POLITECNICO
DI TORINO
Dipartimento
di Automatica e Informatica

Iterative Quadratic Sorting Algorithms

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Insertion Sort

Let N integers be stored in an array A whose indices range from l=0 and r=N-1.

Conceptually array A consists of 2 subarrays:

- the left one: already sorted
- the right one: not yet sorted

Initially the left subarray contains 1 item, the right subarray contains N-1 items.

An array with just 1 item is sorted by definition.

Approach

Incremental paradigm:

- At each step we expand the already sorted left subarray inserting an item taken from the still unsorted right subarray
- Insertion must guarantee that the left subarray remains sorted after inserting the item (invariance of the sorting property)
- termination: all items have been properly inserted, the left subarray contains N items, the right subarray is empty.

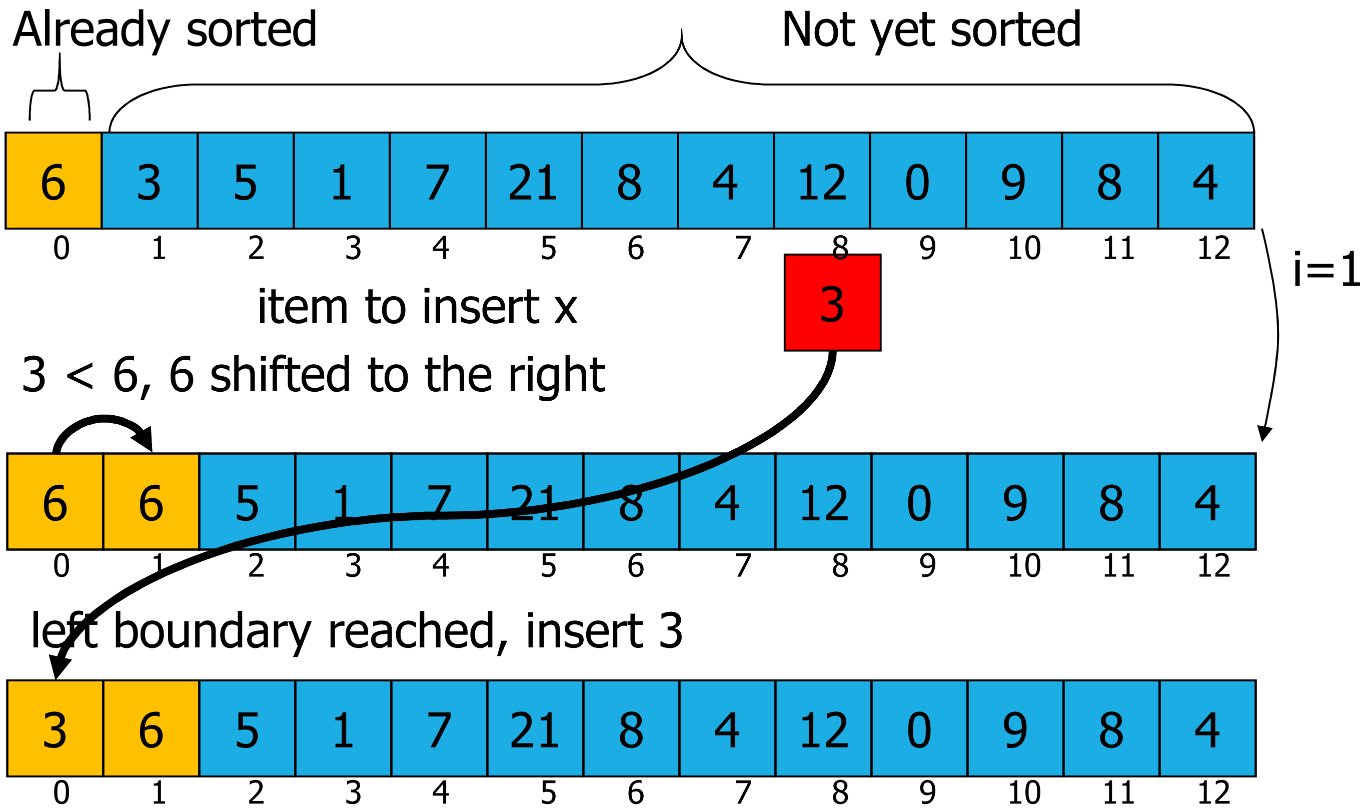
Step i: sorted insertion

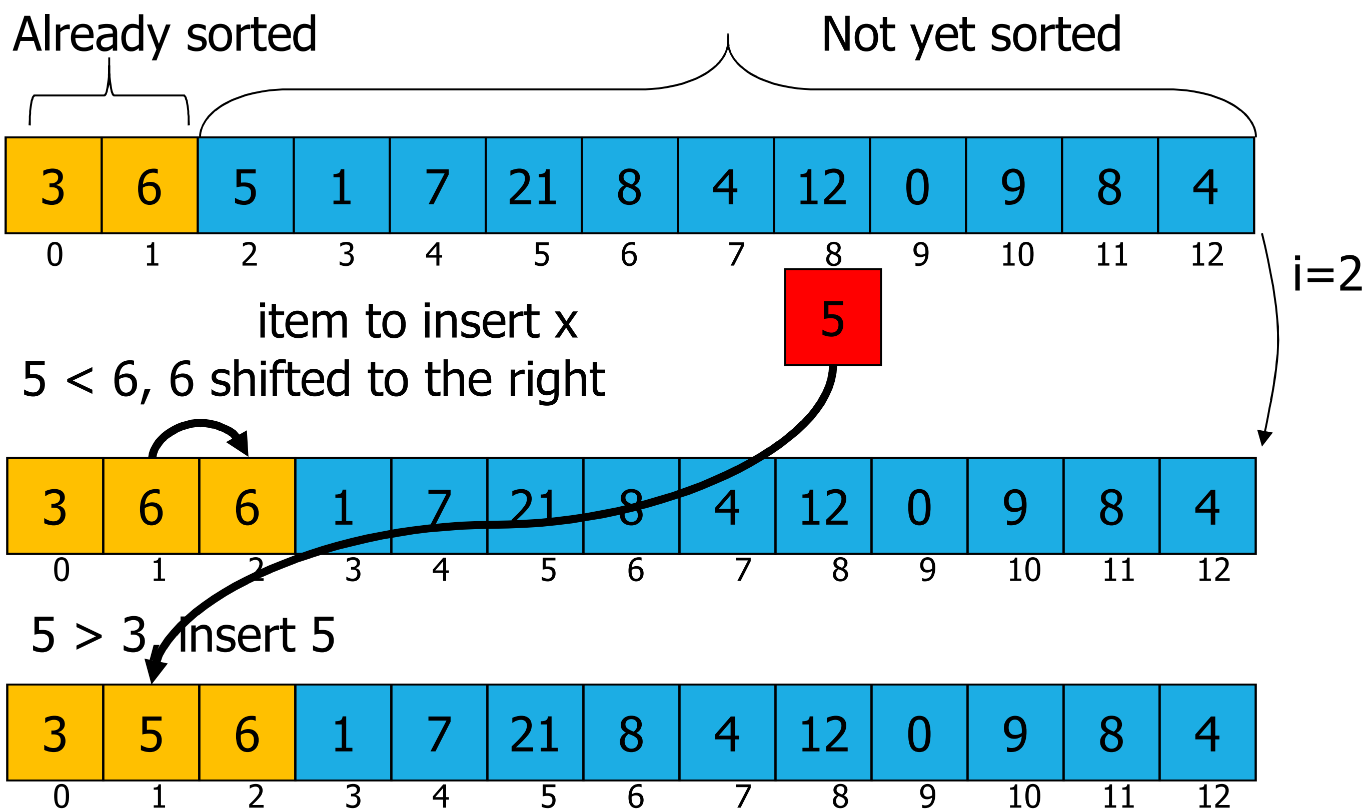
At step i item x = A_i is stored at the correct position in the left subarray:

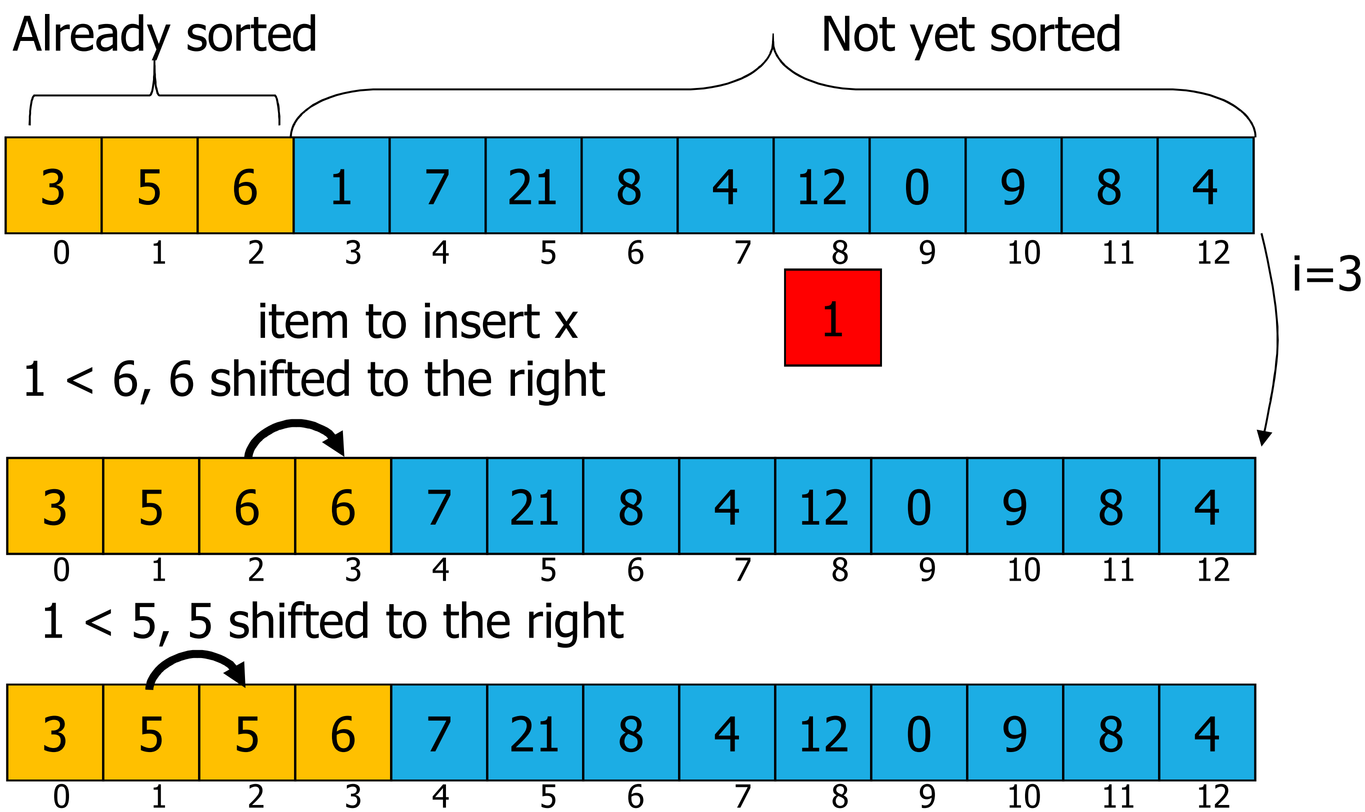
- the left subarray (already sorted and ranging from A_{i-1} to A_0) is scanned until an item is found such that $A_j > A_i$
- during the scan, items in the range from A_j to A_{i-1} are shifted to the right by 1 position

