POLITECNICO
DI TORINO
Dipartimento
di Automatica e Informatica

Iterative Linearithmic Sorting Algorithms

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Linearithmic Sorting Algorithms

- Comparison-based sorting algorithms whose complexity is $\Omega(n \lg n)$ are **OPTIMAL**
- In general they are recursive (topic dealt with in the second year Course): Merge sort, Quick sort, Heap sort
- There is an iterative version of Merge sort: Bottom-up Merge sort

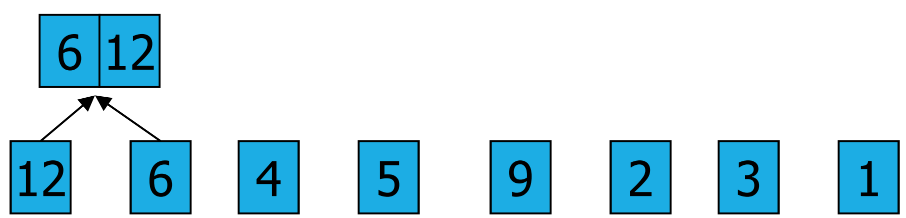
Bottom-up Merge sort

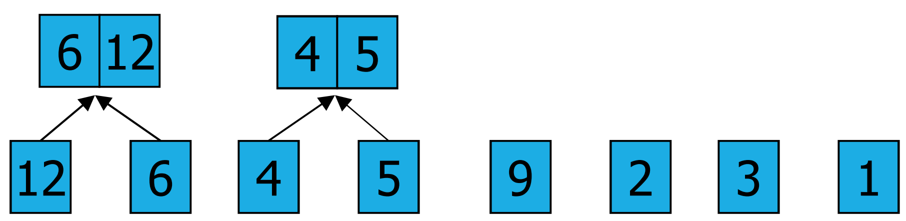
- An array containing a single item is sorted by definition
- Iteration:
  - Merge 2 sorted subarrays into a sorted array whose size equals the sum of the sizes of the 2 subarrays
  - Until size N of the array to be sorted is reached.

Example

- Assumption: size of array to sort is a power of 2 $N = 2^k$
- Starting from subarrays of size 1 (thus sorted by definition), apply Merge to get as a result at each step sorted arrays of double size m
- A temporary array of size N is required to store the result of Merge
- Termination: the temporary sorted array has the same size of the initial array.

