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Sandpile Dynamics on Random Graphs

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An annealed, random neighbor, critical height sandpile model (in which each site is assigned a fixed number k of new randomly chosen neighbors at every time step), can be exactly solved and shown to yield an avalanche size distribution $D(s)\propto s^{-\mu}$ with $\mu=3/2$ and a lifetime distribution $D(T)\propto T^{-\beta}$, with $\beta=2$ in the thermodynamic limit at fixed k.[1)] This exact result is predicted from the theory of branching processes, since each series of avalanches may be considered as a branching process provided that there are no loops. This condition is fulfilled as the number of sites in the system diverges while k remains constant, since in "high dimension", the paths of avalanches have low probabilities to self-intersect, so that the branching process approximation becomes exact.

In this paper, I am interested in a generalization of the random neighbor model with fixed k to random directed graphs (whose nodes are denoted by i) where the coordination numbers $k(i)$ are independent random variables. When k is the same for all nodes, the size distribution of avalanches is given by $D(s)\propto s^{-3/2}$ e^{-s/s_c},[1)] with the characteristic size $s_c\to\infty$ as the level of dissipation decreases. In effect, the size and duration of avalanches in a critical height model can only diverge if no dissipation is allowed since one avalanche would eventually wander from site to site without stopping, because there is no way out for excess matter. Because the power-law results hold up to the cutoff size for any k, one may guess that replacing a fixed value of k by *a distribution* $P(k)$ where every site has an independent probability $P(k)$ of having k outgoing vertices, will lead to qualitatively similar results, provided that P has a finite mean $\langle k\rangle$, but it remained to be checked. Such a model certainly applies to many phenomena ranging from biology to economics (see *e.g.* ref. 2), where no regular structure of interaction exists. The system's dynamics works as follows: (a) N_b nodes are randomly selected as boundaries. Each such site has only $k(i)-n(i)$ $(n(i)\le k(i))$ neighbors in the lattice, and $n(i)$ connections out of the lattice, where excess matter can flow: boundary sites dissipate $n(i)$ grains every time they topple. (b) Each node i possesses exactly $k(i)$ vertices directed towards other nodes ($k(i)-n(i)$ for boundary sites). The probability for the node to throw k connections is given by $P(k)$. In the simulations reported below, $P(k)$ is either gaussian with expectation $\langle k\rangle$ and variance σ, or uniform in the interval $[k_{\min}, k_{\max}]$. This model should look similar to the model with fixed k, with $k=\langle k\rangle$. Yet, due to the fact that k is now a random variable, fluctuations could prevent the process from being critical. (c) The initial condition consists in assigning a height $z=0$ to each site. (d) The driving consists in dropping one unit of height to a randomly selected site. (v) Every site is assigned a local critical height, $z_c(i)=k(i)-1$. The system is unstable if there is at least one site i (or several sites) whose height z_i is strictly greater than $z_c(i)$. The unstable site i and its forward neighbors are then updated as follows: (a) $z_i\to z_i-k(i)$; (b) if i is not a boundary site, then for the $k(i)$ forward neighbors j of i: $z_j\to z_j+1$; (c) if i is a boundary site, then for the $k(i)-n(i)$ forward neighbors j of i: $z_j\to z_j+1$. Relaxation is performed until no site is unstable.

Relaxation time is counted in global updates until the system reaches a stable state again. The size of an avalanche is defined as the number of toppling events. One toppling at site j causes b topplings with a probability $p_b=\sum_{\{\sigma_1,\cdots,\sigma_b\}}P(k(\sigma_1)-1)\cdots P(k(\sigma_b)-1)(1-P(k(\sigma_{b+1})-1))\cdots(1-P(k(\sigma_{k(j)})-1))$, with $0\le b\le k(j)$, where $\{\sigma_1,\cdots,\sigma_b\}$ denotes the set of all possible choices of b sites σ_m among the $k(j)$ neighbors of site j, and $P(k(\sigma_m)-1)$ is the probability that σ_m contains exactly $k(\sigma_m)-1$ grains, which corresponds to the treshold of instability of σ_m. The argument

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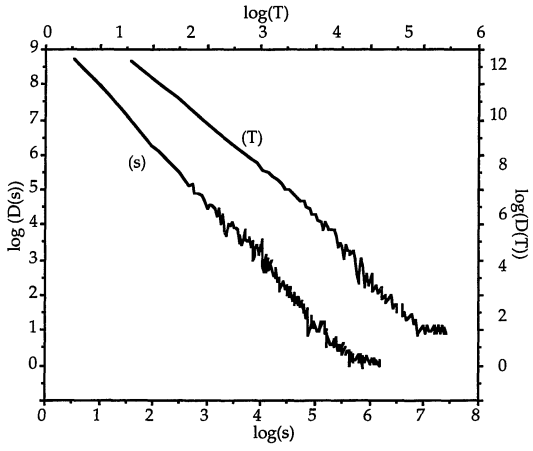

Fig. 1. $D(s)$ for gaussian $P(k)$, of mean 20 and variance 6 and $D(T)$ for $P(k)$ uniform on ref. 3.

justifying the use of the branching process approximation still holds here: there are almost no loops as N goes to ∞, provided $\langle k\rangle$ is finite. Therefore, $D(s)$ can be written as $D(s) \propto s^{-3/2} e^{-s/s_c}$, but the corresponding spatially averaged branching ratio $\langle b\rangle=(1/N)\sum_{j=1}^{N}\sum_{b=0}^{k(j)} bp_b$ cannot be determined analytically. If $k(j)$ is replaced by its expectation $k=\langle k\rangle$, the result of ref. 1 is obtained. Simulations show that indeed this model exhibits the same features as the model with fixed $k=\langle k\rangle$. The simulations have been performed on a quenched graph with $N=20000$. Figure 1 represents $D(s)$ and $D(T)$ for P gaussian ($\langle k\rangle=20$, $\sigma=8$) and uniform (ref. 3) respectively. The results for all other $P(k)$ share the same features. Finally, the cutoff time (in $D(T)$) as a function of dissipation, is the same as the one found with a mean-field prediction ref. 3 where k has been replaced by $\langle k\rangle$. In conclusion, the model with a distribution of $P(k)$ exhibits the same properties as the fixed k model.

References

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