known number of primes less than n = 1,000,000,000 which is known to be 50,847,534 [**4**, p. 35] and then compare that to $\frac{n}{\log(n)} = \frac{1,000,000,000}{\log(1,000,000,000)}$ = 48,254,942.43…, then our guess for the number of primes given by Gauss is off by 2,592,591.56… . We are off by over two million! So, Gauss's guess may have been great in the limit of his ratio, but for finding the number of primes less than a specific number it seems like it may be leaving at least a little to be desired.

Why is it that we point out the problem with Gauss's formulation for the PNT? We do it because if the Riemann hypothesis were true, then we would be able to come up with an even better formulation for the PNT. We will soon introduce the connection between $\pi(n)$ and the zeros of $\zeta(s)$. This connection along with the Riemann hypothesis will then suggest a more accurate approximation for $\pi(n)$ [**6**, p. 29] & [**1**, chapter 5].

Since we have now taken a substantive digression into looking at the Prime Number Theorem, it seems only fitting that we now turn our attention to truly showing the connection between the Riemann hypothesis and the PNT. As promised, we start with the definition of the Liouville function.

Definition: *The Liouville function is defined by*

$$\lambda(n) = (-1)^{\omega(n)}$$

where $\omega(n)$ *is the number of, not necessarily distinct, prime factors of n, counted with multiplicity.*

So we have $\lambda(2) = \lambda(3) = \lambda(5) = \lambda(7) = \lambda(8) = -1$ and $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ for a few examples of the values this function takes. Also, we have that $\lambda(x)$ is multiplicative; which is to say that $\lambda(xy) = \lambda(x)\lambda(y)$ for any $x, y \in \mathbb{N}$.

Next we look at connections between the Liouville function, the Riemann hypothesis, and also the Prime Number Theorem. These connections we are looking at now were explored and established by Landau in his doctoral thesis in 1899.

First, we have the following equivalent statement of the Riemann hypothesis from Borwein et al. [**1**, chapter 1].

Theorem 6: *The Riemann hypothesis is equivalent to the statement that for every fixed ε* > 0,

$$\lim_{n \to \infty} \frac{\lambda(1) + \lambda(2) + ... + \lambda(n)}{n^{\frac{1}{2} + \varepsilon}} = 0 .$$

And, we may want to translate this statement into something perhaps a little more intelligible. This theorem says that the Riemann hypothesis is equivalent to the statement that a given integer has the same chance of having an odd number of distinct prime factors as it does of having an even number of distinct prime factors. Otherwise the numerator isn't going to be tending towards the rather small number in absolute value we will need.

Another way of translating this equivalence is to say, if we look at the sequence

$$\{\lambda(i)\}_{i=1}^{\infty} = \{1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, \ldots\}$$

then this sequence essentially behaves like a random sequence of 1's and -1's. In fact, we could see that the difference between the number of 1's and -1's, a given number of terms into this sequence, should be much smaller than a little more than the square root of the number of terms we are looking at.

Now the connection between the Riemann hypothesis and the Liouville function should be somewhat clear. Next we need to establish the connection between this function and Gauss's PNT. Then we may try and make the connection between the Riemann hypothesis and the PNT. To this end, we have the following theorem, also from Landau.

Theorem 7: *The Prime Number Theorem is equivalent to the statement*

$$\lim_{n \to \infty} \frac{\lambda(1) + \lambda(2) + \ldots + \lambda(n)}{n} = 0 .$$

If we translate this theorem along a similar vein to how we translated the equivalence of the Riemann hypothesis, then we could read this as saying that in the sequence we looked at above the difference between the number of 1’s and -1’s, a given number of terms into this sequence, should be much smaller than the number of terms we are looking at.

Of course, if the difference between the number of 1’s and -1’s is much smaller than the square root of the number of terms we are looking at, then it must also be much smaller than the number of terms, without taking the square root, we are looking at. Thus, through this look at the Riemann hypothesis and the PNT we can clearly see that if the Riemann hypothesis holds, then the PNT must immediately be true as well [**1**, chapter 1].

