Recursion

Theory Aspects of Recursion

Stefano Quer
Dipartimento di Automatica e Informatica
Politecnico di Torino

The stack

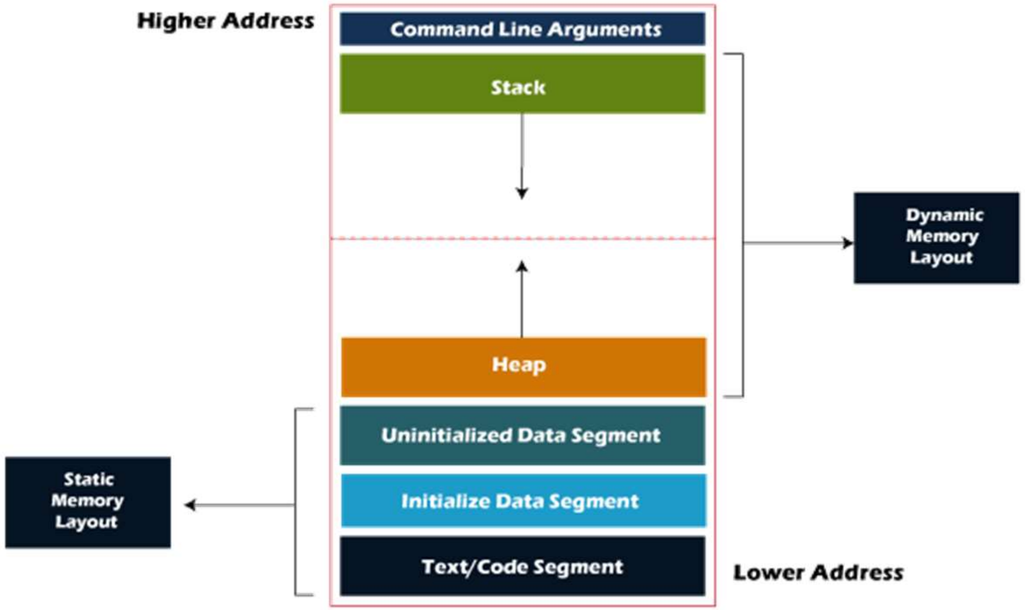
- In computer science a stack in an Abstract Data Type (ADT) that serves as a collection of elements

The **stack** was introduced in the unit on lists

- A stack supports the following operations
  - Push: Inserts objects on top
  - Pop: Extracts objects from the top
  - As both insertions and extractions are performed at the same end of the ADT, the stack follows a LIFO (Last-In First-Out) strategy

The stack

- A programmer can implement its own stacks
- The operating system (or any application) can use its own stack as well



Memory Layout

The stack

- For a C compiler, the stack is the data structure containing at least
  - Formal parameters
  - Local variables
  - The return address when the function execution is over
  - The pointer to the function's code

The stack

- All these pieces of data form a **stack frame**
  - A new stack frame is created when the function is called and the same stack frame is destroyed when the function is over
- Stack frames are stored in the system stack
  - The system stack has a predefined amount of memory available
    - When it goes beyond the space allocated to it, a **stack overflow** occurs
  - The stack grows from larger to smaller addresses (thus upwards)
  - The **stack pointer SP** is a register containing the address of the first available stack frame

