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Notes on identifiability of PTA problem

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Contents

1 Preamble

These notes collect some ideas around how to check if parameters of our PTA-GW state space model are identifiable.

We will deal with just 2 parameters ι and h_0.

2 General identifiability method

How do we tell if parameters are identifiable? Lets use the approach from Karlsson 2012 [1]

Take a state space model

$$\dot{x}(t) = f(x(t), u(t), \theta) \tag{1}$$

$$y(t) = g(x(t), u(t), \theta) \tag{2}$$

where state x has dimension n, θ has dimension d and y has dimension p.

Now define an extended Lie derivative operator along the vector field f

$$\mathcal{L}_f = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{i=0}^{i=\infty} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \tag{3}$$

[2]. We use the notation $\mathcal{L}_f^{(k)}$ to refer to the Lie operator applied k times. We also define a variable

$$\alpha^{(k)} = \mathcal{L}_f^{(k)} g(x, y, \theta) \tag{4}$$

Note that Karlsson 2012 refer to this as $y^{(j)}(0)$.

[1] https://www.sciencedirect.com/science/article/pii/S1474667015380745

[2] Is the vector field f the same as the state space function f? https://journals.plos.org/ploscompbiol/article?id=10.1371/journal.pcbi.1005153 suggests that is IS. We will proceed assuming that they are the same thing.

We now define a vector $\mathcal{Y}$ which contains $\alpha^{(k)}$, $k = 0, ..., n + d - 1$. We construct a Jacobian

$$J = \frac{\mathcal{Y}(x,\theta)}{\partial(x,\theta)} \tag{5}$$

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Rank test for structural identifiability: if J is full rank (see here) then the parameters are identifiable.

[3] See Equation 6 in https://www.sciencedirect.com/science/article/pii/S0025556412000922 for a nice visual example of what this Jacobian looks like.

3 Relation to PTA-GW model

Our model is

$$\dot{f}_{\rm p} = -\gamma f_{\rm p} + u(t) \tag{6}$$

where $u(t) = \gamma f_{\rm EM}(t) + \dot{f}_{\rm EM}(t)$. The measurement equation is

$$f_{\rm m}(t) = [1 - H(t,\theta)]\, f_{\rm p} \tag{7}$$

By comparison with Section 2:

- $x = f_{\rm p}$
- $y = f_{\rm m}$
- function $f(x) = -\gamma x + u(t)$
- function $g(x) = [1 - H(t,\theta)]\, x$
- $p = m = N_{\rm psr}$
- $d = 2$ (We are just dealing with h_0 and ι)

4 Calculating α

Lets get some values for $\alpha^{(k)}$ and see if there is a pattern.

$$\begin{aligned}
\alpha^{(1)} =& \mathcal{L}_f^{(1)} g(x,y,\theta) && (8)\\
=& \mathcal{L}_f \left[1 - H(t,\theta)\right] x && (9)\\
=& \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \left\{\left[1 - H(t,\theta)\right] x\right\} + \sum_{i=0}^{i=\infty} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \left\{\left[1 - H(t,\theta)\right] x\right\} && (10)
\end{aligned}$$