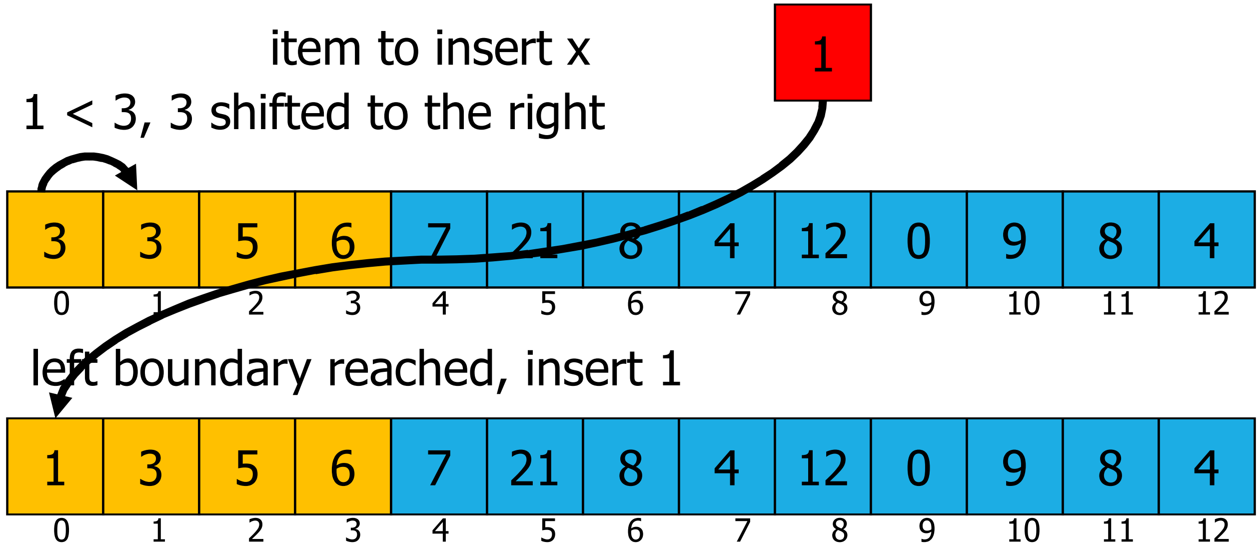
When the loop terminates the correct position for inserting A_i has been found.

Example


Already sorted
Not yet sorted
6 3 5 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
i=1
item to insert x
3
3 < 6, 6 shifted to the right
6 6 5 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
left boundary reached, insert 3
3 6 5 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12


Already sorted
Not yet sorted
3 6 5 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
i=2
item to insert x
5
5 < 6, 6 shifted to the right
3 6 6 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
5 > 3, insert 5
3 5 6 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12


Already sorted
Not yet sorted
3 5 6 1 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
i=3
item to insert x
1
1 < 6, 6 shifted to the right
3 5 6 6 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
1 < 5, 5 shifted to the right
3 5 5 6 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12

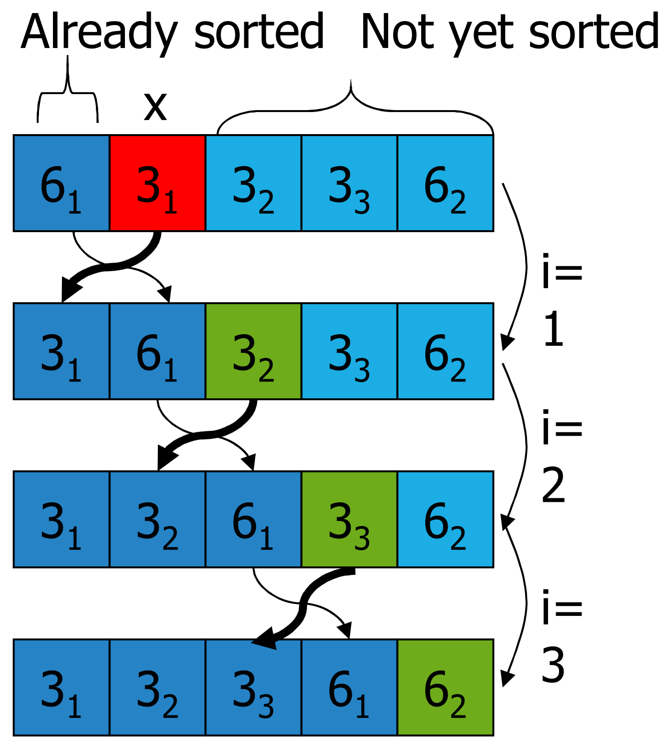


item to insert x
1 < 3, 3 shifted to the right
1
3 3 5 6 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12
left boundary reached, insert 1
1 3 5 6 7 21 8 4 12 0 9 8 4
0 1 2 3 4 5 6 7 8 9 10 11 12

```
void InsertionSort(int A[], int N) {
  int i, j, l=0, r=N-1, x;
  for (i = l+1; i <= r; i++) {
    x = A[i];
    j = i - 1;
    while (j >= l && x < A[j]){
      A[j+1] = A[j];
      j--;
    }
    A[j+1] = x;
  }
}
```

Insertion sort Features

- in place: used only array A and variable x
- stable: if the item to insert is a duplicate key, it can’t jump over to the left a previous occurrence of the same key



Complexity Analysis of Insertion sort

Worst-case asymptotic analysis:

- Based on each step of the statements used
- Based on the high-level behaviour of the statements used

to estimate

- The number of comparisons
- The number of swaps.

