Cubic Interpolation of an Unequally Spaced Function.

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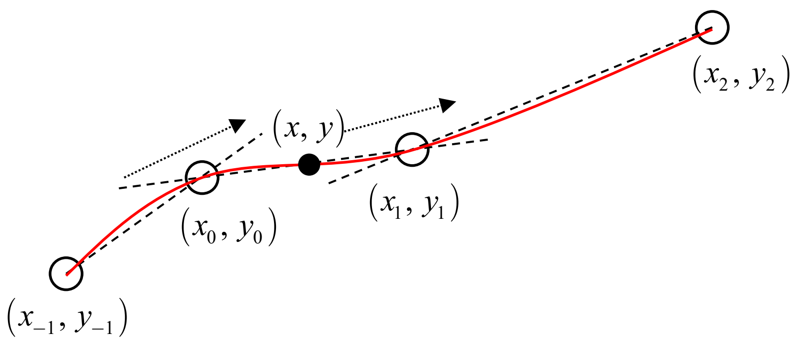

Cubic polynomial to fit points 0 and 1 and match gradients at points 0 and 1.
Gradient tension parameters τ_0 and τ_1.
Summary of result below, with working steps shown at the end:

$$y = y_0 + (x - x_0)\Big(a_1 + (x - x_0)\big(a_2 + a_3(x - x_0)\big)\Big)$$

$$m_{-1} = \frac{y_0 - y_{-1}}{x_0 - x_{-1}}$$

$$m_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\left.\frac{dy}{dx}\right|_0 = \tau_0 m_0 + (1 - \tau_0) m_{-1}$$

$$\left.\frac{dy}{dx}\right|_1 = \tau_1 m_1 + (1 - \tau_1) m_0$$

$$a_1 = \left.\frac{dy}{dx}\right|_0$$

$$a_2 = \frac{3m_0 - \left.\frac{dy}{dx}\right|_0 - \left(\left.\frac{dy}{dx}\right|_0 + \left.\frac{dy}{dx}\right|_1\right)}{x_1 - x_0}$$

$$a_3 = \frac{\left(\left.\frac{dy}{dx}\right|_0 + \left.\frac{dy}{dx}\right|_1\right) - 2m_0}{(x_1 - x_0)^2}$$

For interpolation between the first point pair set $m_{-1} = m_0$.
For interpolation between the last point pair set $m_1 = m_0$.
For standard spline, set $\tau_0 = \tau_1 = \frac{1}{2}$.
To find the indices, first find the value pair at index 0 by using a binary search on an ascending order sorted points list.

Monotonic cubic interpolation:

Define:

$$\alpha = \frac{1}{m_0} \left. \frac{dy}{dx} \right|_0$$

$$\beta = \frac{1}{m_0} \left. \frac{dy}{dx} \right|_1$$

Check the following conditions and apply constraints on the point gradients:

$$\begin{aligned}
&\text{if } (\operatorname{sgn}(m_{-1}) \neq \operatorname{sgn}(m_0)) \\
&\left. \frac{dy}{dx} \right|_0 = 0 \\
&\text{if } (\operatorname{sgn}(m_1) \neq \operatorname{sgn}(m_0)) \\
&\left. \frac{dy}{dx} \right|_1 = 0 \\
&\text{if } (m_0 == 0) \\
&\left. \frac{dy}{dx} \right|_0 = 0 \\
&\left. \frac{dy}{dx} \right|_1 = 0 \\
&\text{else} \\
&\{ \\
&\text{if } (\alpha < 0) \\
&\left. \frac{dy}{dx} \right|_0 = 0 \\
&\text{else if } (\alpha > 3) \\
&\left. \frac{dy}{dx} \right|_0 = 3m_0 \\
& \\
&\text{if } (\beta < 0) \\
&\left. \frac{dy}{dx} \right|_1 = 0 \\
&\text{else if } (\beta > 3) \\
&\left. \frac{dy}{dx} \right|_1 = 3m_0 \\
&\}
\end{aligned}$$

For a given dataset, all the terms can be pre-calculated and stored once (arrays of piecewise cubic polynomial coefficients), thereby leaving only a standard cubic polynomial evaluation at the desired coordinate.

$$y - y_0 = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3$$
$$\frac{dy}{dx} = a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2$$

$$y_0 - y_0 = a_0 + a_1 (x_0 - x_0) + a_2 (x_0 - x_0)^2 + a_3 (x_0 - x_0)^3$$
$$y_1 - y_0 = a_0 + a_1 (x_1 - x_0) + a_2 (x_1 - x_0)^2 + a_3 (x_1 - x_0)^3$$
$$\left.\frac{dy}{dx}\right|_0 = a_1 + 2a_2 (x_0 - x_0) + 3a_3 (x_0 - x_0)^2$$
$$\left.\frac{dy}{dx}\right|_1 = a_1 + 2a_2 (x_1 - x_0) + 3a_3 (x_1 - x_0)^2$$

$$a_0 = 0$$
$$a_1 = \left.\frac{dy}{dx}\right|_0$$

$$y_1 - y_0 = \left.\frac{dy}{dx}\right|_0 (x_1 - x_0) + a_2 (x_1 - x_0)^2 + a_3 (x_1 - x_0)^3$$
$$\left.\frac{dy}{dx}\right|_1 = \left.\frac{dy}{dx}\right|_0 + 2a_2 (x_1 - x_0) + 3a_3 (x_1 - x_0)^2$$

$$a_2 = \frac{\left.\frac{dy}{dx}\right|_1 - \left.\frac{dy}{dx}\right|_0 - 3a_3 (x_1 - x_0)^2}{2(x_1 - x_0)} = \frac{\frac{1}{2}\left(\left.\frac{dy}{dx}\right|_1 - \left.\frac{dy}{dx}\right|_0\right)}{x_1 - x_0} - \frac{3}{2} a_3 (x_1 - x_0)$$

$$y_1 - y_0 = \left.\frac{dy}{dx}\right|_0 (x_1 - x_0) + \frac{1}{2}\left(\left.\frac{dy}{dx}\right|_1 - \left.\frac{dy}{dx}\right|_0\right)(x_1 - x_0) - \frac{1}{2} a_3 (x_1 - x_0)^3$$
$$y_1 - y_0 = \frac{1}{2}\left(\left.\frac{dy}{dx}\right|_1 + \left.\frac{dy}{dx}\right|_0\right)(x_1 - x_0) - \frac{1}{2} a_3 (x_1 - x_0)^3$$
$$a_3 = \left(\left.\frac{dy}{dx}\right|_1 + \left.\frac{dy}{dx}\right|_0\right)\frac{1}{(x_1 - x_0)^2} - 2\frac{(y_1 - y_0)}{(x_1 - x_0)^3}$$

$$a_2 = \frac{\frac{1}{2}\left(\left.\frac{dy}{dx}\right|_1 - \left.\frac{dy}{dx}\right|_0\right) - \frac{3}{2}\left(\left.\frac{dy}{dx}\right|_1 + \left.\frac{dy}{dx}\right|_0\right) + 3\frac{y_1 - y_0}{x_1 - x_0}}{x_1 - x_0}$$
$$= \frac{3\frac{y_1 - y_0}{x_1 - x_0} - \left.\frac{dy}{dx}\right|_0 - \left(\left.\frac{dy}{dx}\right|_1 + \left.\frac{dy}{dx}\right|_0\right)}{x_1 - x_0}$$