Utilization of Reductions and Abstraction Elimination in Typed Genetic Programming

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ABSTRACT

Lambda calculus representation of programs offers a more expressive alternative to traditional S-expressions. In this paper we discuss advantages of this representation coming from use of reductions (beta and eta) and how to overcome disadvantages caused by variables occurring in the programs by use of the abstraction elimination algorithm. We discuss the role of those reductions in the process of generating initial population and compare several crossover approaches including novel approach to crossover operator based both on reductions and abstraction elimination. The design goal of this operator is to turn the disadvantage of abstraction elimination - possibly quadratic increase of program size - into a virtue; our approach leads to more crossover points. At the same time, utilization of reductions provides offspring of small sizes.

Categories and Subject Descriptors

H.4 [**TODO**]: Miscellaneous; D.2.8 [**Software Engineering**]: Metrics—*complexity measures, performance measures*

General Terms

TODO

Keywords

TODO, ToDo, todo

1. PLÁN BOJE

1.1 Jak to celý pojmout?

Pokud chceme dělat GP nad stromama větší vyjadřovací síly než maj klasický S-výrazy, tak máme v záse dvě přirozený možnosti pro obecný řešení: buď to dělat celý v (polymorfních) kombinátorech od začátku (a vyhnout se tak proměnejm a lambda abstrakcím), nebo to skusit v lambda kalkulu a nejak se vypořádat s proměnejma a lambdama. Motivace pro práci s lambda termama je, že za nima košatá teorie, která například popisuje redukce lambda termů (ty zajišt'ujou jak zmenšení samotných stromu, tak to, že prohledávací prostor se zmenší, díky tomu, že se různé stromy redukují na ten samej).

Jak se vypořádat s proměnejma a lambdama? (aka jak křížit)

- Po vy generování převod do kombinátorů pomocí eliminace abstrakcí (tak jak se to dělá v diplomce). To ale působí v něčem neohrabaně, když to porovnáme s generováním přímo v kombinatorech, když vezmeme v potaz to že eliminace abstrakcí má za důsledek až kvadratickej nárůst stromu - čili to co sme nahnali na redukcích stratíme eliminací.

- Jedince držíme jako redukovane malinké-kompaktní-a-elegantní lambda termy. Ve chvíly kdy křížíme provedeme na obou rodičích eliminaci abstrakcí (která ubere proměny a lambdy a namísto toho tam dá kombinátory S,K,I případně i další při fikanějších eliminacích), tím nám sice narostou, ale my z toho máme jedine radost, protože tím se nám zvýšil počet míst ke křížení (což se ukazuje jako dobráv věc, viz s-expr reprezentace vs @-tree reprezentace lambda termů). Po skřížení se vložený kombinátory nahradí odpovídajícim lambda termem (tzn např všude kde je K dám (x y . x) atd) výslednej term zredukuju a dostávam zase malinké-kompaktní-a-elegantní dítě. Nevýhoda toho zahrnout do článku i tohle je v tom, že k tomu nemám ještě žádný pokusy - ale k tomu zbytku mám upřímě v zato taky dost ubohý pokysy, takže toho bych se asi nebál. Většinu potřebnýho kodu bych k tomu ale měl už mít víceméně hotovou, takže pokud by se to na něčem nezaseklo, tak myslim že je realný udělat i pokus do toho dvacátýho. Zvlášt' přitažlivý mi tohle křížení příde i kvůli tomu, že v přírodě se taky rozbalujou a zabaloujou chromozomi při meioze/mitoze.

