Gravitational Wave Kalman filter for phase data

Nicholas O'Neill

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Note: This still needs some checking and thinking, especially about the interpretation of the different times, t_r, t_e, t_{r0}, t and about the coordinate system I am using. I will also try to make the calculation more rigorous, perhaps using coordinate independent quantities. However, I think the calculation is good enough for now and can be written up as a python program for tests.

1 How does the gravitational wave affect pulse arrival times?

Assume Earth is located at $\boldsymbol{x} = \boldsymbol{0}$ and the pulsar is at $\boldsymbol{x} = \boldsymbol{q}d$, where $\boldsymbol{q}$ is the unit vector pointing from Earth to the pulsar. If we assume the Earth and pulsar are at rest, then in the TT-gauge coordinate system their position coordinates do not change with time.

Let l be the distance the light signal from the pulsar has travelled along its path to Earth. Then to zeroth order, the position of the light pulse along the path is $\boldsymbol{x} = \boldsymbol{q}(d - l)$ and the time taken is $t = l$ (I am assuming $c = 1$, really the time taken would be $t = l/c$).

Assume the gravitational wave is travelling in the direction $\boldsymbol{n}$. Let us describe the gravitational wave by the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

$$h_{\mu\nu} = H_{\mu\nu} e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{x} - t) + \phi_0)} \tag{2}$$

with the transverse-traceless gauge (TT-gauge) conditions,

$$H_{\mu 0} = 0, \tag{3}$$

$$H_{ij} n_j = 0, \tag{4}$$

$$H_{ii} = 0. \tag{5}$$

To work out how the gravitational wave affects the measured phases, we must know how it affects the arrival times of pulses. We can work out the time it takes light to travel from the pulsar to Earth by using the metric. If t_e is the time the light is emitted by the pulsar and t_r is the time it is received on Earth then

$$t_r = t_e + \int_0^d \frac{dt}{dl} dl. \tag{6}$$

Then use the metric to calculate dt/dl. Since the proper distance travelled by the light is $ds^2 = 0$, we have

$$
\begin{aligned}
0 &= g_{\mu\nu} dx^\mu dx^\nu && (7)\\
&= (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu && (8)\\
&= -dt^2 + (\delta_{ij} + h_{ij}) dx^i dx^j && (9)\\
dt^2 &= (\delta_{ij} + h_{ij}) dx^i dx^j && (10)
\end{aligned}
$$

The path the light takes has $dx^i = -q^i dl$.

$$\begin{aligned}
dt^2 &= (\delta_{ij} + h_{ij}) dx^i dx^j && (11)\\
&= (\delta_{ij} + h_{ij}) q^i q^j dl^2 && (12)\\
&= (1 + h_{ij} q^i q^j) dl^2 && (13)\\
\left(\frac{dt}{dl}\right)^2 &= 1 + h_{ij} q^i q^j && (14)\\
\frac{dt}{dl} &\approx 1 + \frac{1}{2} h_{ij} q^i q^j && (15)\\
&= 1 + \frac{1}{2} H_{ij} q^i q^j e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{x} - t) + \phi_0)} && (16)\\
&= 1 + \frac{H}{2} e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{x} - t) + \phi_0)} && (17)
\end{aligned}$$

In the last line, for brevity I introduced the notation $H = H_{ij} q^i q^j$. Now,

$$\begin{aligned}
t_r &= t_e + \int_0^d \frac{dt}{dl} dl && (18)\\
&= t_e + \int_0^d 1 + \frac{H}{2} e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{x} - t) + \phi_0)} dl && (19)\\
&= t_e + d + \frac{H}{2} \int_0^d e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{x} - t) + \phi_0)} dl && (20)
\end{aligned}$$

Substitute the parametrisation of the path ($\boldsymbol{x} = \boldsymbol{q}(d - l)$, $t = t_e + l$) into the integral.

$$\begin{aligned}
t_r &= t_e + d + \frac{H}{2} \int_0^d e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{q}(d-l) - t_e - l) + \phi_0)} dl && (21)\\
&= t_e + d + \frac{H}{2} e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{q}d - t_e) + \phi_0)} \int_0^d e^{-i\Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})l} dl && (22)\\
&= t_e + d + \frac{H}{2} e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{q}d - t_e) + \phi_0)} \left[\frac{e^{-i\Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})d} - 1}{-i\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})}\right] && (23)\\
&= t_e + d - \frac{H}{2i\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})} \left(e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{q}d - (1+\boldsymbol{n}\cdot\boldsymbol{q})d - t_e) + \phi_0)} - e^{i(\Omega(\boldsymbol{n}\cdot\boldsymbol{q}d - t_e) + \phi_0)}\right) && (24)\\
&= t_e + d + \frac{iH}{2\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})} \left(e^{i(\Omega(-d - t_e) + \phi_0)} - e^{i(\Omega((1+\boldsymbol{n}\cdot\boldsymbol{q})d - d - t_e) + \phi_0)}\right) && (25)
\end{aligned}$$

