State-space PTA

Kimpson[1], Melatos, O'Leary, Evans, others, etc. etc. ⋆ †
[1] *Royal Astronomical Society, Burlington House, Piccadilly, London W1J 0BQ, UK*

Last updated 2020 June 10; in original form 2013 September 5

ABSTRACT
This is an abstract

Key words: editorials, notices – miscellaneous

1 INTRODUCTION

The detection of high frequency ($\sim$ 1 Hz) gravitational waves (GWs) from coalescing BH binaries with ground-based detectors such as LIGO/Virgo (LIGO Scientific Collaboration et al. 2015; Acernese et al. 2015) is now a routine enterprise (e.g. Abbott et al. 2019, 2021). Gravitational radiation from sources which radiate in the mill-Hz regime are expected to be detectable from $\sim$ 2030 with the space-based Laser Interferometer Space Antenna, (Amaro-Seoane et al. 2017) especially given the early success by the pathfinder mission (Armano et al. 2019). Detecting GWs from systems which evolve over even longer timescales, $\mathcal{O}$(years), has necessitated the development of novel astrophysical methods, since it is practically impossible to engineer interferometric detectors with sufficiently long baselines. The foremost technique for the detection of GWs in this nano-Hz regime is a via timing an ensemble of milliseconds pulsars; a pulsar timing array (PTA) (Verbiest et al. 2021). The presence of a nano-Hz gravitational wave will influence the propagation of the pulsar radio beacon, leaving a characteristic impression on the pulsar timing signal. By measuring the modulation of the received pulsar signal in this way, one can effectively construct a detector with a baseline on the scale of parsecs.

Multiple PTA detectors have now been built over the last few decades, including the North American Nanohertz Observatory for Gravitational Waves (NANOGrav, Arzoumanian et al. 2020), the Parkes Pulsar Timing array (PPTA Kerr et al. 2020), and the European Pulsar Timing Array (EPTA, Ferdman et al. 2010). These previously disparate efforts have now been joined in international collaboration, along with a number of newer PTAs, under the umbrella of the International Pulsar Timing Array (IPTA Perera et al. 2019). The primary target of PTA observations is the gravitational radiation emitted from the inspiral of supermassive black hole binaries (SMBHBs) with masses $\sim \infty r' \mathcal{M}_{\odot}$. These GW signals from SMBHBs can be broadly classified into either deterministic or stochastic. For the former, sufficiently bright and near binaries may be resolvable with PTAs, allowing the very earliest stages of their evolution and coalescence to be investigated. For the latter, the incoherent superposition of multiple weaker SMBHBs sources leads to a stochastic background detectable at nano-Hz frequencies. Other potential sources for PTAs include cosmic strings (e.g. Sanidas et al. 2012) and cosmological phase transitions (e.g. Xue et al. 2021), but the deterministic and stochastic GW signals from SMBHBs remain the primary targets.

The detection of loud, resolved sources with a PTA typically involves a parametrised model for the pulsar timing residuals induced by the modulation of the pulsar signal by a GW. One can then search for evidence that this model describes the data via the usual Bayesian likelihood techniques, and try to estimate the parameters of the model (e.g. Babak et al. 2016). For detecting the stochastic background the approach is different; one measures the correlation in pulsar timing residuals between any two pair of pulsars. The presence of a GW induces a characteristic correlation function as a function of the angular separation between the pulsars; the Hellings-Downs curve (Hellings & Downs 1983). For both classes of source, detection of GW signals in the timing residuals of a PTA is a challenging enterprise, and currently neither a stochastic background nor an individually resolved source has yet been detected (Antoniadis et al. 2022; Hobbs & Dai 2017).

The sensitivity of a PTA to a GW signal is heavily dependent on the total level of noise (Wang 2015). In particular, the intrinsic pulsar timing noise - i.e. random variations in the pulse arrival time - has been identified as a key factor limiting the sensitivity of PTAs to GW signals (Shannon & Cordes 2010; Lasky et al. 2015; Caballero et al. 2016). This pulsar timing noise has multiple potential theorized causes including microglitches (Melatos et al. 2008), glitch recovery (Hobbs et al. 2010), fluctuations in both the internal and external stochastic torques (Antonelli et al. 2023) and superfluid turbulence (Melatos & Link 2014). In order to mitigate the influence of timing noise, PTAs are typically composed of millisecond pulsars which are known to be much more stable rotators with minimal timing noise. However, timing noise in MSPs could be a 'latent' phenomenon (Shannon & Cordes 2010); as we increase the length of observation spans and the timing precision required to detect gravitational waves, timing noise in MSPs may become emergent and more apparent, analogous to that seen in younger pulsars. In addition to timing noise there are also secondary noise sources one must consider such as phase jitter noise and radiometer noise (Cordes & Shannon 2010; Lam et al. 2019; Parthasarathy et al. 2021).