1.2 Osnova

- Jak reprezentovat stromy programu pro GP?
  - Reprezentace v klasickym kozovy je S-expression.
  - Nebo mužem používat kombinátory jako Briggs a O'Neil (to si myslim je jakoby hlavní konkurence, vuči který by to chtělo obhájit)

- Lambda termy a jejich awesomeness

- **Povídaní o redukcích**
- Povídání o lnf
    - že lnf je přirozený rozšíření s-exprešnu do lambda kalkulu vlastnosti termu v lnf
        - proč je eta redukovat (eta redukcí lnf dostanem beta-eta-nf a že nemusíme beta redukovat pač se to tim nerozbyje)
- Generování
    - Generování gramatikou
    - Gramatika pro lnf
    - (?) Inhabitation trees jako intuitivní model takovýhodle lnf generování
- Problémy s proměnýma a jak je řešit.
    - Křížit lambda termy i s proměnejma a abstrakcema, problémy řešit když nastanou (pomluvit a odsoudit)
    - Zmenšit prostor termu (tim že někerý nejsme schopný vygenerovat) s kterým operujeme tak aby křížení už nebyl problém (to dělá Yu - v těle lambda fce dovoluje jen použití proměnných z její hlavy)
    - Převod hned po vygenerování
    - Převod až při křížení
        - eliminace abstrakcí
        - skřížim
        - vložený kombinátory nahradim odpovídajícím termem
        - celý to redukuju
- Pokusy (?!)
- Závěr

2. INTRODUCTION

Genetic programming (GP) represents an efficient method for automatic generating of programs by means of evolutionary techniques [4, 5]. Early attempts to enhance the GP approach with the concept of types include the seminal work [6] where the ideas from Ada programming language were used to define a so-called strongly typed GP. Use of types naturally opens door to enriching S-expressions, the traditional GP representation of individuals, with concepts from lambda calculus, which is simple yet powerful functional mathematical and programming language extensively used in type theory. Such attempts has shown to be successful [8].

The key issue in the lambda calculus approach to enrich GP with types is the method of individual generation. During the expansion phase the set of unfinished terms can be browsed with respect to various search strategies. Our approach to this problem aims to utilize the full arsenal given by the simply typed lambda calculus. Thus, the natural idea is to employ an exhaustive systematic search. On the other hand, if we were to mimic the standard GP approach, a quite arbitrary yet common and successful ramped half-and-half generating heuristic [7] should probably be used. These two search methods in fact represent boundaries between which we will try to position our parameterized solution that allows us to take advantage of both strategies. This design goal also differentiate our approach from the three state of the art proposals for typed GP known to us that are discussed in the following section. Our proposed *geometrical search strategy* described in this paper is such a successful hybrid mixture of random and systematic exhaustive search. Experiments show that it is also very efficient dealing with one of the traditional GP scarecrows - the bloat problem.

The rest of the paper is organized as follows: The next section briefly discusses related work in the field of typed GP, while section 4 introduces necessary notions. Main original results about search strategies in individual generating are described in section 5. Section 6 presents results of our method on three well-known tasks, and the paper is concluded by section 7.

3. RELATED WORK

Yu presents a GP system utilizing polymorphic higher-order functions[1] and lambda abstractions [8]. Important point of interest in this work is use of `foldr` function as a tool for *implicit recursion*, i.e. recursion without explicit recursive calls. The terminal set for constructing lambda abstraction subtrees is limited to use only constants and variables of that particular lambda abstraction, i.e., outer variables are not allowed to be used as terminals in this work. This is significant difference from our approach since we permit all well-typed normalized λ-terms. From this difference also comes different crossover operation. We focus more on term generating process; their term generation is performed in a similar way as the standard one, whereas our term generation also tries to utilize techniques of systematic enumeration.

Briggs and O'Neill present technique utilizing typed GP with combinators [3]. The difference between approach presented in this work and our approach is that in this work terms are generated straight from *library* of combinators and no lambda abstractions are used. They are using more general polymorphic type system than us – the Hindley–Milner type system. They also discuss the properties of exhaustive enumeration of terms and compare it with GP search. They also present interesting concept of *Generalized genetic operator* based on term generation.

Binard and Felty use even stronger type system (*System F*) [2]. But with increasing power of the type system comes increasing difficulty of term generation. For this reason evolution in this work takes interesting and nonstandard shape (fitness is associated with *genes* which are evolved together with *species* which together participate in creation of individuals). This differs from our approach, which tries to be generalization of the standard GP[4].

