State-space PTA

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ABSTRACT
This is an abstract

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1 INTRODUCTION

The detection of high frequency ($\sim$ 1Hz) gravitational waves (GWs) from coalescing BH binaries with ground-based detectors such as LIGO/Virgo (LIGO Scientific Collaboration et al. 2015; Acernese et al. 2015) is now a routine enterprise (e.g. Abbott et al. 2019, 2021). Gravitational radiation from sources which radiate in the mill-Hz regime are expected to be detectable from $\sim$ 2030 with the space-based Laser Interferometer Space Antenna, (Amaro-Seoane et al. 2017) especially given the early success by the pathfinder mission (Armano et al. 2019). Detecting GWs from systems which evolve over even longer timescales, O(years), has necessitated the development of novel astrophysical methods, since it is practically impossible to engineer interferometric detectors with sufficiently long baselines. The foremost technique for the detection of GWs in this nano-Hz regime is a via timing an ensemble of milliseconds pulsars; a pulsar timing array (PTA) (Verbiest et al. 2021). The presence of a nano-Hz gravitational wave will influence the propagation of the pulsar radio beacon and consequently will imprint on the pulsar timing signal. By measuring the modulation of the received pulsar signal in this way, one can effectively construct a detector with a baseline on the scale of parsecs.

Multiple PTA detectors have now been built over the last few decades, including the North American Nanohertz Observatory for Gravitational Waves (NANOGrav, Arzoumanian et al. 2020), the Parkes Pulsar Timing array (PPTA Kerr et al. 2020), and the European Pulsar Timing Array (EPTA, Ferdman et al. 2010). These previously disparate efforts have now been joined in international collaboration, along with a number of newer PTAs, under the umbrella of the International Pulsar Timing Array (IPTA Perera et al. 2019). The primary target of PTA observations is the inspiral of supermassive black hole binaries (SMBHBs) with masses $\sim 10^7 M_{\odot}$. These signals from SMBHBs can be broadly classified into either deterministic or stochastic. For the former, sufficiently bright and near binaries may be resolvable with PTAs, allowing the very earliest stages of their evolution and coalescence to be investigated. For the latter, the incoherent superposition of multiple weaker SMBHBs sources leads to a stochastic background detectable at nano-Hz frequencies. Other potential sources for PTAs include cosmic strings (e.g. Sanidas et al. 2012) and cosmological phase transitions (e.g. Xue et al. 2021), but the deterministic and stochastic GW signals from SMBHBs remain the primary targets.

The detection of loud, resolved sources with a PTA typically involves a parametrised model for the pulsar timing residuals induced by the modulation of the pulsar signal by a GW. One can then search for evidence that this model describes the data via the usual Bayesian likelihood techniques, and try to estimate the parameters of the model (e.g. Babak et al. 2016). For detecting the stochastic background the approach is different; one measures the correlation in pulsar timing residuals between any two pair of pulsars. The presence of a GW induces a characteristic correlation function as a function of the angular separation between the pulsars; the Hellings-Downs curve (Hellings & Downs 1983). For both classes of source, detection of GW signals in the timing residuals of a PTA is a challenging enterprise, and currently neither a stochastic background nor an individually resolved source has yet been detected (Antoniadis et al. 2022; Hobbs & Dai 2017).

Motivated by the difficulties faced by the classic PTA analysis methods, in this work we present a novel approach to formulate PTA analysis and GW detection as a state-space problem. This approach enables the state-space evolution to be tracked optimally, and for a specific realisation of the pulsar process noise (i.e. the spin wandering of the pulsar) infer both the system parameters and the detectability of the GW signal.

For this initial work we will focus exclusively on resolved, monochromatic GW sources. Discussion on

This paper is organised as follows.

In section 2 we will briefly outline the response In section 3 we will then construct a state space form for determinsitic signals (3.1) outline the mehtods for expectaiton likelihood (3.2) and apply this state space kalma nform

In section 4 we will extend to stochastic sources...? OR we will leave thus for a second paper...