⋆ Contact e-mail: mn@ras.ac.uk
† Present address: Science magazine, AAAS Science International, 82-88 Hills Road, Cambridge CB2 1LQ, UK

In the standard approach to PTA-GW analysis, timing noise is nuisance which must be accurately modelled and subtracted and fundamentally limits the sensitivity of the PTA detector to a GW signal and its ability to infer the source parameters. Motivated by these challenges faced by the classic PTA analysis methods, in this work we present a novel approach to formulate PTA analysis and GW detection as a state-space problem. This approach enables the pulsar state-space evolution to be tracked optimally, given a specific realisation of the pulsar process noise (i.e. the spin wandering of the pulsar), and determine both the presence of a GW signal in the pulsar data, and infer the underlying source parameters. For this initial exploratory study we will focus exclusively on resolved, monochromatic GW sources.

This paper is organised as follows:

2 STATE-SPACE MODEL

We want to formulate the PTA analysis as a state-space problem with a separation between the intrinsic pulsar state and the measurement state recorded by an observer. We will take our state to be the pulsar pulse frequency $f(t)$. We will now consider how the intrinsic pulse frequency evolves in time, completely separate from the influence of a GW, and then go on to derive the influence of a GW perturbation on the frequency recorded by an observer at Earth.

2.1 Evolution of the pulsar frequency

We will take as our model of the intrinsic pulsar frequency f a variation of the phenomenological model of (Vargas & Melatos 2023). Within this model, f evolves according to a combination of both deterministic torques (i.e. electromagnetic spin-down) and stochastic torques (i.e. 'spin wandering', achromatic variations in the pulse TOA intrinsic to the star). The deterministic torque is taken to arise from the pulsar magnetic dipole, with braking index $n = 3$ whilst the stochastic torque is a simple white noise process. Specifically, the frequency evolves according to a Ornstein-Uhlenbeck process (equivalently a Langevin equation) with a time-dependent drift parameter:

$$\frac{df}{dt} = -\gamma[f - f_{\rm EM}(t)] + \dot{f}_{\rm EM} + \xi(t) \tag{1}$$

where $f_{\rm EM}$ is the solution of the electromagnetic spindown equation, γ a proportionality constant, and $\xi(t)$ a white noise process that satisfies,

$$\langle \xi(t)\xi(t')\rangle = \sigma^2 \delta(t-t') \tag{2}$$

For PTA analysis, we are concerned with timescales on the order of years. Consequently, we can express the EM spindown straightforwardly as

$$f_{\rm EM}(t) = f_{\rm EM}(0) + \dot{f}_{\rm EM}(0)t \tag{3}$$

Completely, the frequency evolution is then given by the solution of the stochastic differential equation,

$$\frac{df}{dt} = -\gamma[f - f_{\rm EM}(0) - \dot{f}_{\rm EM}(0)t] + \dot{f}_{\rm EM}(0) + \xi(t) \tag{4}$$

As emphasised in Vargas & Melatos (2023), this model for the frequency evolution is a phenomenological model that aims to qualitatively reproduce the typical behaviour of observed pulsars, rather than being derived from a physical model of the neutron star (e.g. a model of the neutron star crust and superfluid components Meyers et al. 2021). However, for our purposes of exploring the detection of GWs via a state space formulation it will prove sufficiently accurate and appropriate.

2.2 Modulation of pulsar frequency due to a GW

In the presence of a GW, the $f(t)$ measured by an observer on Earth is different from that measured by an observer in the local NS reference frame. We want to determine how the GW influences the received frequency

2.2.1 *Plane GW perturbation*

We take a gravitational plane wave that perturbs a background Minkowski spacetime as

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} e^{i(\Omega(\bar{n}\cdot\bar{x}-t)+\Phi_0)} \tag{5}$$

for spatial coordinates $\bar{x}$, where the GW has a constant angular frequency Ω, propagates in the $\bar{n}$-direction and has a phase offset of Φ_0. We emphasise that Ω has no time dependence - we are concerned solely with monochromatic sources. Note that we are free to choose our coordinate system such that Φ_0 is the GW phase at $t = 0$ *at the Earth.*[1]. The amplitude tensor $H_{\mu\nu}$ has zero temporal components ($H_{0\mu} = H_{\mu 0} = 0$) whilst the spatial part is

$$H_{ij} = h_+ e^+_{ij}(\bar{n}) + h_\times e^\times_{ij}(\bar{n}) \tag{6}$$

where $h_{+,\times}$ are the polarisation amplitudes of the gravitational plane wave. The polarisation tensors $e^{+,\times}_{ij}$ are uniquely defined by the principal axes of the wave:

$$e^+_{ab}(\hat{\Omega}) = \hat{m}_a\hat{m}_b - \hat{n}_a\hat{n}_b \tag{7}$$

$$e^\times_{ab}(\hat{\Omega}) = \hat{m}_a\hat{n}_b + \hat{n}_a\hat{m}_b \tag{8}$$

which are in turn specified via the location of the GW source on the sky (via θ, ϕ coordinates) and the polarisation angle ψ

$$\begin{aligned}\vec{m} &= (\sin\phi\cos\psi - \sin\psi\cos\phi\cos\theta)\hat{x} \\ &\quad - (\cos\phi\cos\psi + \sin\psi\sin\phi\cos\theta)\hat{y} \\ &\quad + (\sin\psi\sin\theta)\hat{z} \end{aligned}\tag{9}$$

$$\begin{aligned}\vec{n} &= (-\sin\phi\sin\psi - \cos\psi\cos\phi\cos\theta)\hat{x} \\ &\quad + (\cos\phi\sin\psi - \cos\psi\sin\phi\cos\theta)\hat{y} \\ &\quad + (\cos\psi\sin\theta)\hat{z} \end{aligned}\tag{10}$$