Detailled Analysis of the # of comparisons

Assumption:

- All instructions have unit cost
- i starts from 1 and increases to N-1 (for simplicity no use of l and r)

instructions	# executions
`for(i=1; i<N; i++) {`	2N

- 2 instructions at each step (< and +)
- N executions taking into account the last one as well when the condition i<N fails

instructions	# executions
`for(i=1; i<N; i++) {`	2N
`x = A[i];`	N-1
`j = i-1;`	N-1

N-1 executions only when condition i<N holds

instructions	# executions
`for(i=1; i<N; i++) {`	2N
`x = A[i];`	N-1
`j = i-1;`	N-1
`while (j>=0 && x<A[j]){`	$2\sum_{j=2}^{N} j$

- 2 instructions at each step (>=and <)
- N-1 executions of the loop in the worst case (array already sorted in descending order)
- j operations at each loop execution taking into account as well the last one when condition j>=0 fails

instructions	# executions
`for(i=1; i<N; i++) {`	2N
`  x = A[i];`	N-1
`  j = i-1;`	N-1
`  while (j>=0 && x<A[j]){`	$2\sum_{j=2}^{N} j$
`    A[j+1] = A[j];`	$\sum_{j=2}^{N}(j-1)$
`    j--;`	$\sum_{j=2}^{N}(j-1)$
`  }`	

- N-1 executions of the loop in the worst case (array already sorted in descending order)
- j-1 operations at each loop execution only when the loop condition holds

instructions	# executions
`for(i=1; i<N; i++) {`	2N
`  x = A[i];`	N-1
`  j = i-1;`	N-1
`  while (j>=0 && x<A[j]){`	$2\sum_{j=2}^{N} j$
`    A[j+1] = A[j];`	$\sum_{j=2}^{N}(j-1)$
`    j--;`	$\sum_{j=2}^{N}(j-1)$
`  }`	
`  A[j+1] = x;`	N - 1
`}`	

N-1 executions only when condition i<N holds

Thus:

$T(N) = 2N+(N-1)+(N-1)+2\sum_{j=2}^{N} j+\sum_{j=2}^{N}(j\text{-}1) + \sum_{j=2}^{N}(j\text{-}1)+(N-1)$

Recalling that:

$\sum_{j=2}^{N} j = 2+3+\ldots N = N(N+1)/2 - 1$

$\sum_{j=2}^{N}(j-1) = N(N-1)/2$

$T(N) = 2N + 3(N-1) + 2(N(N+1)/2 - 1) + 2(N(N-1)/2) = 2N^2 + 6N - 6$

$T(N) = O(N^2)$: the number of comparisons in the worst case grows quadratically.

High-level Complexity Analysis

Two nested loops:

- Outer loop: N-1 executions
- Inner loop in the worst case: i executions at the i-th iteration of the outer loop

Complexity:

$$T(N) = 1+2+3+ \ldots +(N-2)+(N-1)$$

$$= \Sigma_{1 \leq i < N} i = N(N-1)/2$$

T(N) grows quadratically with N.

Finite arithmetic progression with ratio 1 (Gauss, end of XVII cent.)

$T(N) = O(N^2)$: the number of comparisons and of swaps grows quadratically.

Bubble/Exchange sort

Let N integers be stored in an array A whose indices range from l=0 and r=N-1.

Conceptually array A consists of 2 subarrays:

- the right one: already sorted
- the left one: not yet sorted

Initially the right subarray is empty, the left one contains N items.

The basic operation is a comparison between adjacent items of array A (A[j] and A[j+1]), swapping them if A[j] > A[j+1].

Approach

Incremental paradigm:

- At each step expand the already sorted right subarray by inserting an item taken from the not yet sorted left subarray
- Insertion must guarantee that the right subarray remains sorted after insertion (invariance of the sorting property)
- At iteration i insert the largest item of the left subarray (A_l ... A_{r-i+l}) into the leftmost position of the right subarray A[r-i+l]. The sorted right subarray grows by 1 in size to the left, dually the unsorted left one decreases in size by 1
- Termination: all items have been properly inserted, the right subarray contains N items, the left one is empty.

