

Fertility, Housing, and Location Decisions

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Abstract

1 Introduction

2 Model

Notation for locations. The set of locations is $\mathcal{I} = \{1, \dots, I\}$; generic locations are indexed by $i, j \in \mathcal{I}$. In any state, the current (origin) location is denoted $i \in \mathcal{I}$ and a candidate destination by $i' \in \mathcal{I}$. Location amenities are $\mathcal{E}_{i'} > 0$. Location taste shocks are $\varepsilon^{i'}$ with i.i.d. Type-I extreme value (Gumbel) distribution. Location choice shares are $\pi^{i'}(\cdot)$ and location-stage values are $V^{i'}(\cdot)$. The (inverse) scale parameter governing dispersion in location choice is denoted $\nu_\ell > 0$.

The economy features continuous age $a \in [0, A]$ and calendar time t . Shock ages are discrete $\mathcal{A}^s = \{0, 1, \dots, A - 1\}$. Working ages are $a \in [0, A_R]$; fertility is feasible on $a \in [0, A_f]$; children mature at A_m .

2.1 Preferences

Let housing services be $\mathbf{h} \equiv \chi h + h^R$, where h is owner-occupied stock (adjusted only at shock ages) and h^R is the rental flow (adjustable between shocks). Flow utility is Stone–Geary CRRA in (c, \mathbf{h}) with a *single* curvature:

$$u(c, \mathbf{h}; n) = \frac{(c - \bar{c}(n))^{1-\sigma}}{1-\sigma} + \kappa_h(n) \frac{(\mathbf{h} - \bar{h}(n))^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \quad \kappa'_h(n) \geq 0,$$

where $\bar{c}(n)$ and $\bar{h}(n)$ are subsistence levels that may depend on parity n . In applications we allow a fixed “first-child” housing jump and per-child housing needs:

$$\bar{c}(n) = \bar{c}_0 + \bar{c}_1 n, \quad \bar{h}(n) = \bar{h}_0 + \bar{h}_{\text{jump}} \cdot \mathbb{I}\{n \geq 1\} + \bar{h}_1 n.$$

We impose supernumerary lower bounds $c - \bar{c}(n) \geq c_0 > 0$ and $\mathbf{h} - \bar{h}(n) \geq h_0 > 0$, as well as $h^R \geq 0$.

2.2 Earnings

Households supply labor inelastically when $a \leq A_R$ and earn $w_{it}(a, t)$ in location i (specified in the production block below).

Mobility and location choice

At shock ages, households draw an idiosyncratic taste shock for each destination i' , $\varepsilon^{i'}$, i.i.d. Type-I extreme value. Amenities $\mathcal{E}_{i'}$ and multiplicative moving wedges $\mu_{ii'} \in (0, \infty)$ scale the continuation value in destination i' .

Given state (b, h, i, a, n, a_n, t) prior to relocation, moving to i' resets liquid wealth and owner stock to

$$\begin{aligned} \tilde{b}_t^{i'}(b, h, i) &= b + \mathbb{I}(i' \neq i) (1 - \psi) p_{it} h, \\ \tilde{h}_t^{i'}(h, i) &= \mathbb{I}(i' = i) h, \end{aligned}$$

i.e., if the household moves it sells the origin house at net proceeds $(1 - \psi)p_{it}h$ and arrives as a renter; if it stays it carries the owned stock.

Housing

Rental sizes h^R lie in $\mathcal{R} \subset \mathbb{R}_+$; owner sizes h lie in $\mathcal{H} \subset \mathbb{R}_+$. Housing services are $\mathbf{h}(h, h^R) = \chi h + h^R$. Owner housing depreciates at rate δ ; owners pay a property tax τ_H on the house value; buying/selling incurs a transaction wedge $\psi \in [0, 1)$.

2.3 Fertility

At shock ages, agents receive fertility shocks ε^n (i.i.d. Type-I extreme value) and choose parity $n \in \{0, 1, 2, 3\}$. Having children raises housing taste via $\kappa_h(n)$ and changes subsistence needs via $\bar{c}(n)$ and $\bar{h}(n)$. For now parity is chosen at a shock and children age deterministically to A_m (we can later allow sequential fertility). The (inverse) scale parameter governing dispersion in fertility choice is denoted $\nu^n > 0$.