In contrast with above mentioned works our approach uses very simple type system (simply typed lambda calculus) and concentrates on process of generation able to generate all possible well-typed normalized lambda terms. In order to do so we use technique based on *inhabitation machines* described by Barendregt [1].

4. PRELIMINARIES

[1]Higher-order function is a function taking another function as input parameter.

In this section, several notions necessary to build a typed GP based on lambda calculus are introduced. First, let us describe a programming language, in which the GP algorithm generates individual programs — the so called λ-terms.

DEFINITION 1. *Let V be infinite countable set of variable names. Let C be set of constant names, $V \cap C = \emptyset$. Then Λ is set of λ-terms defined inductively as follows.*

$$\begin{aligned} x \in V \cup C &\Rightarrow x \in \Lambda \\ M, N \in \Lambda &\Rightarrow (M\ N) \in \Lambda && \textit{(Function application)} \\ x \in V, M \in \Lambda &\Rightarrow (\lambda\, x \,.\, M) \in \Lambda && \textit{(λ-abstraction)} \end{aligned}$$

Function application and λ-*abstraction* are concepts well known from common programming languages. For example in JavaScript $(M\ N)$ translates to expression M`(`N`)` and $(\lambda\, x \,.\, M)$ translates to expression `function(`x`){return` M`;}`. In other words, the function application corresponds to the act of supplying a function with an argument, and the λ-abstraction is equivalent to *anonymous function*[2].

For better readability, $M_1\ M_2\ M_3\ \ldots\ M_n$ is an abbreviation for $(\ldots((M_1\ M_2)\ M_3)\ \ldots\ M_n)$ and $\lambda\, x_1 x_2 \ldots x_n \,.\, M$ for $(\lambda\, x_1 \,.\, (\lambda\, x_2 \,.\, \ldots (\lambda\, x_n \,.\, M) \ldots))$.

4.1 β-reduction

In order to perform computation there must be some mechanism for term evaluation. In λ-calculus there is β-reduction for this reason.

A term of a form $(\lambda\, x \,.\, M)N$ is called *β-redex.* A β-redex can be β-reduced to term $M[x := N]$. This fact is written as *relation* $\rightarrow_\beta$ of those two terms:

$$(\lambda\, x \,.\, M)N \rightarrow_\beta M[x := N] \tag{1}$$

It is also possible to reduce *subterm β-redexes* which can be formally stated as:

$$\begin{aligned} P \rightarrow_\beta Q &\Rightarrow (R\ P) \rightarrow_\beta (R\ Q) \\ P \rightarrow_\beta Q &\Rightarrow (P\ R) \rightarrow_\beta (Q\ R) \\ P \rightarrow_\beta Q &\Rightarrow \lambda\, x \,.\, P \rightarrow_\beta \lambda\, x \,.\, Q \end{aligned}$$

In other words, β-reduction is the process of insertion of arguments supplied to a function into its body.

Another useful relations are $\twoheadrightarrow_\beta$ and $=_\beta$ defined as follows.

1. (a) $M \twoheadrightarrow_\beta M$
   (b) $M \rightarrow_\beta N \Rightarrow M \twoheadrightarrow_\beta N$
   (c) $M \twoheadrightarrow_\beta N, N \twoheadrightarrow_\beta L \Rightarrow M \twoheadrightarrow_\beta L$
2. (a) $M \twoheadrightarrow_\beta N \Rightarrow M =_\beta N$
   (b) $M =_\beta N \Rightarrow N =_\beta M$
   (c) $M =_\beta N, N =_\beta L \Rightarrow M =_\beta L$

We read those relations as follows.

1. $M \twoheadrightarrow_\beta N$ — "M β-reduces to N."
2. $M \rightarrow_\beta N$ — "M β-reduces to N in one step."
3. $M =_\beta N$ — "M is β-convertible to N."

[2]Apart from JavaScript, anonymous functions are common e.g. in Python and Ruby, they were recently introduced to C++, and they are expected to be supported in Java 8.

