Feb 22. Notes

Hazard Rate Function

Let X be a continuous random variable and interpret x as the ***lifetime*** of some item.

Let $f(x)$ be the probability density function of x and let $F(x) = \int f$ be the cumulative density function. We can define:

Def. Hazard Rate / Failure Rate

$$\lambda : [0, \infty) \to \mathbb{R}$$
$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

EX.1

Suppose an item has survived for some time t.

> **Question.** What's the probability that the item won't survive for an additional time dt?

This is given by the conditional probability:

$$P(X \in (t, t + dt) \mid X > t)$$

which expands to:

$$\begin{aligned} &= \frac{P(X \in (t, t + dt) \cap X > t)}{P(X > t)} \\ &= \frac{P(X \in (t, t + dt))}{P(X > t)} \\ &\approx \frac{f(t)}{1 - F(t)} \cdot dt \end{aligned}$$

`@Hint` Use rectangle approximation for $P(X \in (t, t + dt))$ to get the final line.

Thm. Hazard Rate of Exponential Random Variable is its parameter and is constant

Ley X be an exponential random variable: $X \sim \text{Exponential}(\lambda_0)$

The hazard function for X is:

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{\lambda_0 e^{-\lambda_0 t}}{e^{-\lambda_0 t}} = \lambda_0$$

`@Hint` The hazard rate for exponential random variables is constant.

Thm. Hazard rate uniquely determines F

The hazard rate of a continuous random variable X uniquely determines X's cumulative density function $F_X(x)$.

Proof.

We need to show that there exists a relation between $\lambda(t)$ and $F_X(x)$.

Integrate $\lambda(t)$:

$$\begin{aligned}\int_0^t \lambda(s)\mathrm{d}s &= \int_0^t \frac{f(s)}{1-F(s)}\mathrm{d}s \\ &\text{use u-sub with } u = 1 - F(s) \\ &= -\ln(1-F(s))\Big|_0^t \\ &= -\ln(1-F(t)) + \ln(1-\underbrace{F(0)}_{0,\because \int_0^0 f=0}) \\ &= -\ln(1-F(t)) + \ln(1) \\ &= -\ln(1-F(t))\end{aligned}$$

Isolate $F(t)$:

$$\begin{aligned}\int_0^t \lambda(s)\mathrm{d}s &= -\ln(1-F(t)) \\ \implies F(t) &= \boxed{1-\exp\left(-\int_0^t \lambda(s)\mathrm{d}s\right)}\end{aligned}$$

Therefore we have a unique equality the connects $\lambda(s)$ and F_X

EX.2

Let the death rate of a smoker (S) be twice of a non smoker(N).

> **Question.** Compare the probability that an A year old smoker will survive until age $B, B > A$ to that of a non-smoker.

We are given that a smoker is twice as likely to die as a non-smoker, so their respective hazard rates are:

$$\lambda_S(t) = 2\lambda_N(t)$$

We want to know:

$$P(\text{Lifetime of } N \text{ is } > B \mid \text{Already } A \text{ years old})$$

Expand with conditional probability:

$$\frac{P(\text{Lifetime of } N \text{ is } > B \cap \text{Lifetime of } N \text{ is } > A)}{P(\text{Lifetime of } N \text{ is } > A)}$$

$$= \frac{P(\text{Lifetime of } N \text{ is } > B)}{P(\text{Lifetime of } N \text{ is } > A)}$$

The intersection can be simplified because the event $\text{Lifetime of N is} > \text{A}$ is a subset of $\text{Lifetime of N is} > \text{B}$ from the fact that $B > A$.

Apply Def. Hazard Rate to both the numerator and the denominator:

$$\begin{aligned}
&\frac{P(\text{Lifetime of } N \text{ is } > B)}{P(\text{Lifetime of } N \text{ is } > A)} \\
&= \frac{\lambda_N(B)}{\lambda_N(A)} \\
&= \frac{\frac{f(B)}{1-F(B)} \cdot dt}{\frac{f(A)}{1-F(A)} \cdot dt} \\
&\textcolor{red}{\text{Somehow get rid of } f(B) \cdot dt, f(A) \cdot dt?} \\
&= \frac{1 - F(B)}{1 - F(A)}
\end{aligned}$$

Plug in the integrals found in Thm. Hazard rate uniquely determines F:

$$\begin{aligned}
&= \frac{1 - F(B)}{1 - F(A)} \\
&= \frac{1 - 1 + \exp\left(-\int_0^B \lambda_N(s)\mathrm{d}s\right)}{1 - 1 + \exp\left(-\int_0^A \lambda_N(s)\mathrm{d}s\right)} \\
&= \exp\left(-\int_A^B \lambda_N(s)\mathrm{d}s\right)
\end{aligned}$$

Similarly for smokers:

$$\begin{aligned}
&P(\text{Lifetime of } S \text{ is } > B \mid \text{Already } A \text{ years old}) \\
&= \exp(-\int_A^B \lambda_S(s)\mathrm{d}s) \\
&= \exp\left(-\int_A^B 2\lambda_N(s)ds\right) \\
&= \left(\exp(-\int_A^B \lambda_N(s)ds)\right)^2
\end{aligned}$$

So we have this quadratic relationship.