2.2.2 *Pulse frequency as a photon*

We will consider the pulse frequency as a photon with covariant 4-momentum p_μ. Generally, the frequency of a photon recorded by an observer with 4-velocity u^μ is

$$\nu = p_\alpha u^\alpha \tag{11}$$

We consider both our emitter and receiver to be stationary, such that

$$u^\alpha|_{\rm emitter} = u^\alpha|_{\rm receiver} = (1,0,0,0) \tag{12}$$

Consequently the frequency can be directly identified with the temporal component of the covariant 4-momentum,

$$f = p_t \tag{13}$$

[1] **TK: This point is important, since the phase offset is then the same between multiple pulsars. See also Melatos 2022 PT12**

The expression for the evolution of the pulse frequency as measured by the observer on Earth is then,

$$p_t(t_1)|_{\text{Earth}} = p_t(t_0)|_{\text{source}} + \int_{t=t_0}^{t=t_1} \dot{p}_t dt \tag{14}$$

where the overdot denotes a derivative w.r.t. t. Since the influence of the GW perturbation on $\dot{p}_t$ is small, we can relate the source emission and receiver times as $t_1 = t_0 + d$ and consider the photon trajectory to be an unperturbed path. [2]

To complete our expression, we now just need to determine $\dot{p}_t$ and integrate it.

2.2.3 *Hamiltonian Mechanics*

The Hamiltonian in covariant notation can be written as

$$H(x^\mu, p_\mu) = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu, \tag{15}$$

which if we substitute in our expression for the perturbed metric is

$$H = \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + \frac{1}{2} H_{ij} p^i p^j e^{i(\Omega(\bar{n}\cdot\bar{x}-t)+\Phi_0)} \tag{16}$$

Now, Hamilton's equations are

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu} \tag{17}$$

for affine parameter λ. The derivative of the temporal component of the covariant momenta is then,

$$\frac{dp_t}{d\lambda} = -\frac{i\Omega}{2} H_{ij} p^i p^j e^{i(\Omega(\bar{n}\cdot\bar{x}-t))+\Phi_0} \tag{18}$$

[3]

Therefore the derivative w.r.t coordinate time t is,

$$\dot{p}_t = \frac{dp_t}{d\lambda}\left(\frac{dt}{d\lambda}\right)^{-1} = \frac{dp_t}{d\lambda}\left(\frac{1}{p^t}\right) \tag{19}$$

Note that $\dot{p}_t$ is entirely a function of the GW perturbation. In the Minkowski case the spacetime is stationary and so p_t should be conserved along the geodesic. It will prove useful to recognise that $p^\mu = \omega(1, -q^x, -q^y, -q^z)$ where $\bar{q}$ is the unit vector between the Earth and pulsar and ω is the *constant* photon angular frequency. Given the small effect of the GW perturbation, at first order we can identify ω as either the frequency at source or observer [4]. We will consider the pulsar locations to be constant with respect to the Earth. TK: **This may have already been "done" during the barycentreing when pulsar TOAs are generated, in which case q is the vector from the SSB to the pulsar**. Similarly we can express the spatial coordinates as $\bar{x}(t) = -\bar{q}t$.

Bringing this all together we can write $\dot{p}_t$ in a condensed form as,

$$\dot{p}_t = Ae^{i\gamma t+\Phi_0} \tag{20}$$

where

$$\gamma = -\Omega(1 + \bar{n}\cdot\bar{q}) \tag{21}$$

[5] and

$$A = -\frac{i\Omega\omega}{2} H_{ij} q^i q^j \tag{22}$$

[2] **TK: See also e.g. Maggiore, Meltaos who takes the same approach...**

[3] **TK: This is equivalent to Melatos 2018, Eq 5 for the specific case of a GW propagating in the z-direction, with zero phase offset**.

