Explicit Evaluation of All Statistical Auto and Cross-moments (two components) up to Fourth Order

Paul Nathan 08/12/10

The weighted mean, with weights ω of a scalar quantity X stored in a zero-based array is given by

$$\bar{X} = \frac{\sum_{i=0}^{N-1} \omega_i X_i}{\sum_{i=0}^{N-1} \omega_i}$$

The *biased*, weighted double-product moment of specified order p and q of joint quantities X and Y is given by

$$\overline{X^p Y^q} = \frac{\sum_{i=0}^{N-1} \omega_i \left(X_i - \bar{X}\right)^p \left(Y_i - \bar{Y}\right)^q}{\sum_{i=0}^{N-1} \omega_i}$$

The *un-biased*, weighted double-product moment of specified order p and q of joint quantities X and Y is given by

$$\overline{X^p Y^q} = \frac{\sum_{i=0}^{N-1} \omega_i}{\left(\sum_{i=0}^{N-1} \omega_i\right)^2 - \sum_{i=0}^{N-1} \omega_i^{\,2}} \sum_{i=0}^{N-1} \omega_i \left(X_i - \bar{X}\right)^p \left(Y_i - \bar{Y}\right)^q$$

The equivalent un-weighted (i.e. all weights set to unity) expressions are, in order of appearance above

$$\bar{X} = \frac{1}{N} \sum_{i=0}^{N-1} X_i$$

$$\overline{X^p Y^q} = \frac{1}{N} \sum_{i=0}^{N-1} \left(X_i - \bar{X}\right)^p \left(Y_i - \bar{Y}\right)^q$$

$$\overline{X^p Y^q} = \frac{1}{N-1} \sum_{i=0}^{N-1} \left(X_i - \bar{X}\right)^p \left(Y_i - \bar{Y}\right)^q$$

The statistical quantities to be derived are summarised in the following table. Let lower-case denote fluctuations, upper-case denote instantaneous values, and barred upper-case denote mean values

Order					
1	$\bar{U}$	$\bar{V}$			
2	$\overline{u^2}$	$\overline{v^2}$	$\overline{uv}$		
3	$\overline{u^3}$	$\overline{v^3}$	$\overline{u^2 v}$	$\overline{uv^2}$	
4	$\overline{u^4}$	$\overline{v^4}$	$\overline{u^3 v}$	$\overline{uv^3}$	$\overline{u^2 v^2}$

Recall the linear nature of the mean operator and also recall the following basic identities that will be of use

$$
\begin{aligned}
x &= X - \overline{X} \\
X &= \overline{X} + x \\
\overline{x} &= 0 \\
\overline{X\overline{X}^n} &= \overline{\left(\overline{X} + x\right)}\overline{X}^n = \overline{X}^{n+1} + \overline{x}\overline{X}^n = \overline{X}^{n+1} \\
\overline{X\overline{Y}^n} &= \overline{\left(\overline{X} + x\right)}\overline{Y}^n = \overline{X}\overline{Y}^n + \overline{x}\overline{Y}^n = \overline{X}\overline{Y}^n \\
\overline{X\overline{X}^n\overline{Y}^m} &= \overline{\left(\overline{X} + x\right)}\overline{X}^n\overline{Y}^m = \overline{X}\overline{X}^n\overline{Y}^m + \overline{x}\overline{X}^n\overline{Y}^m = \overline{X}^{n+1}\overline{Y}^m
\end{aligned}
$$

Pascal's triangle may be used to obtain the coefficients of the binomial expansion of $\left(X - \overline{X}\right)^n$. The minus sign in the bracket results in the negation of every other term of the expansion. The binomial coefficients up to fourth order are, explicitly

Order:									
2			1		−2		1		
3		1		−3		3		−1	
4	1		−4		6		−4		1

Firstly, the *un-weighted* statistics:

2nd Order

$$
\begin{aligned}
\overline{u^2} &= \overline{\left(U - \overline{U}\right)^2} = \overline{U^2} - 2\overline{U\overline{U}} + \overline{U}^2 \\
&= \overline{U^2} - \overline{U}^2
\end{aligned}
$$

$$
\overline{v^2} = \overline{V^2} - \overline{V}^2
$$

$$
\begin{aligned}
\overline{uv} &= \overline{\left(U - \overline{U}\right)\left(V - \overline{V}\right)} = \overline{UV} - \overline{U\overline{V}} - \overline{\overline{U}V} + \overline{\overline{U}\overline{V}} \\
&= \overline{UV} - \overline{U}\overline{V}
\end{aligned}
$$

