Graphs

Definitions

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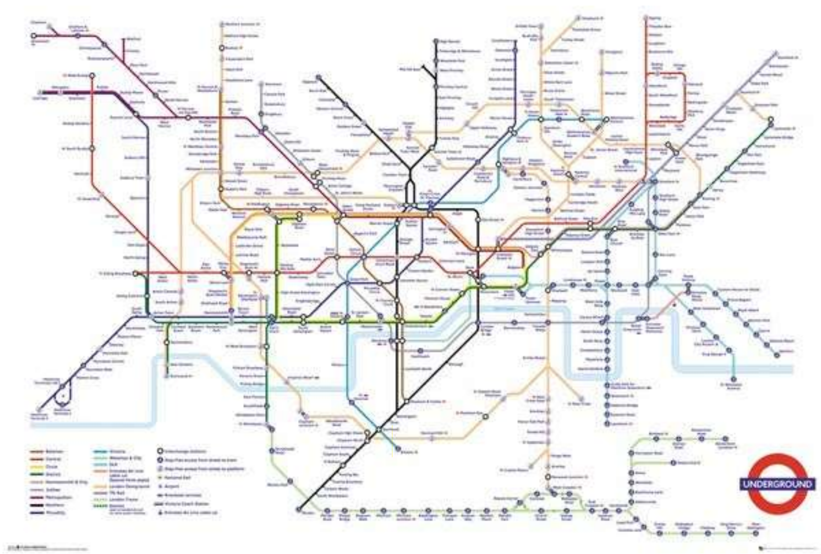
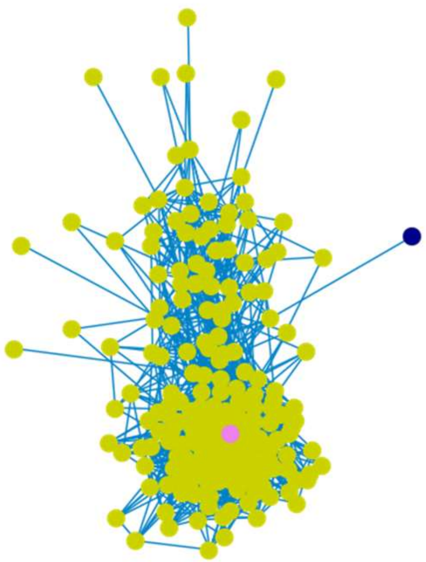
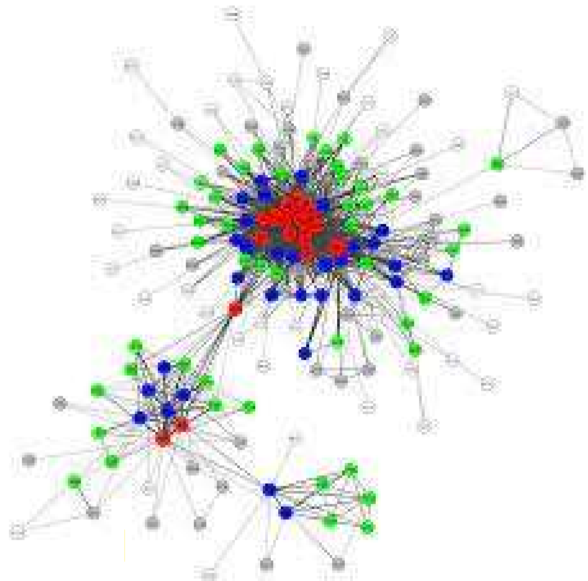
Graphs

- A graph G is a pair $G = (V, E)$ where
  - V is a finite and non empty set of vertices
    - Each vertex represents simple or complex data
  - E is a finite set of edges
    - Edges define a binary relation on V