[4] see Melatos 2022, PT16

[5] compare with Melatos 2022, PT11

2.2.4 *Performing the integral*

The frequency shift experienced by the observer relative to the source due to a GW is then

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau - d)|_{\text{source}} = A\int_{t=\tau-d}^{t=\tau} e^{i\gamma t+\Phi_0} dt \tag{23}$$

$$= \frac{-iA}{\gamma} e^{i\gamma\tau} e^{\Phi_0}[1 - e^{-i\gamma d}] \tag{24}$$

We can write this more explicitly as,

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau - d)|_{\text{source}} = \tag{25}$$

$$\frac{\omega}{2}\frac{H_{ij}q^i q^j}{(1+\hat{n}\cdot\hat{q})} e^{-i\Omega\tau(1+\bar{n}\cdot\bar{q})+\Phi_0}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q})d}] \tag{26}$$

or alternatively, with $h_{ij} = g_{ij} - \eta_{ij}$,

$$p_t(\tau)|_{\text{Earth}} - p_t(\tau - d)|_{\text{source}} = \frac{\omega}{2}\frac{h_{ij}q^i q^j}{(1+\hat{n}\cdot\hat{q})}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q})d}] \tag{27}$$

2.3 State-space structure

We now have a model for the evolution of the pulsar frequency states and the measured frequency timeseries. We can express the intrinsic frequency evolution, Eq. 4, in a alternative form as,

$$df = Afdt + N(t)dt + \sigma dB(t) \tag{28}$$

where $A = -\gamma$, $N(t) = \gamma(f_{\text{EM}}(0) + \dot{f}_{\text{EM}}(0)t) + \dot{f}_{\text{EM}}(0)$ and $dB(t)$ denotes increments of Brownian motion (Wiener process). This equation is easily identified as an Ornstein-Uhlenbeck process which has a general solution given by,

$$f(t) = e^{At}f(0) + \int_0^t e^{A(t-t')}N(t')dt' + \int_0^t e^{A(t-t')}\sigma dB(t') \tag{29}$$

If we move from a solution in continuous time to discrete time then

$$f(t_{i+1}) = Ff(t_i) + T_i + \eta_i \tag{30}$$

where

$$F_i = e^{A(t_{i+1}-t_i)} \tag{31}$$

$$T_i = \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-t')}N(t')dt' \tag{32}$$

$$\eta_i = \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-t')}\sigma dt' \tag{33}$$

Given the discrete solution $f(\bar{t})$ where $\bar{t} = (t_1, t_2, ..., t_N)$ the intrinsic frequency can be related to the measured frequency as

$$f_M = fg(\bar{\theta}, t) + N_M \tag{34}$$

where $g(\theta, t)$ ("measurement function")is a function of some parameters, $\bar{\theta}$ and time t, whilst N_M is a Gaussian measurement noise that satisfies

$$\langle N_M(t)N_M(t')\rangle = \Sigma^2\delta(t - t') \tag{35}$$

The measurement function follows from Eq. 27

$$g(\bar{\theta}, t) = 1 - \frac{1}{2}\frac{h_{ij}(t)q^i q^j}{1+\bar{n}\cdot\bar{q}}[1 - e^{i\Omega(1+\bar{n}\cdot\bar{q})d}] \tag{36}$$

or in a more concise form where we have switched to a trigonometric form of the equations,

$$g(\bar{\theta}, t) = 1 - A\cos(-\Omega t(1 + n\cdot q) + \Phi_0) \tag{37}$$

where

$$A = \frac{1}{2}\frac{H_{ij}q^i q^j}{1+\bar{n}\cdot\bar{q}}[1-\cos\left(\Omega(1+\bar{n}\cdot\bar{q})d\right)] \tag{38}$$

3 DETECTION AND PARAMETER ESTIMATION

Given the proceeding state-space model we will now try to use this model to infer the unknown parameters of the system. Let's first review and categorise all the parameters of model. We can generally separate these into parameters which correspond to the intrinsic frequency evolution of the pulsar, the GW parameters and the noise parameters i.e.

$$\bar{\theta} = \bar{\theta}_{\rm PSR} \cup \bar{\theta}_{\rm GW} \cup \bar{\theta}_{\rm noise} \tag{39}$$

$$\bar{\theta}_{\rm PSR} = [\gamma, f_{\rm EM}(0), \dot{f}_{\rm EM}(0), d] \tag{40}$$

$$\bar{\theta}_{\rm GW} = [h_+, h_\times, \delta, \alpha, \psi, \Omega, \Phi_0] \tag{41}$$

$$\bar{\theta}_{\rm noise} = [\sigma, \Sigma] \tag{42}$$

Whilst the GW parameters are shared between measurements for each pulsar, the pulsar parameters are clearly not. For a PTA dataset of N pulsars we have $7+5N$ parameters to estimate. For now we will consider the measurement noise to be known, although in principle this too could be estimated. Note that whilst this is a large parameter space, in general the pulsar parameters are much better constrained than the GW parameters: for example we have rough estimates for the pulsar distances accurate to $\sim 10\%$, but we have no prior information on the source location.