PTA sources can be braodly characterised into two main buckets: determinsitic and stocahstic. Derterminisitc sources are loud isngle

- Introduce PTAs generally.
- Types of astrophysical source to be detected with PTAs.
- Why we want to do parameter estimation

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- Advantages of doing this with state-space approach

2 EVOLUTION OF THE PULSAR FREQUENCY

We will take as our model of the intrinsic pulsar frequency f a variation of the phenomenological model of (Vargas & Melatos 2023). Within this model, f evolves according to a combination of both deterministic torques (i.e. electromagnetic spin-down) and stochastic torques (i.e. 'spin wandering', achromatic variations in the pulse TOA intrinsic to the star). Specifically, the frequency evolves according to a Ornstein-Uhlenbeck process (equivalently a Langevin equation) with a time-dependent drift parameter:

$$\frac{df}{dt} = -\gamma[f - f_{\rm EM}(t)] + \dot{f}_{\rm EM} + \xi(t) \tag{1}$$

where $f_{\rm EM}$ is the solution of the electromagnetic spindown equation, γ a proportionality constant, and $\xi(t)$ a white noise process that satisfies,

$$\langle \xi(t)\xi(t')\rangle = \sigma^2\delta(t-t') \tag{2}$$

For PTA analysis, we are concerned with timescales on the order or years. Consequently, we can express the EM spindown straightforwardly as

$$f_{\rm EM}(t) = f_{\rm EM}(0) + \dot{f}_{\rm EM}(0)t \tag{3}$$

Completely, the frequency evolution is then given by the solution of the stochastic differential equation,

$$\frac{df}{dt} = -\gamma[f - f_{\rm EM}(0) - \dot{f}_{\rm EM}(0)t] + \dot{f}_{\rm EM}(0) + \xi(t) \tag{4}$$

3 MODULATION OF PULSAR FREQUENCY DUE TO A GW

In the presence of a GW, the $f(t)$ measured by an observer on Earth is different from that measured by an observer in the local NS reference frame. We want to determine how the GW influences the received frequency

3.1 Plane GW perturbation

We take a gravitational plane wave that perturbs a background Minkowski spacetime as

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}e^{i(\Omega(\bar{n}\cdot\bar{x}-t)+\Phi_0)} \tag{5}$$

for spatial coordinates $\bar{x}$, where the GW has a constant angular frequency Ω, propagates in the $\bar{n}$-direction and has a phase offset of Φ_0. We emphasise that Ω has no time dependence - we are concerned solely with monochromatic sources. Note that we are free to choose our coordinate system such that Φ_0 is the GW phase at $t=0$ *at the Earth.*[1]. The amplitude tensor $H_{\mu\nu}$ has zero temporal components ($H_{0\mu} = H_{\mu 0} = 0$) whilst the spatial part is

$$H_{ij} = h_+ e^+_{ij}(\bar{n}) + h_\times e^\times_{ij}(\bar{n}) \tag{6}$$

where $h_{+,\times}$ are the polarisation amplitudes of the gravitational plane wave. The polarisation tensors $e^{+,\times}_{ij}$ are uniquely defined by the principal axes of the wave:

$$e^+_{ab}(\hat{\Omega}) = \hat{m}_a\hat{m}_b - \hat{n}_a\hat{n}_b \tag{7}$$

$$e^\times_{ab}(\hat{\Omega}) = \hat{m}_a\hat{n}_b + \hat{n}_a\hat{m}_b \tag{8}$$

which are in turn specified via the location of the GW source on the sky (via θ, ϕ coordinates) and the polarisation angle ψ

$$\begin{aligned}\vec{m} &= (\sin\phi\cos\psi - \sin\psi\cos\phi\cos\theta)\hat{x} \\ &\quad - (\cos\phi\cos\psi + \sin\psi\sin\phi\cos\theta)\hat{y} \\ &\quad + (\sin\psi\sin\theta)\hat{z}\end{aligned} \tag{9}$$

$$\begin{aligned}\vec{n} &= (-\sin\phi\sin\psi - \cos\psi\cos\phi\cos\theta)\hat{x} \\ &\quad + (\cos\phi\sin\psi - \cos\psi\sin\phi\cos\theta)\hat{y} \\ &\quad + (\cos\psi\sin\theta)\hat{z}\end{aligned} \tag{10}$$