Setting this aside, we can return to Riemann and look at how he tried to explicitly tie his zeta function to the PNT. As was mentioned not too long ago, it should be possible, if the Riemann hypothesis is true, to come up with a better formulation of the prime counting function. We will follow the path Riemann took to get there by looking at some notes put together by Andrew Granville of Université de Montréal [**5**].

Yet, before we begin that we will look at an improvement of the PNT which was also due to Gauss in 1849. It can be shown that the logarithmic integral function, $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$ is a better approximation for $\pi(x)$ than the ratio $\frac{x}{\log(x)}$.

Our original PNT stated that the ratio between $\pi(x)$ and $\frac{x}{\log(x)}$ tended to 1 as x approached infinity. However with some work (see [**3**, chapter 7]) we can show that we have the better Prime Number Theorem:

Theorem 8 (The Prime Number Theorem): *Let* $\pi(x)$ *denote the number of primes less than or equal to x. Then*

$$\pi(x) \sim \mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)} .$$

This is not merely a trivial rephrasing of the Prime Number Theorem. If we were to compare the difference between our earlier ratio and $\pi(x)$to the difference between Li(*x*) and $\pi(x)$ for multiple values of *x*, then we would see that the second difference is consistently much smaller. For instance, given $x = 10^{12}$ we have $\frac{x}{\log(x)} - \pi(x) = -169,923,160$ and $\mathrm{Li}(x) - \pi(x) = 11,588$. Thus we could say that Li(*x*) is a much better estimate for the prime counting function. The first term in the asymptotic expansion of Li(*x*) is $\frac{x}{\log(x)}$, so we might explain the fact that Li(*x*) is closer to $\pi(x)$ is because value generated by the rest of the terms in the expansion. [**4**, p. 114-116].

Riemann's efforts were then made in trying to find a formula for the overcount that Gauss's new PNT was making. Riemann sought a formula for the error term we were getting from the log integral function. To do this, Riemann made a prediction which we can state as

$$|\log(\operatorname{lcm}[1, 2, 3, \ldots, x]) - x| \leq 2\sqrt{x}(\log x)^2 \text{ for all } x \geq 100.$$

Which is to say that we have $\log(\operatorname{lcm}[1, 2, 3, \ldots, x])$ is approximately x.

Now, since the power of a prime p which divides $\operatorname{lcm}[1, 2, 3, \ldots, x]$ is precisely the largest power of p not exceeding x, we have

$$\left(\prod_{p \leq x} p\right) \cdot \left(\prod_{p^2 \leq x} p\right) \cdot \left(\prod_{p^3 \leq x} p\right) \cdot \ldots = \operatorname{lcm}[1, 2, 3, \ldots, x].$$

Now we take the logarithm of both sides of this, and after using the multiplicative property on logs, we have

$$\left(\sum_{p \leq x} \log p\right) + \left(\sum_{p^2 \leq x} \log p\right) + \left(\sum_{p^3 \leq x} \log p\right) + \ldots = \log(\operatorname{lcm}[1, 2, 3, \ldots, x]).$$

So, with Riemann's prediction we get

$$\left(\sum_{p \leq x} \log p\right) + \left(\sum_{p^2 \leq x} \log p\right) + \left(\sum_{p^3 \leq x} \log p\right) + \ldots \text{is approximately } x.$$

And, we notice that the primes in the first sum are exactly those primes which are counted by $\pi(x)$, and the primes in the second sum are those primes counted by $\pi\left(x^{\frac{1}{2}}\right)$, and so on. Then by partial summation we can deduce that

$$\pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \ldots \approx \frac{x}{\log(x)}.$$

Now, since we have two equivalent forms of the PNT we can conclude that

$$\frac{x}{\log(x)} \approx \operatorname{Li}(x).$$

So we finally deduce that

$$\pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \ldots \approx \int_2^x \frac{dt}{\ln t} = \mathrm{Li}(x) .$$

Now we notice that

$\frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) \approx \frac{1}{2}\mathrm{Li}\left(x^{\frac{1}{2}}\right)$, $\frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) \approx \frac{1}{3}\mathrm{Li}\left(x^{\frac{1}{3}}\right)$, and so on.