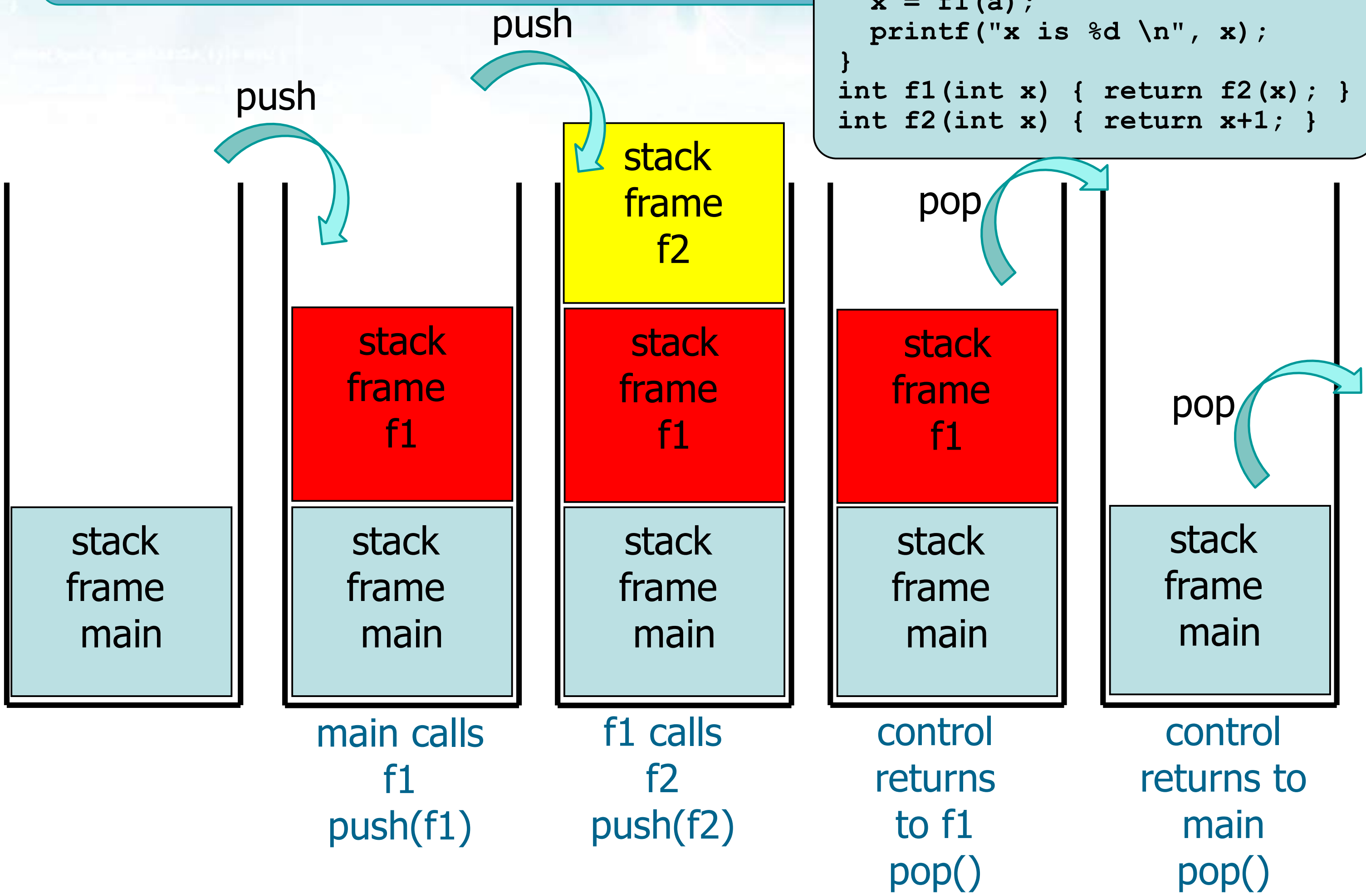
Example

- Let us analyze the stack structure during the execution of the following program

```
int f1(int x);
int f2(int x);

main() {
  int x, a = 10;
  x = f1(a);
  printf("x is %d \n", x);
}
int f1(int x) {
  return f2(x);
}
int f2(int x) {
  return x+1;
}
```

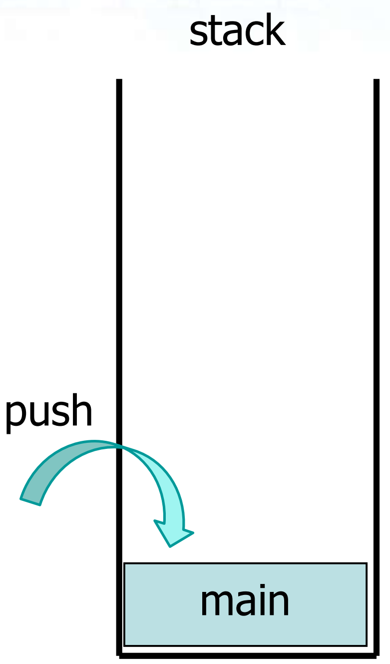
```
int f1(int x);
int f2(int x);
main() {
  int x, a = 10;
  x = f1(a);
  printf("x is %d \n", x);
}
int f1(int x) { return f2(x); }
int f2(int x) { return x+1; }
```