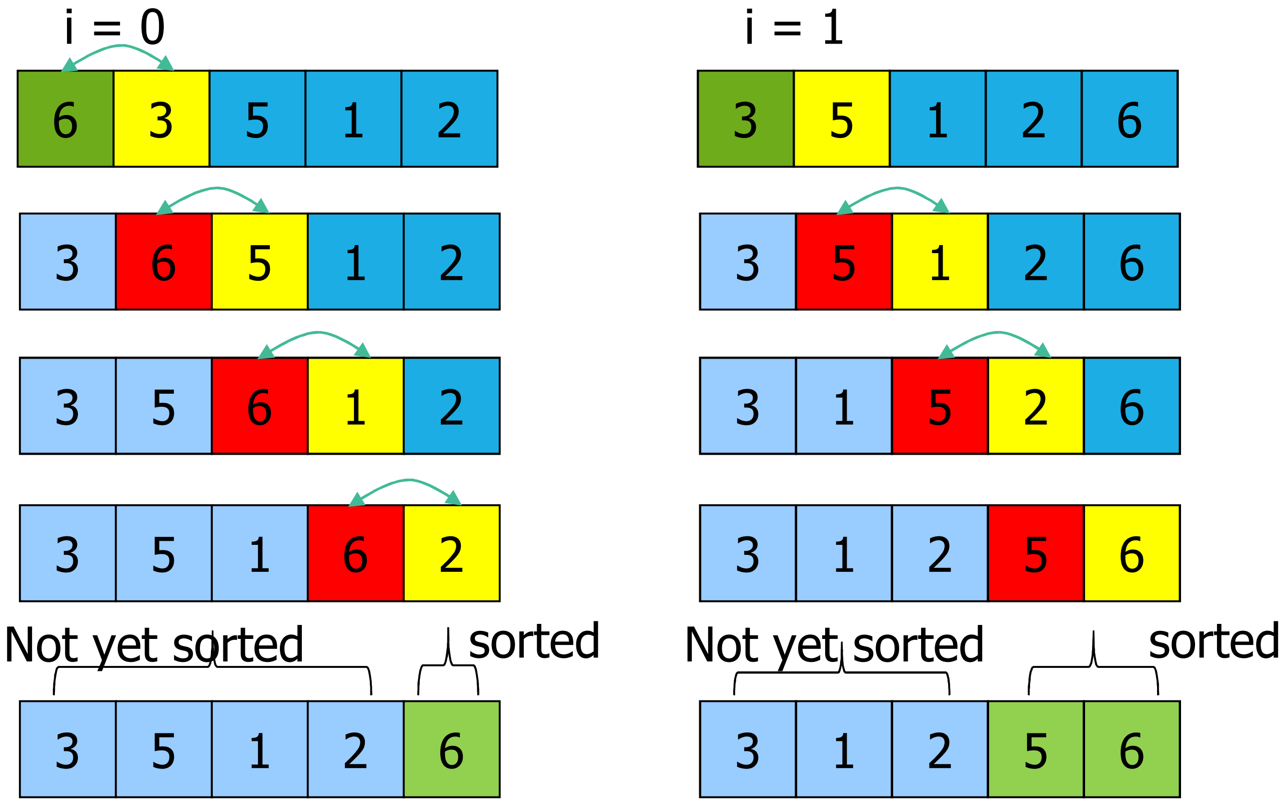
Step i: identifying the maximum

Identification of the largest item in the left subarray at step i

- scan the left subarray comparing item pairs A[j] and A[j+1] and swapping them if A[j] > A[j+1].

When the loop ends, the largest item has «floated» like a «bubble» to the correct position at the leftmost location of the sorted right subarray.

Example

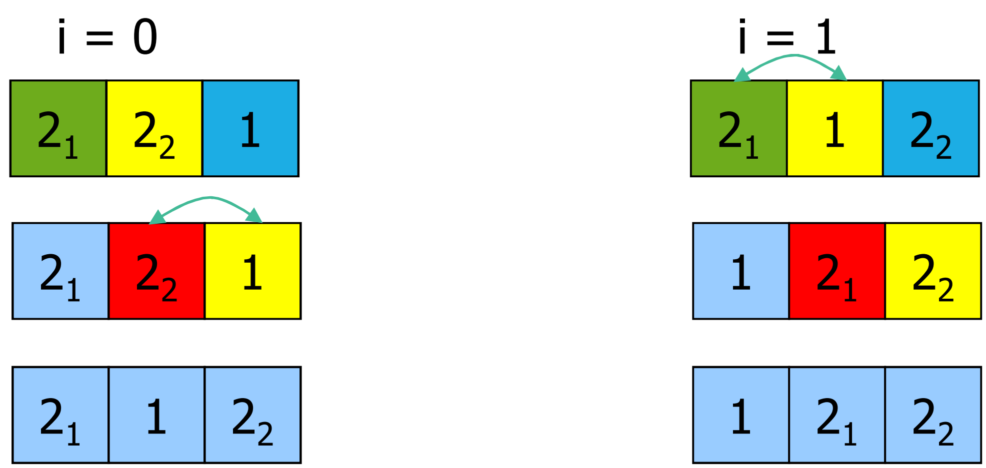

i = 0
6 3 5 1 2
3 6 5 1 2
3 5 6 1 2
3 5 1 6 2
Not yet sorted
sorted
3 5 1 2 6
i = 1
3 5 1 2 6
3 5 1 2 6
3 1 5 2 6
3 1 2 5 6
Not yet sorted
sorted
3 1 2 5 6

```
void BubbleSort(int A[], int N){
  int i, j, l=0, r=N-1;
  int temp;
  for (i = l; i < r; i++) {
    for (j = l; j < r - i +l; j++)
      if (A[j] > A[j+1]) {
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
      }
  }
  return;
}
```

i-l is the size of the already sorted right subarray

Bubble/Exchange sort Features

- in place: used only array A and variable temp
- stable: if there are several duplicate keys the rightmost none takes the rightmost position and no other identical key jumps over it to the right:



High-level Complexity Analysis

Two nested loops:

- Outer loop: always executed N-1 times
- Inner loop: at the i-th iteration executed always N-1-i times

  $T(N) = (N-1) + (N-2) + \ldots 2 + 1$

  $= \Sigma_{1 \leq i < N}\, i = N(N-1)/2$

Finite arithmetic progression with ratio 1 (Gauss, end of XVII cent.)

$T(N) = \Theta(N^2)$.

- Number of swaps in the worst case: $O(N^2)$: a swap doesn't necessarily occur
- Number of comparisons in the worst case: $\Theta(N^2)$: a comparison always takes place

Optimization

- Use a flag to mark whether there has been swaps. Execution continues only if there have been swaps
  - The outer loop is executed at most N-1 times
  - The inner loop at the i-th iteration is executed always N-1-i times
- Number of comparisons in the worst case: $O(N^2)$: a comparison doesn't necessarily take place
- Average case complexity is improved, no change in worst case complexity.