And given this, it is possible to solve for the prime counting function so that we have the equivalent form

$$\pi(x) \approx \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}\left(x^{\frac{1}{2}}\right) + \ldots .$$

Thus, Riemann has yielded a prediction very similar to that of Gauss but with extra terms which should hopefully give us a formula for Gauss's overcount. And, we are going to begin referring to a weighted counting function following from the sum we just used. We define the weighted counting function $\Psi(x)$ as the following, with the *p*'s all primes:

$$\Psi(x) := \left(\sum_{p \le x} \log p\right) + \left(\sum_{p^2 \le x} \log p\right) + \left(\sum_{p^3 \le x} \log p\right) + \ldots = \sum_{p^m \le x,\, m \ge 1} \log p .$$

Now we want to begin linking what we've just done to the Riemann zeta function. To start that end, we begin with taking the logarithmic derivative of Euler's product identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and in turn obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \prod_{p} \frac{\log p}{p^s - 1} = \sum_{p} \sum_{m \geq 1} \frac{\log p}{p^{ms}} \; .$$

We combine this with $\Psi(x)$ to get a new representation for the weighted counting function. We will use the line $\Re(s) = c$, where *c* is taken large enough that everything will converge absolutely, and we get that

$$\Psi(x) := \sum_{p^m \leq x,\, m \geq 1} \log p = \frac{1}{2\pi i} \sum_{p,\, m \geq 1} \log p \int_{s : \Re(s) = c} \left(\frac{x}{p^m}\right)^s \frac{ds}{s}$$

Without loss of generality, we let $x = k + \frac{1}{2}$; that is, we may let x be a half integer. Thus, we have $\frac{1}{2\pi i}\int_{s : \Re(s) = c} \left(\frac{x}{p^m}\right)^s \frac{ds}{s} = 1$ for each *p* and *m*. Now we swap the order of the sum and the integral, which we can do because we chose *c* such that everything converged absolutely, and we have

$$\Psi(x) = -\frac{1}{2\pi i}\int_{s : \Re(s) = c} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}\, ds \; .$$

Next, we want to replace the line $\Re(s) = c$ over which we have taken the integral with a line far to the left. That is, we want to make *c* much smaller and have this all still work. So if we were to look at the difference between the values of the integrals, one with $\Re(s) = c$ taken as is and the other with the line far to the left, then we would find that is given by the sum of residues. And for any meromorphic function *f*, the poles of $\frac{f'(s)}{f(s)}$ are just given by the zeros and poles of *f*, and all will have order one. And the residue is given by the order of that zero of *f* or minus the order of that pole of *f*. Thus, we have a formulation of the counting function due to Von Mangoldt which is the explicit formula

$$\Psi(x) = \sum_{p^m \leq x} \log p = x - \sum_{\mu : \zeta(\mu) = 0} \frac{x^\mu}{\mu} - \frac{\zeta'(0)}{\zeta(0)}.$$

(Here, you may note we have introduced the “mu” which are the zeros of the Riemann zeta function.) In this formulation, if μ is a zero of $\zeta(s)$ of order *k*, then in the above sum there will be *k* terms appearing for that zero. In order to attempt to achieve any sort of partial sum of the above summation we can begin by adding μ of ascending $|\mu|$ values.

However, since there are infinitely many zeros of the Riemann zeta function in the critical strip, the above formulation can be difficult to use practically. Thus, we will tweak our above formulation for the counting function to only include a finite number of the zeros of $\zeta(s)$, specifically those in the box

$$\mathscr{B}(T) := \{\mu : \zeta(\mu) = 0, 0 \leq \Re(\mu) \leq 1, -T \leq \Im(\mu) \leq T\}$$

However, since this will cause us to have an approximation and not an exact formulation we will be needing an error term, and this error will depend on the height *T* of the box that we choose. Given $1 \leq T \leq x$ we have that

$$\Psi(x) = x - \sum_{\mu \in \mathscr{B}(T)} \frac{x^\mu}{\mu} + O\left(\frac{x \log x \log T}{T}\right).$$

To see why this is, we refer to a good deal of work attempting to explain this by Davenport [**3**, chapter 17].

