Beyond Degree-Choosability
Toward a Local Version of Reed's ω, Δ, χ conjecture

Tom Kelly
Joint work with Luke Postle

Department of Combinatorics & Optimization
University of Waterloo

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Coloring

Definition

A graph G is k-colorable if there is an assignment of the "colors" $1, \ldots, k$ to $V(G)$ such that adjacent vertices receive different colors.
The chromatic number of G, denoted $\chi(G)$, is the smallest k such that G is k-colorable.

- $\Delta(G) =$ max degree of a vertex in G, and
- $\omega(G) =$ max size of a clique in G.

Trivial bounds:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

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Theorem (Brooks (1941))

For every connected graph G that is not a clique or odd cycle,

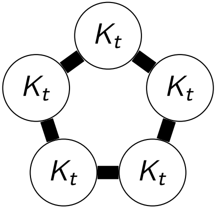
$$\chi(G) \leq \Delta(G).$$

Reed's ω, Δ, χ Conjecture

Reed's Conjecture (1998)

For every graph G,

$$\chi(G) \leq \left\lceil \frac{1}{2}\left(\Delta(G) + 1 + \omega(G)\right)\right\rceil.$$



5-cycle blowup

$$\Delta = 3t - 1$$
$$\omega = 2t$$
$$\left\lceil \tfrac{1}{2}\left(\Delta + 1 + \omega\right)\right\rceil = \left\lceil \tfrac{5t}{2}\right\rceil.$$
$$\alpha = 2.$$

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For every graph G,

$$\chi(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil.$$

As evidence, Reed proved his conjecture holds when $\Delta(G)$ is sufficiently large and

$$\omega(G) \geq \left(1 - 7 \cdot 10^{-7}\right) \Delta(G).$$

Corollary (Reed)

There exists $\varepsilon > 0$ such that for every graph G,

$$\chi(G) \leq (1 - \varepsilon)(\Delta(G) + 1) + \varepsilon\omega(G).$$

List-Coloring

Definition

For a graph G, $L = (L(v) : v \in V(G))$ is a list-assignment if each $L(v) \subset \mathbb{N}$ is a "list of colors", and G is L-colorable if there is an assignment of colors to $V(G)$ such that adjacent vertices receive different colors and each $v \in V(G)$ receives a color from $L(v)$.

The list-chromatic number of G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Clearly,

$$\chi(G) \leq \chi_\ell(G)$$

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It is natural to ask:

- does Brooks' Theorem hold for χ_ℓ?
- is Reed's Conjecture true for χ_ℓ?

ω, Δ, and χ/χ_ℓ

Conjecture (List-Coloring Version of Reed's)

For every graph G,

$$\chi_\ell(G) \leq \left\lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G)) \right\rceil .$$

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	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 1.4 \cdot 10^{-8}$	Reed 98	N

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	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 830^{-1}$	Bruhn & Joos 15+	N

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	$(1-\varepsilon)(\Delta+1)+\varepsilon\omega$, $\varepsilon = 26^{-1}$	Bonamy, Perrett & Postle 16+	N

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$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Molloy 17+	Y
$= 2$	$(67 + o(1))\frac{\Delta}{\ln\Delta}$	Jamall 11	Y

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$\leq r$	$O(r)\frac{\Delta \ln\ln\Delta}{\ln\Delta}$	Molloy 17+	Y
$= 2$	$(4 + o(1))\frac{\Delta}{\ln\Delta}$	Pettie & Su 15	Y

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The Local Paradigm

Theorem (Erdős, Rubin, Taylor, 1979)

Every connected graph G is L-colorable if $|L(v)| \geq d(v)$ for all $v \in V(G)$, unless every block of G is a clique or odd cycle.

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Our main result:

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$, if $\Delta(G)$ sufficiently large and for all $v \in V(G)$,

1. $|L(v)| \geq (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v)$, *and*
2. $|L(v)| - \omega(v) \geq \log^{14}(\Delta(G))$,

then G has an L-coloring.

An Application

King's Conjecture (2009)

For every graph G,

$$\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \right\rceil .$$

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- In 2015, King and Reed proved Reed's Conjecture for claw-free graphs and King's Conjecture for claw-free graphs with a three-colorable complement.

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Our result implies

Corollary (K,P)

Let $\varepsilon \leq \frac{1}{52}$. If $\Delta(G)$ sufficiently large and $\omega(G) \leq (1 - \varepsilon - o(1))\Delta(G)$, then

$$\chi_\ell(G) \leq \max_{v \in V(G)} (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v).$$

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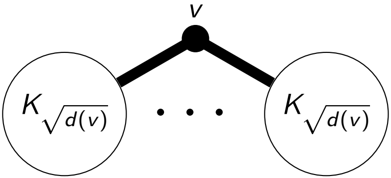
Ideally we would show that in some instance of G' and L', for each $v \in V(G')$, $|L'(v)| > d_{G'}(v)$, but this doesn't work.

Local Difficulties



$$L = \{1, \ldots, \sqrt{d(v)}\}, \quad \{d(v) - \sqrt{d(v)}, \ldots, d(v)\}$$

A pathological case:

$$d_{G'}(v) - |L'(v)| \approx d_G(v) - |L(v)|.$$

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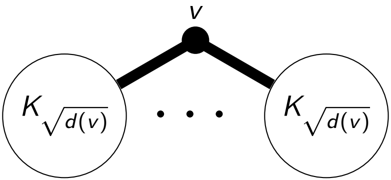

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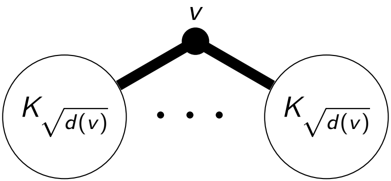

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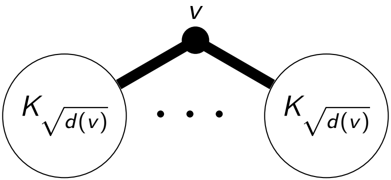

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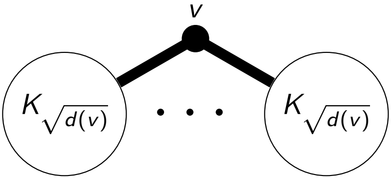

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Say v is lordly if v has many subservient neighbors.

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To deal with lordly vertices, color G' in the order of original list size, i.e. color vertices before their subservient neighbors.

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- aberrant if it has many neighbors u with $L(u)$ significantly different from $L(v)$:
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Key Fact: If a vertex is lordly, egalitarian-sparse, or aberrant, the expected number of "saved" colors is $\Omega(d(v) - \omega(v))$.

Proof Overview

Theorem (K, Postle (2017+))

For $\varepsilon \leq \frac{1}{52}$*, if* Δ *sufficiently large,* $\Delta(G) \leq \Delta$*, and for all* $v \in V(G)$*,*

1. $|L(v)| \geq (1 - \varepsilon)(d(v) + 1) + \varepsilon\omega(v)$*, and*
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Main Structural Lemma

Every vertex of a minimum counterexample is either aberrant, lordly, or egalitarian-sparse.

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Main Probabilistic Lemma

If a vertex v is either aberrant, lordly, or egalitarian-sparse, then with high enough probability,

$$|L'(v)| > |\{u \in N(v) \cap V(G') : |L(u)| \geq |L(v)|\}|.$$

Thanks!