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# On Satellite Orbits with Very Small Eccentricities

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On the basis of a recent theory of this author, satellite orbits with very small eccentricities have been investigated. It was found that such orbits can be represented by the formula

$$r=a_*(1-e\cos v+\tfrac{1}{2}e^2\cos 2v+\tfrac{1}{4}\nu^2 s^2\cos 2w)+O(\nu^3),$$

where $w=(1+\epsilon)v+\omega$. Graphical illustrations of this result are given.

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FOR several practical purposes it is advantageous to have satellite orbits with very small eccentricities. In the cases of satellites 1959 gamma (Discoverer II) and 1960 $\beta2$ (Tiros I) the eccentricities were 0.008 and 0.003, respectively, and among the pre-launch nominal elements we even found the value $e=0$. Unfortunately, most of the satellite theories developed during recent years exhibit some difficulties if the eccentricity is a very small quantity, because it appears as a divisor in the formulae expressing the perturbations in the osculating argument of perigee $\omega$, and mean anomaly $M$. See, in particular, the papers by Brouwer (1959), Garfinkel (1959), and Kozai (1959). The difficulties, however, can be easily removed from these theories, as pointed out by the authors themselves. In the application of Hansen's method to artificial satellites (Musen 1959), on the other hand, it is difficult to see the meaning of the otherwise so very useful approximation by a rotating ellipse, if the eccentricity becomes extremely small. As a matter of fact, even the geometrical meaning of a very small eccentricity is not quite clear, because the perturbations play an important role in its definition as a mean orbital element. The determination of satellite orbits also presents some difficulties in the case of very small eccentricities.

One would expect more mathematical simplicity, rather than more complexity, for nearly circular orbits, so that the difficulties just mentioned must be of an artificial nature. Indeed, the problem of such orbits is not a new one in celestial mechanics, and there are several possible ways to avoid the small divisor $e$. For instance, instead of using the osculating elements $e$ and $\omega$, introduce their combinations, $e\sin\omega$ and $e\cos\omega$, and compute the perturbations in $M+\omega$, or $v+\omega$, where $v$ is the true anomaly. If one prefers to work with canonical equations, Poincaré's canonical elements

$$(\mu a)^{\frac{1}{2}} \qquad\qquad M+\omega$$
$$\{2(\mu a)^{\frac{1}{2}}[1-(1-e^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}\sin\omega \quad \{2(\mu a)^{\frac{1}{2}}[1-(1-e^2)^{\frac{1}{2}}]\}^{\frac{1}{2}}\cos\omega$$

can serve the same purpose. Or determine the perturbations in the elements $a$, $e$, $\omega$, and $M$ separately, and combine them into expressions of perturbations in the radius vector $r$, and in the argument of latitude $v+\omega$; the small divisor $e$ will cancel out, as indicated in the papers of Brouwer, Garfinkel, and Kozai. Because this is so, one would naturally think of developing a perturbation theory using coordinates directly instead of orbital elements, as was done, for instance, by King-Hele (1958), Strubel (1960), and Izsak (1960). Strubel paid special attention to the problem of small eccentricities (and also to that of the critical inclination), and the results to be presented here are partially contained in his work. His views on the merits of classical methods in celestial mechanics, however, are somewhat too pessimistic, in my opinion.

The point is the following. Osculating elements are introduced in celestial mechanics because one would expect them to have much slower variations than the coordinates, and because the differential equations describing their variations are usually easier to deal with. In the case of very small eccentricities, however, this is no longer true. But, in any theory using orbital elements we are obliged to introduce certain mean elements, and the question is: How well adapted are these mean orbital elements to describe the disturbed motion?