12	6	4	5	9	2	3	1

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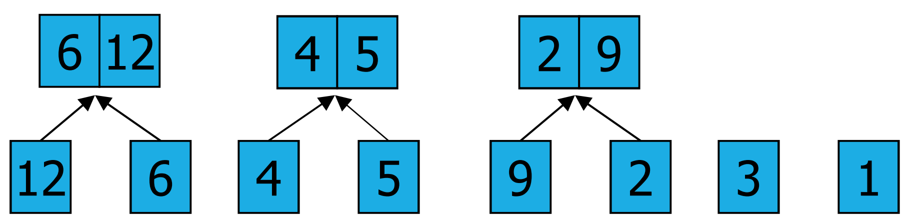
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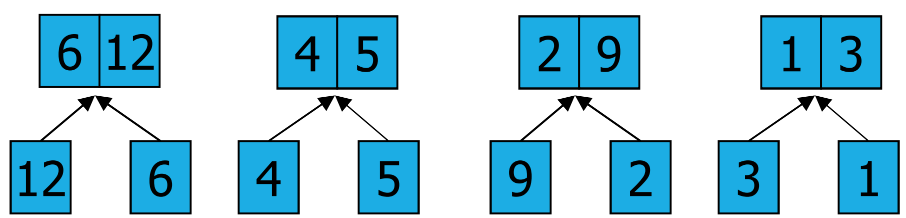
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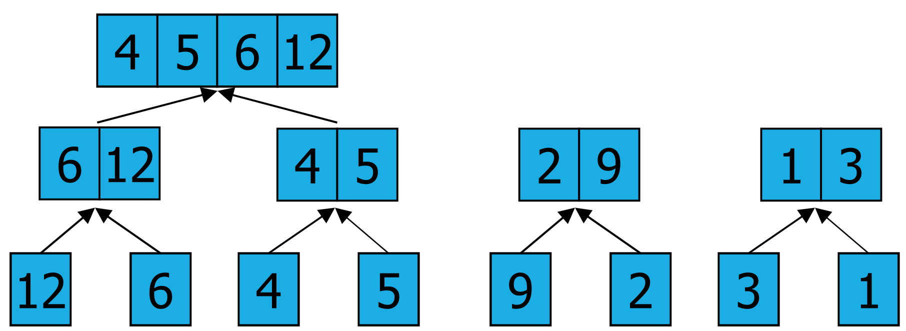
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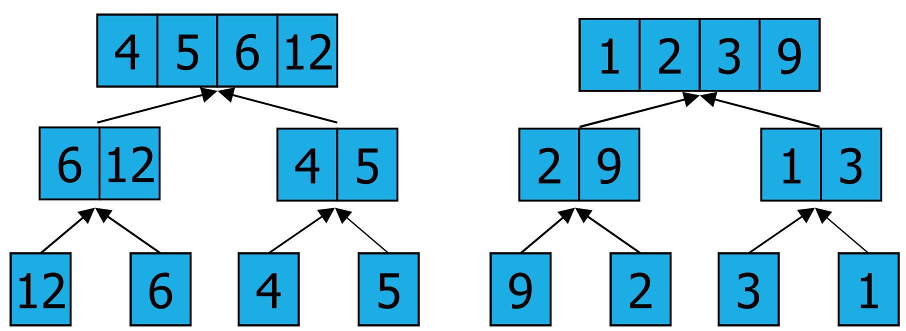
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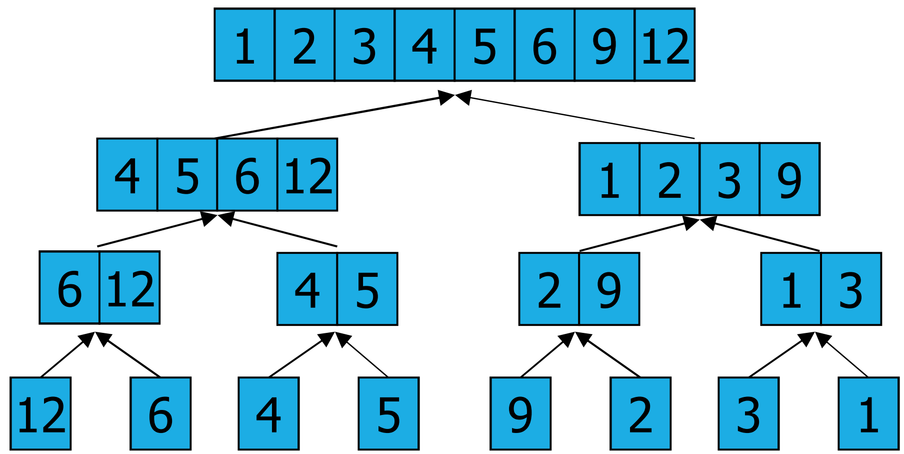
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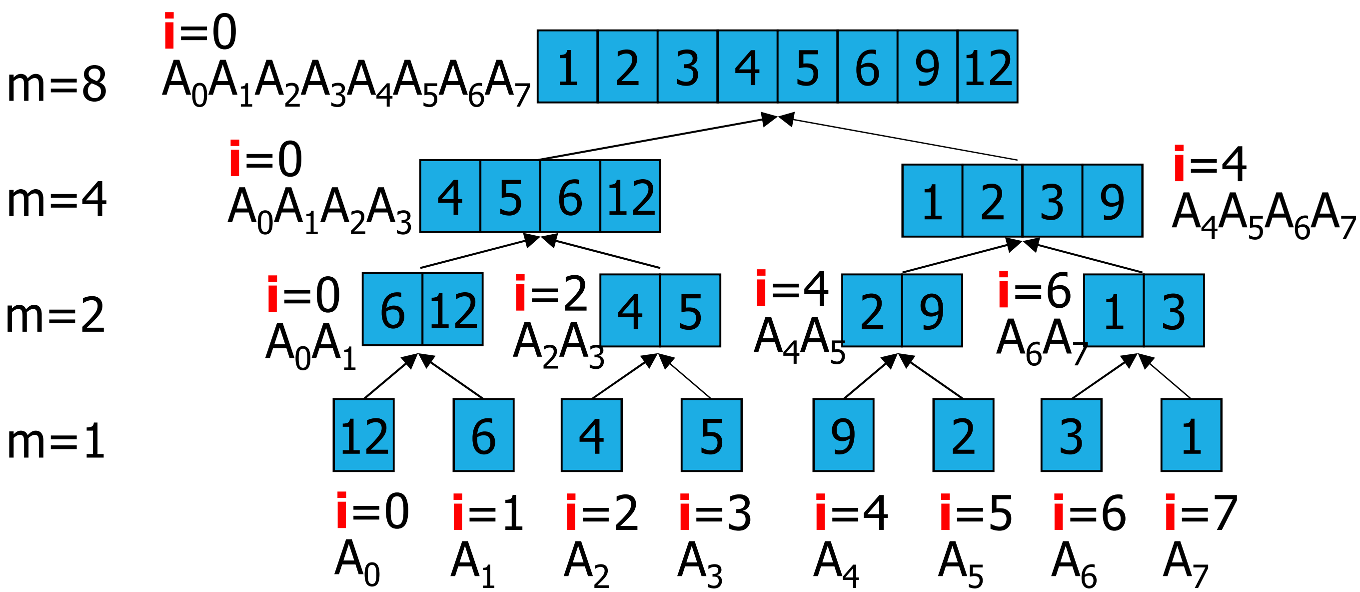
1