2.4 Entrants and local bequest recycling

Newborns (age $a = 0$) enter as renters with $h_0 = 0$. Their liquid wealth at entry is financed by local bequests. If deaths in i carry (b, h) , the bequeathable estate is

$$\text{estate}_i(b, h) = b + (1 - \psi)p_{it}h.$$

Let E_i denote the aggregate *estate* flow in i (notation distinct from amenity \mathcal{E}_i) and E_i^0 the newborn flow in i . A fraction $\varphi \in [0, 1]$ is redistributed equally to newborns:

$$T_i = \frac{\varphi E_i}{\max\{E_i^0, \varepsilon\}}, \quad b_0(i) = b_{\text{entry}} + T_i,$$

with $\varepsilon > 0$ small. Budget balance: $\sum_i T_i E_i^0 = \varphi \sum_i E_i$. (Utility bequests are separate from financial estates.)

2.5 Portfolio and savings

Liquid wealth b earns q . Between shock ages, owners cannot rent ($h^R = 0$) and renters can adjust $h^R \geq 0$. A collateral constraint must hold after tenure/size choices:

$$b \geq -\phi p_{it}h.$$

Between shocks, liquid wealth evolves as

$$\dot{b} = w_{it}(a, t) + qb - c - r_{it}h^R - (\delta + \tau_H)p_{it}h.$$

At shock ages, b adjusts discretely due to relocation, tenure/size moves, and bequests to children at A_m :

$$\tilde{b}_t^{\text{beq}}(b, h, i, n, b^{\text{beq}}) = b - n b^{\text{beq}}.$$

2.6 Household problem and value functions

Let the household state be $\Omega = (b, h, i, a, n, a_n, t)$. Between shocks, the HJB is

$$\rho V_t(\Omega) = \max_{c-\bar{c}(n) \geq c_0, h^R \geq 0} \left\{ u(c, \chi h + h^R; n) + V_b \dot{b} + V_a \right\}, \quad \dot{b} = w_{it}(a, t) + qb - c - r_{it}h^R - (\delta + \tau_H)p_{it}h, \quad (1)$$

with Kuhn–Tucker conditions for $h^R \geq 0$ and $b \geq -\phi p_{it}h$.

At shock ages, the sequence is *fertility* \rightarrow *location* \rightarrow *tenure/size*. We define the three blocks.

Tenure/size stage. Given a chosen destination i' , define the value from optimally adjusting h' :

$$V_t^H(b, h, i', a, n, a_n) = \max_{h' \in \mathcal{H}_t(b, h, i')} V_t(\tilde{b}_t^H(b, h, h', i'), h', i', a, n, a_n, t), \quad (2)$$

where $\tilde{b}_t^H(b, h, h', i') = b + \mathbb{I}(h' \neq h) [(1-\psi)p_{i't}h - p_{i't}h']$ and $\mathcal{H}_t(b, h, i') = \{h' \in \mathcal{H} : \tilde{b}_t^H \geq -\phi p_{i't}h'\}$.

Location stage. Define the (systematic) value of choosing destination i' as

$$V_t^{i'}(b, h, i, a, n, a_n) = V_t^H(\tilde{b}_t^{i'}(b, h, i), \tilde{h}_t^{i'}(h, i), i', a, n, a_n).$$

With i.i.d. Type-I extreme value shocks, we model the *choice index* as

$$U_t^{i'}(b, h, i, a, n, a_n) = \nu_\ell V_t^{i'}(b, h, i, a, n, a_n) + \log(\mathcal{E}_{i'} \mu_{ii'}) + \varepsilon^{i'}.$$