3rd Order

$$
\begin{aligned}
\overline{u^3} &= \overline{\left(U - \overline{U}\right)^3} = \overline{U^3} - 3\overline{U^2\overline{U}} + 3\overline{U\overline{U}^2} - \overline{U}^3 \\
&= \overline{U^3} - 3\overline{U^2}\overline{U} + 2\overline{U}^3
\end{aligned}
$$

$$
\overline{v^3} = \overline{V^3} - 3\overline{V^2}\overline{V} + 2\overline{V}^3
$$

$$
\begin{aligned}
\overline{u^2v} &= \overline{\left(U^2 - 2U\overline{U} + \overline{U}^2\right)\left(V - \overline{V}\right)} = \overline{U^2V} - 2\overline{UV\overline{U}} + \overline{V\overline{U}^2} - \overline{U^2\overline{V}} + 2\overline{U\overline{U}\overline{V}} - \overline{U}^2\overline{V} \\
&= \overline{U^2V} - 2\overline{UV}\overline{U} - \overline{U^2}\overline{V} + 2\overline{U}^2\overline{V}
\end{aligned}
$$

$$
\overline{uv^2} = \overline{UV^2} - 2\overline{UV}\overline{V} - \overline{V^2}\overline{U} + 2\overline{V}^2\overline{U}
$$

4th Order

$$\overline{u^4} = \overline{\left(U-\overline{U}\right)^4} = \overline{U^4} - 4\overline{U^3\overline{U}} + 6\overline{U^2\overline{U}^2} - 4\overline{U\overline{U}^3} + \overline{U}^4$$
$$= \overline{U^4} - 4\overline{U^3}\,\overline{U} + 6\overline{U^2}\,\overline{U}^2 - 3\overline{U}^4$$

$$\overline{v^4} = \overline{V^4} - 4\overline{V^3}\,\overline{V} + 6\overline{V^2}\,\overline{V}^2 - 3\overline{V}^4$$

$$\overline{u^3v} = \overline{\left(U^3 - 3U^2\overline{U} + 3U\overline{U}^2 - \overline{U}^3\right)\left(V-\overline{V}\right)}$$
$$= \overline{U^3V} - 3\overline{U^2V\overline{U}} + 3\overline{UV\overline{U}^2} - \overline{V\overline{U}^3} - \overline{U^3\overline{V}} + 3\overline{U^2\overline{U}\,\overline{V}} - 3\overline{U\overline{U}^2\overline{V}} + \overline{U}^3\overline{V}$$
$$= \overline{U^3V} - 3\overline{U^2V}\,\overline{U} + 3\overline{UV}\,\overline{U}^2 - \overline{U^3}\,\overline{V} + 3\overline{U^2}\,\overline{U}\,\overline{V} - 3\overline{U}^3\overline{V}$$

$$\overline{uv^3} = \overline{UV^3} - 3\overline{UV^2}\,\overline{V} + 3\overline{UV}\,\overline{V}^2 - \overline{V^3}\,\overline{U} + 3\overline{V^2}\,\overline{U}\,\overline{V} - 3\overline{V}^3\overline{U}$$

$$\overline{u^2v^2} = \overline{\left(U-\overline{U}\right)^2\left(V-\overline{V}\right)^2} = \overline{\left(U^2 - 2U\overline{U} + \overline{U}^2\right)\left(V^2 - 2V\overline{V} + \overline{V}^2\right)}$$
$$= \overline{U^2V^2} - 2\overline{UV^2\overline{U}} + \overline{V^2\overline{U}^2} - 2\overline{U^2V\overline{V}} + 4\overline{UV\overline{U}\,\overline{V}} - 2\overline{V\overline{U}^2\overline{V}} + \overline{U^2\overline{V}^2} - 2\overline{U\overline{U}\,\overline{V}^2} + \overline{U}^2\overline{V}^2$$
$$= \overline{U^2V^2} - 2\overline{UV^2}\,\overline{U} + \overline{V^2}\,\overline{U}^2 - 2\overline{U^2V}\,\overline{V} + 4\overline{UV}\,\overline{U}\,\overline{V} + \overline{U^2}\,\overline{V}^2 - 3\overline{U}^2\overline{V}^2$$