Example

flag = 1

i = 0 outer loop executed

flag = 0

j = 0 inner loop executed	4	2	1	5	3	⇨	2	4	1	5	3	flag = 1
j = 1 inner loop executed	2	4	1	5	3	⇨	2	1	4	5	3	flag = 1
j = 2 inner loop executed	2	1	4	5	3	⇨	2	1	4	5	3	flag = 1
j = 3 inner loop executed	2	1	4	5	3	⇨	2	1	4	3	5	flag = 1

i = 1 outer loop executed

flag = 0

j = 0 inner loop executed	2	1	4	3	5	⇨	1	2	4	3	5	flag = 1
j = 1 inner loop executed	1	2	4	3	5	⇨	1	2	4	3	5	flag = 1
j = 2 inner loop executed	1	2	4	3	5	⇨	1	2	3	4	5	flag = 1
j = 3 inner loop executed	1	2	3	4	5	⇨	1	2	3	4	5	flag = 1

i = 2 outer loop executed

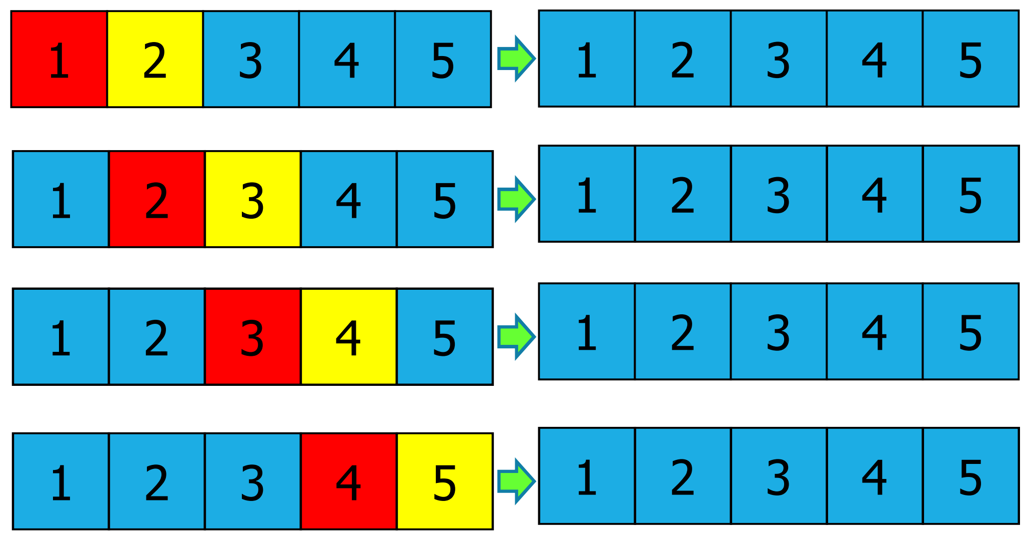
flag = 0

j = 0 inner loop executed

j = 1 inner loop executed

j = 2 inner loop executed

j = 3 inner loop executed



i = 3 but flag = 0 as there has been no swaps, outer loop not executed

```
void OptBubbleSort(int A[], int N) {
  int i, j, l=0, r=N-1, flag=1;
  int temp;

  for (i = l; i < r && flag==1; i++) {
    flag = 0;
    for (j = l; j < r - i + l; j++)
      if (A[j] > A[j+1]) {
        flag = 1;
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
    }
  }
  return;
}
```

Selection sort

Let N integers be stored in an array A whose indices range from l=0 and r=N-1.

Conceptually array A consists of 2 subarrays:

- Left subarray: already sorted
- Right subarray: not yet sorted

Initially the left subarray is empty, the right one contains N items.

Approach

Incremental paradigm:

- At each step expand the already sorted left subarray inserting an item taken from the not yet sorted right subarray
- Insertion must guarantee that the left subarray remains sorted after insertion (invariance of the sorting property). This is guaranteed by identifying the smallest item in the right subarray (A_i ... A_r) and assigning it to A[i]. The sorted left subarray grows by 1 in size to the right, dually the not yet sorted right one decreases in size by 1
- Termination: all items have been properly inserted, the left subarray contains N items, the right one is empty.

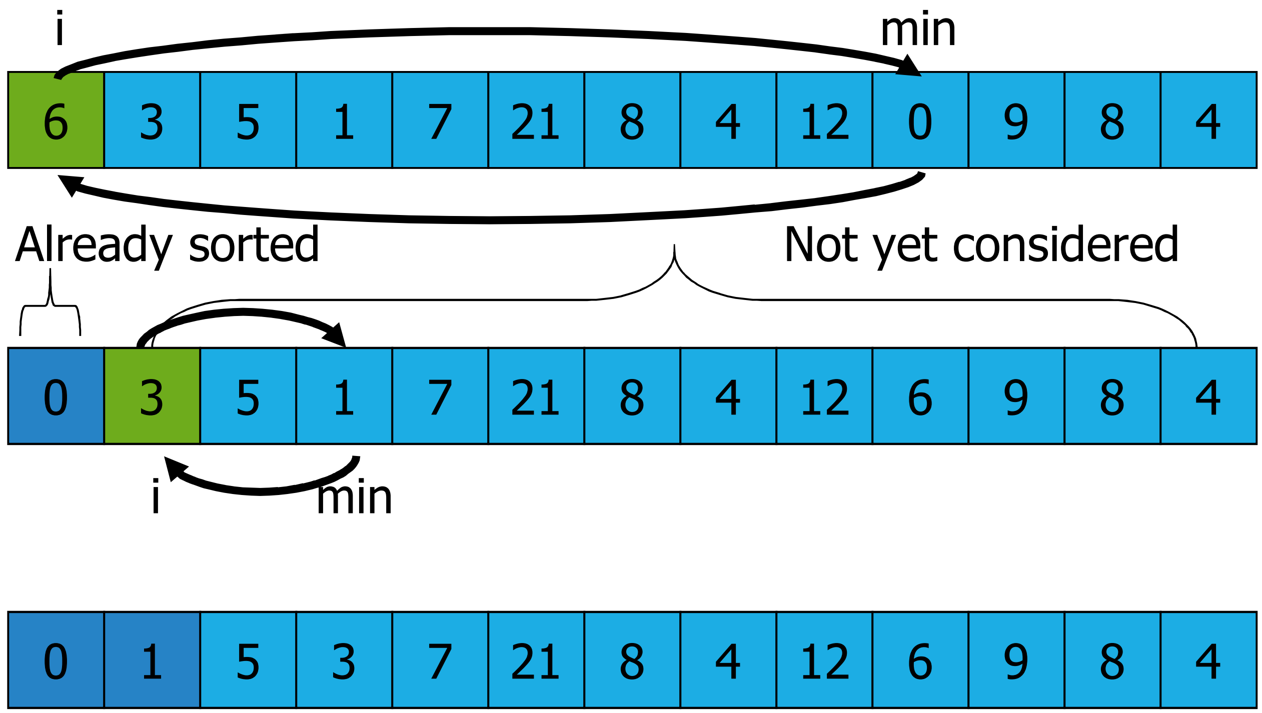
Step i: identifying the minimum

Identification of the smallest item of the right subarray at the i-th step:

- Scan the right subarray, assuming that the smallest item is in A[i] and updating the smallest item at each comparison with the following items

When the loop ends, the smallest item has been identified by means of its position (index in A) and swapped with A[i].

Example

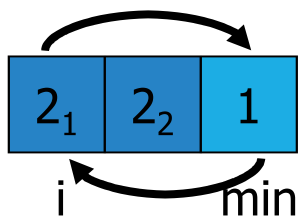


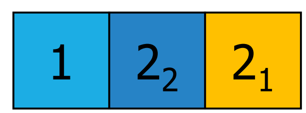
i
min
6 3 5 1 7 21 8 4 12 0 9 8 4
Already sorted
Not yet considered
0 3 5 1 7 21 8 4 12 6 9 8 4
i
min
0 1 5 3 7 21 8 4 12 6 9 8 4

```c
void SelectionSort(int A[], int N) {
  int i, j, l=0, r=N-1, min;
  int temp;
  for (i = l; i < r; i++) {
    min = i;
    for (j = i+1; j <= r; j++)
      if (A[j] < A[min])
         min = j;
    if (min != i) {
       temp = A[i];
      A[i] = A[min];
      A[min] = temp;
     }
  }
  return;
}
```

Selection sort Features

- in place: only array A and variable temp are used
- non stable: a swap of 2 «far away» items may allow a duplicate key to jump to the left of a previous identical key:





High-level Complexity Analysis

Two nested loops:

- Outer loop: always executed N-1 times
- Inner loop: at the i-th iteration always executed N-1-i times

$T(N) = (N-1) + (N-2) + \ldots 2 + 1$

$= \Sigma_{1 \leq i < N}\, i = N(N-1)/2$

Finite arithmetic progression with ratio 1 (Gauss, end of XVII cent.)

$T(n) = \Theta(N^2)$.

- Number of swaps in the worst case: $O(N)$: it may happen to always swap the current item with the current smallest one, but this occurs at most N times
- Number of comparisons in the worst case: $\Theta(N^2)$: comparisons always occur
- The algorithm is quadratic, as complexity depends on the number of comparisons, not on the number of swaps.

Shell sort (Shell, 1959)

Limit of Insertion sort: only adjacent items are compared and possibly swapped.

Rationale of Shell sort:

- Compare and possibly swap items at distance h
- Define a decreasing sequence of integers h that end at 1. The last step is Insertion sort.

Linear Sequences

- Finite set of consecutive items. Each item is uniquely associated to an index

$$a_0, a_1, \dots , a_i, \dots, a_{n-1}$$

- A predecessor/successor relationship is defined on pairs of consecutive items:

$$a_{i+1} = succ(a_i) \qquad\qquad a_i = pred(a_{i+1})$$

- Storage and access:
  - Array: contiguous data in memory with direct access:
    - Given index i, it is possible to access item a_i without scanning the linear sequence
    - Cost of access doesn't depend on the position of the item in the linear sequence, thus it is O(1)
  - list: non contiguous data in memory with sequential access. Topic of the second year course.

Subarrays and Subsequences

Given a linear sequence of N integers stored in array A

$$A = (a_0, a_1, \dots a_{N-1})$$

- a **subsequence** of A of length k ($k \leq N$) is any tuple Y of k items of A with increasing and not necessarily contiguous indices $i_0, i_1, \cdots, i_{k-1}$
- a **subarray** of A of length k ($k \leq N$) is any tuple Y of k items of A with increasing and contiguous indices $i_0, i_1, \cdots, i_{k-1}$.

Example

A

6	3	5	1	7	21	8	4	12	0	9	8	4
0	1	2	3	4	5	6	7	8	9	10	11	12

A

6	3	5	1	7	21	8	4	12	0	9	8	4
0	1	2	3	4	5	6	7	8	9	10	11	12

Subarray of A with k=4 items with increasing and contiguous indices i_4, i_5, i_6, i_7