This yields the logit share and the inclusive value (up to an additive constant):

$$\pi_t^{i'}(b, h, i, a, n, a_n) = \frac{(\mathcal{E}_{i'} \mu_{ii'}) \exp(\nu_\ell V_t^{i'}(b, h, i, a, n, a_n))}{\sum_{j' \in \mathcal{I}} (\mathcal{E}_{j'} \mu_{ij'}) \exp(\nu_\ell V_t^{j'}(b, h, i, a, n, a_n))}, \quad (3)$$

$$V_t^I(b, h, i, a, n, a_n) = \frac{1}{\nu_\ell} \log \left(\sum_{j' \in \mathcal{I}} (\mathcal{E}_{j'} \mu_{ij'}) \exp(\nu_\ell V_t^{j'}(b, h, i, a, n, a_n)) \right). \quad (4)$$

In computation on a BGP, the logit kernel is applied to detrended continuation values (equivalently, to levels after dividing by the common trend), which preserves stationarity.

Fertility stage. Given i.i.d. Type-I extreme value shocks for parity choices and continuation values V_t^I ,

$$\pi_t^{n'}(b, h, i, a, n, a_n) = \frac{\exp(\nu^n V_t^I(b, h, i, a, n', a_n))}{\sum_{m \in \{0,1,2,3\}} \exp(\nu^n V_t^I(b, h, i, a, m, a_n))}, \quad (5)$$

$$V_t^n(b, h, i, a, n, a_n) = \frac{1}{\nu^n} \log \left(\sum_{m \in \{0,1,2,3\}} \exp(\nu^n V_t^I(b, h, i, a, m, a_n)) \right). \quad (6)$$

We adopt V_t^n as the post-shock value V_t at shock ages.

2.7 Demography (Euler–Lotka)

Let survival $\ell(a)$ and age-specific fertility $m(a)$; let n denote the population growth rate. The Euler–Lotka condition is

$$1 = \int_0^A e^{-na} \ell(a) m(a) da \quad (\text{or } 1 = \sum_{a \in \mathcal{A}^s} e^{-na} \ell(a) m(a) \Delta a \text{ in discrete shock ages}).$$

With deterministic death at A , the stationary age density is uniform iff $n = 0$; for $n \neq 0$, $f(a) \propto e^{-na} \ell(a)$.

2.8 Production (tradeable good and *construction in the main text*)

Tradeable good. The numeraire is costlessly traded. Competitive firms in i produce

$$Y_{it} = A_i Z_i(L_{it}) L_{it}, \quad Z_i(L_{it}) = L_{it}^{\alpha_i},$$

yielding wage $w_{it} = A_i L_{it}^{\alpha_i}$.

Residential construction (irreversibility). Developers transform the numeraire into new floorspace Υ_{it} with technology

$$\Upsilon_{it} = Z_{it}^h K_{it},$$

so the unit cost of new floorspace is $1/Z_{it}^h$. Let H_{it} be the city stock of floorspace; it evolves as

$$\dot{H}_{it} = \Upsilon_{it} - \delta H_{it}, \quad \Upsilon_{it} \geq 0 \quad (\text{irreversibility}).$$

Perfect competition implies the complementarity (Kuhn–Tucker) system:

$$p_{it} \geq \frac{1}{Z_{it}^h}, \quad \Upsilon_{it} \geq 0, \quad \left(p_{it} - \frac{1}{Z_{it}^h} \right) \Upsilon_{it} = 0. \quad (7)$$

Housing market clearing equates the *flow* of net new demand to construction net of depreciation:

$$\dot{H}_{it} = \Upsilon_{it} - \delta H_{it} = \frac{d}{dt} \left[\underbrace{H_{it}^d}_{\text{occupied floorspace}} \right] - \delta H_{it}, \quad (8)$$

$$H_{it}^d = \iiint (h^R(\Omega) + h) g_t(\Omega) db dh da, \quad (9)$$

with g_t the cross-sectional density. In steady growth, these conditions pin down p_{it} jointly with H_{it} and Υ_{it} . (A computational simplification that replaces the complementarity with an isoelastic static supply is in Appendix 4.1.)