At this point, our goal is to be able to show that $\Psi(x) \sim x$. To this end, we choose $T \geq (\log x)^2$ and bound our sum over the zeros of $\zeta(s)$. So we bound each term in absolute value to get

$$\left| \sum_{\mu \in \mathcal{B}(T)} \frac{x^{\mu}}{\mu} \right| \leq \sum_{\mu \in \mathcal{B}(T)} \left| \frac{x^{\mu}}{\mu} \right|$$

$$\leq \max_{\mu \in \mathcal{B}(T)} x^{\Re(\mu)} \sum_{\mu \in \mathcal{B}(T)} \frac{1}{|\mu|}$$

And if $\beta(T)$ is the largest real part of any zero in $\mathcal{B}(T)$, we have

$$\ll x^{\beta(T)} (\log T)^2$$

Here we use the fact that we know there are about $\frac{T}{2\pi} \log\left(\frac{T}{2e} \right)$ zeros of $\zeta(s)$ in $\mathcal{B}(T)$ for all T.

It can then be shown that we may take $\beta(T) = 1 - \frac{c}{\log T}$ for some constant $c > 0$. This is to say that we could show that the greatest the real part can be for any zero of the zeta function is not 1, but rather something less than 1. Thus, we may choose T so that $\log T = (\log x)^{\frac{1}{2}}$ and get

$$\Psi(x) = x + O\left(\frac{x}{e^{c' (\log x)^{1/2}}} \right)$$

for some constant $c' > 0$. And this implies the Prime Number Theorem,

$$\pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{e^{c' (\log x)^{1/2}}} \right).$$

While Jacques Hadamard and Charles de la Vallée-Poussin may have proved that we don't have to worry about the real part of any zeros of the Riemann zeta function being as great as 1, if were are able to extend the zero-free region for $\zeta(s)$, then we will be able to improve our error term in the PNT. For instance, if the Riemann hypothesis is true, then we have that all nontrivial zeros of the zeta function lie on the critical line, which gives us that $\beta(T) = \frac{1}{2}$ for all T. Thus we may take $T = \sqrt{x}$ and obtain

$$\Psi(x) = x + O\left(x^{\frac{1}{2}} (\log x)^2\right)$$

and so we also get that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x} \log x).$$

[**5**].

Thus, if the Riemann hypothesis is true we have a very nice formulation for the prime counting function which accounts for the error we were getting out of Gauss's guess.

"A simpler variant of his formula," due to Conrey [**2**, p. 343], is

$$\Psi(x) := \sum_{n \leq x} \Lambda(n)$$

$$= x - \sum_{\mu} \frac{x^{\mu}}{\mu} - \log 2\pi - \frac{1}{2} \log\left(1 - x^{-2}\right).$$

This is only valid for x not a power of a prime and the von Mangoldt function $\Lambda(n)$ = log p if $n = p^k$ for some prime p and some k and $\Lambda(n) = 0$, otherwise. It is worthwhile to note that $|x^{\mu}| = |x^{\Re(\mu)}|$.

With this above representation of the prime counting function, we now have an explicit way of showing that the count of primes less than a certain number, our x, is dependent upon the zeros of the Riemann zeta function. Now we can see that if Riemann's hypothesis is true, and all of the zeros μ have real part equal to one half, then the sum above doesn't have to worry about other values for $|x^{\mu}|$. So if Riemann's hypothesis is true, understanding that sum becomes far easier than if it were not true and the values of $\Re(\mu)$ jumped around all over the place. So, if Riemann's hypothesis is actually true, then we have an explicit definition for the prime counting function, and should we wish to know the number of primes less than a given number, we can find that number with this function and no longer have to worry about an amount of error.