Applications

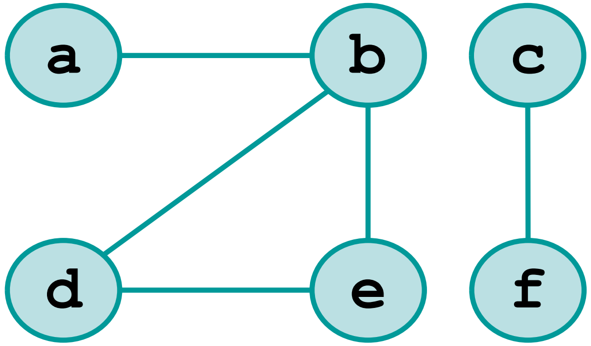
Domain	Vertex	Edge
communications	phone, computer	fiber optic, cable
circuits	gate, register, processor	wire
mechanics	joint	spring
finance	stocks, currencies	transactions
transports	airoport, station	air corridor, railway line
games	position on board	legal move
social networks	person	friendship
neural networks	neuron	synapsis
chemical compounds	molecules	link

Type of graphs

- Graphs can be
  - Undirected
    - Each edge is an unsorted pair of vertices $(u, v) \in E$ and $u, v \in V$
  - Directed
    - Each edge is a sorted pair of vertices $(u, v) \in E$ and $u, v \in V$
  - Weighted
    - Edges (undirected or directed) have a weight
      - $\exists w : E \rightarrow R \backslash \mathrm{w}(u, v) = weigth\ \ of\ edge\ (u, v)$
      - In practice, weights may be integers, reals, positive or negative values, etc.

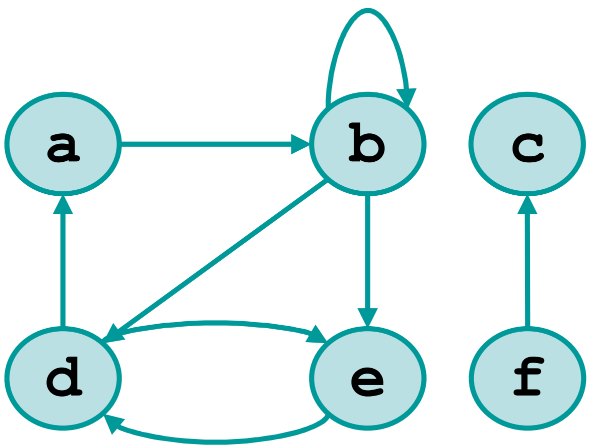
Examples

- Undirected graph




```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(b,d),(b,e),(d,e),
     (c,f)}
```

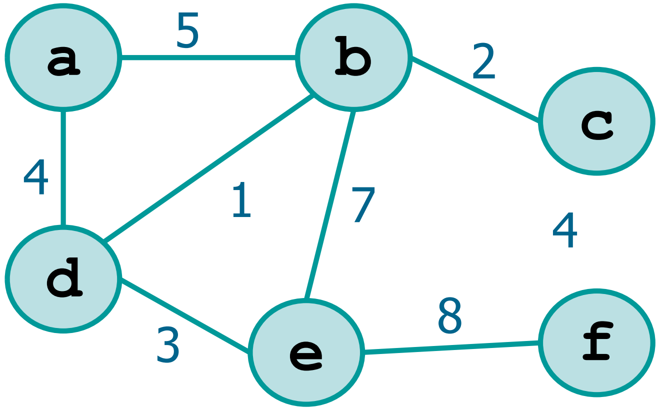
- Directed graph




```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(b,b),(b,d),(b,e),
     (d,a),(d,e),(e,d),(f,c)}
```

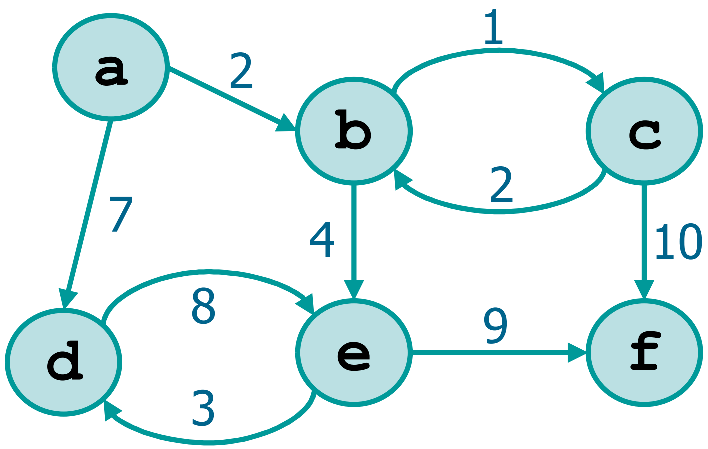
Examples

- Weighted undirected graph




```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(a,d),(b,c),(b,d),
     (b,e),(d,e),(e,f)
```