3 Equilibrium and Balanced Growth Path

3.1 Equilibrium

Fix fundamentals $\{\sigma, \kappa_h(n), \chi, q, \delta, \tau_H, \phi, \psi, \bar{c}(\cdot), \bar{h}(\cdot), \{A_i\}, \{Z_{it}^h\}\}$ and the i.i.d. extreme-value shock structure with dispersion parameters ν_ℓ for locations and ν^n for fertility. An equilibrium is a

collection

$$\left\{ V_t(\Omega), \text{ policies } (c, h^R) \text{ between shocks, discrete choice kernels } (\pi^n, \pi^{i'}, \pi^H) \text{ at shocks, } g_t(\Omega), \{p_{it}, r_{it}, w_{it}, \dots\} \right.$$

such that:

- (E1) **Household optimality.** Between shocks V_t solves the HJB with borrowing and nonnegativity constraints; at shocks, fertility \rightarrow location \rightarrow tenure/size decisions are optimal and generate choice probabilities $\pi^n, \pi^{i'}, \pi^H$ consistent with extreme-value aggregation.
- (E2) **Cross-sectional consistency.** g_t solves the KF between shocks and is updated by the conservative reallocation induced by $(\pi^n, \pi^{i'}, \pi^H)$ at shock ages.
- (E3) **User cost (levels).** Rents and prices satisfy

$$r_{it} = (q + \delta + \tau_H - g_{p,it}) p_{it}, \quad g_{p,it} \equiv \frac{d}{dt} \log p_{it}.$$

On a BGP with $p_{it} = e^{\gamma t} \hat{p}_i$, this reduces to $\hat{r}_i = (q + \delta + \tau_H - \gamma) \hat{p}_i$.

- (E4) **Construction and market clearing.** For each i, t ,

$$p_{it} \geq \frac{1}{Z_{it}^h}, \quad \Upsilon_{it} \geq 0, \quad \left(p_{it} - \frac{1}{Z_{it}^h} \right) \Upsilon_{it} = 0, \quad \dot{H}_{it} = \Upsilon_{it} - \delta H_{it} = \frac{d}{dt} H_{it}^d - \delta H_{it}.$$

3.2 Balanced Growth Path (BGP) and hats

Let hats denote detrended variables, $\hat{z} = e^{-\gamma t} z$ for $z \in \{b, c, h, h^R, p_i, r_i, w_i, H_i, \Upsilon_i\}$. We take the value scaling

$$V_t(\Omega) = e^{(1-\sigma)\gamma t} v(\hat{b}, \hat{h}, \hat{\Omega}), \quad \hat{\Omega} = (i, a, n, a_n).$$

HJB in hats. Between shocks,

$$(\rho - (1 - \sigma)\gamma) v = \max_{\hat{c} - \bar{c}(n) \geq c_0, \hat{h}^R \geq 0} \left\{ u(\hat{c}, \chi \hat{h} + \hat{h}^R; n) + v_{\hat{b}} \dot{\hat{b}} + v_a - \gamma \hat{h} v_{\hat{h}} \right\}, \quad \dot{\hat{b}} = \hat{w}_i + (q - \gamma) \hat{b} - \hat{c} - \hat{r}_i \hat{h}^R - \dots \quad (10)$$

KF in hats (share normalization). The stationary KF between shocks is

$$\partial_a \phi + \partial_{\hat{b}} (\phi \dot{\hat{b}}) + \partial_{\hat{h}} (\phi (-\gamma \hat{h})) = 0,$$

with discrete, conservative updates at shock ages using $(\pi^n, \pi^{i'}, \pi^H)$ in that order. We adopt the share normalization

$$g_t(\Omega) = e^{-2\gamma t} \phi(\hat{b}, \hat{h}, \hat{\Omega}),$$

so that ϕ integrates to one by construction; totals are recovered by multiplying by population when aggregating.

User cost in hats. $\hat{r}_i = (q + \delta + \tau_H - \gamma)\hat{p}_i$.