4.2 η-reduction

Similarly as for β-reduction we can define η-reduction except that instead of 1 we use:

$$(\lambda\, x \,.\, (M\ x)) \rightarrow_\eta M \qquad \textbf{if } x \notin FV(M)$$

Analogically, a term of a form $(\lambda\, x \,.\, (M\ x))$ is called *η-redex.*

Relation $\rightarrow_{\beta\eta}\ =\ \rightarrow_\beta \cup \rightarrow_\eta$. (Relation $R = \{\ (a, b) \mid a\ R\ b\ \}$.)

Similarly as for $\twoheadrightarrow_\beta$ and $=_\beta$ we can define relations $\twoheadrightarrow_\eta$, $=_\eta$, $\twoheadrightarrow_{\beta\eta}$ and $=_{\beta\eta}$.

4.3 η^{-1}-reduction

η^{-1}-reduction (also called η-expansion) is the reduction converse to η-reduction. Again it may be obtained by replacing 1, now with:

$$M \rightarrow_{\eta^{-1}} (\lambda\, x \,.\, (M\ x)) \qquad \textbf{if } x \notin FV(M)$$

4.4 Normal forms

1. A λ-term is a *β-normal form* (*β-nf*) if it does not have a β-redex as subterm.
2. A λ-term M *has* a *β-nf* if $M =_\beta N$ and N is a *β-nf*.

A normal form may be thought of as a result of a term evaluation.

Similarly we can define *η-nf* and *$\beta\eta$-nf*.

4.5 Types etc

A λ-term as described above corresponds to a program expression with no type information included. Now we will describe *types* (or *type terms*).

DEFINITION 2. *Let A be set of atomic type names. Then $\mathbb{T}$ is set of types inductively defined as follows.*

$$\begin{aligned} \alpha \in A &\Rightarrow \alpha \in \mathbb{T} \\ \sigma, \tau \in \mathbb{T} &\Rightarrow (\sigma \rightarrow \tau) \in \mathbb{T} \end{aligned}$$

Type $\sigma \rightarrow \tau$ is type for functions taking as input something of a type σ and returning as output something of a type τ. $\tau_1 \rightarrow \tau_2 \rightarrow \ldots \rightarrow \tau_n$ is an abbreviation for $\tau_1 \rightarrow (\tau_2 \rightarrow (\ldots \rightarrow (\tau_{n-1} \rightarrow \tau_n) \ldots))$. The system called *simply typed λ-calculus* is now easily obtained by combining the previously defined *λ-terms* and *types* together.

DEFINITION 3.

1. *Let Λ be set of λ-terms. Let $\mathbb{T}$ be set of types. A statement $M : \sigma$ is a pair $(M, \sigma) \in \Lambda \times \mathbb{T}$. Statement $M : \sigma$ is vocalized as "M has type σ". The term M is called the subject of the statement $M : \sigma$.*
2. *A declaration is a statement $x : \sigma$ where $x \in V \cup C$.*
3. *A context is set of declarations with distinct variables as subjects.*

Context is a basic type theoretic concept suitable as a typed alternative for terminal and function set in standard GP. Notation $\Gamma, x : \sigma$ denotes $\Gamma \cup \{(x : \sigma)\}$ such that Γ does not contain any declaration with x as subject. We also write $x : \sigma \in \Gamma$ instead of $(x, \sigma) \in \Gamma$.

Definition 4. *A statement $M : \sigma$ is derivable from a context Γ (notation $\Gamma \vdash M : \sigma$) if it can be produced by the following rules.*

$$\begin{aligned} x : \sigma \in \Gamma \;&\Rightarrow\; \Gamma \vdash x : \sigma \\ \Gamma \vdash M : \sigma \to \tau \;,\; \Gamma \vdash N : \sigma \;&\Rightarrow\; \Gamma \vdash (M\ N) : \tau \\ \Gamma, x : \sigma \vdash M : \tau \;&\Rightarrow\; \Gamma \vdash (\lambda\, x \,.\, M) : \sigma \to \tau \end{aligned}$$

Přeformulovat aby se to hodilo do tohodle kontextu.... Our goal in term generation is to produce terms M for a given pair $\langle \tau ; \Gamma \rangle$ such that for each M is $\Gamma \vdash M : \tau$.