A

6	3	5	1	7	21	8	4	12	0	9	8	4
0	1	2	3	4	5	6	7	8	9	10	11	12

Subsequence of A with k=4 items

with increasing and non contiguous indices i_1, i_4, i_6, i_{11}

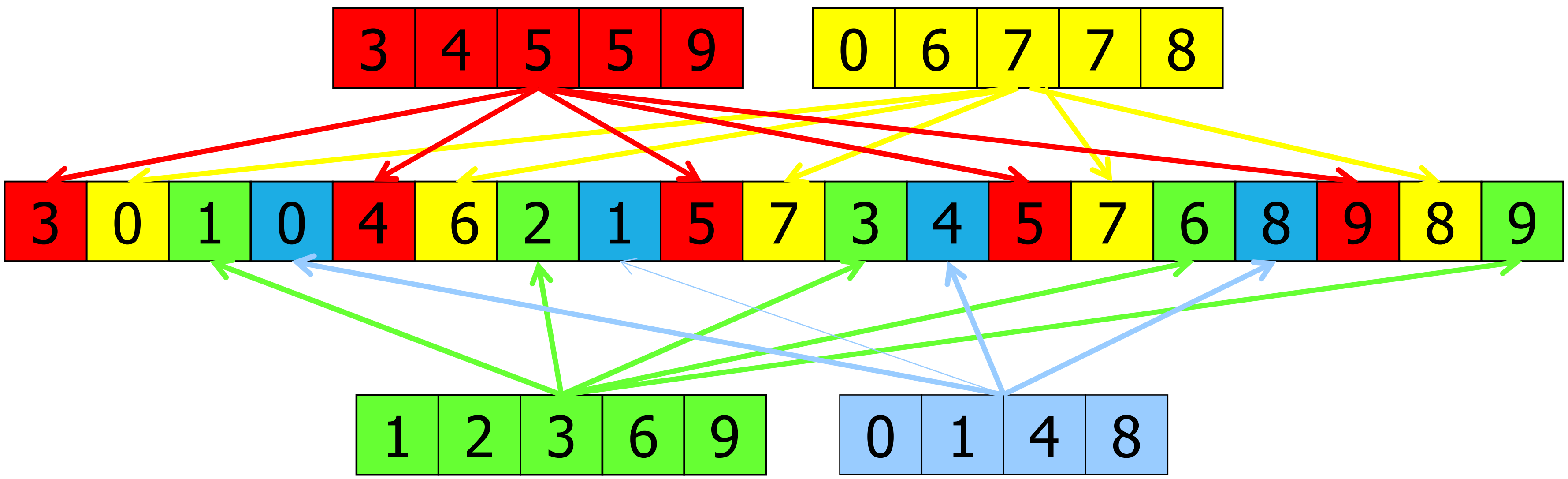
Subsequences in Shell sort

Shell sort works with subsequences composed by array items whose indices at distance h from each other.
Example: h=4

5	8	9	0	4	6	3	1	3	7	6	4	5	7	1	8	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

If the subsequences containing items at distance h are sorted, the array is h-sorted.

Example: h=4



For each subsequence apply Insertion sort. The items of the subsequence are the items at distance h from the current one.

```
for i = l+h; i <= r; i++) {
  j = i; x = A[i];
  while (j>= l+h && x < A[j-h]) {
    A[j] = A[j-h];
    j -=h;
  }
  A[j] = x;
}
```

Example

sequence h: 13, 4, 1

Step 1: h=13

5	8	9	0	4	6	3	1	3	7	6	4	5	7	1	8	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

5	8	9	0	4	6	3	1	3	7	6	4	5	7	1	8	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

5	1	9	0	4	6	3	1	3	7	6	4	5	7	8	8	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

5	1	8	0	4	6	3	1	3	7	6	4	5	7	8	9	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

5	1	8	0	4	6	3	1	3	7	6	4	5	7	8	9	9	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

5	1	8	0	0	6	3	1	3	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 2: h=4

5	1	8	0	0	6	3	1	3	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	5	6	3	1	3	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Step 2: h=4

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 2: h=4

0	1	8	0	3	6	3	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	8	1	5	7	6	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 2: h=4

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 3: h=1

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	1	3	0	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	3	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	3	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	3	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 3: h=1

0	0	1	3	3	6	6	1	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	6	6	5	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	5	6	6	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	5	6	6	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	5	6	6	7	8	4	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	5	6	6	7	8	5	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

sequence h: 13, 4, 1

Step 3: h=1

0	0	1	1	3	3	4	5	5	6	6	7	8	7	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	5	5	6	6	7	7	8	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	5	5	6	6	7	7	8	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	5	5	6	6	7	7	8	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	5	5	6	6	7	7	8	8	9	9	4
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

0	0	1	1	3	3	4	4	5	5	6	6	7	7	8	8	9	9
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Sequences

- Shell's sequence (1959) $h_i = 2^{i-1}$

$h_1 = 1, h_2 = 2, h_3 = 4, h_4 = 8, h_5 = 16, \ldots$

- Hibbard's sequence (1963): $h_i = 2^i - 1$

$h_1 = 1, h_2 = 3, h_3 = 7, h_4 = 15, h_5 = 31, \ldots$

- Pratt's sequence (1971):

$1, 2, 3, 4, 6, 8, 9, 12, 16, \ldots, 2^p 3^q, \ldots$

- Knuth's (1973): initially $h_0=0$ then $h_i = 3h_{i-1}+1$

$h_1 = 1, h_2 = 4, h_3 = 13, h_4 = 40, h_5 = 121, \ldots$

- Sedgewick's sequence (1986):

1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, ...

```
void ShellSort(int A[], int N) {
  int i, j, x, l=0, r=N-1, h=1;
  while (h < N/3)
    h = 3*h+1;
  while(h >= 1) {
    for (i = l + h; i <= r; i++) {
      j = i;
      x = A[i];
      while(j >= l + h  && x < A[j-h]) {
        A[j] = A[j-h];
        j -=h;
      }
      A[j] = x;
    }
    h = h/3;
  }
}
```

Knuth's sequence

INSERTION SORT

Shell sort Features

- in place: apart from array A only variable x is used
- non stable: a swap between «distant» items can force a duplicate key to jump to the lest of a previous occurrence:

2_1	2_2	2_3	2_4	2_5	0

2_1	2_2	2_3	2_4	2_5	0

2_1	0	2_3	2_4	2_5	2_2

0	2_1	2_3	2_4	2_5	2_2

High-level Complexity Analysis

- With Shell's sequence 1 2 4 8 16 … T(N) = $O(N^2)$
- With Hibbard's sequence 1 3 7 15 31 … T(N) = $O(N^{3/2})$
- With Pratt's sequence 1 2 3 4 6 8 9 12 … T(N) = $O(N\log^2 N)$
- With Knuth's sequence: 1 4 13 40 121 … T(n) = $O(N^{3/2})$
- With Sedgewick's sequence 1, 5, 19, 41, 109, 209, … T(N) = $O(N^{4/3})$