Values of π(x) and Δ(x) for various x's

The tables were compiled by Andrey V. Kulsha. See below the explanation of Δ(x).

Values of x	Tables			Local minima of Δ(x)	Local maxima of Δ(x)
1 to 10	step 10^{-3}	step 10^{-4}	step 10^{-5}	Δ(5-0) = -0.3952461978	Δ(1+0) = +1.0000000000
10^1 to 10^2	step 10^{-2}	step 10^{-3}	step 10^{-4}	Δ(11-0) = -0.5492343329	Δ(19+0) = +0.5607597113
10^2 to 10^3	step 10^{-1}	step 10^{-2}	step 10^{-3}	Δ(223-0) = -0.6051733874	Δ(113+0) = +0.7848341482
10^3 to 10^4	step 1	step 10^{-1}	step 10^{-2}	Δ(1423-0) = -0.7542604400	Δ(1627+0) = +0.6754517455
10^4 to 10^5	step 10^1	step 1	step 10^{-1}	Δ(19373-0) = -0.7278356754	Δ(24137+0) = +0.7457431860
10^5 to 10^6	step 10^2	step 10^1	step 1	Δ(302831-0) = -0.6995719492	Δ(355111+0) = +0.7008073861
10^6 to 10^7	step 10^3	step 10^2	step 10^1	Δ(1090697-0) = -0.6389660809	Δ(3445943+0) = +0.6809987397
10^7 to 10^8	step 10^4	step 10^3	step 10^2	Δ(36917099-0) = -0.7489165055	Δ(30909673+0) = +0.7157292126
10^8 to 10^9	step 10^5	step 10^4	step 10^3	Δ(516128797-0) = -0.6775687236	Δ(110102617+0) = +0.7878100197
10^9 to 10^{10}	step 10^6	step 10^5	step 10^4	Δ(7712599823-0) = -0.6889577485	Δ(1110072773+0) = +0.6833192028
10^{10} to 10^{11}	step 10^7	step 10^6	step 10^5	Δ(11467849447-0) = -0.7251609705	Δ(10016844407+0) = +0.6386706267
10^{11} to 10^{12}	step 10^8	step 10^7	step 10^6	Δ(110486344211-0) = -0.7355462679	Δ(330957852107+0) = +0.7533813432
10^{12} to 10^{13}	step 10^9	step 10^8	step 10^7	Δ(1635820377397-0) = -0.6892596608	Δ(2047388353069+0) = +0.6808028098
10^{13} to 10^{14}	step 10^{10}	step 10^9	step 10^8	Δ(36219717668609-0) = -0.8360329846	Δ(21105695997889+0) = +0.6896466780
10^{14} to 10^{15}	step 10^{11}	step 10^{10}	step 10^9	Δ(348323506633621-0) = -0.6494959371	Δ(117396942462053+0) = +0.6789107425
10^{15} to 10^{16}	step 10^{12}	step 10^{11}	step 10^{10}	Δ(1212562524413153-0) = -0.7750460589	Δ(1047930291039067+0) = +0.7042622330
10^{16} to 10^{17}	step 10^{13}	step 10^{12}	step 10^{11}	Δ(18019655286689201-0) = -0.5710665212	Δ(16452596773450399+0) = +0.7144542025
10^{17} to 10^{18}	step 10^{14}	step 10^{13}	step 10^{12}	Δ(266175790131587543-0) = -0.7599282036	Δ(125546149553907317+0) = +0.6572554320
10^{18} to 10^{19}	step 10^{15}			Δ(5805523423155128399-0) = -0.6804259482	Δ(1325005986250807813+0) = +0.7839983342
10^{19} to 10^{20}	step 10^{16}			Δ(55496658217283199013-0) = -0.8042730098	Δ(11538454954199984761+0) = +0.7574646817
10^{20} to 10^{21}	step 10^{18}			(to do)	(to do)
10^{21} +	214 entries			Δ(x) has no global minimum	Δ(x) has no global maximum

The values of π(x) for $x < 10^{17}$ were mostly computed with primesieve written by Kim Walisch and are also available with a step of 10^9 (fully double-checked). The data is encoded as a stream of 2-byte differences between the successive rounded values of $(\pi(x)-\mathrm{li}(x))\cdot 3/2$, and a small delphi program is provided to get a plain text file. The multiplier of 3/2 doesn't bring the differences out of 2-byte range [-32768 ... +32767] and allows to compute li(x) with an absolute error of up to 1/6, so the Ramanujan method with extended precision works well at least for $x < 10^{18}$.