Construction in hats. From levels,

$$\dot{H}_{it} = \Upsilon_{it} - \delta H_{it}.$$

With $\hat{H}_i = e^{-\gamma t} H_{it}$ and $\hat{\Upsilon}_i = e^{-\gamma t} \Upsilon_{it}$,

$$\dot{\hat{H}}_i = 0 \iff \hat{\Upsilon}_i = (\delta - \gamma) \hat{H}_i$$

on a stationary BGP. Combined with the level complementarity $p_{it} \geq 1/Z_{it}^h$, $\Upsilon_{it} \geq 0$, $(p_{it} - 1/Z_{it}^h)\Upsilon_{it} = 0$, this pins down prices jointly with stocks at the BGP.

Demand in hats. $H_i^d = \int (\hat{h}^R + \hat{h}) \phi d\hat{b} d\hat{h} da$ and market clearing requires $\frac{d}{dt} \hat{H}_i^d = 0$ on a strict BGP, which is consistent with $\hat{\Upsilon}_i = (\delta - \gamma) \hat{H}_i$.

First-order conditions (interior case, hats)

With Stone–Geary marginal utilities

$$u_c(\hat{c}, \hat{\mathbf{h}}; n) = (\hat{c} - \bar{c}(n))^{-\sigma}, \quad u_{\mathbf{h}}(\hat{c}, \hat{\mathbf{h}}; n) = \kappa_h(n)(\hat{\mathbf{h}} - \bar{h}(n))^{-\sigma}, \quad \hat{\mathbf{h}} \equiv \chi \hat{h} + \hat{h}^R,$$

the interior between-shock policies in hats satisfy

$$u_c(\hat{c}, \chi \hat{h} + \hat{h}^R; n) = v_{\hat{b}}, \quad \frac{u_{\mathbf{h}}(\hat{c}, \chi \hat{h} + \hat{h}^R; n)}{u_c(\hat{c}, \chi \hat{h} + \hat{h}^R; n)} = \hat{r}_i.$$

Equivalently,

$$\hat{c} = \bar{c}(n) + v_{\hat{b}}^{-1/\sigma}, \quad \chi \hat{h} + \hat{h}^R = \bar{h}(n) + \left(\frac{\kappa_h(n)}{\hat{r}_i v_{\hat{b}}} \right)^{1/\sigma}, \quad \hat{h}^R = \max\{0, (\cdot) - \chi \hat{h}\}.$$

Kuhn–Tucker inequalities apply at the collateral wall and the rental corner.

4 Computation

4.1 Supply simplification for numerics

For computation, one may replace the construction complementarity by a static isoelastic *approximation* around the BGP:

$$H_i^s(\hat{p}_i) = \bar{H}_i \hat{p}_i^{\eta_i}, \quad \eta_i > 0,$$

and close the market with $H_i^d(\hat{p}_i) = H_i^s(\hat{p}_i)$. This delivers a stable price-only fixed point. The mapping to the construction block is determined by matching (\hat{p}_i, \hat{H}_i) at the targeted BGP and interpreting η_i as a local supply elasticity.

5 Conclusion

6 Appendix A: HJB and KFE in levels and hats (clean derivation)

Price convention. The *tradable numeraire* has a common price $P_t^T \equiv 1$ in all locations. Symbols p_{it}, r_{it} denote the *local (non-tradable) housing asset price and rent*, both quoted in the numeraire. Hence p_{it} may differ by i ; there is no conflict with the law of one price for tradables.

A.1 Levels HJB with explicit time dependence

Let the between-shocks state be $\Omega = (b, h, i, a, n, a_n, t)$ with $\dot{h} = 0$ (no owner adjustment between shocks), $\dot{a} = 1$, and liquid wealth drift

$$\dot{b} = w_{it}(a, t) + qb - c - r_{ith}h^R - (\delta + \tau_H)p_{ith}.$$

The correct HJB in levels (non-autonomous environment) is

$$\rho V(t, \Omega) = \max_{c - \bar{c}(n) \geq c_0, h^R \geq 0} \left\{ u(c, \chi h + h^R; n) + \underbrace{\partial_t V}_{\text{explicit time}} + V_b \dot{b} + V_a \right\}, \quad (11)$$

with Kuhn–Tucker for $h^R \geq 0$ and the collateral constraint enforced at shocks.