While it may certainly be interesting to look at the Riemann hypothesis as providing a nice way of obtaining a proof for the Prime Number Theorem that explicitly takes care of the error we get, as stated, we already have several proofs of the PNT. And while Riemann's way might be best, it doesn't do much good to talk about the consequences of the Riemann hypothesis if it turns out that the Riemann hypothesis is just a false conjecture. So let's look for moment at what reasonable suspicions we should have to believe that the Riemann hypothesis is true and why we shouldn't believe that it is actually false.

Some individuals such as Aleksandar Ivić have written a great deal on the analytic consequences of the Riemann hypothesis and what reason we should have to believe or remain skeptical about the hypothesis. And yet individuals such as these remain unconvinced either way and are unwilling to commit themselves to the belief or disbelief of the Riemann hypothesis [**1**, pp. 130-160]. But for now, let us look at the evidence which we have to believe in the Riemann hypothesis. We can at least start by looking at what is known about the zeros we have been able to find.

There is much computational evidence to suggest to us that the Riemann hypothesis should be true. While of course computational evidence should not be taken for proof, since it could easily be that the first zero not to lie on the critical line $\Re(s) = \frac{1}{2}$ is just outside of our computational range. It could be that the first zero not to lie on the critical line is the $10^{10^{1000000000}}$ th zero. So we should by no means suggest that this computational evidence is tantamount to proof, but we can say that it provides reasonable reason to believe our instincts should they tell us that the Riemann hypothesis is true [**2**, p. 344].

Recent work by van de Lune calculated that the first 10 billion zeros lie on the critical line [**2**, p. 344]. Gourdon in 2004 was able to go even further to show that the first 10 trillion zeros lie on the critical line [**10**]. Further down, where the continuous count stops, Andrew Odlyzko calculated millions of zeros near the 10^{20}, 10^{21}, and 10^{22} zero and also found that these were on the critical line [**2**, p. 344]. So the computational evidence is in numbers that go well beyond the ability of the majority of people to think about, let alone really picture.

Of course, the computational evidence would be worthless if we were only finding the first relative few zeros were on the critical line and it turned out the vast majority that followed were not. So work has also been done to at least show that the proportion of zeros on the critical line isn't zero. It was Selberg who proved that there was a positive proportion of the zeros of zeta on the critical line. Then N. Levinson has done us one better by showing that at least one-third of all of the zeros must be on the critical line. That proportion has even been improved since then to show that we have at least 40% of the zeros on the critical line [**2**, pp. 344-5].

Conrey [**2**, p345] suggests another argument for believing that the Riemann hypothesis should be true. It is perhaps the most compelling argument, on an instinctual level, that the Riemann hypothesis is something we should believe in. For this argument we relate what Riemann's hypothesis would say if it were false to what that in turn says about the primes. Should the Riemann hypothesis not be true, then there must be at least one non-trivial zero of $\zeta(s)$ not on the critical line. Since we've already calculated trillions of zeros, it already seems odd that a zero would just pop up that wasn't on the line. And we can think of how interesting that zero would seem in comparison to its brothers that did live on the line. Any zeros not lying on the line would make the distribution of primes become far less regular as we understand them. It would mean that somewhere down the count of primes some of them were jumping out of the nice regular order that the rest of them seemed to follow by following the zeros of the critical line.

The first prime to jump out of that nice order that makes the distribution of the primes so much nicer to picture would be a true black sheep among the primes. But suppose that the other 60% of the zeros that Conrey did not yet show were on the critical line turned out not to actually be on the critical line. Just how chaotic would that mean the distribution of the primes was? Nature seems so much nicer and more orderly than that. We see such wonderful orderly patterns appearing in the natural world. It would be a true shame if such wonderfully interesting objects as the prime numbers ended up not having the nice distribution that Riemann suggested.