We will use a Kalman filter in order to obtain likelihood estimates of the data given a set of parameters. We will now review the Kalman filter and the associated likelihood in Section before going on to discuss how to use nested sampling methods in conjunction with the filter for detection (model selection) and parameter estimation

3.1 Kalman Filtering

The Kalman filter (Kalman 1960) is a algorithmic technique for recovering a set of system state variables, $\bar{x}$, given some noisy measurements, $\bar{z}$. It is a common technique in signal processing that has also been applied more recently with great success in astrophysics (e.g. Meyers et al. 2021; Melatos et al. 2021). The linear Kalman filter operates on measurements that are related to states via a linear transformation

$$z = Hx + v \tag{43}$$

where H is the measurement matrix and v a Gaussian measurement noise. The underlying states are the solutions to the state-space equation

$$\dot{x} = Fx + Gu + w \tag{44}$$

where x are the state variables, F the system dynamics matrix, Gu the control matrix/vector and w a stochastic zero-mean process which . By comparison with the preceding equations, Eqs. 28 - 38, it is immediately obvious how our state space model maps onto the Kalman filter structure. Specifically, our states are just the N intrinsic pulsar frequencies $\bar{x} = (f_1, f_2, ...f_N)$ whilst our measurements are the N measured pulse frequencies $\bar{z} = (f_1^{(M)}, f_2^{(M)}, ...f_N^{(M)})$. If we specialize to the case of constant time sampling between our observations,Δt, then for our formulation the components that make up the Kalman filter are as follows:

$$F_i = F_{i+1} = e^{-\gamma\Delta t} \tag{45}$$

$$T_i = \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-t')}N(t')dt' \tag{46}$$

$$= f_{\rm EM}(0) + \dot{f}_{\rm EM}(0)(\Delta t + t_i) - e^{-\gamma\Delta t}(f_{\rm EM}(0) + \dot{f}_{\rm EM}(0)t_i) \tag{47}$$

$$H_i = 1 - A(\theta_{\rm GW})\cos(-\Omega t_i(1+n\cdot q) + \Phi_0) \tag{48}$$

and A is a constant that is given by Eq. 38. The Kalman filter includes the effect of process noise w and the measurement noise v via the definition of a process noise matrix $Q = E[ww^T]$ and a measurement noise matrix $R = E[vv^T]$, which have the discrete form,

$$Q_i\delta_{ij} = \langle \eta_i \eta_j^T \rangle = \frac{-\sigma^2}{2\gamma}\left(e^{-2\gamma\Delta t} - 1\right) \tag{49}$$

$$R_i = R_{i+1} = \Sigma^2 \tag{50}$$

For our formulation we are concerned only with the linear Kalman filter since the measurements are a linear function of the states and the state transitions are also linear. Extension to non-linear problems is straightforward using either an extended Kalman filter, the unscented Kalman filter or the particle filter. For a full review of the Kalman filter equations which may bee familiar to some, we refer the reader to the appendices of Melatos et al. (2021) and Meyers et al. (2021).

In order for the Kalman filter to function successfully, the parameters of the model, which appear in the various Kalman matrices, must be accurate. Erroneous or inaccurate parameters lead to inaccurate predictions of the underlying states (e.g. Fig 1).

The filter tracks the error in its predictions of the underlying states by projecting the state predictions back into measurement space $\hat{z}$. These measurement predictions can then be compared against the true observed measurements. The Gaussian log-likelihood as a function of the parameters is,

$$\log\mathcal{L} = -\frac{1}{2}(N\log 2\pi + \log\sigma^2 + \frac{y^2}{x}) \tag{51}$$

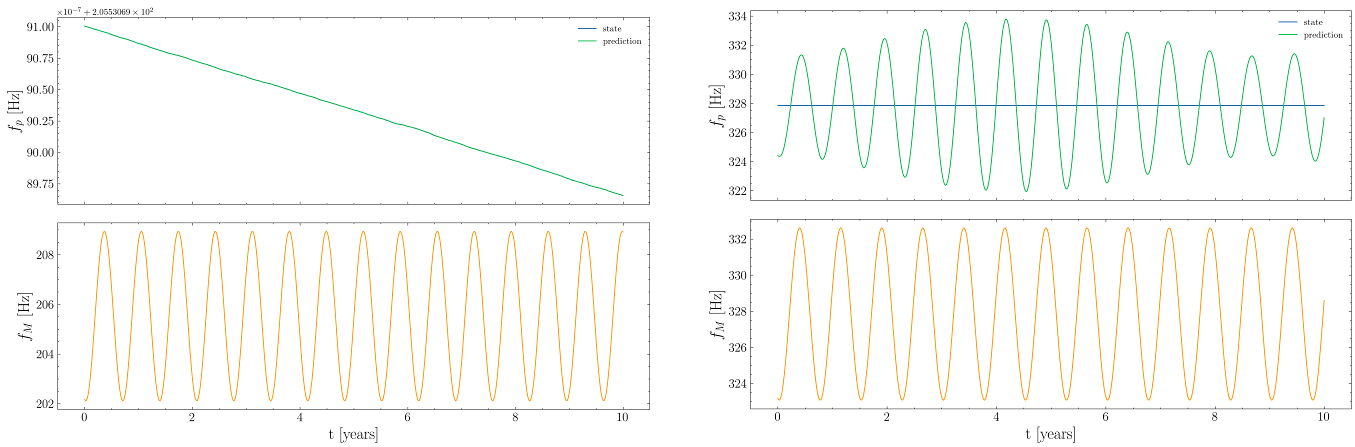
where this is this is that.... This likelihood provided by the Kalman filter can then be used for Bayesian inference based techniques of the underlying system parameters, which we will now explore in more detail.

