

A New Perspective on Statistics: Convolution as the Fundamental Principle

Abstract

Statistics, at its core, is the study of uncertainty and data patterns. Traditional approaches often emphasize combinatorial probability and continuous distributions as separate concepts. We propose a unifying framework where **convolution**—the mathematical operation that governs how independent random variables combine—forms the foundation of statistical thinking. By recognizing convolution as the driving force behind many statistical distributions, we gain a deeper understanding of probability theory, inferential statistics, and data science applications.

1. Introduction

Statistics is traditionally introduced through descriptive measures and probability theory. However, we argue that the entire discipline can be **reconstructed** using a single fundamental mathematical operation: **convolution**. Convolution is the process by which the distributions of independent random variables are

combined to produce new distributions. This operation underlies both discrete and continuous probability distributions, forming the basis of inferential statistics, machine learning, and Bayesian inference.

2. Convolution: The Mathematics of Combining Distributions

Convolution governs how independent random variables combine. If X and Y are independent random variables, their sum $Z = X + Y$ has a probability distribution given by the convolution of their individual distributions. For discrete random variables, the convolution is:

$$P(Z = z) = (f * g)(z) = \sum_x f(x) \cdot g(z - x),$$

where f and g are the PMFs of X and Y , respectively. For continuous random variables, the convolution is:

$$f_Z(z) = (f * g)(z) = \int_{-\infty}^{\infty} f(x)g(z - x) dx,$$

where f and g are the PDFs of X and Y , respectively. This operation is the **fundamental principle** behind many statistical distributions and processes.

3. Convolution in Discrete Distributions

3.1 The Binomial Distribution as Repeated Convolution

The binomial distribution arises from the sum of n independent Bernoulli random variables. Each Bernoulli random variable has a PMF:

$$f(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0. \end{cases}$$

The PMF of the sum $X = X_1 + X_2 + \cdots + X_n$ is the result of the n -fold discrete convolution of the Bernoulli PMF with itself:

$$P(X = k) = (f * f * \cdots * f)(k),$$

where the convolution is performed n times. This repeated convolution operation produces the binomial PMF:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

Here, the binomial coefficient $\binom{n}{k}$ arises naturally from the convolution process, as it counts the number of ways to arrange k successes in n trials. Thus, the binomial distribution is fundamentally a result of **repeated discrete convolution**.

3.2 The Poisson Distribution as a Limit of Convolution

The Poisson distribution can be derived as a limit of the binomial distribution when $n \rightarrow \infty$ and $p \rightarrow 0$, with $\lambda = np$ held constant. This limiting process is a consequence of the convolution operations underlying the binomial distribution. The Poisson PMF:

$$P(N = k) = \frac{\overset{\text{stats_approach}}{\lambda^k e^{-\lambda}}}{k!}$$

reflects the accumulation of many rare events, a process inherently governed by convolution.

4. Convolution in Continuous Distributions

4.1 The Normal Distribution from Convolution

By the Central Limit Theorem, the sum of a large number of independent and identically distributed (i.i.d.) random variables converges to a normal distribution:

$$X_1 + X_2 + \cdots + X_n \approx \mathcal{N}(\mu, \sigma^2).$$

This is a direct consequence of repeated convolution operations. Each additional variable convolves with the existing sum, smoothing the distribution and driving it toward normality. Mathematically, this can be expressed as:

$$f_Z(z) = (f * f * \cdots * f)(z),$$

where f is the PDF of the individual random variables, and the convolution is performed n times.

4.2 The Gamma Distribution as Repeated Convolution

The gamma distribution models the waiting time for the k -th event in a Poisson process. It arises from the sum of k independent exponential random variables, each representing the waiting time between events. The convolution of exponential distributions produces the gamma PDF:

$$f_{T_k}(t) = (f * f * \dots * f)(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!},$$

where f is the PDF of the exponential distribution, and the convolution is performed k times. This demonstrates how convolution generates distributions that describe cumulative processes.

5. Convolution in Bayesian Inference

Bayesian inference relies on updating prior beliefs with new data. This process can be viewed as a convolution of prior and likelihood distributions to produce the posterior distribution. For example, when the prior and likelihood are conjugate distributions (e.g., beta-binomial or gamma-Poisson), the posterior distribution is obtained through a convolution-like operation:

$$\text{Posterior} \propto (\text{Prior} * \text{Likelihood}).$$

6. Unified View of Statistics

Recognizing **convolution** as the fundamental principle of statistics allows us to:

- **Derive classical probability distributions systematically.**
- **Explain the Central Limit Theorem and its applications.**
- **Model real-world data using a unified framework.**

This perspective connects discrete and continuous probability theory, offering a more intuitive and mathematically rigorous approach to learning statistics.

7. Conclusion

We propose that statistics should be taught and understood through the lens of **convolution**, the mathematical operation that governs how distributions combine. This shift in perspective aligns statistics more closely with raw mathematics, making it not only more intuitive but also more powerful in applications such as Bayesian inference, deep learning, and stochastic modeling. By emphasizing convolution as the driving force behind statistical distributions, we provide a deeper and more unified understanding of statistical theory.