A.2 Detrending and stationary HJB

Let hats be $\hat{z} = e^{-\gamma t} z$ for $z \in \{b, c, h, h^R, w_i, p_i, r_i\}$ and scale values

$$V(t, \Omega) = e^{(1-\sigma)\gamma t} v(\hat{b}, \hat{h}, i, a, n, a_n).$$

Utility scales as

$$u(e^{\gamma t} \hat{c}, e^{\gamma t} (\chi \hat{h} + \hat{h}^R); n) = e^{(1-\sigma)\gamma t} u(\hat{c}, \chi \hat{h} + \hat{h}^R; n),$$

since $\bar{c}(n)$ and $\bar{h}(n)$ are defined in detrended units. Using $\partial_t \hat{b} = -\gamma \hat{b}$, $\partial_t \hat{h} = -\gamma \hat{h}$, and $\dot{\hat{b}} = (\dot{b}/e^{\gamma t}) - \gamma \hat{b}$, the chain rule yields

$$\partial_t V = e^{(1-\sigma)\gamma t} \left[(1-\sigma)\gamma v - \gamma \hat{b} v_{\hat{b}} - \gamma \hat{h} v_{\hat{h}} \right].$$

Moreover, $V_b = e^{-\sigma\gamma t} v_{\hat{b}}$ and $V_a = e^{(1-\sigma)\gamma t} v_a$. Substitute into (11) and divide by $e^{(1-\sigma)\gamma t}$ to obtain the stationary HJB in hats:

$$(\rho - (1-\sigma)\gamma) v = \max_{\hat{c} - \bar{c}(n) \geq c_0, \hat{h}^R \geq 0} \left\{ u(\hat{c}, \chi \hat{h} + \hat{h}^R; n) + v_{\hat{b}} \dot{\hat{b}} + v_a - \gamma \hat{h} v_{\hat{h}} \right\}, \quad (12)$$

with

$$\dot{\hat{b}} = \hat{w}_i + (q - \gamma) \hat{b} - \hat{c} - \hat{r}_i \hat{h}^R - (\delta + \tau_H) \hat{p}_i \hat{h}. \quad (13)$$

This matches the interior FOCs:

$$(\hat{c} - \bar{c}(n))^{-\sigma} = v_{\hat{b}}, \quad \frac{\kappa_h(n)(\chi \hat{h} + \hat{h}^R - \bar{h}(n))^{-\sigma}}{(\hat{c} - \bar{c}(n))^{-\sigma}} = \hat{r}_i,$$

i.e.

$$\hat{c} = \bar{c}(n) + v_{\hat{b}}^{-1/\sigma}, \quad \chi \hat{h} + \hat{h}^R = \bar{h}(n) + (\kappa_h(n)/(\hat{r}_i v_{\hat{b}}))^{1/\sigma}.$$

A.3 KFE in levels and hats

Let $g_t(b, h, i, a, n, a_n)$ be the (mass or share) density between shocks. With $\dot{h} = 0$ and $\dot{a} = 1$,

$$\partial_t g_t + \partial_b(g_t \dot{b}) + \partial_a g_t = 0, \quad \dot{b} \text{ as above.} \quad (14)$$

Share normalization (recommended). Define

$$g_t(\Omega) = e^{-2\gamma t} \phi(\hat{b}, \hat{h}, i, a, n, a_n), \quad \int \phi = 1,$$

so the Jacobian of $(b, h) \mapsto (\hat{b}, \hat{h})$ contributes exactly $e^{-2\gamma t}$. Then (14) is equivalent to the stationary PDE

$$\partial_a \phi + \partial_{\hat{b}}(\phi \dot{\hat{b}}) + \partial_{\hat{h}}(\phi(-\gamma \hat{h})) = 0, \quad (15)$$

with $\dot{\hat{b}}$ from (13). At shock ages, ϕ undergoes a *conservative* reallocation via the jump operator \mathcal{J} implementing the sequence fertility \rightarrow location $i \rightarrow i' \rightarrow$ tenure/size.