3.2 Nested Sampling

Nested Sampling Skilling (2006) is an integration algorithm used for evaluating integrals of the Bayesian form

$$Z = \int L\pi(\theta)d\theta \tag{52}$$

where this is this is that... i.e. the model evidence. It is a highly advantageous approach over the usual MCMC-type approaches which are used solely for calculating the posterior but can be highly computationally expensive, does not return an evidence, and can struggle on multi-modal problems.



(a) (b)

Figure 1. TK: caption needed

Within the astrophysical community, it was first adopted in cosmology, CITE CITE but has become a widespread technique used wihin astronomy, astrophysics, particle physics and materials science CITE

Nested sampling genearlly works by algorithmically evolving a set of "live points" that seek to describe an underlying probability distribution. For a full review we refer the reader to original works by Skilling (2006) and a recent review from a physics sciences perspective from **ASHTON**

Multiple nested sampling libraries exist. For gravitational astrophysics it is common to use the dynesty sampler CITE, via the Bilby gravitational wave inference library and we continue to follow this precedent.

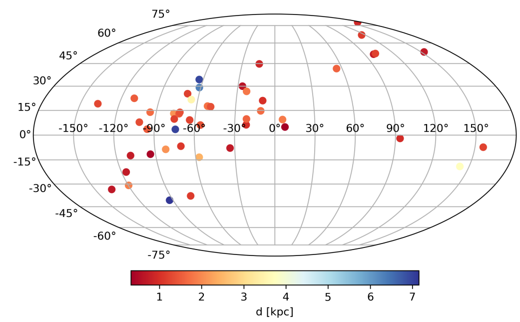


Figure 2. Spatial distribution and distances of NANOGrav pulsars

3.3 PTA pulsars

In order to proceed, it is now necessary to specify a PTA configuration.

As discussed, multiple separate PTA detectors exist under the umbrella of the IPTA. Going forward we will take the 47 pulsars that make up the NANOGrav PTA (Arzoumanian et al. 2020). NANOGrav is selected simply as a well-representative example of the typical pulsars that make up a PTA. Our results and formulation are not contingent on the choice of PTA, and naturally extend to other PTAs or PTAs with more pulsars.

Within the state-space forumulation, the pulsar evolution is governed by a set of 5 parameters $\bar{\theta}_{\rm PSR}$ for each pulsar. The parameters $f_{\rm EM}(0), \dot{f}_{\rm EM}(0)$ and d are well specified via existing pulsar datasets. We take the frequency and frequency derivative as returned from the pulsar datasets to simply be the values now at $t = 0$. The pulsar distances are also known, though less well constrained. Going forward we take the distances retuned from the datasets as the true values of the pulsars that make up our synthetic PTA.

The specification of γ and σ are more involved. γ specifies an effective timescale of reversion to the mean

TK: discussion on how to select these two parameters c.f. Andres

In order to proceed and explore how well this state-space formulation works, we will need to specify a selection of pulsars to make up our PTA. We will take the 47 pulsars that make up the NANOGrav PTA.

For each pulsar, we need to specify the complete set of $\bar{\theta}_{\rm PSR}$ as well as σ. The parameters $f_{\rm EM}(0), \dot{f}_{\rm EM}(0), d$ are straightforward to set and can be read directly from the current "present day" best estimates of the pulsar frequency, derivative and distance.

3.4 Parameter estimation on synthetic data

We are now in a position to try to infer the parameters of our system

3.5 Detection on synthetic data

The general structure of a Kalman state-space problem is

$$\dot{x} = f(x) + w \tag{53}$$

where x are the state variables, $f()$ a non-linear function of the states, and w a stochastic zero-mean process. The process noise matrix $Q = E(ww^T)$. The states are related to the measurement z via a non-linear measurement function $h()$

$$z = h(x) + v \tag{54}$$

where v is a stochastic zero-mean process and measurement noise matrix $R = E(vv^T)$.

This structure maps onto the preceding equations nicely. Our hidden state is just the intrinsic pulsar frequency, which evolves as,