What We Can Take Away from It All

The brilliance of Bernhard Riemann is something I haven't always had an appreciation for. While we may all look at the results that Riemann has given to us, going back in my mind all the way to talking about him in my first calculus course as we began to consider the area under curves as we integrated, we don't always appreciate the brilliance and revolutionary thought processes that he must have had to get his results. We use them in our work and study with full confidence that they are true and hold, perhaps mainly because Riemann was so methodical and such a perfectionist that there is little room to question him. And yet, in perhaps the most brilliant and original contribution of his life Riemann did not maintain that same discipline.

There is a lot of reason to be grateful then that we ever saw the Riemann hypothesis at all. It seems Riemann was so moved by the adventurous spirit of his discoveries and predictions that he was moved to speak about his hypothesis and related conclusions without the well thought out proof that accompanies so much else of his work. His work followed the changing tide in the study of Mathematics. Study and discovery for Mathematics' own sake was becoming the style of the day. And as such, the dipping into the river of Cauchy's complex numbers by Riemann led to such a wonderful and revolutionary discovery in the Riemann hypothesis. Riemann had carried on the work of his mentors Gauss and Dirichlet to a point where he thought he had found the missing connections in their guesses and predictions. We may not have evidence that Riemann had any kind of proof of this, but he certainly seemed to deeply believe in his conclusions.

Riemann had taken the zeta function Dirichlet had so aptly used and subjected it to the new wondrous landscape of Cauchy's complex numbers. And in so doing, Riemann gave birth to Analytic Number Theory. Though it went unrecognized and unappreciated for years, we appreciate it now. Riemann's hypothesis has resisted proof for more than a hundred years despite tempting some of the best minds of generations of mathematicians.

Part of the reason, as we saw, that Riemann's hypothesis has been so difficult to prove is that it uses his zeta function which is such an analytically complicated beast. It took us a lot of work just to extend the simple series to something which we could view analytically on the whole complex plane, minus its pole. We had to go and look at the poles of the gamma function just to understand where the trivial zeros of Riemann's zeta function were coming from. And while we may have narrowed down the non-trivial zeros to a critical strip, even Riemann had failed to prove Gauss's form of the PNT by showing that none of the non-trivial zeros had a real part equal to 1. It took the minds of Hadamard and de la Vallée-Poussin, some thirty years later, to manage even this.

After examining the zeta function analytically we were finally able to walk through how the prime counting function could be rewritten if the Riemann hypothesis were true. We would no longer have the error that either of Gauss's predictions had. We could have an explicit formulation for the prime counting function giving us a better PNT than Gauss could have ever hoped for.

In an ironic way, I now appreciate how I was so unappreciative of the Riemann hypothesis before. All the work tying complex function theory and number theory together is quite a load. It is not an easy task to examine such an innocent looking series, as the zeta function started out as, and end up expanding it into an analytic monstrosity. I can understand now how it wouldn't have been worthwhile to spend any significant amount of time digressing from a complex analysis course just to look at this zeta function. And the level of analysis needed certainly makes it obvious to why elementary number theory course would steer clear of Riemann's hypothesis.

However, the beauty of Riemann's hypothesis should not be ignored. If the Riemann hypothesis is true, we would have such a nice representation of the prime counting function. No longer would young mathematics students' brains be riddled with questions about how randomly the prime numbers are distributed. If all of the non-trivial zeros of the zeta function lie on the critical line, we would then know precisely how well distributed the prime numbers were amongst their natural number brethren.

What is even more, the consequences of the Riemann hypothesis extend far beyond what I have done here. Many a theorem in the world of number theory starts with the simple statement “If the Riemann hypothesis is true, then…;” see chapter 7 in [**1**] for some results that rely on the Riemann hypothesis or its extensions. In many ways, the Riemann hypothesis is almost its own branch of study. The multitude of approaches at proof, the number of subjects needing to be stirred together just to understand the statement of it, and the consequences it would bear are staggering.

David Hilbert was once asked that if he were to be revived after 500 years what the first question he would ask was. His answer was the question, “Has somebody proved the Riemann Hypothesis?” [**6**, p. 69]. I think if I were to be revived after 500 years, my first question might be something about the availability of jetpacks. But then I think Hilbert’s question would be a great second one to ask.

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