Recursive functions

- With recursive functions
  - Calling and called functions coincide, but operate on different data
  - The system stack is used as in any other function call
- Too many recursive calls may result in a stack overflow

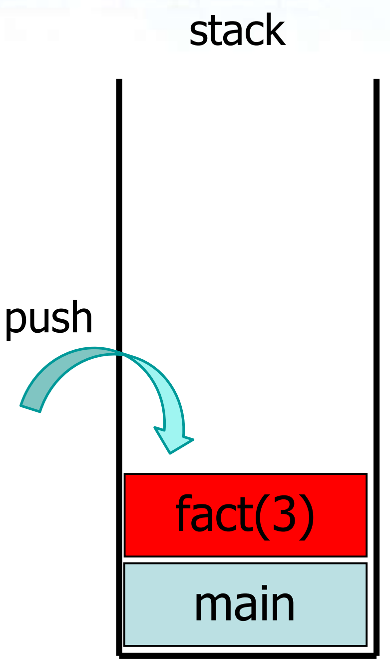
Example 1



Initial configuration

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

Example 1

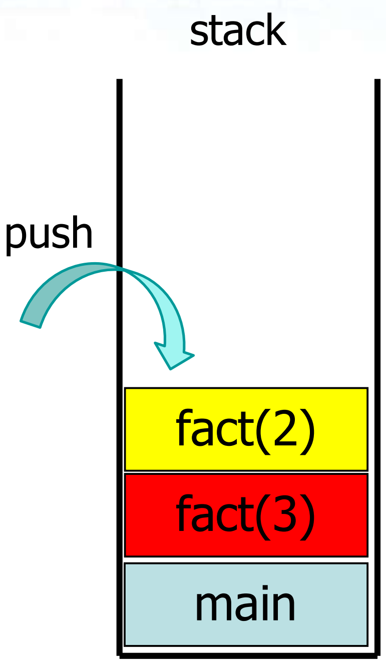


main calls fact(3)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

3! = 3*2!

Example 1



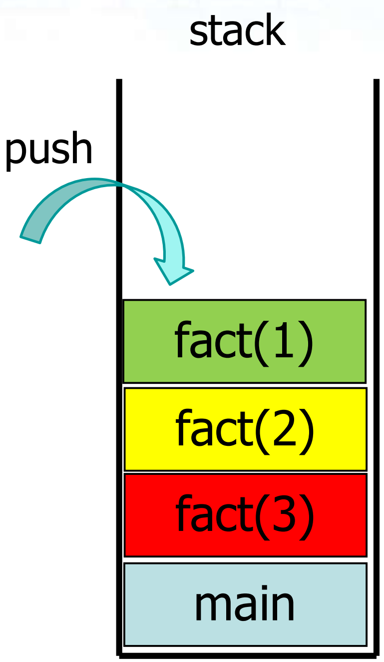
fact(3) calls fact(2)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