Summary of Un-weighted Statistics up to 4th Order

2nd Order

$$\overline{u^2} = \overline{U^2} - \overline{U}^2$$
$$\overline{v^2} = \overline{V^2} - \overline{V}^2$$
$$\overline{uv} = \overline{UV} - \overline{U}\,\overline{V}$$

3rd Order

$$\overline{u^3} = \overline{U^3} - 3\overline{U^2}\,\overline{U} + 2\overline{U}^3$$
$$\overline{v^3} = \overline{V^3} - 3\overline{V^2}\,\overline{V} + 2\overline{V}^3$$
$$\overline{u^2v} = \overline{U^2V} - 2\overline{UV}\,\overline{U} - \overline{U^2}\,\overline{V} + 2\overline{U}^2\overline{V}$$
$$\overline{uv^2} = \overline{UV^2} - 2\overline{UV}\,\overline{V} - \overline{V^2}\,\overline{U} + 2\overline{V}^2\overline{U}$$

4th Order

$$\overline{u^4} = \overline{U^4} - 4\overline{U^3}\,\overline{U} + 6\overline{U^2}\,\overline{U}^2 - 3\overline{U}^4$$
$$\overline{v^4} = \overline{V^4} - 4\overline{V^3}\,\overline{V} + 6\overline{V^2}\,\overline{V}^2 - 3\overline{V}^4$$
$$\overline{u^3v} = \overline{U^3V} - 3\overline{U^2V}\,\overline{U} + 3\overline{UV}\,\overline{U}^2 - \overline{U^3}\,\overline{V} + 3\overline{U^2}\,\overline{U}\,\overline{V} - 3\overline{U}^3\overline{V}$$
$$\overline{uv^3} = \overline{UV^3} - 3\overline{UV^2}\,\overline{V} + 3\overline{UV}\,\overline{V}^2 - \overline{V^3}\,\overline{U} + 3\overline{V^2}\,\overline{U}\,\overline{V} - 3\overline{V}^3\overline{U}$$
$$\overline{u^2v^2} = \overline{U^2V^2} - 2\overline{UV^2}\,\overline{U} + \overline{V^2}\,\overline{U}^2 - 2\overline{U^2V}\,\overline{V} + 4\overline{UV}\,\overline{U}\,\overline{V} + \overline{U^2}\,\overline{V}^2 - 3\overline{U}^2\overline{V}^2$$

Next, the ***weighted*** statistics, with scalar weights denoted by ω :

2nd Order

$$\begin{aligned}
\overline{u^2} &= \overline{\omega\left(U-\bar{U}\right)^2} = \overline{\omega U^2} - 2\overline{\omega U\bar{\bar{U}}} + \overline{\omega\bar{U}^2} \\
&= \overline{\omega U^2} - 2\overline{\omega U}\bar{U} + \bar{\omega}\bar{U}^2
\end{aligned}$$

$$\overline{v^2} = \overline{\omega V^2} - 2\overline{\omega V}\bar{V} + \bar{\omega}\bar{V}^2$$

$$\begin{aligned}
\overline{uv} &= \overline{\omega\left(U-\bar{U}\right)\left(V-\bar{V}\right)} = \overline{\omega UV} - \overline{\omega U\bar{\bar{V}}} - \overline{\omega V\bar{\bar{U}}} + \overline{\omega\bar{\bar{U}}\bar{\bar{V}}} \\
&= \overline{\omega UV} - \overline{\omega U}\bar{V} - \overline{\omega V}\bar{U} + \bar{\omega}\bar{U}\bar{V}
\end{aligned}$$

3rd Order

$$\begin{aligned}
\overline{u^3} &= \overline{\omega\left(U-\bar{U}\right)^3} = \overline{\omega U^3} - 3\overline{\omega U^2\bar{\bar{U}}} + 3\overline{\omega U\bar{\bar{U}}^2} - \overline{\omega\bar{\bar{U}}^3} \\
&= \overline{\omega U^3} - 3\overline{\omega U^2}\bar{U} + 3\overline{\omega U}\bar{U}^2 - \bar{\omega}\bar{U}^3
\end{aligned}$$

$$\overline{v^3} = \overline{\omega V^3} - 3\overline{\omega V^2}\bar{V} + 3\overline{\omega V}\bar{V}^2 - \bar{\omega}\bar{V}^3$$