The paper by Izsak (1960) contains a complete second-order solution of Vinti's (1959) dynamical problem; for our present purpose, however, we need consider only the first-order part of the relevant expressions. Let us introduce the oblate-spheroidal coordinates $\rho$, $\sigma$, and $\alpha$, with a linear eccentricity $c$, so that the rectangular coordinates become

$$x=(\rho^2+c^2)^{\frac{1}{2}}(1-\sigma^2)^{\frac{1}{2}}\cos\alpha,$$
$$y=(\rho^2+c^2)^{\frac{1}{2}}(1-\sigma^2)^{\frac{1}{2}}\sin\alpha,$$
$$z=\rho\sigma,$$

$\alpha$ being the right ascension of the satellite. The constant $c$ is defined by the relation

$$c=J_2^{\frac{1}{2}}a_E,$$

where $J_2$ is the coefficient of the second-degree Legendre polynomial in the earth's gravitational potential,

$$V=-\frac{\mu}{r}\left[1-\sum_{n=2}^{\infty}J_n\left(\frac{a_E}{r}\right)^n P_n\left(\frac{z}{r}\right)\right],$$

and $a_E$ is the equatorial radius of the earth. Also, for the radius vector of the satellite, we have the expression

$$r=[\rho^2+c^2(1-\sigma^2)]^{\frac{1}{2}}. \tag{1}$$

The semimajor axis $a$, the eccentricity $e$, and the sine of the inclination $s=\sin I$ were defined as follows. In the exact solution of Vinti's dynamical problem the coordinates $\rho$ and $\sigma$ turn out to be even and odd periodic functions of the single variables $v$ and $w$, respectively; therefore their minimum and maximum values,

$$\rho_1,\ \rho_2 \text{ of } \rho$$

and

$$-\sigma_1,\ \sigma_1 \text{ of } \sigma$$

will remain the same during all revolutions. Let us define

$$a=\tfrac{1}{2}(\rho_1+\rho_2),\quad e=\frac{\rho_2-\rho_1}{\rho_1+\rho_2},\quad \text{and}\quad s=\sigma_1.$$

The argument $v$ is analogous to the true anomaly in a Keplerian motion, and $w=(1+\epsilon)v+\omega$ corresponds to the argument of latitude; here the quantities $\epsilon$ and $\omega$ are constants, the first denoting the motion of perigee. We shall also use the abbreviation

$$\nu=c/a,$$

and an eccentric anomaly $E$. Then the first-order solution of Vinti's dynamical problem can be written in the simple form

$$\rho=a(1-e\cos E) \tag{2}$$

$$\sigma=s\,\mathrm{sn}\left(\frac{2L}{\pi}w\,|\,l^2\right) \tag{3}$$

$$\begin{aligned}\alpha-\Omega_*=\tan^{-1}\left[(1-s^2)^{\frac{1}{2}}\,\mathrm{tn}\left(\frac{2L}{\pi}w\,|\,l^2\right)\right]\\ -\frac{3\nu^2(1-s^2)^{\frac{1}{2}}}{2(1-e^2)^2}w-\frac{2\nu^2(1-s^2)^{\frac{1}{2}}}{(1-e^2)^2}e\sin v\\ -\frac{\nu^2(1-s^2)^{\frac{1}{2}}}{4(1-e^2)^2}e^2\sin 2v+O(\nu^4)\end{aligned} \tag{4}$$

$$\begin{aligned}M_*=\hat{n}(t-t_*)=E-\left[1-\frac{\nu^2(1-s^2)}{1-e^2}\right]e\sin E\\ -\frac{\nu^2s^2}{4(1-e^2)^{\frac{1}{2}}}\sin 2w+O(\nu^4),\end{aligned} \tag{5}$$

where $\Omega_*$ and $t_*$ are constants of integration, $M_*$ and $\hat{n}$ denote the mean anomaly and the anomalistic mean motion of the satellite, and $\mathrm{sn}[(2L/\pi)w\,|\,l^2]$ and $\mathrm{tn}[(2L/\pi)w\,|\,l^2]$ stand for Jacobian elliptic functions with parameters $l^2$. The quarter real period of the function $\mathrm{sn}(u\,|\,l^2)$ is

$$L=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2l^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2l^4+\cdots\right],$$

so that the period of

$$\mathrm{sn}\left(\frac{2L}{\pi}w\,|\,l^2\right)=\left(1+\frac{l^2}{16}\right)\sin w+\frac{l^2}{16}\sin 3w+O(\nu^4)$$

becomes $2\pi$. The connection between the true and eccentric anomalies is given by the formulae