$3! = 3*2!$

$2! = 2*1!$

Example 1



fact(2) calls fact(1)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

$3! = 3*2!$

$\Downarrow$

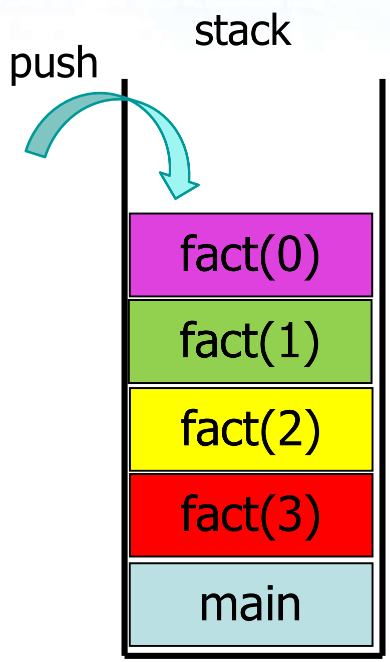
$2! = 2*1!$

$\Downarrow$

$1! = 1*0!$

Example 1

Hit the termination condition



fact(1) calls fact(0)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

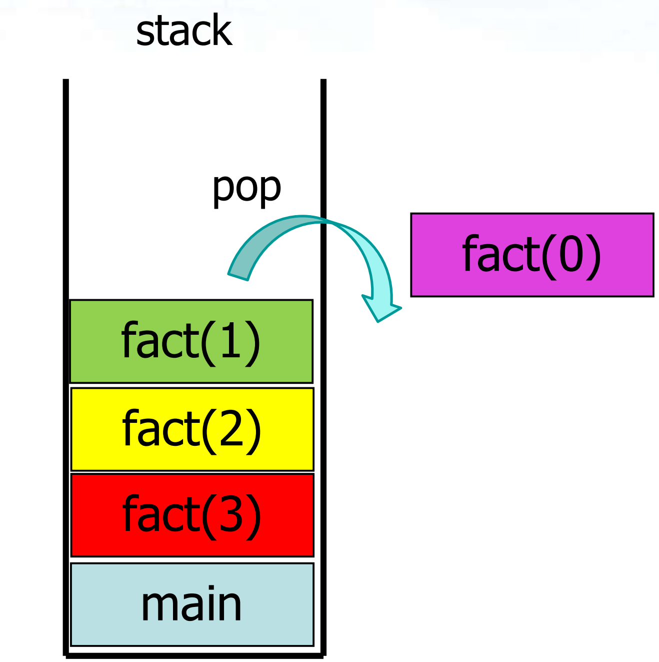
3! = 3*2!

2! = 2*1!

1! = 1*0!

0! = 1

Example 1



fact(0) terminates,
returns value 1 and
returns control
to fact(1)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

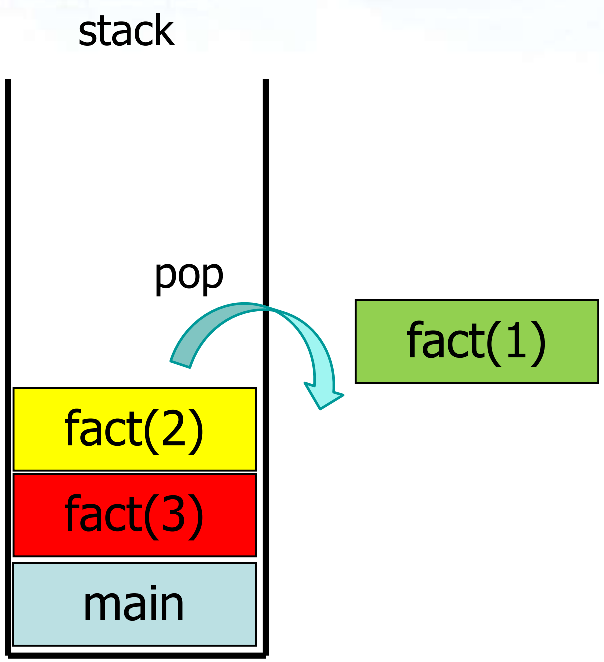
3! = 3*2!

2! = 2*1!

1! = 1*0!

0! = 1

Example 1



fact(1) terminates, returns value 1 and returns control to fact(2)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

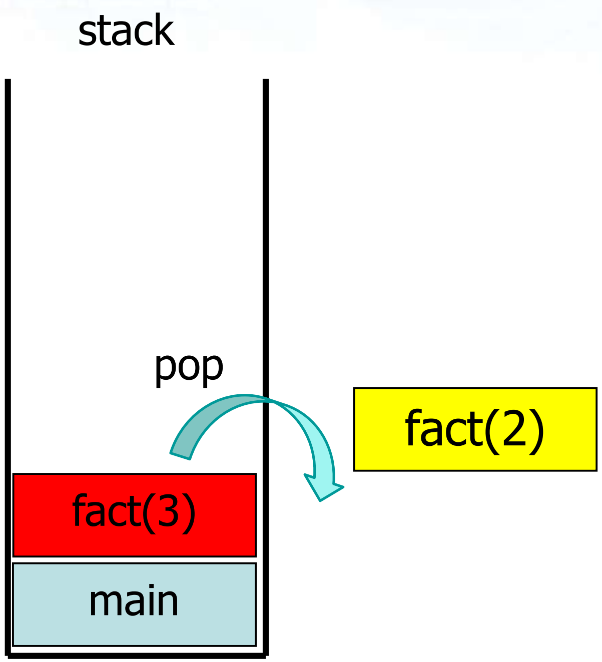
3! = 3*2!