3.2 Pulse frequency as a photon

We will consider the pulse frequency as a photon with covariant 4-momentum p_μ. Generally, the frequency of a photon with 4-momentum p_μ recorded by an observer with 4-velocity u^μ is

$$\nu = p_\alpha u^\alpha \tag{11}$$

We consider both our emitter and receiver to be stationary, such that

$$u^\alpha|_{\rm emitter} = u^\alpha|_{\rm receiver} = (1,0,0,0) \tag{12}$$

Consequently the frequency can be identified with the temporal component of the covariant 4-momentum,

$$\nu = p_t \tag{13}$$

The expression for the evolution of the pulse frequency as measured by the observer on Earth is then,

$$p_t(t_1)|_{\rm Earth} = p_t(t_0)|_{\rm source} + \int_{t=t_0}^{t=t_1} \dot{p}_t\, dt \tag{14}$$

where the overdot denotes a derivative w.r.t. t. Since the influence of the GW perturbation on $\dot{p}_t$ is small, we can relate the source emission and receiver times as $t_1 = t_0 + d$ and consider the photon trajectory to be an unperturbed path. [2]

To complete our expression, we now just need to determine $\dot{p}_t$ and integrate it.

3.3 Hamiltonian Mechanics

The Hamiltonian in covariant notation can be written as

$$H(x^\mu, p_\mu) = \frac{1}{2}g_{\mu\nu}p^\mu p^\nu, \tag{15}$$

which if we substitute in our expression for the perturbed metric is

$$H = \frac{1}{2}\eta_{\mu\nu}p^\mu p^\nu + \frac{1}{2}H_{ij}p^i p^j e^{i(\Omega(\bar{n}\cdot\bar{x}-t)+\Phi_0)} \tag{16}$$

Now, Hamilton's equations are

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu} \tag{17}$$

for affine parameter λ. The derivative of the temporal component of the covariant momenta is then,

$$\frac{dp_t}{d\lambda} = -\frac{i\Omega}{2}H_{ij}p^i p^j e^{i(\Omega(\bar{n}\cdot\bar{x}-t))+\Phi_0} \tag{18}$$

[1] This point is important, since the phase offset is then the same between multiple pulsars. See also Melatos 2022 PT12

[2] **TK: See also e.g. Maggiore, Meltaos who takes the same approach...**

[3]
Therefore the derivative w.r.t coordinate time t is,

$$\dot{p}_t = \frac{dp_t}{d\lambda}\left(\frac{dt}{d\lambda}\right)^{-1} = \frac{dp_t}{d\lambda}\left(\frac{1}{p^t}\right) \tag{19}$$

To make contact with Melatos 2022 it will prove useful to recognise that $p^\mu = \omega(1, -q^x, -q^y, -q^z)$ where $\bar{q}$ is the unit vector between the Earth and pulsar [4] and ω is the *constant* photon angular frequency. Given the small effect of the GW perturbation, at first order we can identify ω as either the frequency at source or observer (see Melatos 2022, PT16).

Note that $\dot{p}_t$ is entirely a function of the GW perturbation. In the Minkowski case the spacetime is stationary and so p_t should be conserved along the geodesic.

Bringing this all together we can write $\dot{p}_t$ in a condensed form as,

$$\dot{p}_t = Ae^{i\gamma t + \Phi_0} \tag{20}$$

where

$$\gamma = -\Omega(1 + \bar{n}\cdot\bar{q}) \tag{21}$$

[5] and

$$A = -\frac{i\Omega\omega}{2} H_{ij} q^i q^j \tag{22}$$

where the ω term in the numerator results from a ω^2 terms that comes from the $p^i p^j$ term and a ω^{-1} term from $(p^t)^{-1}$.