$$\begin{aligned}
\overline{u^2 v} &= \overline{\omega\left(U^2-2U\bar{U}+\bar{U}^2\right)\left(V-\bar{V}\right)} = \overline{\omega U^2 V} - 2\overline{\omega UV\bar{\bar{U}}} + \overline{\omega V\bar{\bar{U}}^2} - \overline{\omega U^2\bar{\bar{V}}} + 2\overline{\omega U\bar{\bar{U}}\bar{\bar{V}}} - \overline{\omega\bar{\bar{U}}^2\bar{\bar{V}}} \\
&= \overline{\omega U^2 V} - 2\overline{\omega UV}\bar{U} + \overline{\omega V}\bar{U}^2 - \overline{\omega U^2}\bar{V} + 2\overline{\omega U}\bar{U}\bar{V} - \bar{\omega}\bar{U}^2\bar{V}
\end{aligned}$$

$$\overline{uv^2} = \overline{\omega UV^2} - 2\overline{\omega UV}\bar{V} + \overline{\omega U}\bar{V}^2 - \overline{\omega V^2}\bar{U} + 2\overline{\omega V}\bar{U}\bar{V} - \bar{\omega}\bar{V}^2\bar{U}$$

4th Order

$$\overline{u^4} = \overline{\omega\left(U-\overline{U}\right)^4} = \overline{\omega U^4} - 4\overline{\omega U^3\overline{U}} + 6\overline{\omega U^2\overline{U}^2} - 4\overline{\omega U\overline{U}^3} + \overline{\omega\overline{U}^4}$$
$$= \overline{\omega U^4} - 4\overline{\omega U^3}\,\overline{U} + 6\overline{\omega U^2}\,\overline{U}^2 - 4\overline{\omega U}\,\overline{U}^3 + \varpi\overline{U}^4$$

$$\overline{v^4} = \overline{\omega V^4} - 4\overline{\omega V^3}\,\overline{V} + 6\overline{\omega V^2}\,\overline{V}^2 - 4\overline{\omega V}\,\overline{V}^3 + \varpi\overline{V}^4$$

$$\overline{u^3 v} = \overline{\omega\left(U^3 - 3U^2\overline{U} + 3U\overline{U}^2 - \overline{U}^3\right)\left(V-\overline{V}\right)}$$
$$= \overline{\omega U^3 V} - 3\overline{\omega U^2 V\overline{U}} + 3\overline{\omega UV\overline{U}^2} - \overline{\omega V\overline{U}^3} - \overline{\omega U^3\overline{V}} + 3\overline{\omega U^2\overline{U}\,\overline{V}} - 3\overline{\omega U\overline{U}^2\overline{V}} + \overline{\omega\overline{U}^3\overline{V}}$$
$$= \overline{\omega U^3 V} - 3\overline{\omega U^2 V}\,\overline{U} + 3\overline{\omega UV}\,\overline{U}^2 - \overline{\omega V}\,\overline{U}^3 - \overline{\omega U^3}\,\overline{V} + 3\overline{\omega U^2}\,\overline{U}\,\overline{V} - 3\overline{\omega U}\,\overline{U}^2\overline{V} + \varpi\overline{U}^3\overline{V}$$

$$\overline{uv^3} = \overline{\omega UV^3} - 3\overline{\omega UV^2}\,\overline{V} + 3\overline{\omega UV}\,\overline{V}^2 - \overline{\omega U}\,\overline{V}^3 - \overline{\omega V^3}\,\overline{U} + 3\overline{\omega V^2}\,\overline{U}\,\overline{V} - 3\overline{\omega V}\,\overline{V}^2\overline{U} + \varpi\overline{V}^3\overline{U}$$

$$\overline{u^2v^2} = \overline{\omega\left(U-\overline{U}\right)^2\left(V-\overline{V}\right)^2} = \overline{\omega\left(U^2 - 2U\overline{U} + \overline{U}^2\right)\left(V^2 - 2V\overline{V} + \overline{V}^2\right)}$$
$$= \overline{\omega U^2V^2} - 2\overline{\omega UV^2\overline{U}} + \overline{\omega V^2\overline{U}^2} - 2\overline{\omega U^2V\overline{V}} + 4\overline{\omega UV\overline{U}\,\overline{V}} - 2\overline{\omega V\overline{U}^2\overline{V}} + \overline{\omega U^2\overline{V}^2} - 2\overline{\omega U\overline{U}\,\overline{V}^2} + \overline{\omega\overline{U}^2\overline{V}^2}$$
$$= \overline{\omega U^2V^2} - 2\overline{\omega UV^2}\,\overline{U} + \overline{\omega V^2}\,\overline{U}^2 - 2\overline{\omega U^2V}\,\overline{V} + 4\overline{\omega UV}\,\overline{U}\,\overline{V} - 2\overline{\omega V}\,\overline{U}^2\overline{V} + \overline{\omega U^2}\,\overline{V}^2 - 2\overline{\omega U}\,\overline{U}\,\overline{V}^2 + \varpi\overline{U}^2\overline{V}^2$$