4.6 Long normal form

Definition 5. *Let $\Gamma \vdash M : \sigma$ where $\sigma = \tau_1 \to \ldots \to \tau_n \to \alpha, n \geq 0$.*

1. *Then M is in long normal form (lnf) if following conditions are satisfied.*
   - (a) *M is term of the form $\lambda\, x_1 \ldots x_n \,.\, f\ M_1\ \ldots\ M_m$ (specially for $n = 0$, M is term of the form f).*
   - (b) *Each M_i is in lnf.*
2. *M has a lnf if $M =_{\beta\eta} N$ and N is in lnf.*

As is shown in [1], *lnf* has following nice properties.

Proposition 1. *If M has a β-nf, then it also has a unique lnf, which is also its unique $\beta\eta^{-1}$-nf.*

Proposition 2. *Every B in β-nf has a lnf L such that $L \twoheadrightarrow_\eta B$.*

4.7 Grammar producing λ-terms in *lnf*

možná dát do těla, v preliminarýs by mohlo stačit def lnf

In [1] is shown term generating grammar with following rules (our notation is used, but we will not highlighted terminals anymore).

$$\begin{aligned} (\alpha, \Gamma) &\longmapsto (\ f\ (\rho_1, \Gamma)\ \ldots\ (\rho_m, \Gamma)\ ) \\ &\quad \textbf{if } \alpha \in A, (f : \rho_1 \to \ldots \to \rho_m \to \alpha) \in \Gamma \\ (\sigma \to \tau, \Gamma) &\longmapsto (\ \lambda\ x\ .\ (\tau ; \Gamma, x : \sigma)\ ) \end{aligned}$$

The second rule can be replaced by more effective one. $(\tau_1 \to \ldots \to \tau_n \to \alpha, \Gamma) \longmapsto$
$(\ \lambda\ x_1\ \ldots\ x_n\ .\ (\alpha ; \Gamma, x_1 : \tau_1, \ldots, x_n : \tau_n)\ )$ $\quad$ **if** $n > 0$
This rule packs consecutive uses of the second rule into one use. This is valid since the use of the second rule is deterministic; it is used if and only if the non-terminal's type is not atomic.

4.8 Abstraction elimination

Abstraction elimination is a process of transforming an arbitrary λ-term into λ-term that contains no lambda abstractions and no bound variables. The newly produced λ-term may contain function applications, free symbols from former λ-term and some new symbols standing for combinators **S**, **K** and **I**.

Those combinators are defined as:

$$\begin{aligned} \mathbf{S} &= \lambda\, f\, g\, x \,.\, f\, x\, (g\, x) \\ \mathbf{K} &= \lambda\, x\, y \,.\, x \\ \mathbf{I} &= \lambda\, x \,.\, x \end{aligned}$$

Let us describe transformation performing this process.

$$\begin{aligned} [x] &= x \\ [\,(M\,N)\,] &= ([M]\ \ [N]) \\ [\lambda\, x \,.\, x] &= \mathbf{I} \\ [\lambda\, x \,.\, M] &= (\mathbf{K}\ \ [M]) && \textbf{if } x \notin \mathrm{FV}(M) \\ [\lambda\, x \,.\, (\lambda\, y \,.\, M)] &= [\lambda\, x \,.\ [\lambda\, y \,.\, M]] && \textbf{if } x \in \mathrm{FV}(M) \\ [\lambda\, x \,.\, (M\,N)] &= (\mathbf{S}\ \ [\lambda\, x \,.\, M]\ \ [\lambda\, x \,.\, N]) && \textbf{if } x \in \mathrm{FV}(M) \\ & && \vee\ x \in \mathrm{FV}(N) \end{aligned}$$

This is simple version of this process. More optimized version, in the means of the size of resulting term and its performance is following one, presented in [**?**].