The values of π(x) for $10^{17} \le x \le 10^{20}$ were taken from [1].

The values of π(x) for $10^{20} < x < 10^{21}$ were computed with primecount written by Kim Walisch.

The values of π(x) for $x \ge 10^{21}$ were taken from [1], [2] and from Sloane's A006988, A007097. See also the results of Jan Büthe [3], David J. Platt [4], Thomas R. Nicely [5] and Xavier Gourdon [6].

Some extreme regions where |Δ(x)| exceeds 0.75

Values of x	Tables	# of entries
From $1.100\cdot10^8$ to $1.102\cdot10^8$ with a step of 10^1	max08(01)09.txt	20 001
From $3.309\cdot10^{11}$ to $3.310\cdot10^{11}$ with a step of 10^4	max11(04)12.txt	10 001
From $3.309578\cdot10^{11}$ to $3.309580\cdot10^{11}$ with a step of 10^1	max11(01)12.txt	20 001
From $3.590\cdot10^{13}$ to $3.625\cdot10^{13}$ with a step of 10^7	min13(07)14.txt	35 001
From $3.62194\cdot10^{13}$ to $3.62200\cdot10^{13}$ with a step of 10^4	min13(04)14.txt	60 001
From $3.62197176\cdot10^{13}$ to $3.62197178\cdot10^{13}$ with a step of 10^1	min13(01)14.txt	20 001
From $1.212\cdot10^{15}$ to $1.214\cdot10^{15}$ with a step of 10^8	min15(08)16.txt	20 001
From $1.212556\cdot10^{15}$ to $1.212565\cdot10^{15}$ with a step of 10^5	min15(05)16.txt	90 001
From $1.212562517\cdot10^{15}$ to $1.212562526\cdot10^{15}$ with a step of 10^2	min15(02)16.txt	90 001
From $3.2949\cdot10^{15}$ to $3.2957\cdot10^{15}$ with a step of 10^7	min15(07)16.txt	80 001
From $2.6615\cdot10^{17}$ to $2.6635\cdot10^{17}$ with a step of 10^{10}	min17(10)18.txt	20 001
From $2.661751\cdot10^{17}$ to $2.661760\cdot10^{17}$ with a step of 10^7	min17(07)18.txt	90 001
From $2.6617579011\cdot10^{17}$ to $2.6617579017\cdot10^{17}$ with a step of 10^3	min17(03)18.txt	60 001
From $1.3245\cdot10^{18}$ to $1.3260\cdot10^{18}$ with a step of 10^{11}	max18(11)19.txt	15 001

From $1.3250059 \cdot 10^{18}$ to $1.3250067 \cdot 10^{18}$ with a step of 10^7	max18(07)19.txt	80 001
From $1.32500598624 \cdot 10^{18}$ to $1.32500598626 \cdot 10^{18}$ with a step of 10^3	max18(03)19.txt	20 001
From $1.1536 \cdot 10^{19}$ to $1.1542 \cdot 10^{19}$ with a step of 10^{11}	max19(11)20.txt	60 001
From $1.15384544 \cdot 10^{19}$ to $1.15384551 \cdot 10^{19}$ with a step of 10^7	max19(07)20.txt	70 001
From $1.153845497419 \cdot 10^{19}$ to $1.153845497421 \cdot 10^{19}$ with a step of 10^3	max19(03)20.txt	20 001
From $5.5496655 \cdot 10^{19}$ to $5.5496662 \cdot 10^{19}$ with a step of 10^8	min19(08)20.txt	70 001
From $5.54966582172 \cdot 10^{19}$ to $5.54966582174 \cdot 10^{19}$ with a step of 10^4	min19(04)20.txt	20 001

These results confirm some previously made computations [7] [8]. See also [9] and [10] about the oscillations of Δ(x) at larger x's.

Where did Δ(x) come from?