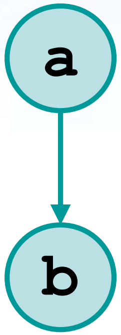
- Weigthed directed graph



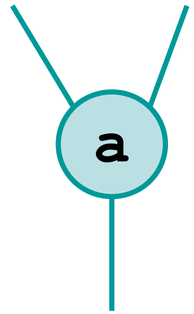

```
G = {V, E}
V = {a, b, c, d, e, f}
E = {(a,b),(a,d),(b,c),(b,e),
     (c,b),(c,f),(d,e),(e,d),
     (e,f)}
```

Edges

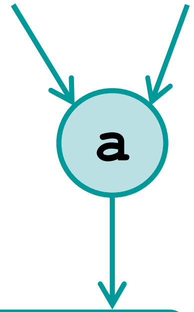
- Given an edge (a, b)
  - Vertices a and b are adjacent
    - $a \rightarrow b \quad \leftrightarrow \quad (a, b) \in E$
  - Edge (a, b) is incident
    - From vertex a, in vertex b, on vertices a and b



$Degree\ (a) = 3$



- For
  - Undirected graph the degree of a node is the number of incident edges
  - Directed graph the
    - In-degree (out-degree) of a node is the number of incoming (outgoing) edges
    - The degee of a node is its in-degree plus its out-degree



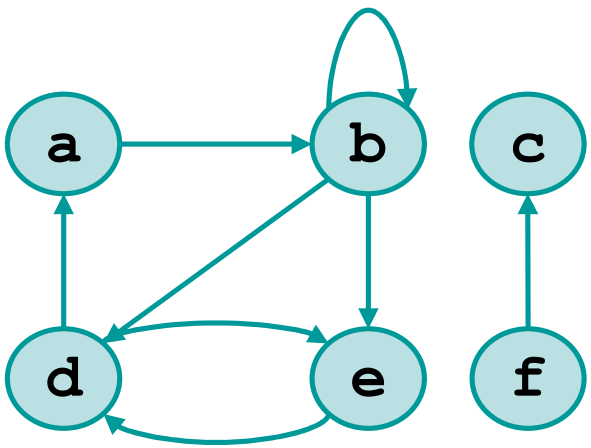
$In - degree\ (a) = 2$
$Out - degree\ (a) = 1$
$Degree\ (a) = 3$

Paths

- A path $p\text{: u} \rightarrow_p \hat{u}$ is defined in $G = (V, E)$ as the sequence of vertices leading from u top $\hat{u}$

$$\exists(v_0, v_1, \cdots, v_k) |\ u = v_0, \hat{u} = v_k, \forall i = 1,2, \cdots, k\ (v_{i-1}, v_i) \epsilon E$$

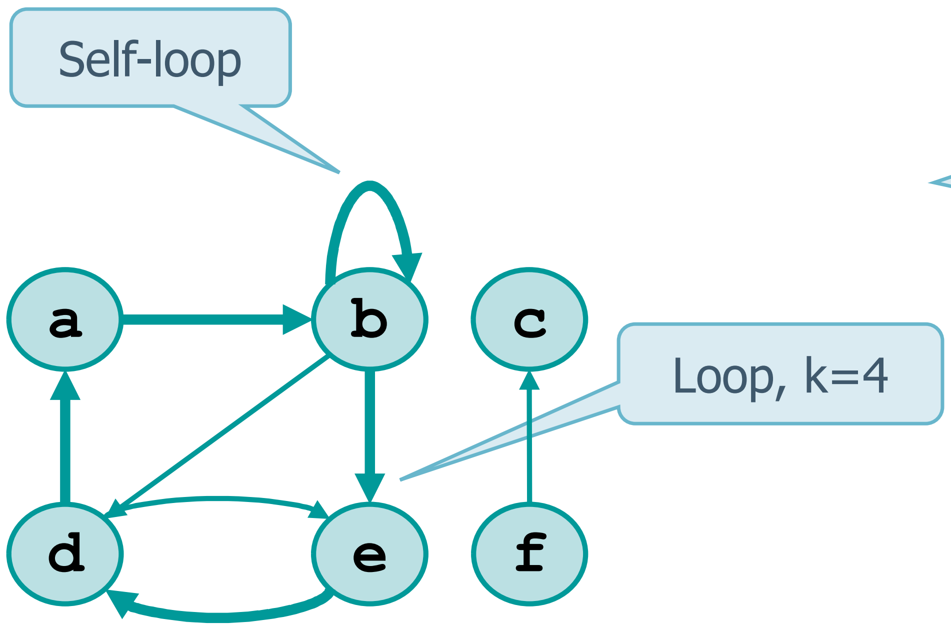
  - k is the length of the path
  - $\hat{u}$ is reachable from u iff $\exists p\text{: } u \rightarrow_p \hat{u}$
  - A path p is simple if $(v_0, v_1, \cdots, v_k) \in p$ are dinstinct



