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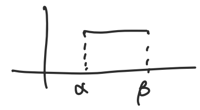
Properties:

1. Density: $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } x \in [\alpha, \beta], \\ 0 & \text{otherwise.} \end{cases}$



2. $EX = \frac{\alpha+\beta}{2}$.

3. $\text{Var}(X) = \frac{(\beta-\alpha)^2}{12}$.

Example 6.3. Assume that X is uniform on $[0,1]$. What is $P(X \in \mathbb{Q})$? What is the probability that the binary expansion of X starts with 0.010?

As $\mathbb{Q}$ is countable, it has an enumeration, say, $\mathbb{Q} = \{q_1, q_2, \dots\}$. By Axiom 3 of Chapter 3:

$$P(X \in \mathbb{Q}) = P(\underbrace{\cup_i \{X = q_i\}}_{\text{countable union, disjoint.}}) = \sum_i P(X = q_i) = 0.$$

Note that you cannot do this for sets that are not countable or you would "prove" that $P(X \in \mathbb{R}) = 0$, while we, of course, know that $P(X \in \mathbb{R}) = P(\Omega) = 1$. As X is, with probability 1, irrational, its binary expansion is uniquely defined, so there is no ambiguity about what the second question means.

$[0$ 0.0·· | 0.0·· | 0.1·· | 0.1·· $)1$
0.00·· 0.01·· 0.10 0.11

Divide $[0,1)$ into 2^n intervals of equal length. If the binary expansion of a number $x \in [0,1)$ is $0.x_1x_2\dots$, the first n binary digits determine which of the 2^n subintervals x belongs to: if you know that x belongs to an interval I based on the first $n-1$ digits, then nth digit 1 means that x is in the right half of I and nth digit 0 means that x is in the left half of I. For example, if the expansion starts with 0.010, the number is in $[0, \frac{1}{2}]$, then in $[\frac{1}{4}, \frac{1}{2}]$, and then finally in $[\frac{1}{4}, \frac{3}{8}]$.

Our answer is $\frac{1}{8}$, but, in fact, we can make a more general conclusion. If X is uniform on $[0,1]$, then any of the 2^n possibilities for its first n binary digits are equally likely. In other words, the binary digits of X are the result of an infinite sequence of independent fair coin tosses. Choosing a uniform random number on $[0,1]$ is thus equivalent to tossing a fair coin infinitely many times.

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Example 6.4. A uniform random number X divides $[0,1]$ into two segments. Let R be the ratio of the smaller versus the larger segment. Compute the density of R.

As R has values in $[0,1]$, the density $f_R(r)$ is nonzero only for $r \in [0,1]$ and we will deal only with such r's.

For a fixed r.
P(ratio ≤ r)

$$\begin{aligned} F_R(r) &= P(R \le r) = P\left(X \le \frac{1}{2}, \frac{X}{1-X} \le r\right) + P\left(X > \frac{1}{2}, \frac{1-X}{X} \le r\right) \\ &= P\left(X \le \frac{1}{2}, \boxed{X \le \frac{r}{r+1}}\right) + P\left(X > \frac{1}{2}, \boxed{X \ge \frac{1}{r+1}}\right) \\ &= P\left(X \le \frac{r}{r+1}\right) + P\left(X \ge \frac{1}{r+1}\right) \quad \left(\text{since } \underbrace{\frac{r}{r+1} \le \frac{1}{2}}_{\max\ r=1} \text{ and } \underbrace{\frac{1}{r+1} \ge \frac{1}{2}}\right) \\ &= \frac{r}{r+1} + 1 - \frac{1}{r+1} = \frac{2r}{r+1} \end{aligned}$$

same derivation

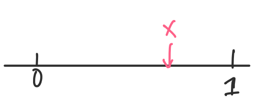
$$\frac{X}{1-X} \le r$$
$$r \le r - rX$$
$$X + rX \le r$$
$$X(1+r) \le r$$
$$X \le \frac{r}{1+r}$$

$$= \int_0^{r/(r+1)} f_X(s)\,ds + \int_{\frac{1}{r+1}}^1 f_X(s)\,ds$$

$\Rightarrow \max \frac{r}{r+1} < \frac{1}{1+1} = \frac{1}{2}$ $\quad \min \frac{1}{r+1} = \frac{1}{2}$