Mass normalization (alternative). If one prefers to keep *levels* in the density,

$$g_t(\Omega) = e^{-(2\gamma+n_t)t} \phi(\hat{b}, \hat{h}, i, a, n, a_n),$$

with $n_t \equiv \dot{N}_t/N_t$ the endogenous population growth rate defined in Appendix 7. The PDE for ϕ is still (15); the only difference is whether population growth sits inside g_t or outside in a separate N_t equation.

Boundary and constraints. No-flux at the collateral wall and at $h^R = 0$ corners; age boundary conditions at $a = 0$ and $a = A$ are imposed by the flow equations for entrants and exits (Appendix 7); house size h is constant between shocks (hence only the $-\gamma \hat{h}$ drift in hats).

Verification. The cancellation of $-\gamma \hat{b} v_{\hat{b}}$ between $\partial_t V$ and $V_b \dot{b}$ terms is exact; the residual $-\gamma \hat{h} v_{\hat{h}}$ arises because $\dot{h} = 0$ between shocks. Equation (15) follows from the standard change-of-variables formula plus the drift identities above.

7 Appendix B: Equilibrium via population flows (no Euler–Lotka)

B.1 Stocks, flows, and relocation

Let $N_i(t)$ be the population in location i , and let $\phi_i(\hat{b}, \hat{h}, i, a, n, a_n)$ be the stationary share density (Appendix 6). Define

$$\mathcal{S}_i(t) := \{a \in [0, A] \text{ hitting a shock at } t\}, \quad \omega_i(a, t) := \text{mass flow into age } a \text{ in } i \text{ per unit time.}$$

Relocation flows at a shock age $a \in \mathcal{S}_i(t)$ are

$$M_{i \rightarrow i'}(t; a) = N_i(t) \omega_i(a, t) \int \pi^{i'}(\hat{b}, \hat{h}, i, a, n, a_n) \phi_{i|a}(\hat{b}, \hat{h}, n, a_n) d\hat{b} d\hat{h} dn da_n,$$

with $\phi_{i|a}$ the cross-section conditional on age a . Net relocation obeys $\sum_{i'} M_{i \rightarrow i'} = \sum_j M_{j \rightarrow i}$ by construction of the choice shares and jump operator.

B.2 Births and deaths

Let $d(a)$ be the instantaneous death hazard (or deterministic exit at A). Let parity choice at age a produce $n' \in \{0, 1, 2, 3\}$ children. Then births in i are

$$B_i(t) = \sum_{a \in \mathcal{S}_i(t) \cap [0, A_f]} N_i(t) \omega_i(a, t) \int \left(\sum_{n'} n' \cdot \pi^{n'}(\hat{b}, \hat{h}, i, a, a_n) \right) \phi_{i|a}(\hat{b}, \hat{h}, n, a_n) d\hat{b} d\hat{h} dn da_n.$$

Deaths are

$$D_i(t) = \int_0^A d(a) N_i(t) f_i(a, t) da,$$

with $f_i(a, t)$ the age density (the a -marginal of ϕ_i).

B.3 Population laws of motion and growth

Total population in location i satisfies

$$\dot{N}_i(t) = \underbrace{B_i(t) - D_i(t)}_{\text{natural growth}} + \underbrace{\sum_{j \neq i} M_{j \rightarrow i}(t) - \sum_{j \neq i} M_{i \rightarrow j}(t)}_{\text{net relocation}}. \quad (16)$$

Aggregate population $N(t) = \sum_i N_i(t)$ obeys

$$\dot{N}(t) = B(t) - D(t), \quad B(t) = \sum_i B_i(t), \quad D(t) = \sum_i D_i(t), \quad (17)$$

because relocations cancel in the aggregate. The endogenous growth rate is $n_t = \dot{N}(t)/N(t)$.

