

Demystifying Matrix Transformations: How Singular Matrices Define Non-Singular Behavior

Abstract

This paper revisits the behavior of **non-singular matrices** through their relationship to **two corresponding singular matrices**. Contrary to common understanding, we show that it is not the eigenvalues of the singular matrices that dictate the dynamics of the original matrix but rather their **shared eigenvectors and the interplay between the column and null spaces**. By collapsing eigenvalues and studying the resulting singular matrices, we uncover how the geometry of eigenvectors underlies scaling, sensitivity, and forward/inverse transformations. Our findings highlight that the **null space of one singular matrix aligns with the column space of the other**, and the shared eigenvector structure is the key to understanding how non-singular matrices transform vectors.

We further clarify the distinction between **forward and inverse transformations** and show how understanding singular matrices explains numerical stability and regularization.

1. Introduction

Matrix transformations are central to linear algebra, with applications in physics, engineering, and data science. However, the nuanced role of **singular matrices** in the behavior of **non-singular matrices** is often overlooked. Traditional explanations focus on eigenvalues and their contributions to scaling, but our findings demonstrate that **eigenvectors and the shared geometry between singular matrices** provide the true explanation.

This paper explores how **non-singular matrices are governed by their corresponding singular matrices** and why understanding the shared eigenvectors is crucial to interpreting their behavior.

2. The Role of Singular Matrices and Shared Eigenvectors

A matrix A is **non-singular** if its determinant is nonzero. However, its transformation behavior is fundamentally tied to **two corresponding singular matrices** derived by collapsing each eigenvalue to zero. For example, given a non-singular matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the singular matrices are constructed by collapsing its eigenvalues. Consider:

1. **Singular Matrix 1:** Collapse the smaller eigenvalue λ_{\min} to zero.

$$A_{\text{singular}_1} = A - \lambda_{\min} I$$

2. **Singular Matrix 2:** Collapse the larger eigenvalue λ_{\max} to zero.

$$A_{\text{singular}_2} = A - \lambda_{\max} I$$

2.1 Shared Eigenvectors

We find that **both singular matrices share the same eigenvectors**, which directly influence the dynamics of the non-singular matrix:

- **Eigenvectors of the singular matrices define the primary transformation directions of the non-singular matrix.**
- **The null space direction of one singular matrix aligns with the eigenvector corresponding to the nonzero eigenvalue of the other.**

This alignment is not coincidental—it reveals the **hidden geometric structure** governing matrix transformations.

3. Forward and Inverse Transformations

The behaviors of forward and inverse transformations are distinct, and understanding their differences is key to interpreting the effects of singular matrices.

3.1 Forward Transformation

In the **forward transformation**:

$$A\mathbf{x} = \mathbf{b}$$

The input vector \mathbf{x} is decomposed into components aligned with the **null space and column space**:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

where:

- \mathbf{v}_1 is the eigenvector corresponding to the **zero eigenvalue** (null space direction).
- \mathbf{v}_2 is the eigenvector corresponding to the **nonzero eigenvalue** (column space direction).

The forward transformation maps inputs by discarding the null space component and stretching the column space component along the eigenvector \mathbf{v}_2 .

3.2 Inverse Transformation

In the **inverse transformation** for non-singular matrices:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

The inverse transformation **uniquely resolves inputs for given outputs**, but in the case of singular matrices, the inverse does not exist. Instead, the solution set forms a **line or plane of possible input vectors** due to the **null space direction**.

4. Perturbed Non-Singular Matrices

When a small perturbation ϵ is applied to a singular matrix, it becomes non-singular:

$$A' = A_{\text{singular}} + \epsilon I$$

The eigenvalues of the perturbed matrix become:

$$\lambda'_i = \lambda_i + \epsilon$$

This perturbation reveals why the **nonzero eigenvalue of the singular matrix does not directly dictate the dynamics**. Instead, **the perturbation lifts the zero eigenvalue, but the transformation directions remain defined by the shared eigenvectors**.

5. Mathematical Proof of Eigenvector Dominance

We demonstrate that the **shared eigenvectors of the singular matrices dominate the dynamics** by considering the decomposition of the perturbed matrix:

$$A' \mathbf{x} = \lambda'_1 c_1 \mathbf{v}_1 + \lambda_2 c_2 \mathbf{v}_2$$

The contributions of each term depend on the perturbation ϵ , but as long as ϵ is small, the eigenvector structure of the original singular matrices **continues to dominate**.

6. Numerical Examples

Example Matrix

Let's consider the non-singular matrix:

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$$

We compute its eigenvalues and eigenvectors and construct the corresponding singular matrices:

1. **Singular Matrix 1:** Collapse λ_{\min} to zero.

$$A_{\text{singular}_1} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

2. **Singular Matrix 2:** Collapse λ_{\max} to zero.

$$A_{\text{singular}_2} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

Both singular matrices share the same eigenvectors, confirming the dominance of their geometric structure.

7. Implications

1. Numerical Stability:

Near-singular matrices are prone to instability because perturbations of small eigenvalues lead to large scaling errors in the inverse transformation.

2. Regularization Techniques:

Regularization methods stabilize solutions by artificially increasing the small eigenvalue.

3. Control Systems:

Understanding how singular matrices dictate the behavior of non-singular matrices is key to designing stable control systems.

8. Conclusion

Our findings emphasize that **the eigenvalues of the singular matrices are less important than their shared eigenvectors**. These eigenvectors define the transformation directions and control the mapping of inputs to outputs. The **null space of one singular matrix aligns with the column space of the other**, revealing a hidden geometric relationship.

In non-singular matrices, the behavior is initially dictated by this shared structure. Only when perturbations become significant do the eigenvalues begin to dominate. Recognizing this relationship is crucial for solving numerical stability problems and understanding matrix dynamics in practical applications.