3.4 Performing the integral

The frequency shift experienced by the observer relative to the source due to a GW is then

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau - d)|_{\text{source}} = A\int_{t=\tau-d}^{t=\tau} e^{i\gamma t + \Phi_0} dt \tag{23}$$

$$= \frac{-iA}{\gamma} e^{i\gamma\tau} e^{\Phi_0}[1 - e^{-i\gamma d}] \tag{24}$$

3.5 Explicit expression and comparison with Melatos 22

We can expand the concise expression of Eq. 24 as,

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau-d)|_{\text{source}} = \frac{\omega}{2}\frac{H_{ij}q^i q^j}{(1+\hat{n}\cdot\hat{q})} e^{-i\Omega\tau(1+\bar{n}\cdot\bar{q})+\Phi_0}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q})d}] \tag{25}$$

This expression is equivalent to Melatos 2022, PT13 where the dependence on time t is made explicit. Using the Melatos notation where $h_{ij} = g_{ij} - \eta_{ij}$

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau-d)|_{\text{source}} = \frac{\omega}{2}\frac{h_{ij}q^i q^j}{(1+\hat{n}\cdot\hat{q})}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q})d}] \tag{26}$$

[3] This is equivalent to Melatos 2018, Eq 5 for the specific case of a GW propagating in the z-direction, with zero phase offset.
[4] similarly $\bar{x}(t) = -\bar{q}t$
[5] compare with Melatos 2022, PT11

4 STATE-SPACE STRUCTURE

The general structure of a Kalman state-space problem is

$$\dot{x} = f(x) + w \tag{27}$$

where x are the state variables, $f()$ a non-linear function of the states, and w a stochastic zero-mean process. The process noise matrix $Q = E(ww^T)$. The states are related to the measurement z via a non-linear measurement function $h()$

$$z = h(x) + v \tag{28}$$

where v is a stochastic zero-mean process and measurement noise matrix $R = E(vv^T)$.

This structure maps onto the preceding equations nicely. Our hidden state is just the intrinsic pulsar frequency, which evolves as,

$$\dot{f}_P = -\gamma f_P^n + \xi(t) \tag{29}$$

where n is a braking index (typical values $\sim 1-5$, we take a canonical value = 3) [6] γ a proportionality constant, and $\xi(t)$ represents a stochastic white noise process. Equation 33 can be used to relate the intrinsic pulsar frequency to that measured by an observer on Earth:

$$f_M = f_P(1 - X) + N_M \tag{30}$$

where X is the RHS of Eq. 33 and N_M a Gaussian measurement noise. An example of the evolution of the state and the measurement frequencies for our PTA pulsars can be seen in Fig. **??**

5 DISCRETISATION

This intrinsic frequency can be related to a measured frequency as

$$f_M = fg(\bar{\theta}, t) + N_M \tag{31}$$

where $g(\theta, t)$ ("measurement function")is a function of some parameters, $\bar{\theta}$ and time t, whilst N_M is a Gaussian measurement noise that satisfies

$$\langle N_M(t) N_M(t')\rangle = \Sigma^2\delta(t - t') \tag{32}$$

Explicitly, it can be shown that the measurement function is,

$$g(\bar{\theta}, t) = 1 - \frac{1}{2}\frac{h_{ij}(t)q(t)^i q(t)^j}{1 + \bar{n}\cdot\bar{q}(t)}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q}(t))d}] \tag{33}$$

There are a few terms to unpack and define in this equation. $q(t)$ is the vector from the Earth to the pulsar, n the propagation direction of the GW, Ω the (constant) angular frequency of the GW and d the Earth-pulsar distance. $h_{ij}(t)$ is the metric perturbation due to the GW = $g_{ij} - \eta_{ij}$. It is given by the familiar plane-wave relation:

$$h_{ij}(t) = H_{ij}e^{(i(\bar{k}\cdot\bar{x} - \Omega t + \Phi_0))} \tag{34}$$

where $\bar{k}$ is the GW 3-wavevector = $\Omega\bar{n}$ and $x = -\bar{q}t$ (photon propagates from pulsar to Earth), and for phase offset (GW phase at Earth when $t = 0$) Φ_0. We can also write this as,

$$h_{ij}(t) = H_{ij}e^{(-i\Omega t(1+\bar{n}\cdot\bar{q}) + \Phi_0))} \tag{35}$$

The amplitude tensor H_{ij} is

$$H_{ij} = h_+ e^+_{ij}(\bar{n}, \psi) + h_\times e^\times_{ij}(\bar{n}, \psi) \tag{36}$$

[6] See A. Vargas talk re anomalous braking indices. P.S. is there an arXiv preprint?

where $h_{+,\times}$ are the constant amplitudes of the gravitational plane wave, and $e_{ij}^{+,\times}(\bar{n},\psi)$ are the polarisation tensors which are uniquely defined by the principal axes of the GW. ψ is the polarisation angle of the GW.

