Modern Portfolio Theory

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1 Introduction

Let $0 < i \leq N$ be an integer that indexes the set of N assets to be considered in a candidate portfolio. Let w_i denote the weighting of asset i in the portfolio, μ_i denote its expected return, and σ_i denote its standard deviation.

With these variables, the return of the portfolio, R_p, and its variance, σ_p^2, are given by

$$\begin{aligned} R_p &= \sum_{i=1}^{N} w_i \mu_i = \boldsymbol{w} \cdot \boldsymbol{\mu}, \\ \sigma_p^2 &= \boldsymbol{w} \boldsymbol{\Sigma} \boldsymbol{w}, \end{aligned} \tag{1}$$

respectively, where $\boldsymbol{\Sigma}$ is the covariance matrix of returns between the assets included in the portfolio.

2 The Efficient Frontier

The efficient frontier is the optimal set of portfolio weights that generate the maximum return for any given risk (or volatility) of the portfolio. The optimal portfolio weights are thus found by minimizing the portfolio variance, σ_p^2, subject to the constraints

$$\begin{aligned} \sum_{i=1}^{N} w_i \mu_i &= R_p, \\ \sum_{i=1}^{N} w_i &= 1. \end{aligned} \tag{2}$$

Since the problem is a constrained minimization problem, it can be solved using a Lagrangian with the constraints implemented as Lagrange multipliers. This gives the Lagrangian

$$\mathcal{L} = \frac{1}{2} \boldsymbol{w} \boldsymbol{\Sigma} \boldsymbol{w} + \lambda_1 (\boldsymbol{w} \cdot \boldsymbol{\mu} - R_p) + \lambda_2 (\boldsymbol{w} \cdot \boldsymbol{e} - 1) \tag{3}$$

where λ_1 and λ_2 are the constraint parameters, and $\boldsymbol{e}$ denotes a vector of ones of length N.

Minimizing the Lagrangian with respect to the dynamical variables yields the equations

$$\frac{\partial \mathcal{L}}{\partial w_i} = \Sigma_{ij} w_j + \lambda_1 \mu_i + \lambda_2 = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \boldsymbol{w} \cdot \boldsymbol{\mu} - R_p = 0, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = \boldsymbol{w} \cdot \boldsymbol{e} - 1 = 0, \tag{6}$$

where we have used that the correlation matrix, $\boldsymbol{\Sigma}$, is symmetric. Solving for the weights, w_i, using Eq. (4) gives

$$w_i = \lambda_1 \Sigma_{ij}^{-1} \mu_j + \lambda_2 \Sigma_{ij}^{-1} e_j, \tag{7}$$

where Σ_{ij}^{-1} is the ijth element of $(\boldsymbol{\Sigma})^{-1}$ and $e_j = 1$ for all j. Substituting Eq. (7) into Eq. (5), and then Eq. (7) into Eq. (6), gives

$$R_p = \lambda_1 \mu_i \Sigma_{ij}^{-1} \mu_j + \ \lambda_2 \mu_i \Sigma_{ij}^{-1} e_j, \tag{8}$$

$$1 = \lambda_1 \Sigma_{ij}^{-1} \mu_j e_i + \lambda_2 \mu_i \Sigma_{ij}^{-1} e_j. \tag{9}$$

This pair of linear equations can be written as

$$\begin{pmatrix} R_p \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} & \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{e} \\ \boldsymbol{e}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} & \boldsymbol{e}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{e} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a & c \\ c & f \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{10}$$

where variables $a = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, $c = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{e}$, and $f = \boldsymbol{e}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{e}$ have been introduced to simplify the notation. Let d denote the determinant of this matrix, $d = af - cc$. Solving for λ_1 and λ_2 gives

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} f & -c \\ -c & a \end{pmatrix} \begin{pmatrix} R_p \\ 1 \end{pmatrix}, \tag{11}$$

thus

$$\begin{aligned} \lambda_1 &= \frac{1}{d}(f R_p - c), \\ \lambda_2 &= -\frac{1}{d}(c R_p - a). \end{aligned} \tag{12}$$

Substituting these expressions for λ_1 and λ_2 into Eq. (7), and using the definitions for the return and variance of the portfolio to simplify the resulting expression gives

$$\sigma_p^2 = \frac{1}{d}(f R_p^2 - 2c R_p + a), \tag{13}$$

which is the equation of the minimum-variance frontier.

If the portfolio weightings along the efficient frontier are required, we can calculate these using Eq. (7) along with the values for λ_1 and λ_2 from Eq. (12).