6 Continuous Random Variables

Def. Continuous Random Var.

A random variable X is *continuous* if there exists a nonnegative function f so that, for every interval B,

$$P(X \in B) = \int_B f(x)\,dx,$$

The function $f = f_X$ is called the *density* of X. $\approx$ pmf for discrete

We will assume that a density function f is continuous, apart from finitely many (possibly infinite) jumps. Clearly, it must hold that

$\Rightarrow$ integrable

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \qquad \approx \sum_x \text{pmf}(x) = 1$$

Note also that

$$P(X \in [a,b]) = P(a \le X \le b) = \int_a^b f(x)\,dx,$$

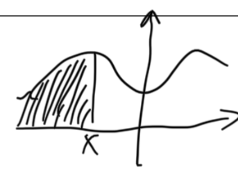
$$P(X = a) = 0, \quad = \int_a^a f(x)\,dx \quad = \frac{1}{\infty}$$

$$P(X \le b) = P(X < b) = \int_{-\infty}^b f(x)\,dx.$$

Def. Probability Density Function (p.d.f.)

The function $F = F_X$ given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(s)\,ds$$



is called the *distribution function* of X. On an open interval where f is continuous,

$$F'(x) = f(x).$$

Density has the same role as the probability mass function for discrete random variables: it tells which values x are relatively more probable for X than others. Namely, if h is very small, then

from:

$$P(X \in [x, x+h]) = F(x+h) - F(x) \approx F'(x) \cdot h = f(x) \cdot h. \qquad \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = F'(x)$$

By analogy with discrete random variables, we define, Discrete Version:

$$EX = \int_{-\infty}^{\infty} x \cdot f(x)\,dx, \qquad \sum_i p(x_i) \cdot x_i$$

for $g: \mathbb{R} \to \mathbb{R}$

$$Eg(X) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx, \qquad \sum_i p(x_i)\, g(x_i)$$

and variance is computed by the same formula: $\mathrm{Var}(X) = E(X^2) - (EX)^2$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

Highlighted examples are covered in class

Example 6.1. Let

$$f(x) = \begin{cases} cx & \text{if } 0 < x < 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine c. (b) Compute $P(1 \le X \le 2)$. (c) Determine EX and $\mathrm{Var}(X)$.

For (a), we use the fact that density integrates to 1, so we have $\int_0^4 cx\,dx = 1$ and $c = \frac{1}{8}$. For (b), we compute

Directly integrate $f(x)$
$$\int_1^2 \frac{x}{8}\,dx = \frac{3}{16}.$$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
since $f(x) = 0 \ \forall x \notin (0,4)$
we only need to find c from:
$\int_0^4 cx\,dx = 1$
$\left.\frac{cx^2}{2}\right|_0^4 = 8c = 1$
$c = \frac{1}{8}$

Finally, for (c) we get

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^4 x f(x)\,dx \implies EX = \int_0^4 \frac{x^2}{8}\,dx = \frac{8}{3}$$

and

$$E[X^2] = \int_0^4 x^2 f(x)\,dx \implies E(X^2) = \int_0^4 \frac{x^3}{8}\,dx = 8.$$

So, $\mathrm{Var}(X) = 8 - \frac{64}{9} = \frac{8}{9}$.

Example 6.2. Assume that X has density

$$f_X(x) = \begin{cases} 3x^2 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Compute the density f_Y of $Y = 1 - X^4$.

In a problem such as this, compute first the distribution function F_Y of Y. Before starting, note that the density $f_Y(y)$ will be nonzero only when $y \in [0,1]$, as the values of Y are restricted to that interval. Now, for $y \in (0,1)$,

$$\int_{(1-y)^{\frac{1}{4}}}^{\infty} 3x^2\,dx$$

$$F_Y(y) = P(Y \le y) = P(\underbrace{1 - X^4}_{Y=1-X^4} \le y) \xrightarrow{\text{rearrange}} P(1 - y \le X^4) = P((1-y)^{\frac{1}{4}} \le X) = \int_{(1-y)^{\frac{1}{4}}}^{1} 3x^2\,dx\,. = \left.\frac{3x^3}{3}\right|_{(1-y)^{\frac{1}{4}}}^{1} = \boxed{1 - (1-y)^{\frac{3}{4}}}$$

It follows that

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -3((1-y)^{\frac{1}{4}})^2 \frac{1}{4}(1-y)^{-\frac{3}{4}}(-1) = \frac{3}{4}\frac{1}{(1-y)^{\frac{1}{4}}},$$

$\frac{d}{dy}$

for $y \in (0,1)$, and $f_Y(y) = 0$ otherwise. Observe that it is immaterial how $f(y)$ is defined at $y = 0$ and $y = 1$, because those two values contribute nothing to any integral.

As with discrete random variables, we now look at some famous densities.

6.1 Uniform random variable

Such a random variable represents the choice of a random number in $[\alpha, \beta]$. For $[\alpha, \beta] = [0,1]$, this is ideally the output of a computer random number generator.

Feb 13 Final Example

Let the probability density function be:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

> **Question.** Find the expected value: $E[e^X]$

We first wrap e^X with $Y = e^x$, then let:

$$\begin{aligned} F_Y(y) = P(Y = y) &= P(e^X \leq y) \\ &= P(X \leq \ln(y)) \\ &= \int_{-\infty}^{\ln(y)} f(s)ds \\ &= \int_{0}^{\ln(y)} ds \\ &= \ln(y) \end{aligned}$$

The probability density function can be found by taking the derivative:

$$f_Y(y) = F_Y'(y) = \frac{1}{y} \qquad y \in [1, e]$$

`@Hint` Since $x \in [0,1], y \in [e^0, e^1] = [1, e]$

Now we can compute the expectation:

$$\begin{aligned} E[Y] &= E[e^X] \\ &= \int_{-\infty}^{\infty} x f_Y(x)dx \\ &= \int_{1}^{e} x \frac{1}{x} dx \\ &= e - 1 \end{aligned}$$