2! = 2*1!

1! = 1*0! = 1

0! = 1

Example 1



fact(2) terminates, returns value 2 and returns control to fact(3)

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

3! = 3*2!

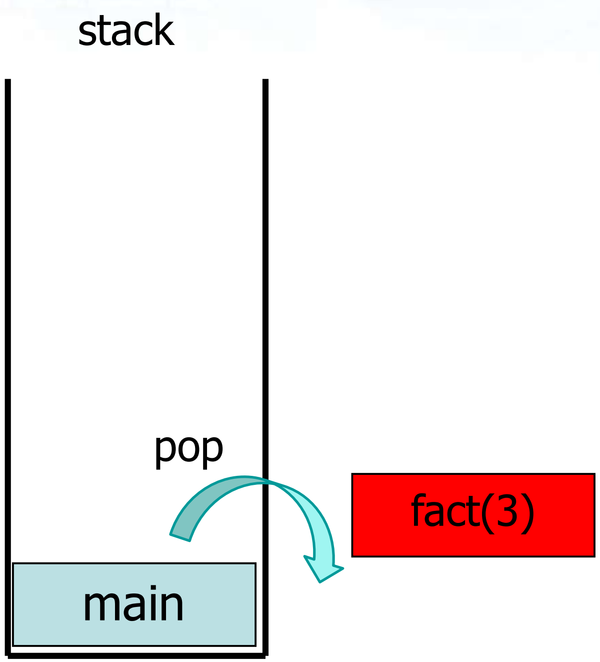
2! = 2*1! = 2

1! = 1*0! = 1

0! = 1

Example 1

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```



fact(3) terminates, returns value 6 and returns control to main

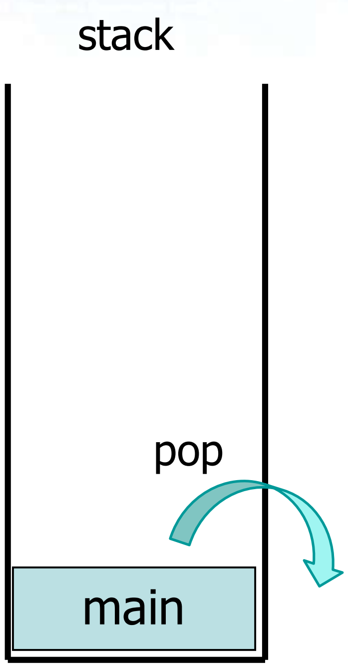
3! = 3*2! = 6

2! = 2*1! = 2

1! = 1*0! = 1

0! = 1

Example 1



fact(3) terminates, returns value 6 and returns control to main

```
main() {
  long n;
  printf("Input n:  ");
  scanf("%d", &n);
  printf("%d %d \n",n, fact(n));
}
long fact(long n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

3! = 3*2! = 6

2! = 2*1! = 2

1! = 1*0! = 1

0! = 1

Recursion versus iteration

- Recursion
  - May be memory-consuming
  - Is somehow equivalent to looping
- All recursive programs may be implemented in iterative form as well
  - There is a duality between recursion and iteration
- The best solution (efficiency and clarity of code) depends on the problem
- Try to remain at the highest possible abstraction level

Duality recursion - iteration

- Factorial iterative computation
  - 5! = 1*2*3*4*5 = 120
  - The implementation may be iterative and recursive as well
  - There is no need to use a stack

```
long fact(long n) {
  long tot = 1;
  int i;

  for (i=2; i<=n; i++)
    tot = tot * i;

  return(tot);
}
```

Duality recursion - iteration

- Fibonacci iterative computation
  - 0 1 1 2 3 5 8 13 21 …
    - F(0) = 0
    - F(1) = 1
    - F(2) = F(0) + F(1) = 1
    - F(3) = F(1) + F(2) = 2
    - F(4) = F(2) + F(3) = 3
    - F(5) = F(3) + F(4) = 5
  - The implementation may be iterative and recursive as well
  - There is no need to use a stack