$$\dot{f}_P = -\gamma f_P^n + \xi(t) \tag{55}$$

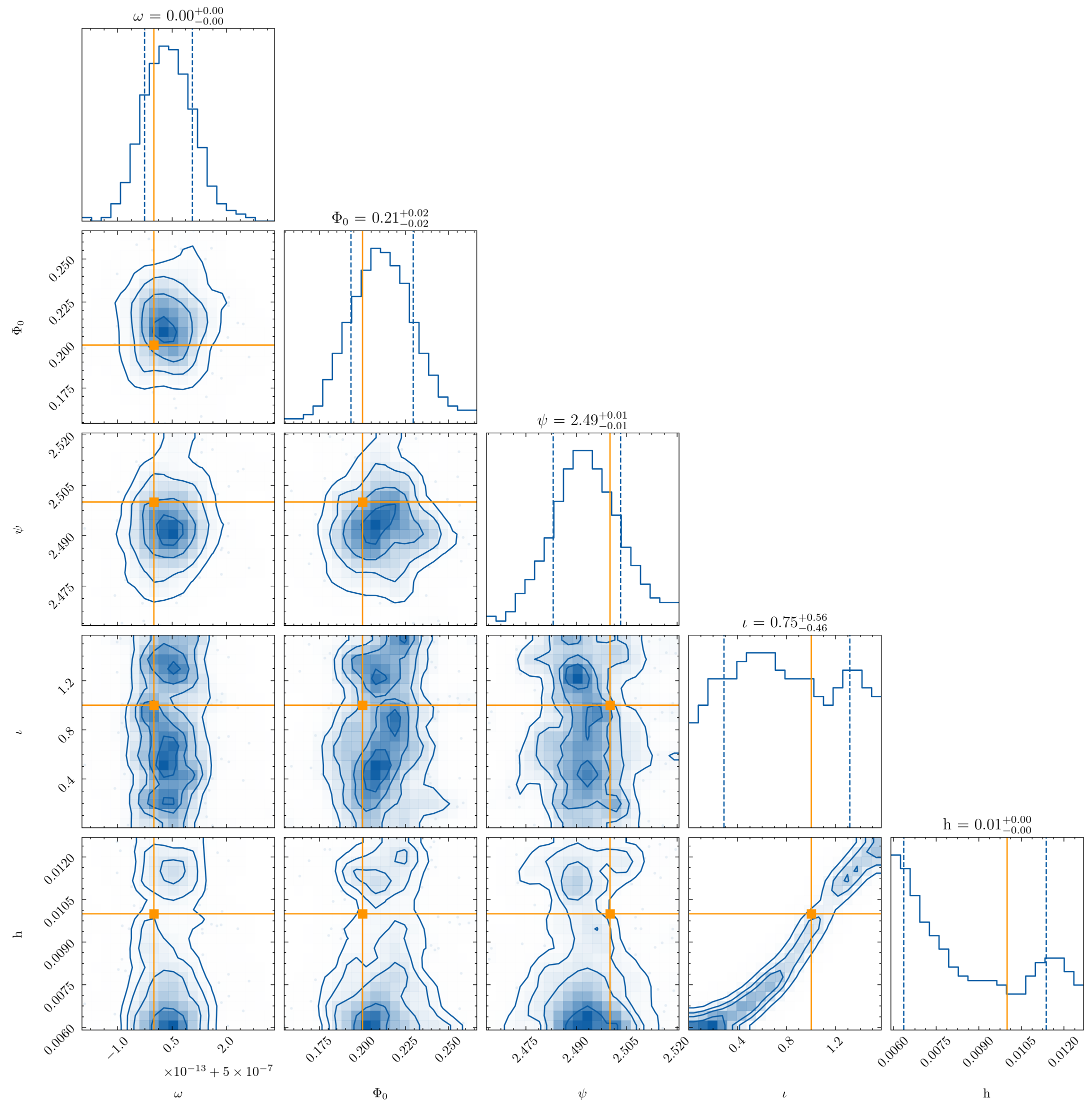


Figure 3. Example corner plot

where n is a braking index (typical values $\sim 1-5$, we take a canonical value = 3) [6] γ a proportionality constant, and $\xi(t)$ represents a stochastic white noise process. Equation 36 can be used to relate the intrinsic pulsar frequency to that measured by an observer on Earth:

$$f_M = f_p(1 - X) + N_M \tag{56}$$

where X is the RHS of Eq. 36 and N_M a Gaussian measurement noise. An example of the evolution of the state and the measurement frequencies for our PTA pulsars can be seen in Fig. **??**

Quality vs quantity: This is important as it measn we dont need to do this for all pulsars, just a few.

4 DISCRETISATION

This intrinsic frequency can be related to a measured frequency as

[6] See A. Vargas talk re anomalous braking indices. P.S. is there an arXiv preprint?

There are a few terms to unpack and define in this equation. $q(t)$ is the vector from the Earth to the pulsar, n the propagation direction of the GW, Ω the (constant) angular frequency of the GW and d the Earth-pulsar distance. $h_{ij}(t)$ is the metric perturbation due to the GW = $g_{ij} - \eta_{ij}$. It is given by the familiar plane-wave relation:

$$h_{ij}(t) = H_{ij} e^{(i(\bar{k}\cdot\bar{x} - \Omega t + \Phi_0))} \tag{57}$$

where $\bar{k}$ is the GW 3-wavevector = $\Omega\bar{n}$ and $x = -\bar{q}t$ (photon propagates from pulsar to Earth), and for phase offset (GW phase at Earth when $t = 0$) Φ_0. We can also write this as,

$$h_{ij}(t) = H_{ij} e^{(-i\Omega t(1+\bar{n}\cdot\bar{q}) + \Phi_0))} \tag{58}$$

The amplitude tensor H_{ij} is

$$H_{ij} = h_+ e^+_{ij}(\bar{n}, \psi) + h_\times e^\times_{ij}(\bar{n}, \psi) \tag{59}$$

where $h_{+,\times}$ are the constant amplitudes of the gravitational plane wave, and $e^{+,\times}_{ij}(\bar{n}, \psi)$ are the polarisation tensors which are uniquely defined by the principal axes of the GW. ψ is the polarisation angle of the GW.