Going forward, for now we will take $q(t) = q$ i..e. the pulsar locations are constant with respect to the Earth. This may have already been "done" during the barycentreing when pulsar TOAs are generated, in which case q is the vector from the SSB to the pulsar

Bringing this together, and adopting a trigonometric form of the equations, we can express the measurement equation as

$$g(\bar{\theta}, t) = 1 - A\cos(-\Omega t(1 + n \cdot q) + \Phi_0) \tag{37}$$

where

$$A = \frac{1}{2}\frac{H_{ij}q^i q^j}{1 + \bar{n}\cdot\bar{q}}\left[1 - \cos\left(\Omega(1 + \bar{n}\cdot\bar{q})d\right)\right] \tag{38}$$

5.1 Parameters of the model

Lets review and categorise all the parameters the the above model. We can generally separate these into parameters which correspond to the intrinsic frequency evolution of the pulsar, the GW parameters and the noise parameters i.e.

$$\bar{\theta} = \bar{\theta}_{\rm PSR} \cup \bar{\theta}_{\rm GW} \cup \bar{\theta}_{\rm noise} \tag{39}$$

$$\bar{\theta}_{\rm PSR} = [\gamma, f_{\rm EM}(0), \dot{f}_{\rm EM}(0), d] \tag{40}$$

$$\bar{\theta}_{\rm GW} = [h_+, h_\times, \delta, \alpha, \psi, \Omega, \Phi_0] \tag{41}$$

$$\bar{\theta}_{\rm noise} = [\sigma, \Sigma] \tag{42}$$

We can also express the measurement equation generally as,

6 PTA PULSARS

In order to proceed and explore how well this state-space formulation works, we will need to specify a selection of pulsars to make up our PTA. We will take the 47 pulsars that make up the NANOGrav PTA. For each pulsar, we need to specify the complete set of $\bar{\theta}_{\rm PSR}$ as well as σ. The parameters $f_{\rm EM}(0), \dot{f}_{\rm EM}(0), d$ are straightforward to set and can be read directly from the current "present day" best estimates of the pulsar frequency, derivative and distance.

7 KALMAN FILTERING

8 METHODS

There are two potential methods we can take

(i) Let the states just be the frequencies $\bar{x} = (f)$. We have a linear equation for both the state evolution and the measurement matrix. Both of these depend on the parameters θ. Use the likelihood output by a standard Kalman filter to run a nested sampler and try to recover θ.

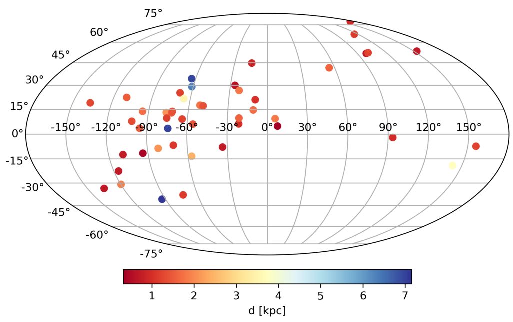

Figure 1. Spatial distribution and distances of NANOGrav pulsars

(ii) Let the states be the frequencies and all the unknown parameters $\bar{x} = (f, \bar{\theta}_{\rm GW}, \bar{\theta}_{\rm PSR}, \bar{\theta}_{\rm noise})$. Our state evolution is now linear but our measurement matrix is non-linear - a function of the states(parameters). Use a non-linear estimator such as a EKF/UKF to try to recover all of the states at once.

8.1 Method 1

In the case where the states are just the intrinsic pulsar frequencies, we can write the ODEs in matrix form as

$$d\bar{X} = \bar{A}\bar{X}dt + \bar{N}(t)dt + \bar{\Sigma}d\bar{B}(t) \tag{43}$$

where $\bar{A} = \mathrm{diag}(\gamma_1, \gamma_2, ...)$, $\bar{X} = \mathrm{diag}(f_1, f_2, ...)$, $\bar{N} = \mathrm{diag}(\gamma_1[a_1 + b_1 t] + b_1, ...)$ and we have let $a = f_{EM}(0)$, $b = \dot{f}_{\rm EM}(0)$.

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