As is stated in [**?**], the biggest disadvantage of this technique is that the translated term is often much larger than in its lambda form — the size of the translated term can be proportional to the square of the size of the original term.
But the advantage is also tempting — no need to deal with variables and lambda heads.

5. TěLO čLÁNKU, JAKOBY OUR APPROACH

5.1 Introduction

(Tématem je utilization redukcí a eliminace abstrakcí.)
Nejprve si ukážerme jak využít eta redukci při konstrukci počáteční populace.
Hlavním tématem článku je operátor křížení. Jeho hlavním problémem je výskyt proměnných v individualt trees. To jak to můžem řešit je několik možností:
- Vůbec nepoužívat lambda abstrakce, (asi v tomdle seznamu vubec neuvádět, ale tohle přesunout někam hodně na začátek, tuhle kapitolu už začít s tim že chceme navrhnout co nejlepší reprezentaci programů jakožto jedinců pro GP tak že involvuje lambda abstrakce) generovat už rovnou kombinatorový termy (dělá Briggs). [Argumenty] proč má smysl zabejvat se programama s lam abstrakcema a ne čístě jen v kombinátorech (pak přesunout možná někam jinam) - Možná to podat takle, "zdá se že použití kombinátorů má řadu výhod (vymenovat naKY co jmenuje briggs), ale konzistentní prozkoumání obecnýho řešení lambdovýho chgarakteru určitě stojí taky za zkoužku, což podporují nasledující argumenty": (a) úspornost - abstrakce eliminací produkuje termy až kvadraticky větší než je původní term. Dá se očekávat, že i při přímém generování je použití kombinátorů míň úsporný (proměná se může bez velký námahy objevit v hluboko vnořeném podtermu velmi jednoduše, zatímco pomocí kombinátorů se její hodnota musí do takto vnořeného podtermu složitě dopravit). (b) použití kombů vyžaduje silnější typovej sytém - kombinátory musej bejt polymorfní a přidáváme je do stavební sady, lambdy v pohodě běžej na simply typed lc. (c) použití proměnných je typické pro lidi, proto by se mohlo sát že i programy generovaný systémem kterej s proměnejma počítá budou o něco "lidštější"
- Použít omezenější strukturu která nepovoluje všechny možné lambda termy (vyfootnoutit nebo někam vejš říct, že všma možnejma termama myslíme i redukovaný termy)

5.2 Benefits of generating λ-terms in *lnf*

By generating λ-terms in *lnf* we avoid generating λ-terms M,N such that $M \neq N$, but $M =_{\beta\eta} N$. In other words, we avoid generating two programs with different source codes, but performing the same computation.

Every λ-term M such that $\Gamma \vdash M : \sigma$ has its unique *lnf* L, for which $L =_{\beta\eta} M$. Therefore the computation performed by λ-term M is not omitted, because it is the same computation as the computation performed by λ-term L.

Generating λ-terms in *lnf* is even better than generating λ-terms in β-*nf*. Since *lnf* is the same thing as $\beta\eta^{-1}$-*nf*, every λ-term in *lnf* is also in β-*nf*.

This comes straight from the definition of $\beta\eta^{-1}$-*nf*, but one can also see it from observing method for generating terms in β-*nf*. As is shown in [1], this method is obtained simply by replacing the first grammar rule by slightly modified rule, resulting in the following grammar.

$$\begin{aligned} (\pi, \Gamma) &\longmapsto (\ f\ (\rho_1, \Gamma)\ \dots\ (\rho_m, \Gamma)\ ) \\ &\quad \textbf{if } (f : \rho_1 \to \dots \to \rho_m \to \pi) \in \Gamma \\ (\sigma \to \tau, \Gamma) &\longmapsto (\ \lambda\, x\ .\ (\tau; \Gamma, x : \sigma)\ ) \end{aligned}$$

The difference lies in that f's type is no longer needed to be fully expanded ($\pi \in \mathbb{T}$ instead of $\alpha \in A$). This makes the grammar less deterministic, resulting in a bigger search space. The new rule is generalization of the old one, thus all terms in *lnf* will be generated, along with many new terms in β-*nf* that are not in *lnf*.