- Outer loop: m is initially 1, it doubles at each step until it becomes N
- Inner loop: run Merge on each pair of sorted and adjacent subarrays of size m, obtaining as a result a sorted subarray of size 2m

- Identification of sorted and adjacent subarrays:



In the inner loop:
$q = i + m - 1$
Left subarray
$A_i \dots A_q$
Right subarray
$A_{q+1} \dots A_{q+m}$

```
void BottomUpMergeSort(int A[], int B[], int N)
  int i, q, m, l=0, r=N-1;
  for (m = 1; m <= r - l; m = m + m)
    for (i = l; i <= r - m; i += m + m) {
      q = i+m-1;
      Merge(A, B, i, q, r);
    }
}
```

boundaries

temporary array

size of sorted subarray doubles

merging sorted and adjacent pairs of subarrays $A_i \ldots A_q$, $A_{q+1} \ldots A_{q+m}$

identification of the starting index for the next pair of sorted and adjacent subarrays of size m

2-way Merge

- Assumption: size of array A is a power of 2 $N = 2^k$
- Merging 2 sorted subarrays of A (2-way) of size m to get a sorted subarray of size 2m
- Possible to generalize to k arrays (k-way Merge)
- Index q to split in half subarrays in A, the result being a left and a right subarray $q = i + m - 1$
- Left subarray with index i in the range $l \leq i \leq q$
- Right subarray with index j in the range $q+1 \leq j \leq r$
- Temporary array B of size N with index k in the range $l \leq k \leq r$ to store result of merging process. Array B is passed as a parameter.

Approach:

- Walk through left and subarrays with indices i and j and through array B with index k
- If left subarray empty, copy in B remaining items from right subarray
- Else if right subarray empty, copy in B remaining items from left subarray
- Else compare current item A[i] of left subarray to current item A[j] in right subarray
  - if A[i] ≤ A[j], copy A[i] in B and increment i, j unchanged
  - else copy A[j] in B and increment j, i unchanged.