G = (V, E)
p: a $\rightarrow_p$ d : (a, b), (b, c), (c, d)
k = 3
d is reachable from a
p is a simple path

Loops

- A loop is defined as a path where $v_0 = v_k$, i.e., the starting and arrival vertices do coincide
- A graphs without loops is called acyclic
- Self-loops are loops whose length is 1



In some contexts self-loops may be forbidden.
If the context allows loops, but the graph is self-loop-free, it is called **simple**

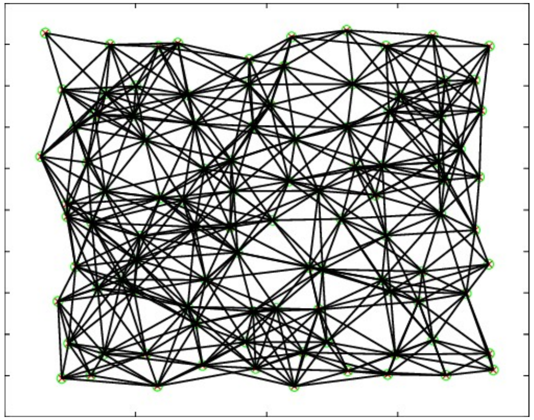
Dense and sparse graphs

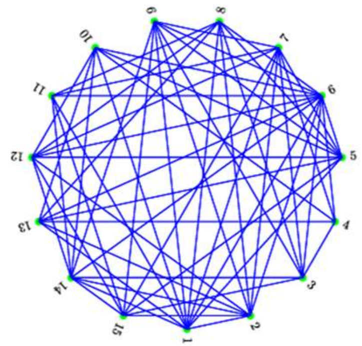
- Given a graph $G = (V, E)$

$|V|$ = Number of vertices, cardinality of the set V
$|E|$ = Number of edges, cardinality of the set E

