Feb 15. Notes

Thm. Fubini's Theorem

Let a region $R \subseteq \mathbb{R}^2$, function $f : R \to \mathbb{R}$ be a continuous function such that

$$\iint_R |f| \mathrm{d}x\mathrm{d}y < \infty$$

Then the integral of f over R is equal to the iterated integral:

$$\iint_R f(x, y)\mathrm{d}x\mathrm{d}y = \int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y$$

Lemma. Expected value of non-negative random variable

For a non-negative random variable Y, the expected value is:

$$E[Y] = \int_0^\infty P(Y > y)\mathrm{d}y$$

Proof.

Let f_Y be the probability density of Y, by Thm. Fubini's Theorem:

$$\int_0^\infty P(Y > y)\mathrm{d}y = \int_0^\infty \int_y^\infty f_Y(x)\mathrm{d}x\mathrm{d}y = \int_0^\infty \int_0^{x} f_Y(x)\mathrm{d}y\mathrm{d}x$$

- Fubini's theorem applies here because f_Y is continuous and the integral $\int_{-\infty}^\infty f_Y(s)ds = 1 < \infty$ is finite.

Evaluate the iterated integral:

$$\begin{aligned} \int_0^\infty \int_0^x f_Y(x)\mathrm{d}y\mathrm{d}x &= \int_0^\infty \left(\int_0^x \mathrm{d}y\right) f_Y(x)\mathrm{d}x \\ &= \int_0^\infty x f_Y(x)\mathrm{d}x \\ &= E[Y] \end{aligned}$$

Prop. Wrapped Expected Value

If X is a continuous random variable with p.d.f $f(x)$, then for all real valued function $g : \mathbb{R} \to \mathbb{R}, g(x) > 0, \forall x \in \mathbb{R}$, the expected value of $g(X)$ is:

$$E[g(X)] = \int_{-\infty}^\infty g(x)f(x)\mathrm{d}x$$

Proof Sketch.

Since $g > 0$, Lemma. Expected value of non-negative random variable applies here.

$$\begin{aligned} E[g(X)] &= \int_0^\infty P(g(X) > y)\mathrm{d}y \\ &= \int_0^\infty \int_{g(x)>y}^\infty f(x)\mathrm{d}x\mathrm{d}y \end{aligned}$$

Apply Thm. Fubini's Theorem:

$$\int_0^\infty \int_{g(x)>y}^\infty f(x)\mathrm{d}x\mathrm{d}y = \int_0^\infty \int_0^{g(x)} f(x)\mathrm{d}y\mathrm{d}x$$