By generating λ-terms in *lnf* we avoid generating λ-terms M,N such that $M \neq N$ and $M =_\eta N$; but by generating in β-*nf* we do not avoid it.

The disadvantage of the *lnf*, as the name suggests, is that it is long. Terms in *lnf* are said to be *fully η-expanded* [1]. Relevant property of η-reduction is that it always shortens the term that is being reduced be it. And conversely, η-expansion prolongs.

$$(\lambda\, x\, .\, (M\ x)) \to_\eta M \qquad \textbf{if } x \notin FV(M)$$

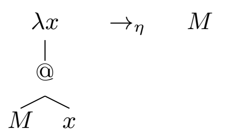
Now we show that for every λ-term M every sequence of η-reductions is finite and leads to unique η-*nf* N.

1. Every application of η-reduction shortens the term. Since every term has finite size, this process must end at some point. Thus every λ-term has η-*nf*.
2. Since η-reduction is *Church–Rosser*, η-*nf* is unique (see [?]).

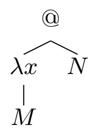
So we can take every generated λ-term M in *lnf* and transform it to shorter term in η-*nf*. The question is whether it remains in β-*nf*, thus being in $\beta\eta$-*nf*. The answer is yes; it can be proven by showing that no new β-redex is created by η-reduction.

Proposition 3. *Let P be in β-nf and $P \to_\eta Q$. Then Q is in β-nf.*

Proof. For better clarity let us show the η-reduction and β-redex using trees.
η-reduction:



And β-redex:



Let us assume that $P \to_\eta Q$ creates a new β-redex B in Q.

Since η-reduction only destroys and never creates *function applications* (i.e. @), the root @ of B must be present in P. But since P contains no β-redex, the left subterm L of this root @ is not λ-abstraction. Only possible way for L to be changed by $\to_\eta$ into a λ-abstraction is that L is the reduced subterm (so that L is changed for its subterm). But that is in contradiction with P not containing any β-redex, because it would cause L be a λ-abstraction.

$\square$

Notable property of *lnf* and $\beta\eta$-*nf* is that there is *bijection* (i.e. one-to-one correspondence) of the set of simply typed λ-terms in *lnf* and the set of simply typed λ-terms in $\beta\eta$-*nf*.

Proposition 4. *Reduction to η-nf is bijection between the set of simply typed λ-terms in lnf and the set of simply typed λ-terms in $\beta\eta$-nf.*

Proof. Since reduction to η-*nf* always leads to an unique term, it is a function. In previous proposition is shown that η-reduction of *lnf* leads to a term in $\beta\eta$-*nf*.

In order to show that a function is bijection it is sufficient to show that it is both *injection* and *surjection.*

Suppose it is not injection.

So there must be M_1, M_2 in *lnf* such that $M_1 \neq M_2$ and N in $\beta\eta$-*nf* such that $M_1 \twoheadrightarrow_\eta N$, $M_2 \twoheadrightarrow_\eta N$. Therefore $M_1 =_\eta M_2$, so $M_1 =_{\beta\eta^{-1}} M_2$. This contradicts with M_1, M_2 being distinct *lnf*s.

Every M in β-*nf* has a *lnf* N such that $N \twoheadrightarrow_\eta M$ (proposition from 4.6). Term M in $\beta\eta$-*nf* is in β-*nf*, thus it has desired *lnf* N which reduces to it.

Therefore it is surjection. $\square$

Suppose we have systematic (i.e. gradually generating all terms, but no term twice) method for generating terms in *lnf*, we may transform it to systematic method for generating terms in $\beta\eta$-*nf* by simply reducing each generated term to its η-*nf*.

5.3 Crossover

Design goal behind our approach to crossover operation is to try to generalize standard tree swapping crossover. This is not because of dislike of more specialized crossovers; it comes from systematic approach for exploring the domain of typed functional GP — thus I would like to start with the most common one.