- We define
  - Dense graph
    - $|E| \cong |V|^2$

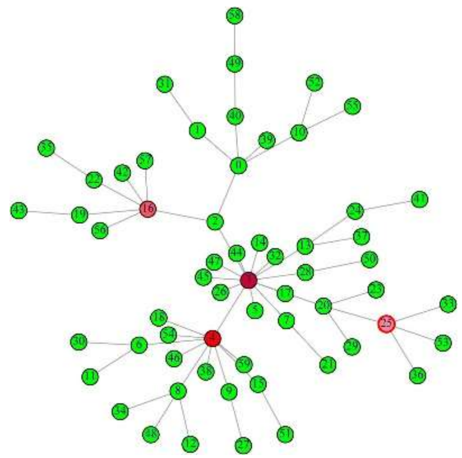
A lot of edges

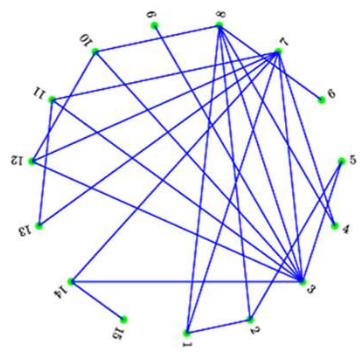




  - Sparse graph
    - $|E| \ll |V|^2$

Few edges

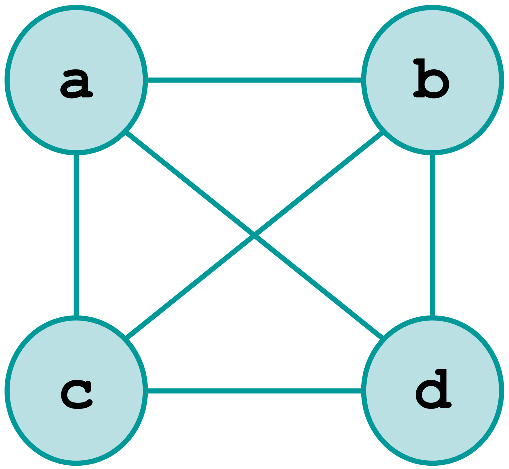


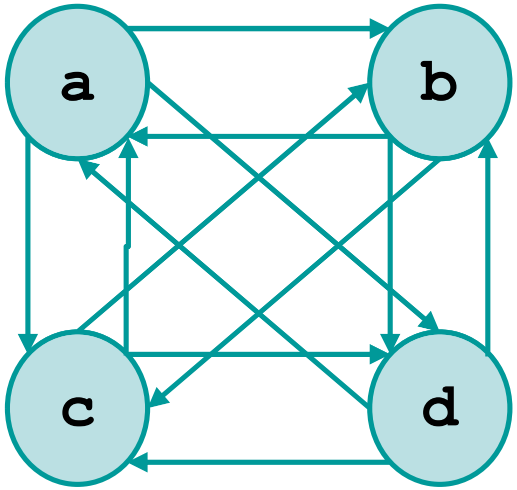


Complete graphs

- Given a graph $G = (V, E)$ the graph is complete iff

$$\forall v_i, v_j \in V \quad \rightarrow \quad \exists (v_i, v_j) \in E$$





Complete graph

- How many edges there are in a complete
  - Undirected graph?
    - $|E|$ is given by the number of simple **simple combinations** of $|V|$ elements taken 2 by 2
      - $|E| = \frac{|V|!}{(|V|-2)! \cdot 2!} = \frac{|V| \cdot (|V|-1) \cdot (|V|-2)!}{(|V|-2)! \cdot 2!} = \frac{|V| \cdot (|V|-1)}{2}$

Combinations
Order does not matter

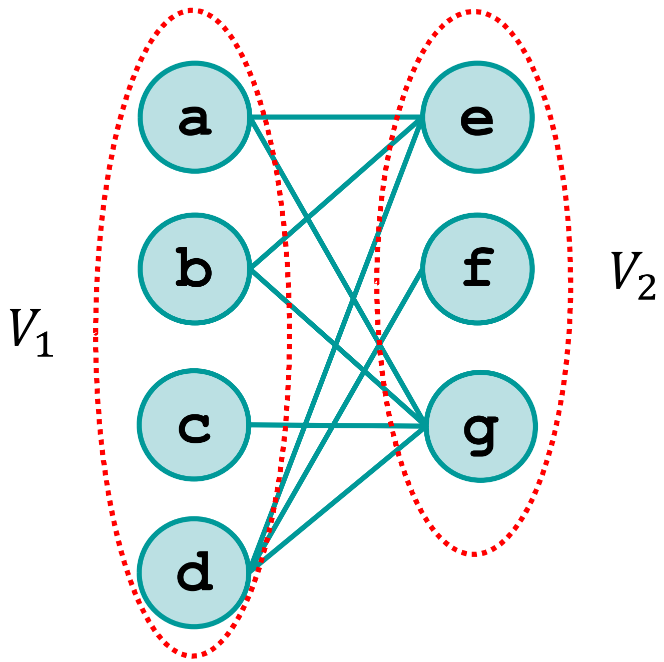
  - Directed graph?
    - $|E|$ is the number of **simple arrangements** of $|V|$ elements taken 2 by 2
      - $|E| = \frac{|V|!}{(|V|-2)!} = \frac{|V| \cdot (|V|-1) \cdot (|V|-2)!}{(|V|-2)!} = |V| \cdot (|V|-1)$

Arrangements
Order matters

Bipartite graph

- A bipartite graph is an undirected graph wher
  - The V set may be partitioned in two subsets V_1 and V_2, such that

$$\forall (v_i, v_j) \in E \quad \textbf{\textit{then}} \quad (v_i \in V_1 \ and \ v_j \in V_2) \ \textbf{\textit{or}} \ (v_i \in V_2 \ and \ v_j \in V_1)$$



Classification