$$\begin{aligned}\mathrm{cn}\left(\frac{2K}{\pi}v\,|\,k^2\right)&=\frac{\cos E-e_*}{1-e_*\cos E},\\ \mathrm{sn}\left(\frac{2K}{\pi}v\,|\,k^2\right)&=\frac{(1-e_*^2)^{\frac{1}{2}}\sin E}{1-e_*\cos E},\end{aligned} \tag{6}$$

where $e_*$ is another eccentricity, namely

$$e_*=e\left[1+\frac{\nu^2}{1-e^2}(1-2s^2)+O(\nu^4)\right]. \tag{7}$$

Finally, the developments of the quantities $\epsilon$, $\hat{n}$, $k^2$, and $l^2$ are, up to $O(\nu^4)$:

$$\begin{aligned}\epsilon=\frac{3\nu^2}{4(1-e^2)^2}(4-5s^2)+\frac{3\nu^4}{64(1-e^2)^4}\\ \times[(96-432s^2+345s^4)-(48+96s^2-170s^4)e^2]\end{aligned}$$

$$\begin{aligned}\hat{n}=\left(\frac{\mu}{a^3}\right)^{\frac{1}{2}}\Bigg\{1-\frac{3\nu^2(1-s^2)}{2(1-e^2)}+\frac{3\nu^4(1-s^2)}{8(1-e^2)^3}\\ \times[(1+11s^2)-(1-5s^2)e^2]\Bigg\}\end{aligned}$$

$$k^2=\frac{\nu^2e^2}{(1-e^2)^2}s^2-\frac{\nu^4e^2}{(1-e^2)^4}[(1-10s^2+11s^4)+s^4e^2]$$

$$l^2=\frac{\nu^2s^2}{1-e^2}-\frac{4\nu^4s^2}{(1-e^2)^3}(1-s^2);$$

the motion of the node was found to be

$$\eta=-\frac{3\nu^2(1-s^2)^{\frac{1}{2}}}{2(1-e^2)^2}+\frac{3\nu^4(1-s^2)^{\frac{1}{2}}}{16(1-e^2)^4}[(18-13s^2)+24s^2e^2].$$

In all these formulae given above the eccentricity never appears as a divisor, even in the mean anomaly (because $M_*$ is not the osculating mean anomaly, but just a linear function of time). Hence they are valid for arbitrarily small values of $e$, the smallest possible value being $e=0$. Because the eccentricity was defined in a geometrical way, it is easy to see the meaning of the condition $e=0$, that is, $\rho(t)\equiv a$. The radius vector $r$, on the other hand, whose behavior determines the consecutive perigees and apogees of the satellite, is by no means constant; rather, we have according to Eqs. (1), (2), and (3) the expression

$$r=a\left\{1+\nu^2\left[1-s^2\,\mathrm{sn}^2\left(\frac{2L}{\pi}w\,|\,l^2\right)\right]\right\}^{\frac{1}{2}}. \tag{8}$$

This formula already shows the fact that the orbit has two apogees with a distance of

$$r_{\max}=a(1+\nu^2)^{\frac{1}{2}}$$

at the ascending ($w=0$) and descending nodes ($w=\pi$), and two perigees with a distance of

$$r_{\min}=a[1+\nu^2(1-s^2)]^{\frac{1}{2}}$$

at the highest ($w=\pi/2$) and lowest points of the orbit above the equator ($w=3\pi/2$). (Only for equatorial orbits, that is, for $s=0$, does $e=0$ mean a circular solution.) Such an orbit is very nearly an ellipse with its center (not a focus) at the center of the earth, having a semimajor axis of $a_0=a(1+\frac{1}{2}\nu^2)$ in the direction of the line of nodes, and an eccentricity of $e_0=\nu s$. Note that this ellipse does not rotate in the orbital plane.