```
long fib(long int n) {
  long int f1p=1, f2p=0, f;
  int i;
  if(n == 0 || n == 1)
    return(n);
  f = f1p + f2p;
  for(i=3; i<= n; i++) {
    f2p = f1p;
    f1p = f;
    f = f1p+f2p;
  }
 return(f);
}
```

Duality recursion - iteration

- Binary search
  - The implementation may be iterative and recursive as well
  - There is no need to use a stack

```
int BinarySearch (
  int v[], int l, int r, int k) {
  int c;

  while (l<=r){
    c = (int) ((l+r) / 2);
    if (k == v[c]) {
      return(c);
    }
    if (k < v[c]) {
      r = c-1;
    } else {
      l = c+1;
    }
  }
  return(-1);
}
```

Emulating recursion

- Recursion may be emulated explicitly dealing with a stack
  - Recursion is realized using the system stack to store and restore the local status
  - It is always possible to emulate recursion through iterations using a user-defined stack
    - The user stack mimics the system stack
    - It is manipulated by the programmer to store and restore information (function stack frames) as the system does into the system stack

Emulating recursion

```
long fact(long int n) {
  if(n == 0)
    return(1);
  return(n * fact(n-1));
}
```

Original recursive function

The ADT stack is a user-defined list_t object

```
long fact(long int n) {
  long fact = 1; int status;
  list_t *top;
  top = NULL;
  while (n>0) {
    top = push (top, n);
    n--;
  }
  do {
    top = pop (top, &n, &status);
    if (status!=FAILURE)
      fact = n * fact;
  } while (status!=FAILURE);
  return fact;
}
```

Non recursive (iterative) function emulating recursion using a user stack

Tail-recursive functions

- In the traditional recursive functions
  - Recursive calls are performed first
  - Then, the return value is used to compute the result
  - The final result is obtained after all calls have terminated, i.e., the program has returned from every recursive call
- Tail-recursion (or tail-end recursion) is a particular case of recursion

Tail-recursive functions

- In tail recursive function, the recursive call is the last operation to be executed, except for return

```
long fact(long int n) {
  if (n == 0)
    return(f);
  return fact(n*fact(n-1));
}
```

The system stack is required

This function is not tail-recursive because the product can be executed only after returning from the recursive call

```
fact(3)
3 * fact(2)
3 * (2 * fact(1))
3 * (2 * (1 * fact(0)))
3 * (2 * (1 * 1))
```

Tail-recursive functions

- Tail-recursive version of the factorial function

```
long fact(long int n) {
  if (n == 0)
    return(f);
  return fact(n*fact(n-1));
}
```

This function is tail-recursive because the product is executed before the recursive call

The system stack is **not** required

```
fact_tr(3,1)
fact_tr(2,3)
fact_tr(1,6)
fact_tr(0,6)
```

The caller sets n = 1

```
long fact_tr(long n, long f){
  if (n==0)
    return f;
  return fact_tr(n-1,n*f);}
```

Tail-recursive functions

- In tail recursive functions
  - Calculations are performed first
  - Recursive calls are done after
  - Current results are passed to future calls
    - The return value of any given recursive step is the same as the return value of the next recursive call
    - The consequence of this is that once you are ready to perform your next recursive step, you do not need the current stack frame any more

Tail-recursive functions

- Current stack frame is not needed anymore
  - Recursion can be substituted by a simple jump (**tail call elimination**)
  - A proper compiler or language (Prolog, Lisp, etc.) may recognize tail recursive functions and it may optimize their code
  - Stack overflows does not happen anymore and there is no limit to the number of recursive calls that can be made
- Tail recursion is essentially equivalent to looping
- Tail recursion only applies if there are no instructions that follow the recursive call

Solution

```
void print (char *s) {
  if (*s == '\0') {
    return;
  }
  printf ("%c", *s);
  print (s+1);
  return;
}
```

Printing a string:
There are no instructions that follow the recursive call.
The compiler may understand this and it may avoid the stack. This function is tail recursive.

```
void reverse_print (char *s) {
  if (*s == '\0') {
    return;
  }
  reverse_print(s+1);
  printf ("%c", *s);
  return;
}
```

Reverse printing a string:
There are instructions that follow the recursive call.
The stack cannot be avoided.
This function is not tail recursive.

Limits of the recursion

- Disadvantages
  - The number of recursions is limited by the stack size
    - The stack consume memory
  - Sub-problems may not be independent, and recomputations may occur leading to inefficiency