Let $t_{r0} = t_e + d$. This is the time the pulse would arrive without gravitational waves, or in other words, the zeroth order solution for t_r.

$$t_r = t_{r0} + \frac{iH}{2\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})} \left(e^{i(-\Omega t_{r0} + \phi_0)} - e^{i(-\Omega t_{r0} + \Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})d + \phi_0)}\right) \tag{26}$$

Taking the real part gives

$$\begin{aligned}
t_r &= t_{r0} + \frac{iH}{2\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})} \Big(i \sin(-\Omega t_{r0} + \phi_0) - i \sin(-\Omega t_{r0} + \Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})d + \phi_0)\Big) && (27)\\
t_r &= t_{r0} - \frac{H}{2\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})} \Big(\sin(-\Omega t_{r0} + \phi_0) - \sin(-\Omega t_{r0} + \Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})d + \phi_0)\Big) && (28)
\end{aligned}$$

We can replace t_{r0} with t_r inside the sines if we want to, since the effect of this change would be second order in H.

2 Measurement equation for pulsar phase

How does the gravitational-wave-induced time offset affect the measured phase? Define two phase functions $\phi_0(t)$ and $\phi(t)$. $\phi_0(t)$ is the pulsar's phase as a function of time, measured at the pulsar. This is the true phase. $\phi(t)$ is the phase of the pulsar we measure on Earth as a function of Earth time. In the absence of a gravitational wave, $H = 0$ and we would have

$$\phi(t_r) = \phi_0(t_r - d), \tag{29}$$

but if there is a gravitational wave, the relation between the true and measured phase time series is

$$\phi(t_r) = \phi_0(t_e) \tag{30}$$

and we can see that this reduces to equation (29) if $H = 0$ because $t_e = t_r - d$ in that case.

Now write the time-modulation equation (28) as

$$t_r = t_e + d + Hg(t_e) \tag{31}$$

where g is the complicated function involving sines. Rearranging this equation to get t_e in terms of t_r gives

$$\begin{aligned} t_e &= t_r - d - Hg(t_e) && (32)\\ &= t_r - d - Hg(t_r - d - Hg(t_e)) && (33)\\ &\approx t_r - d - Hg(t_r - d) && (34) \end{aligned}$$

We can then put this into equation (30) to get

$$\begin{aligned} \phi(t_r) &= \phi_0(t_e) && (35)\\ &= \phi_0(t_r - d - Hg(t_r - d)) && (36)\\ &\approx \phi_0(t_r - d) - Hg(t_r - d)\frac{d\phi_0(t)}{dt}\Big|_{t_r - d} && (37)\\ &\approx \phi_0(t_r - d) - Hg(t_r - d)f_0(t_r - d), && (38) \end{aligned}$$

where $f_0(t)$ is the true pulsar frequency.

This can be written more clearly. t_r is the time at the Earth, so just call it t. Also, $Hg(t_s)$ is the perturbation in the pulse arrival times so let $Hg(t_s) = \Delta t_{\mathrm{GW}}$. So the measured phase is

$$\phi(t) = \phi_0(t - d) - f_0(t - d)\Delta t_{GW}, \tag{39}$$

Let us introduce some new functions, $\phi_{m0}(t)$ and $f_{m0}(t)$ which are the pulsar phase and frequency we would measure without a gravitational wave. As we already know,

$$\begin{aligned} \phi_{m0}(t) &= \phi_0(t - d) && (40)\\ f_{m0}(t) &= f_0(t - d) && (41) \end{aligned}$$

So we measure the phase to be

$$\phi(t) = \phi_{m0}(t) - f_{m0}(t)\Delta t_{GW}. \tag{42}$$

This formula agrees with my intuition. If the signal from the pulsar takes an extra time Δt_{GW} to arrive then the signal is delayed and we observe the pulsar as it was at an earlier time, which is approximately its current phase minus the rotational frequency times the time delay. So a positive Δt_{GW} results in a negative $\Delta\phi$. So I think it all makes sense, although the interpretation of what we are measuring and how we are using the time variable is a bit tricky and will need some more careful thought.

3 Kalman filter equations

Now we must write this in the form needed for Kalman filtering. We know

$$\phi_{\text{measured}} = \phi - f\Delta t_{\mathrm{GW}} \tag{43}$$

We want this in the form

$$\begin{aligned} \boldsymbol{Y_n} &= \boldsymbol{C}\boldsymbol{X_n} + \boldsymbol{v_n} && (44)\\ \boldsymbol{v}_n &\sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_n) && (45) \end{aligned}$$