Example

m=4, q=3

A	4	5	6	12	1	2	3	9
	0	1	2	3	4	5	6	7

i=0 j=4

$i \le q$ && $j \le r$ && $A[0] > A[4]$

B	1
	0

k=0

A	4	5	6	12	1	2	3	9
	0	1	2	3	4	5	6	7

i=0 j=5

$i \le q$ && $j \le r$ && $A[0] > A[5]$

B	1	2
	0	1

k=1

A	4	5	6	12	1	2	3	9
	0	1	2	3	4	5	6	7

i=0 j=6

$i \le q$ && $j \le r$ && $A[0] > A[6]$

B	1	2	3
	0	1	2

k=2

A	4	5	6	12	1	2	3	9
	0	1	2	3	4	5	6	7

i=0 j=7

$i \le q$ && $j \le r$ && $A[0] \le A[7]$

B	1	2	3	4
	0	1	2	3

k=3

$m=4, q=3$

A

0	1	2	3	4	5	6	7
4	5	6	12	1	2	3	9

$i=1$ $j=7$

$i \le q$ && $j \le r$ && $A[1] \le A[7]$

B

0	1	2	3	4
1	2	3	4	5

$k=4$

A

0	1	2	3	4	5	6	7
4	5	6	12	1	2	3	9

$i=2$ $j=7$

$i \le q$ && $j \le r$ && $A[2] \le A[7]$

B

0	1	2	3	4	5
1	2	3	4	5	6

$k=5$

A

0	1	2	3	4	5	6	7
4	5	6	12	1	2	3	9

$i=3$ $j=7$

$i \le q$ && $j \le r$ && $A[3] > A[7]$

B

0	1	2	3	4	5	6
1	2	3	4	5	6	9

$k=6$

A

0	1	2	3	4	5	6	7
4	5	6	12	1	2	3	9

$i=3$ $j=8$

$i \le q$ && $j > r$

B

0	1	2	3	4	5	6	7
1	2	3	4	5	6	9	12

$k=7$

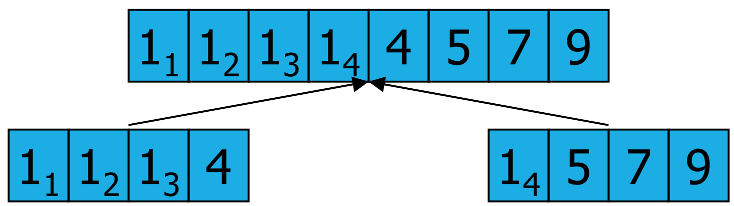
```
void Merge(int A[], int B[], int l, int q, int r) {
  int i, j, k;
  i = l;
  j = q+1;
  for (k = l; k <= r; k++)
    if (i > q)
      B[k] = A[j++];
    else if (j > r)
      B[k] = A[i++];
    else if ((A[i]< A[j]) || (A[i]== A[j]))
      B[k] = A[i++];
    else
      B[k] = A[j++];
  for ( k = l; k <= r; k++ )
    A[k] = B[k];
  return;
}
```

Left subarray empty

Right subarray empty

Merge sort Features

- not in place, a temporary array B of size N is required
- stable: the Merge function copies from the left subarray in case of duplicate keys:



If we omit condition A[i] == A[j] in statement

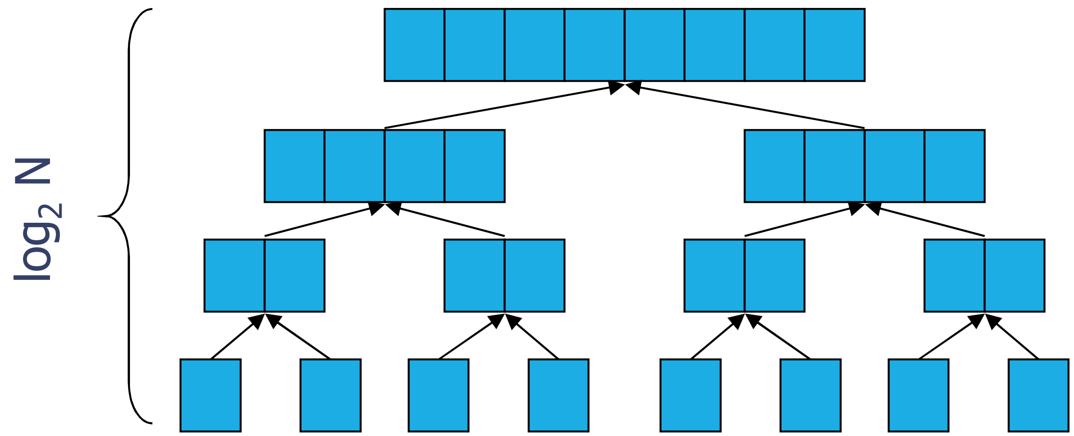
```
if ((A[i]< A[j]) || (A[i]== A[j]))
```

the algorithm becomes unstable!

Complexity Analysis of Merge sort

Informal analysis under the assumption $N = 2^k$

- At each level N operations are globally executed by the calls to Merge
- Initially sorted subarrays have size 1
- At each level the size of the sorted subarrays doubles, thus at the i-th step size is 2^i
- Termination occurs when $2^i = N$, thus $i = \log_2 N$ levels are needed
- As the cost of each level is N, global cost is $N\log_2 N$
- Complexity is linearithmic $T(n) = O(N\log N)$



Levels: $\log_2 N$ Operations at each level: N

Total # of operations: $N \log_2 N$

Generalization to any N $(\neq 2^k)$

The rightmost array boundary during Merge is the smaller value between r and the value we would have if size were a power of 2 i + m + m -1

```
void BottomUpMergeSort(int A[], int B[], int N)
  {
   int i, q, m, l=0, r=N-1;
for (m = 1; m <= r - l; m = m + m)
    for (i = l; i <= r - m; i += m + m) {
      q = i+m-1;
      Merge(A, B, i, q, min(i+m+m-1,r));
    }
}
```

Example

N=7

12	6	4	5	9	2	3