The crossover operation in standard GP is performed by swapping randomly selected subtrees in each parent S-expression (see **??**).

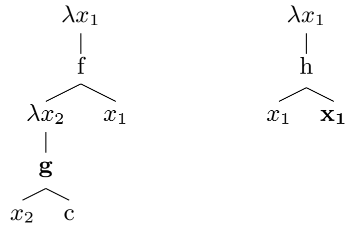
For typed lambda terms two difficulties arise: Types and variables.

We will show how to crossover *typed λ-term trees* in both @-tree and sexpr-tree notation.

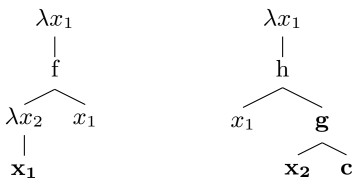
As in standard GP our crossover will be performed by swapping two subtrees. But now with constraint that both subtrees have the same type. This is discussed in grater detail in 5.3.1.

Variables bring more difficulties then types do. This problem arises from variables that are free in subterms corresponding to swapped subtrees.

Following example illustrates the problem. Let us have these two parent trees with selected nodes in bold.



The swap of subtrees results in following trees:



The problem is that variable x_2 in second tree is not bound by any λ-head and since it is not element of Γ, the second tree is not well-typed λ-term.

First approach tried at this problem was to repair the broken term, but since this method is not currently implemented we will describe it in just a few words. If the moved free variable ends in the scope where is defined variable with the same name and type, then it is OK. If it is not this case, we may rename this variable to some other variable "visible" in its new scope if such a variable has the same type. If there is no such a suitable variable we may generate new value to replace the variable by term generating method.

As you can see, this is not much elegant solution.

But we can avoid dealing with this problem by avoiding use of variables. This can be achieved by process called abstraction elimination.

5.3.1 *Typed subtree swapping*

First thing to do in standard subtree swapping is to select random node in the first parent.

We modify this procedure so that we allow selection only of those nodes with such a type that there exists a node in the second parent with the same type.

Standard subtree swapping crossover as a first thing selects whether the selected node will be inner node (usually with probability $p_{ip} = 90\%$) or leaf node (with probability 10%).

We are in a more complicated situation, because one of those sets may be empty, because of allowing only nodes with possible "partner" in the second parent. Thus we do this step only if both sets are nonempty.

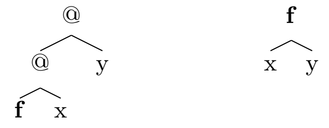
After selecting a node in the first parent we select node in the second parent such that type of that node must by the same as the type of the first node. Again, this may eliminate the "90-10" step of first deciding whether the selected node will be internal node or leaf node.

When both nodes are selected we may swap the trees.

If the abstraction elimination was performed, then since the trees are of the same type and there are no variables to be moved from their scope, the offspring trees are well typed.

Both sexpr-tree and @-tree are able to be crossed by this mechanism. But @-tree has more possibilities then @-tree. This comes from the fact that every subtree of the sexpr-tree corresponds to a subtree of @-tree, but there are subtrees of @-tree that do not correspond to a subtree of a sexpr-tree.

Following example should clarify this.



In @-tree, **f** is leaf thus subtree, whereas in sexpr-tree it is internal node thus not a subtree.

Another nice property of sexpr-trees with no lambdas is that they are the same representation as S-expressions used by standard GP.

Again, similarly as for standard version, a maximum permissible depth $D_{created}$ for offspring individuals is defined (e.g. $D_{created} = 17$). If one of the offspring has greater depth than this limit, then this offspring is replaced by the first parent in the result of the crossover operator. If both offspring exceeds this limit, than both are replaced by both parents.

For @-tree the $D_{created}$ must be larger since @-tree (without lambdas) is a binary tree. This enlargement is approximately proportionate to average number of function arguments. We use generous $D_{created} = 17 \times 3$.

5.4 Unpacking crossover

6. EXPERIMENTS

7. CONCLUSIONS

8. ACKNOWLEDGMENTS

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