

# Rethinking How Taylor Series Should Be Taught: A New Perspective

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## Abstract

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The Taylor series is a cornerstone of calculus, providing a powerful tool to approximate functions as infinite sums of their derivatives. However, its traditional teaching often obscures the deeper conceptual insights that underpin its elegance. This paper proposes an alternative approach to teaching the Taylor series, emphasizing its integral-based interpretation and its foundation in the finite-range equation:

$$f(b) - f(a) = \sum_a^b \frac{df}{dx} \cdot dx.$$

Furthermore, it draws an analogy to the calculation of distance from acceleration using integrations, bridging the gap between physical intuition and mathematical abstraction. By reframing the Taylor series in this way, students can gain a more intuitive and conceptual understanding of its structure and mechanics.

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## Connecting Taylor Series to Finite-Range Calculations

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## 1. Fundamental Equation of Finite Changes

The Taylor series fundamentally builds on the equation:

$$f(b) - f(a) = \sum_a^b \frac{df}{dx} \cdot dx,$$

which expresses the total change in a function  $f(x)$  as the sum of infinitesimal contributions  $\frac{df}{dx} \cdot dx$  over the finite range  $[a, b]$ .

This equation serves as the foundation for the Taylor series by:

1. Accumulating contributions term by term.
2. Extending the concept to higher-order derivatives and their integrals.

### Interpretation:

- $f(b) - f(a)$ : The total change in  $f(x)$  over the interval  $[a, b]$ .
- $\frac{df}{dx} \cdot dx$ : The contribution to  $f(x)$  from the first derivative over a small interval  $dx$ .

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## 2. Extending to Higher-Order Terms

The Taylor series builds on this idea by recursively including contributions from higher-order derivatives:

### 1. First-Order Contribution:

$$f(b) - f(a) = \int_a^b \frac{df}{dx} dx.$$

This corresponds to the linear term in the Taylor series:  $f(a) + f'(a)(b - a)$ .

## 2. Second-Order Contribution:

The second derivative contributes through nested integrals:

$$f(b) - f(a) = \int_a^b \frac{df}{dx} dx + \int_a^b \int_a^{x_1} \frac{d^2 f}{dx^2} dx_2 dx_1.$$

This matches  $\frac{f''(a)}{2!}(b - a)^2$ , the quadratic term.

## 3. Higher-Order Contributions:

The  $n$ -th derivative contributes through  $n$ -fold nested integrals:

$$f(b) - f(a) = \sum_{n=1}^{\infty} \int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-1}} \frac{d^n f}{dx^n} dx_n \cdots dx_1.$$

This explicitly connects the Taylor series to repeated integration.

## 3. The Role of Factorials

The factorial  $n!$  arises naturally as a scaling factor in the Taylor series due to the repeated integration over the same variable range  $[a, b]$ . For example:

- A single integration contributes a factor of  $(b - a)$ .

- A double integration contributes  $\frac{1}{2!}(b - a)^2$ , and so on.

This scaling ensures that each higher-order term is appropriately weighted.

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## Connecting Taylor Series to Distance Calculations

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To build intuition, consider how the position  $s(t)$  of an object is calculated when the initial position, velocity, and acceleration are known. This process mirrors the Taylor series by accumulating contributions term by term.

### Step 1: Initial Distance

The initial position  $s(0)$  is constant and does not depend on time. Its contribution to the total position is:

$$s(t) = s(0) + \dots$$

### Step 2: Contribution from Velocity

Velocity  $v(0)$  is the first derivative of position. To calculate its contribution to position, integrate  $v(0)$  over time:

$$\int_0^t v(0) dt = v(0)t.$$

Adding this to the initial position, we have:

$$s(t) = s(0) + v(0)t + \dots$$

### Step 3: Contribution from Acceleration

Acceleration  $a(0)$  is the second derivative of position. To calculate its contribution, apply a double integral to  $a(0)$ :

1. Integrate  $a(0)$  once to calculate the velocity contribution:

$$\int_0^t a(0) dt = a(0)t.$$

2. Integrate the result again to calculate the position contribution:

$$\int_0^t \int_0^{t_1} a(0) dt_2 dt_1 = \frac{1}{2}a(0)t^2.$$

Adding this to the previous terms, we get the final expression for position:

$$s(t) = s(0) + v(0)t + \frac{1}{2}a(0)t^2.$$

This process mirrors the accumulation of higher-order terms in the Taylor series.

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## Relating to the Taylor Series

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Using the integral-based approach, the Taylor series for  $f(x)$  is expressed as:

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.$$

Each term  $\frac{f^{(n)}(a)}{n!} (x - a)^n$  represents the contribution of the  $n$ -th derivative to the total function. Using the relationship:

$$(x - a)^n = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} 1 \, dx_n \, dx_{n-1} \cdots dx_1,$$

the Taylor series can be reframed as:

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} 1 \, dx_n \, dx_{n-1} \cdots dx_1.$$

This formulation highlights the recursive nature of integration in the Taylor series.

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## Advantages of the Integral Perspective

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### 1. Foundation in Finite-Range Changes:

- The equation  $f(b) - f(a) = \sum_a^b \frac{df}{dx} \cdot dx$  provides a clear starting point for understanding the Taylor series.

### 2. Bridging Differentiation and Integration:

- This approach highlights the duality between differentiation and integration, showing how the Taylor series builds higher powers of  $x$  through accumulation.

### 3. Connecting Mathematics to Physics:

- The analogy to distance calculations grounds abstract mathematical ideas in tangible, real-world scenarios.

### 4. Highlighting the Role of Constants:

- By factoring  $f^{(n)}(a)$  out of the integral, students can understand it as a constant derived from the function's behavior at  $x = a$ .

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## Conclusion

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Reframing the Taylor series as a summation of higher-order integrals provides students with a deeper, more intuitive understanding of its structure and mechanics. Anchoring the series in the finite-range equation  $f(b) - f(a) = \sum_a^b \frac{df}{dx} \cdot dx$  offers a unified perspective that connects differentiation, integration, and physical intuition. By adopting this approach, educators can demystify the Taylor series and inspire greater appreciation for its elegance and applications.