Finding the Best Fit Axis-aligned Ellipse through a Set of Points

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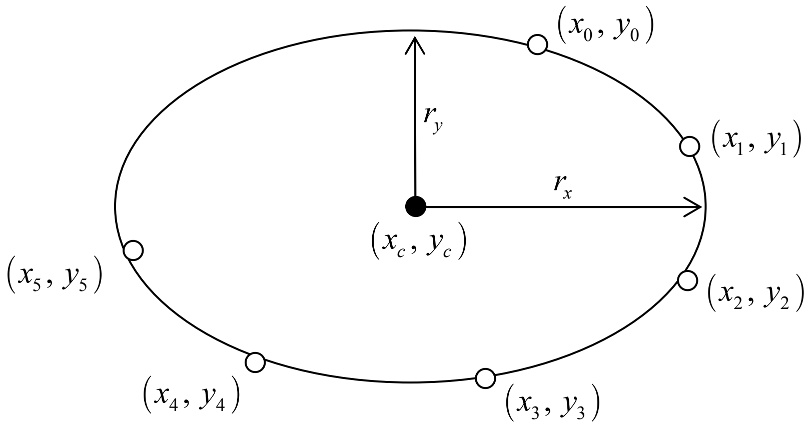
Consider a set of n points on the plane, what axis-aligned ellipse with radii along the x and y axes of r_x and r_y respectively and centre (x_c, y_c) best fits these points?



The equation for the axis-aligned ellipse is given by

$$\frac{(x-x_c)^2}{r_x^{\,2}}+\frac{(y-y_c)^2}{r_y^{\,2}}=1$$

Setting $\alpha=\dfrac{r_x}{r_y}$ and multiplying through by $r_x^{\,2}$ gives

$$(x-x_c)^2+\alpha^2(y-y_c)^2=r_x^{\,2}$$

This may be expanded as follows, with a view to formulating a matrix equation for the four unknowns

$$2xx_c+2\alpha^2 yy_c-\alpha^2y^2+r_x^{\,2}-x_c^{\,2}-\alpha^2y_c^{\,2}=x^2$$

In matrix form

$$\begin{pmatrix} x_0 & y_0 & -y_0^{\,2} & 1 \\ & & & \\ & & & \\ x_{n-1} & y_{n-1} & -y_{n-1}^{\,2} & 1 \end{pmatrix}\begin{pmatrix} 2x_c \\ 2\alpha^2 y_c \\ \alpha^2 \\ r_x^{\,2}-x_c^{\,2}-\alpha^2y_c^{\,2} \end{pmatrix}=\begin{pmatrix} x_0^{\,2} \\ \\ \\ x_{n-1}^{\,2} \end{pmatrix}$$

A minimum of four points is required to solve this exactly; otherwise (and more generally) the least-squares solution can be found by first multiplying through by the transpose of the coefficient matrix to turn **A** into a 4x4 square matrix, and then solving in the usual manner

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\mathbf{X}=\mathbf{A}^{\mathrm{T}}\mathbf{B}$$
$$\mathbf{X}=\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)$$

The ellipse centre coordinates, semi-major and semi-minor axes are obtained from the solution vector **X**

$$\begin{aligned}
\alpha &= \sqrt{X_2} \\
(x_c, y_c) &= \left(\frac{X_0}{2}, \frac{X_1}{2\alpha^2}\right) \\
r_x &= \sqrt{X_3 + x_c^{\ 2} + \alpha^2 y_c^{\ 2}} \\
r_y &= \frac{r_x}{\alpha}
\end{aligned}$$

An alternative iterative method that yields the same result is the Gauss-Newton method. This non-linear regression method uses the gradient of the level set function F to find its minimum

$$F(x_c, y_c, r_x, r_y) = \frac{(x - x_c)^2}{r_x^{\ 2}} + \frac{(y - y_c)^2}{r_y^{\ 2}} - 1 = 0$$

$$\nabla F(x_c, y_c, r_x, r_y) = \begin{pmatrix} \dfrac{\partial F}{\partial x_c} \\ \dfrac{\partial F}{\partial y_c} \\ \dfrac{\partial F}{\partial r_x} \\ \dfrac{\partial F}{\partial r_y} \end{pmatrix} = \begin{pmatrix} -\dfrac{2(x - x_c)}{r_x^{\ 2}} \\ -\dfrac{2(y - y_c)}{r_y^{\ 2}} \\ -\dfrac{2(x - x_c)^2}{r_x^{\ 3}} \\ -\dfrac{2(y - y_c)^2}{r_y^{\ 3}} \end{pmatrix}$$

The iteration step is as follows, for vector function $\mathbf{F}$ (consisting of n equations ($n \geq 4$) for each set of points (x_i, y_i) that will have the ellipse fitted to them) and parameter vector $\mathbf{X} = \begin{pmatrix} x_c & y_c & r_x & r_y \end{pmatrix}^{\mathrm{T}}$

$$\mathbf{X}_{j+1} = \mathbf{X}_j - \left(\mathbf{J}^{\mathrm{T}}\mathbf{J}\right)^{-1}\left(\mathbf{J}^{\mathrm{T}}\mathbf{F}(\mathbf{X}_j)\right)$$

The Jacobian matrix $\mathbf{J}$ is

$$\mathbf{J}_j = \begin{pmatrix} (\nabla F_0)^{\mathrm{T}} \\ \vdots \\ \vdots \\ (\nabla F_{n-1})^{\mathrm{T}} \end{pmatrix}_j = \begin{pmatrix} \dfrac{\partial F_0}{\partial x_c} & \dfrac{\partial F_0}{\partial y_c} & \dfrac{\partial F_0}{\partial r_x} & \dfrac{\partial F_0}{\partial r_y} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial F_{n-1}}{\partial x_c} & \dfrac{\partial F_{n-1}}{\partial y_c} & \dfrac{\partial F_{n-1}}{\partial r_x} & \dfrac{\partial F_{n-1}}{\partial r_y} \end{pmatrix}_j$$

Where F_i is evaluated using the current (j^{th}) estimate of the parameters. Iterations may stop once the absolute relative difference of each parameter (between successive iterations) is below some threshold