$\Rightarrow (X \le \frac{1}{2}) \supseteq (X \le \frac{r}{r+1})$ $\quad (X \ge \frac{1}{2}) \supseteq (X \ge \frac{1}{r+1})$

$\Rightarrow$ Intersection is just $(X \le \frac{r}{r+1})$



$R = \frac{[X, 1]}{[0, X]}$

$R(r) \in (0, 1]$

For $r \in (0,1)$, the density, thus, equals

$$f_R(r) = \frac{d}{dr}F_R(r) = \frac{2}{(r+1)^2}.$$

part (2)

We have computed the density of R, but we will use this example to make an additional point. Let $S \overset{\text{short}}{=} \min\{X, 1-X\}$ be the smaller of the two segments and $L \overset{\text{long}}{=} \max\{X, 1-X\}$ the larger. Clearly $R = S/L$. Is $ER = ES/EL$? To check that this equation does not hold we compute

$$\int_0^1 r \cdot f_R(r)\,dr = ER = \int_0^1 r \cdot \frac{2}{(r+1)^2}\,dr = 2\ln(2) - 1 \approx 0.3863.$$

Moreover, we can compute ES by

$$= \int_0^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^1 (1-x)\,dx$$

$$ES = \int_0^1 \min\{x, 1-x\}\,dx = \frac{1}{4} = E[\min\{x, 1-x\}]$$

or by checking (by a short computation which we omit) that S is uniform on $[0, 1/2]$. Finally, as $S + L = 1$,

$$EL = 1 - ES = \frac{3}{4}.$$

Thus $ES/EL = 1/3 \neq ER$.

6.2 Exponential random variable

A random variable is Exponential(λ), with parameter $\lambda > 0$, if it has the probability density function given below. This is a distribution for the *waiting time* for some random event, for example, for a lightbulb to burn out or for the next earthquake of at least some given magnitude.

Properties:

1. Density: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$
2. $EX = \frac{1}{\lambda}$.
3. $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.
4. $P(X \geq x) = e^{-\lambda x}$.
5. Memoryless property: $P(X \geq x + y | X \geq y) = e^{-\lambda x}$.

The last property means that, if the event has not occurred by some given time (no matter how large), the distribution of the remaining waiting time is the same as it was at the beginning. There is no "aging."

Proofs of these properties are integration exercises and are omitted.

Feb 17. Final Example

Prop. Distribution of Distribution is Uniform

Let X be a random variable with a cumulative density function F_X where F_X is both continuous and strictly increasing.

Let $Y = F_X(x)$, then Y has a uniform distribution over $[0, 1]$

Proof.

The cumulative density of Y is:

$$\begin{aligned} F_Y(y) &= F_{F_X(x)}(y) \\ &= P(F_X(x) \leq y) \end{aligned}$$

Recall that a strictly increasing function is injective. So $F_X^{-1}(x)$ exists.

Apply F_X^{-1} to the inequality:

$$\begin{aligned} P(F_X(x) \leq y) &= P\Big(F_X^{-1}(F_X(x)) \leq F_X^{-1}(y)\Big) \\ &= P(X \leq F_X^{-1}(y)) \\ &= \int_{-\infty}^{F_X^{-1}(y)} f_X(x)\,\mathrm{d}x \end{aligned}$$

Now we use u-substitution to integrate this.

$$\frac{\mathrm{d}F_X}{\mathrm{d}x} = f_X \implies \mathrm{d}F_X = f_X\,\mathrm{d}x$$

Substitute back in and change bounds of integration:

$$\int_{-\infty}^{F_X^{-1}(y)} f_X(x)\,\mathrm{d}x = \int_{-\infty}^{F_X(F_X^{-1}(y))} \mathrm{d}F_X = \int_{-\infty}^{y} \mathrm{d}F_X$$

F_X only ranges from $[0, 1]$, so we can replace the lower bound with 0.

$$\int_{-\infty}^{y} \mathrm{d}F_X = \int_{0}^{y} \mathrm{d}F_X = y$$

which is uniform over the interval $[0, 1]$.