The prime-counting function, π(x), may be computed analytically. The explicit formula for it, valid for x > 1, looks like

$$\pi_0(x) = \mathrm{R}(x) - \sum_{\rho} \mathrm{R}(x^{\rho}) - \frac{1}{\ln x} + \frac{1}{\pi} \arctan \frac{\pi}{\ln x}$$

where

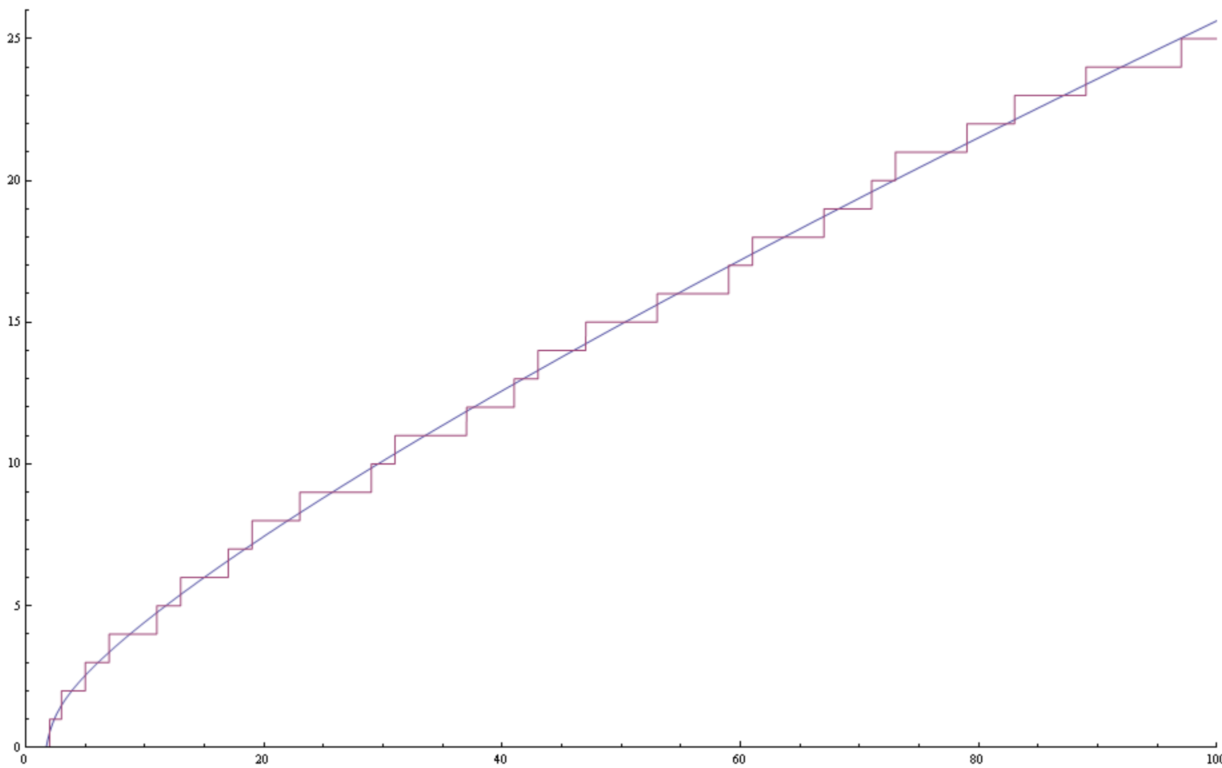
$$\pi_0(x) = \lim_{\varepsilon \to 0} \frac{\pi(x-\varepsilon) + \pi(x+\varepsilon)}{2}$$

$$\mathrm{R}(x^{\rho}) = 1 + \sum_{k=1}^{\infty} \frac{(\rho \ln x)^k}{k! k \zeta(k+1)}$$

and the sum runs over the non-trivial (i.e. with positive real part) zeros of Riemann ζ-function in order of increasing the absolute value of the imaginary part. This sum describes the fluctuations of π(x), while the remaining terms give the «smooth» part of it and may be used as a very good estimator of π(x):

```
In[1]:= Plot[{RiemannR[x] - 1/Log[x] + ArcTan[Pi/Log[x]]/Pi, PrimePi[x]}, {x, 1, 100}, PlotRange → {{0, 100}, {0, 26}}, PlotPoints → 200]
```

Out[1]=



Here you can see the plot of π(x) (the purple line) compared to the blue line of

$$\mathrm{R}(x) - \frac{1}{\ln x} + \frac{1}{\pi} \arctan \frac{\pi}{\ln x}$$

The difference between these two heuristically oscillates with an amplitude of about