Summary of Weighted Statistics up to 4th Order

2nd Order

$$\overline{u^2} = \overline{\omega U^2} - 2\overline{\omega U}\,\overline{U} + \varpi\overline{U}^2$$
$$\overline{v^2} = \overline{\omega V^2} - 2\overline{\omega V}\,\overline{V} + \varpi\overline{V}^2$$
$$\overline{uv} = \overline{\omega UV} - \overline{\omega U}\,\overline{V} - \overline{\omega V}\,\overline{U} + \varpi\overline{U}\,\overline{V}$$

3rd Order

$$\overline{u^3} = \overline{\omega U^3} - 3\overline{\omega U^2}\,\overline{U} + 3\overline{\omega U}\,\overline{U}^2 - \varpi\overline{U}^3$$
$$\overline{v^3} = \overline{\omega V^3} - 3\overline{\omega V^2}\,\overline{V} + 3\overline{\omega V}\,\overline{V}^2 - \varpi\overline{V}^3$$
$$\overline{u^2v} = \overline{\omega U^2V} - 2\overline{\omega UV}\,\overline{U} + \overline{\omega V}\,\overline{U}^2 - \overline{\omega U^2}\,\overline{V} + 2\overline{\omega U}\,\overline{U}\,\overline{V} - \varpi\overline{U}^2\overline{V}$$
$$\overline{uv^2} = \overline{\omega UV^2} - 2\overline{\omega UV}\,\overline{V} + \overline{\omega U}\,\overline{V}^2 - \overline{\omega V^2}\,\overline{U} + 2\overline{\omega V}\,\overline{U}\,\overline{V} - \varpi\overline{V}^2\overline{U}$$

4th Order

$$\overline{u^4} = \overline{\omega U^4} - 4\overline{\omega U^3}\,\overline{U} + 6\overline{\omega U^2}\,\overline{U}^2 - 4\overline{\omega U}\,\overline{U}^3 + \varpi\overline{U}^4$$
$$\overline{v^4} = \overline{\omega V^4} - 4\overline{\omega V^3}\,\overline{V} + 6\overline{\omega V^2}\,\overline{V}^2 - 4\overline{\omega V}\,\overline{V}^3 + \varpi\overline{V}^4$$
$$\overline{u^3 v} = \overline{\omega U^3 V} - 3\overline{\omega U^2 V}\,\overline{U} + 3\overline{\omega UV}\,\overline{U}^2 - \overline{\omega V}\,\overline{U}^3 - \overline{\omega U^3}\,\overline{V} + 3\overline{\omega U^2}\,\overline{U}\,\overline{V} - 3\overline{\omega U}\,\overline{U}^2\overline{V} + \varpi\overline{U}^3\overline{V}$$
$$\overline{uv^3} = \overline{\omega UV^3} - 3\overline{\omega UV^2}\,\overline{V} + 3\overline{\omega UV}\,\overline{V}^2 - \overline{\omega U}\,\overline{V}^3 - \overline{\omega V^3}\,\overline{U} + 3\overline{\omega V^2}\,\overline{U}\,\overline{V} - 3\overline{\omega V}\,\overline{V}^2\overline{U} + \varpi\overline{V}^3\overline{U}$$
$$\overline{u^2v^2} = \overline{\omega U^2V^2} - 2\overline{\omega UV^2}\,\overline{U} + \overline{\omega V^2}\,\overline{U}^2 - 2\overline{\omega U^2V}\,\overline{V} + 4\overline{\omega UV}\,\overline{U}\,\overline{V} - 2\overline{\omega V}\,\overline{U}^2\overline{V} + \overline{\omega U^2}\,\overline{V}^2 - 2\overline{\omega U}\,\overline{U}\,\overline{V}^2 + \varpi\overline{U}^2\overline{V}^2$$