It is more difficult to visualize the nature of orbits with very small, but not vanishing eccentricities. Let us consider again Eqs. (1), (2), and (3), from which, neglecting quantities of $O(\nu^4)$, we obtain first the expression

$$r=a\left(1-e\cos E+\frac{\nu^2}{2}\,\frac{1-s^2\sin^2 w}{1-e\cos E}\right).$$

Furthermore, if we assume that $e=O(\nu)$, then Eqs. (6) and (7) give

$$e\cos E=\tfrac{1}{2}e^2+e\cos v-\tfrac{1}{2}e^2\cos 2v+O(\nu^3),$$

so that we can write

$$r=a_*(1-e\cos v+\tfrac{1}{2}e^2\cos 2v+\tfrac{1}{4}\nu^2 s^2\cos 2w)+O(\nu^3), \quad (9)$$

where $a_*=a[1-\frac{1}{2}e^2+\frac{1}{2}\nu^2(1-\frac{1}{2}s^2)]$ and $w=(1+\epsilon)v+\omega$.

There is no simple geometrical meaning that can be attached to this equation, as shown by numerical examples such as those illustrated in Fig. 1. The several curves in this diagram represent the deviations of the radius vector from its mean value $a_*$, for different values of $e$ and $\omega$. It should be noted that $\omega$ is not the argument of perigee in the usual sense, but only a constant of integration in the theory; the linear function $\tilde{\omega}=\epsilon v+\omega$ can considered to be the mean argument of perigee. The reason for varying $\omega$ is the following. As the true anomaly $v$ increases by the amount $2\pi/\epsilon$, the phase difference $w-v=\tilde{\omega}$ takes all values between 0 and $2\pi$, and the shape of the orbit changes from revolution to revolution, except for $e=0$, or $s=0$. These changes can be simulated by letting $v$ increase from 0 to $2\pi$ only, but varying the constant $\omega$.

In the construction of Fig. 1 the following constants were used:

$$a=7100\text{ km},\quad a_*=7102\text{ km},$$

$$I=48\overset{\circ}{.}39,\qquad s=\sin I=0.7477,$$

$$J_2=1.0822\times10^{-3},\quad \nu^2=0.000873,\quad \tfrac{1}{4}\nu^2 s^2=0.000122.$$

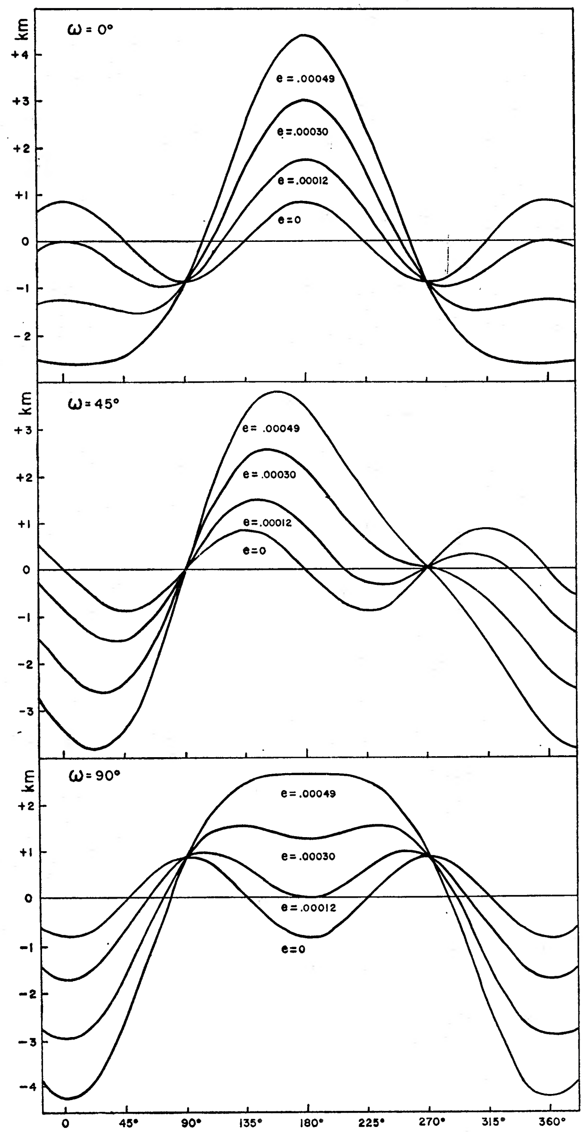


FIG. 1. The variations of the radius vector as functions of the true anomaly, for different values of $e$ and $\omega$.

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