

Reframing Convolution: A Blend of Transformation and Operation

Abstract

Convolution, a foundational concept in signal processing, probability, and statistics, is traditionally treated as a mathematical operation. However, this paper argues that convolution is not merely an operation but a blend of transformation and operation. Like the Fourier transform, convolution generates an intermediate function in an augmented domain and integrates to project into the desired representation. By comparing the steps of convolution to those of the Fourier transform, this paper highlights their structural similarities. To illustrate the interplay between convolution and Fourier transforms, it also explains how convolution in the time domain becomes multiplication in the Fourier domain, specifically showing how the deduction term in convolution becomes addition in the Fourier exponential, and subsequently, a product of exponentials. This perspective reveals convolution's dual nature and its role as both a transformative and operational tool.

1. Introduction

Convolution appears ubiquitously in fields like signal processing, probability, and statistical modeling. It is often described as a smoothing or blending operation, but this characterization overlooks its deeper structure. Unlike pure operations such as addition or multiplication, convolution involves transformation-like steps: it creates a new function in an augmented domain and integrates to project the result into the target domain. This makes convolution a hybrid process—part transformation, part operation.

This paper reframes convolution as a blend of transformation and operation. It compares convolution to the Fourier transform, which is a pure transformation, and highlights their shared structural steps. Furthermore, it examines how convolution and Fourier transforms interact, demonstrating their complementary nature.

2. Convolution: A Blend of Transformation and Operation

2.1 Traditional Definition of Convolution

Mathematically, the convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$$

This definition suggests convolution is an operation that combines two functions through shifting, multiplication, and integration. However, this view obscures its underlying transformative structure. To understand this, consider its key steps:

1. Generating a New Function:

Convolution generates an intermediate function, $f(\tau)g(t - \tau)$, by multiplying one function with a shifted version of the other. This intermediate function depends on both τ and t , effectively operating in an augmented domain t_1 and t_2 (where $t = t_1 + t_2$).

2. Projecting Into the Output Domain:

Integration collapses the intermediate function over τ into the desired output domain t .

These steps closely parallel the mechanics of a transformation, such as the Fourier transform, as discussed in Section 3.

2.2 Convolution's Dual Nature

Convolution is a hybrid process, combining:

- **Transformative Aspects:**

Like the Fourier transform, convolution involves generating a new function in an augmented domain before collapsing it into the target domain.

- **Operational Aspects:**

Unlike pure transformations, convolution involves a form of multiplication during the intermediate step. This makes it an operation as well, rather than a purely reversible transform.

Notably, convolution is not invertible, distinguishing it from pure transformations such as the Fourier or Laplace transforms.

3. Steps of Convolution and Comparison to Fourier Transform

Convolution shares structural similarities with the Fourier transform, as both follow a two-step process:

3.1 Generate a New Function

- **Convolution:**

The intermediate function is generated by multiplying $f(\tau)$ and a shifted version of $g(t)$:

$$f(\tau)g(t - \tau)$$

This function depends on both the original variable τ and the output variable t .

- **Fourier Transform:**

The intermediate function is generated by multiplying $f(t)$ with a complex exponential $e^{-i\omega t}$:

$$f(t)e^{-i\omega t}$$

This function depends on both the original time variable t and the frequency variable ω .

3.2 Integrate Over the Original Variable

- **Convolution:**

The intermediate function $f(\tau)g(t - \tau)$ is integrated over τ to produce a function of t :

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$$

- **Fourier Transform:**

The intermediate function $f(t)e^{-i\omega t}$ is integrated over t to produce a function of ω :

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$$

This structural similarity highlights convolution's transformative nature, even though its output remains tied to the original domain.

4. Convolution and Fourier Transform Interaction

4.1 Convolution in the Fourier Domain

Convolution in the time domain corresponds to multiplication in the Fourier domain. Mathematically:

$$\mathcal{F}[(f * g)(t)] = \mathcal{F}[f](\omega) \cdot \mathcal{F}[g](\omega)$$

Derivation:

Starting with the definition of convolution:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$$

Taking the Fourier transform:

$$\mathcal{F}[(f * g)(t)] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \right) e^{-i\omega t} dt$$

Swapping the order of integration:

$$\mathcal{F}[(f * g)(t)] = \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} g(t - \tau) e^{-i\omega t} dt \right) d\tau$$

Change variables in the inner integral, setting $u = t - \tau$ (so $t = u + \tau$ and $dt = du$):

$$\int_{-\infty}^{\infty} g(t - \tau) e^{-i\omega t} dt = e^{-i\omega\tau} \int_{-\infty}^{\infty} g(u) e^{-i\omega u} du$$

Here, **the deduction term** $t - \tau$ in convolution becomes **addition** $e^{-i\omega t} = e^{-i\omega u} e^{-i\omega\tau}$ due to the linearity of the exponential in the Fourier transform. This addition transforms into a **product of exponentials**.

Recognize the Fourier transform of $g(t)$:

$$\mathcal{F}[g](\omega) = \int_{-\infty}^{\infty} g(u) e^{-i\omega u} du$$

Substitute back:

$$\mathcal{F}[(f * g)(t)] = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \mathcal{F}[g](\omega) d\tau$$

Recognize the Fourier transform of $f(t)$:

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} d\tau$$

Thus:

$$\mathcal{F}[(f * g)(t)] = \mathcal{F}[f](\omega) \cdot \mathcal{F}[g](\omega)$$

5. Conclusion

Convolution is not merely an operation; it is a blend of transformation and operation. The deduction term in convolution ($t - \tau$) transforms into addition in the Fourier exponential ($e^{-i\omega t}$), and this addition becomes a product of exponentials. This process highlights convolution's dual nature. By interacting with Fourier transforms, convolution simplifies into multiplication in the frequency domain. This complementary relationship emphasizes its significance across mathematics, statistics, and engineering.