So we write it as

$$\begin{aligned} \phi_{\text{measured},n} &= \begin{pmatrix} 1 & -\Delta t_{\mathrm{GW,n}} \end{pmatrix} \begin{pmatrix} \phi_n \\ f_n \end{pmatrix} + v_n, && (46)\\ \Delta t_{\mathrm{GW},n} &= -\frac{H}{2\Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})}\Big(\sin\left(-\Omega t_n + \phi_0\right) - \sin\left(-\Omega t_n + \Omega(1 + \boldsymbol{n}\cdot\boldsymbol{q})d + \phi_0\right)\Big), && (47)\\ v_n &\sim \mathcal{N}(0, R_n), && (48) \end{aligned}$$

where R_n is the measurement error variance of the phase measurements.
The equations of motion are

$$\frac{d\phi}{dt} = f \tag{49}$$

$$\frac{df}{dt} = -\gamma(f - \alpha - \beta t) + \beta + \xi \tag{50}$$

$$\langle \xi(t)\xi(t') \rangle = \sigma^2 \delta(t - t') \tag{51}$$

The update equations are

$$\boldsymbol{X}_{n+1} = \boldsymbol{F}_n \boldsymbol{X}_n + \boldsymbol{B}_n + \boldsymbol{w}_n \tag{52}$$

$$\boldsymbol{w}_n \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_n) \tag{53}$$

with

$$\boldsymbol{F}_n = \begin{pmatrix} 1 & \frac{1-e^{-\gamma\Delta t_n}}{\gamma} \\ 0 & e^{-\gamma\Delta t_n} \end{pmatrix} \tag{54}$$

$$\boldsymbol{Q}_n = \begin{pmatrix} \frac{\sigma^2}{\gamma^3}\left(\gamma\Delta t_n - (1-e^{-\gamma\Delta t_n}) - \frac{1}{2}(1-e^{-\gamma\Delta t_n})^2\right) & \frac{\sigma^2}{2\gamma^2}(1-e^{-\gamma\Delta t_n})^2 \\ \frac{\sigma^2}{2\gamma^2}(1-e^{-\gamma\Delta t_n})^2 & \frac{\sigma^2}{2\gamma}(1-e^{-2\gamma\Delta t_n}) \end{pmatrix} \tag{55}$$

$$\boldsymbol{B}_n = \begin{pmatrix} \alpha\Delta t_n + \frac{1}{2}\beta\Delta t_n^2 - (\alpha + \beta t_{n-1})\frac{1-e^{-\gamma\Delta t_n}}{\gamma} \\ \alpha + \beta t_n - (\alpha + \beta t_{n-1})e^{-\gamma\Delta t_n} \end{pmatrix}, \tag{56}$$

where $\Delta t_n = t_{n+1} - t_n$.

4 Check the frequency measurement equation

In this section I check that the result I obtained above reproduces the same measurement equation for frequencies that has been used in our previous papers.

Let the time between two emitted pulses be dt_e and the time between the received pulses be dt_r. The emitted frequency is $f_e = 1/dt_e$ and the received frequency is $f_r = 1/dt_r$. So $dt_r/dt_e = f_e/f_r$ and $f_r = f_e/(dt_r/dt_e)$. Using the result in equation (25), we find

$$t_r = t_e + d + \frac{iH}{2\Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})}\left(e^{i(\Omega(-d-t_e)+\phi_0)} - e^{i(\Omega((1+\boldsymbol{n}\cdot\boldsymbol{q})d-d-t_e)+\phi_0)}\right) \tag{57}$$

$$\frac{dt_r}{dt_e} = 1 + \frac{H}{2(1+\boldsymbol{n}\cdot\boldsymbol{q})}\left(e^{i(\Omega(-d-t_e)+\phi_0)} - e^{i(\Omega((1+\boldsymbol{n}\cdot\boldsymbol{q})d-d-t_e)+\phi_0)}\right) \tag{58}$$

$$f_r = \frac{f_e}{dt_r/dt_e} \approx f_e\left(1 - \frac{H}{2(1+\boldsymbol{n}\cdot\boldsymbol{q})}\left(e^{i(\Omega(-d-t_e)+\phi_0)} - e^{i(\Omega((1+\boldsymbol{n}\cdot\boldsymbol{q})d-d-t_e)+\phi_0)}\right)\right) \tag{59}$$

Taking the real part gives

$$f_r = f_e\left(1 - \frac{H}{2(1+\boldsymbol{n}\cdot\boldsymbol{q})}\Big(\cos\left(\Omega(-d-t_e)+\phi_0\right) - \cos\left(\Omega((1+\boldsymbol{n}\cdot\boldsymbol{q})d-d-t_e)+\phi_0\right)\Big)\right) \tag{60}$$

Let $t_{r0} = t_e + d$

$$f_r = f_e\left(1 - \frac{H}{2(1+\boldsymbol{n}\cdot\boldsymbol{q})}\Big(\cos(-\Omega t_{r0}+\phi_0) - \cos(-\Omega t_{r0} + \Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})d+\phi_0)\Big)\right) \tag{61}$$

Since the perturbation term is already first order in H we can set $t_{r0} = t$ to get

$$f_r = f_e\left(1 - \frac{H}{2(1+\boldsymbol{n}\cdot\boldsymbol{q})}\Big(\cos(-\Omega t+\phi_0) - \cos(-\Omega t + \Omega(1+\boldsymbol{n}\cdot\boldsymbol{q})d+\phi_0)\Big)\right) \tag{62}$$

Which I believe is the result in Tom's first paper.