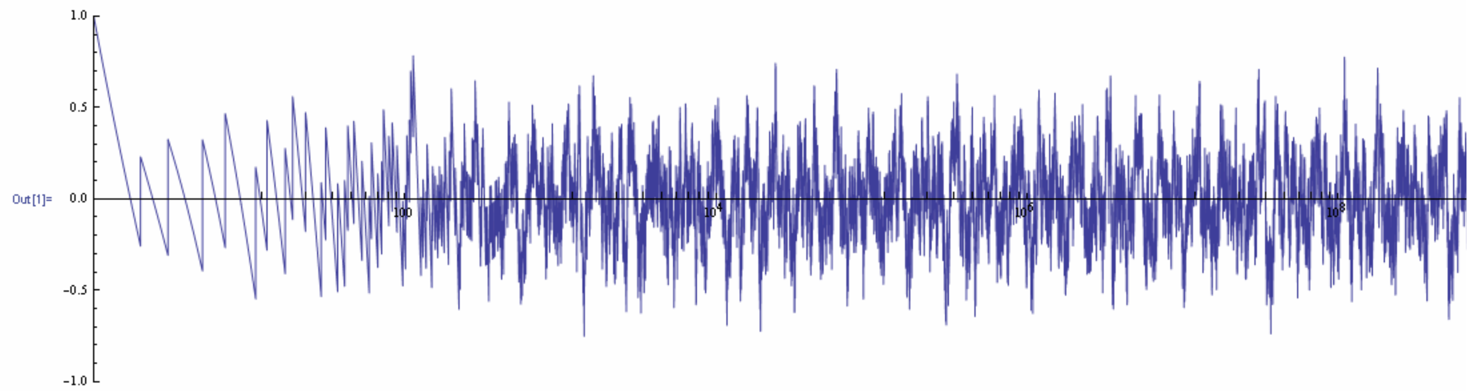
$$\frac{\sqrt{x}}{\ln x}$$

so we have the following expression for Δ(x), the function which clearly represents the fluctuations of the distribution of primes:

$$\Delta(x) = \left(\pi_0(x) - \mathrm{R}(x) + \frac{1}{\ln x} - \frac{1}{\pi}\arctan\frac{\pi}{\ln x}\right)\frac{\ln x}{\sqrt{x}}$$

There's a plot of Δ(x) on the log scale:

```
In[1]:= LogLinearPlot[(PrimePi[x] - RiemannR[x] + 1 / Log[x] - ArcTan[Pi / Log[x]] / Pi) * Log[x] / Sqrt[x], {x, 1, Exp[27]}, PlotPoints → 10 000, PlotRange → {{1, Exp[27]},
```



Some estimations of the logarithmic density of Δ(x) are given in [11] and [12].

On the explicit formula for π(x)

Curiously, this formula seems to be never seen in literature [13], so let's describe its origin. The formula comes from the Möbius inversion

$$\pi_0(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\Pi_0(x^{\frac{1}{n}})$$

of

$$\Pi_0(x) = \lim_{\varepsilon\to 0}\frac{\Pi(x-\varepsilon)+\Pi(x+\varepsilon)}{2}$$

where

$$\Pi(x) = \sum_{p^m\le x}\frac{1}{m} = \sum_{n=1}^{\infty}\frac{1}{n}\pi(x^{\frac{1}{n}})$$

is so-called Riemann prime-counting function (the first sum runs over the powers of primes). We have the following expression for $\Pi_0(x)$ [13]:

$$\Pi_0(x) = \mathrm{li}(x) - \sum_{\rho}\mathrm{li}(x^{\rho}) - \ln 2 + \int_x^{\infty}\frac{dt}{t(t^2-1)\ln t}$$

where li is the logarithmic integral; $\mathrm{li}(x^{\rho})$ should be considered as Ei(ρlnx), where Ei is the analytic continuation of the exponential integral function from positive reals to the complex plane with branch cut along the negative reals. The sum runs, as before, over the non-trivial zeros of ζ-function in the same manner. Thus, the formula immediately follows from these four equalities:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{\frac{1}{n}}) = \mathrm{R}(x)$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(-\frac{n}{2\ln x} + \int_{x^{1/n}}^{\infty}\frac{dt}{t(t^2-1)\ln t}\right) = \frac{1}{\pi}\arctan\frac{\pi}{\ln x}$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\sum_{\rho}\mathrm{li}(x^{\frac{\rho}{n}}) - \frac{n}{2\ln x}\right) = \sum_{\rho}\mathrm{R}(x^{\rho}) + \frac{1}{\ln x}$$

The first two of them are well-known [14]; the third one comes straightly from (32) in [15], while the last one is obvious if we allow generalized summation

$$\sum_{n=1}^{\infty}\mu(n) = \frac{1}{\zeta(0)} = -2$$

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