Going forward,

Bringing this together, and adopting a trigonometric form of the equations, we can express the measurement equation as

4.1 Parameters of the model

We can also express the measurement equation generally as,
Kalman filtering works as follows
you can see that it maps nicel onto our state separation

5 METHODS

There are two potential methods we can take

(i) Let the states just be the frequencies $\bar{x} = (f)$. We have a linear equation for both the state evolution and the measurement matrix. Both of these depend on the parameters θ. Use the likelihood output by a standard Kalman filter to run a nested sampler and try to recover θ.

(ii) Let the states be the frequencies and all the unknown parameters $\bar{x} = (f, \bar{\theta}_{\rm GW}, \bar{\theta}_{\rm PSR}, \bar{\theta}_{\rm noise})$. Our state evolution is now linear but our measurement matrix is non-linear - a function of the states(parameters). Use a non-linear estimator such as a EKF/UKF to try to recover all of the states at once.

5.1 Method 1

In the case where the states are just the intrinsic pulsar frequencies, we can write the ODEs in matrix form as

$$d\bar{X} = \bar{A}\bar{X}dt + \bar{N}(t)dt + \bar{\Sigma}d\bar{B}(t) \quad (60)$$

where $\bar{A} = \mathrm{diag}(\gamma_1, \gamma_2, ...)$, $\bar{X} = \mathrm{diag}(f_1, f_2, ...)$, $\bar{N} = \mathrm{diag}(\gamma_1[a_1 + b_1 t] + b_1, ...)$ and we have let $a = f_{EM}(0)$, $b = \dot{f}_{\rm EM}(0)$.

6 DISCUSSION

7 CONCLUSION

PTAs are all about MSPs for small timing noise. Maybe we can get away with large timing noise, and use more pulsars?

Radiomenter noise. In theory well known but this could also be estimated via this approach

7.1 References

REFERENCES

Abbott B. P., et al., 2019, Physical Review X, 9, 031040
Abbott R., et al., 2021, Physical Review X, 11, 021053
Acernese F., et al., 2015, Classical and Quantum Gravity, 32, 024001
Amaro-Seoane P., et al., 2017, arXiv e-prints, p. arXiv:1702.00786
Antonelli M., Basu A., Haskell B., 2023, MNRAS, 520, 2813
Antoniadis J., et al., 2022, MNRAS, 510, 4873
Armano M., et al., 2019, arXiv e-prints, p. arXiv:1903.08924
Arzoumanian Z., et al., 2020, ApJ, 905, L34
Babak S., et al., 2016, MNRAS, 455, 1665
Caballero R. N., et al., 2016, MNRAS, 457, 4421
Cordes J. M., Shannon R. M., 2010, arXiv e-prints, p. arXiv:1010.3785
Ferdman R. D., et al., 2010, Classical and Quantum Gravity, 27, 084014
Hellings R. W., Downs G. S., 1983, ApJ, 265, L39
Hobbs G., Dai S., 2017, National Science Review, 4, 707
Hobbs G., Lyne A. G., Kramer M., 2010, MNRAS, 402, 1027
Kalman R. E., 1960, Journal of Basic Engineering, 82, 35
Kerr M., et al., 2020, Publ. Astron. Soc. Australia, 37, e020
LIGO Scientific Collaboration et al., 2015, Classical and Quantum Gravity, 32, 074001
Lam M. T., et al., 2019, ApJ, 872, 193
Lasky P. D., Melatos A., Ravi V., Hobbs G., 2015, MNRAS, 449, 3293
Melatos A., Link B., 2014, MNRAS, 437, 21
Melatos A., Peralta C., Wyithe J. S. B., 2008, ApJ, 672, 1103
Melatos A., O'Neill N. J., Meyers P. M., O'Leary J., 2021, MNRAS, 506, 3349
Meyers P. M., O'Neill N. J., Melatos A., Evans R. J., 2021, MNRAS, 506, 3349
Parthasarathy A., et al., 2021, MNRAS, 502, 407
Perera B. B. P., et al., 2019, MNRAS, 490, 4666
Sanidas S. A., Battye R. A., Stappers B. W., 2012, Phys. Rev. D, 85, 122003
Shannon R. M., Cordes J. M., 2010, ApJ, 725, 1607
Skilling J., 2006, Bayesian Analysis, 1, 833
Vargas A., Melatos A., 2023, TBD, 1, 1
Verbiest J. P. W., Osłowski S., Burke-Spolaor S., 2021, in , Handbook of Gravitational Wave Astronomy. p. 4, doi:10.1007/978-981-15-4702-7_4-1
Wang Y., 2015, in Journal of Physics Conference Series. p. 012019 (arXiv:1505.00402), doi:10.1088/1742-6596/610/1/012019
Xue X., et al., 2021, Phys. Rev. Lett., 127, 251303

This paper has been typeset from a TEX/LATEX file prepared by the author.