Notes for Tom: Kalman filter phase formulation

Equations of motion (continuous space)

$$\frac{d\Phi}{dt} = f, \qquad (1)$$

$$\frac{df}{dt} = -\gamma(f - f_{em}) + \dot{f}_{em} + \xi \qquad (2).$$

Consider (2)

$$\frac{d}{dt}\{f - f_{em}\} = -\gamma(f - f_{em}) + \xi, \qquad (2A)$$

Let $f^* \equiv f - f_{em}$

Equation (2) becomes

$$\frac{df^*}{dt} = -\gamma f^* + \xi. \qquad (2B)$$

Consider (1)

$$\frac{d\Phi}{dt} = f^* + \frac{d}{dt}\Phi_{em} \qquad (1A)$$

Let $\Phi^* = \Phi - \Phi_{em}$

Equation (1) becomes

$$\frac{d\Phi^*}{dt} = f^* \qquad (1B)$$

(Shifed) Equations of motion (continuous):

$$\frac{d\Phi^*}{dt} = f^* \quad (3)$$

$$\frac{df^*}{dt} = -\gamma f^* + \xi \quad (4)$$

Beautifully simple.

$$\frac{d}{dt}\begin{pmatrix}\Phi^* \\ f^*\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 0 & -\gamma\end{pmatrix}\begin{pmatrix}\Phi^* \\ f^*\end{pmatrix} + \begin{pmatrix}0 \\ \xi\end{pmatrix}$$

$$\underline{\dot{X}} = \underline{A} \quad \underline{X} + \underline{\xi}$$

✓ looks good

Solution given by

Integrating factor!

$$\underline{X}(t_1) = e^{\underline{A}(t_1 - t_0)}\underline{X}(t_0) + \int_{t_0}^{t_1} e^{\underline{A}(t_1 - t_0')}\underline{\xi}(t_0')\, dt_0'$$

$$e^{\underline{A}(t_1 - t_0)} = e^{\underline{A}\Delta t} = \begin{pmatrix}1 & (1-e^{-\gamma\Delta t})/\gamma \\ 0 & e^{-\gamma\Delta t}\end{pmatrix}$$

i.e. the components are 1, $\frac{1-e^{-\gamma\Delta t}}{\gamma}$, 0, $e^{-\gamma\Delta t}$.

→ Same as Nicholas' results, as expected.

Note: Even with heterodyning, the formal structure in the terms multiplying the state-space variables is equivalent, i.e $\underline{Q}$ will be the same. To see this:

$$\underline{Q}_n = \int_0^{\Delta t} e^{\underline{A}s} \, \underline{\Sigma} \, e^{\underline{A}^T s} \, ds,$$

where $\underline{\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}$

The term inside the integrand, i.e, is given by $e^{\underline{A}s} \, \underline{\Sigma} \, e^{\underline{A}^T s}$ (i.e it is a projection)

$$= \sigma^2 e^{-2\gamma s} \begin{pmatrix} \dfrac{[e^{\gamma s}-1]^2}{\gamma^2} & \dfrac{[e^{\gamma s}-1]}{\gamma} \\ \dfrac{[e^{\gamma s}-1]}{\gamma} & 1 \end{pmatrix} \quad (*)$$

→ Integrate (*) between 0 & Δt to get (discrete) $\underline{Q}_n$.

→ Integrate each component between $0 \rightarrow \Delta t$:

$$Q_{00} = \frac{\sigma^2}{\gamma^2}\int_0^{\Delta t} e^{-2gs}\left(e^{gs}-1\right)^2 ds \quad , \quad g = \gamma$$

bad notation - sorry !!

$$= \frac{\sigma^2}{\gamma^2}\int_0^{\Delta t} \left\{ e^{-2gs}\left[e^{2gs} - 2e^{gs} + 1 \right] \right\} ds$$

$$= \frac{\sigma^2}{\gamma^2}\int_0^{\Delta t} \left[1 - 2e^{-gs} + e^{-2gs} \right] ds$$

$$= \frac{\sigma^2}{\gamma^2}\left[s + \frac{2}{g}e^{-gs} - \frac{e^{-2gs}}{2g} \right]_0^{\Delta t}$$

$$= \frac{\sigma^2}{\gamma^2}\left[(\Delta t - 0) + \frac{2}{g}\left(e^{-g\Delta t} - 1\right) - \frac{1}{2g}\left(e^{-2g\Delta t} - 1\right) \right]$$

$$= \frac{\sigma^2}{\gamma^3}\left[\gamma\Delta t + 2\left(e^{-g\Delta t} - 1\right) + \frac{1}{2}\left(1 - e^{-2g\Delta t}\right) \right]$$

Note $g = \gamma$, sorry for the bad notation.

Same as Nicholas Q_{00} ✓

$$Q_{01} = Q_{10} = \frac{\sigma^2}{\gamma}\int_0^{\Delta t}\left[e^{-gs} - e^{-2gs}\right]ds$$

$$= \frac{\sigma^2}{\gamma}\left[\frac{1}{2g}e^{-2gs} - \frac{1}{g}e^{-gs}\right]_0^{\Delta t}$$

$$= \frac{\sigma^2}{\gamma^2}\left[\frac{1}{2}\left(e^{-2g\Delta t} - 1\right) - \left(e^{-g\Delta t} - 1\right)\right]$$

$$= \frac{\sigma^2}{2\gamma^2}\left[\left(e^{-2g\Delta t} - 1\right) + 2\left(1 - e^{-g\Delta t}\right)\right]$$

$$= \frac{\sigma^2}{2\gamma^2}\left[1 - 2e^{-g\Delta t} + e^{-2g\Delta t}\right]$$

$$= \frac{\sigma^2}{2\gamma^2}\left(1 - e^{-g\Delta t}\right)^2 \quad \checkmark$$

$$Q_{22} = \sigma^2 \int_0^{\Delta t} e^{-2gs}\, ds$$

$$= -\frac{\sigma^2}{2\gamma} e^{-2gs} \Big|_0^{\Delta t}$$

$$= -\frac{\sigma^2}{2\gamma}\left(e^{-2gs} - 1\right)$$

$$= \frac{\sigma^2}{2\gamma}\left(1 - e^{-2gs}\right).$$

Model Summary:

$$\frac{d\Phi^*}{dt} = f^*, \quad \frac{df^*}{dt} = -\gamma f^* + \xi.$$

$$F_n = \begin{pmatrix} 1 & \gamma^{-1}\left[1 - e^{-\gamma \Delta t}\right] \\ 0 & e^{-\gamma \Delta t} \end{pmatrix}$$

$Q_n \equiv$ same as Nicholas Eq (54)

$$\Phi^*_M = \Phi^* - (f^* + f_{em})\Delta t_{GW}.$$