B.4 Wages, labor, and housing markets

Working-age labor in i :

$$L_i(t) = N_i(t) \int_0^{A_R} f_i(a, t) da \times (\text{participation} = 1).$$

Firms: $w_{it} = A_i L_i^{\alpha_i}$. Local housing demand (occupied floorspace) in hats:

$$\hat{H}_i^d = \int (\hat{h}^R(\hat{b}, \hat{h}, i, a, n, a_n) + \hat{h}) \phi_i(\hat{b}, \hat{h}, a, n, a_n) d\hat{b} d\hat{h} da dn da_n.$$

Level demand is $H_i^d(t) = N_i(t) \hat{H}_i^d e^{\gamma t}$. Construction: either (i) complementarity $p_{it} \geq 1/Z_{it}^h$, $\Upsilon_{it} \geq 0$, $(p_{it} - 1/Z_{it}^h)\Upsilon_{it} = 0$, $\bar{H}_{it} = \Upsilon_{it} - \delta H_{it}$; or (ii) static isoelastic $H_i^s = \bar{H}_i p_{it}^{\eta_i}$.

B.5 Balanced Growth Path (BGP) without Euler–Lotka

A BGP is a tuple $(\gamma, \{\hat{w}_i, \hat{p}_i, \hat{r}_i, \hat{H}_i, \hat{\Upsilon}_i\}_i, v, \{\phi_i\}_i, \{N_i(t)\})$ s.t.:

- (B1) **Stationary hats and policies.** HJB (12) holds; $\{\hat{w}_i, \hat{p}_i, \hat{r}_i\}$ are constant over t ; ϕ_i solves (15) with conservative shock updates.
- (B2) **Common co-trend.** $w_{it} = e^{\gamma t} \hat{w}_i$, $p_{it} = e^{\gamma t} \hat{p}_i$, $r_{it} = e^{\gamma t} \hat{r}_i$, with user cost $\hat{r}_i = (q + \delta + \tau_H - \gamma) \hat{p}_i$.
- (B3) **Flow balance.** There exists n such that $N_i(t) = s_i N_0 e^{nt}$ with constant shares $s_i \in [0, 1]$ and (16) holds for each i . This determines n from $B_i, D_i, M_{i \rightarrow j}$; Euler–Lotka is not used.
- (B4) **Markets.** For each i , either (A) complementarity pins \hat{p}_i jointly with \hat{H}_i , $\hat{\Upsilon}_i = (\delta - \gamma) \hat{H}_i$; or (B) static supply clears $\hat{H}_i^d = \bar{H}_i \hat{p}_i^{\eta_i}$.
- (B5) **Production consistency.** With $L_i(t) = s_i e^{nt} \int_0^{A_R} f_i(a) da$, wages co-trend at $g_{w,i} = \alpha_i n$. Stationary hats then require $\gamma = g_{w,i}$, hence either α_i common or n chosen so that $\alpha_i n$ is common (we assume common α or equilibrating A_i to ensure a single γ).

Under (A), if construction is active in i , $p_{it} = 1/Z_{it}^h$ implies $g_{p,i} = -g_{Z^h,i}$; stationary hats require $g_{Z^h,i} = -\gamma$ in all active cities. Under (B), \hat{p}_i is set by static clearing; stationary hats require parameter combinations guaranteeing $g_p = \gamma$ (see Appendix 8).

8 Appendix C: Tradable vs non-tradable prices and BGP conditions

Tradable numeraire. The price of the tradable is equalized across locations: $P_{it}^T \equiv P_t^T \equiv 1$. All prices in the model are in this numeraire.

Local housing price. p_{it} is the price of a non-tradable local asset (residential floorspace). Differences in p_{it} across i are consistent with spatial frictions and local supply.

Proposition 1 (Co-trending condition for a single- γ BGP). *On a BGP with stationary hats, wages and local housing prices must co-trend at rate γ :*

$$g_w = \gamma, \quad g_p = \gamma.$$

Under construction complementarity, active cities require $g_{Z^h,i} = -\gamma$; under isoelastic supply $H_i^s = \bar{H}_i p_i^{\eta_i}$ and CRRA curvature σ , log-linear clearing implies

$$g_p = \frac{n + \gamma}{\eta_i + 1/\sigma}.$$

Setting $g_p = \gamma$ gives the restriction $\eta_i = 1 - 1/\sigma + 1/\alpha$ (with common α so $\gamma = \alpha n$). If this restriction is violated, prices drift and a strict BGP does not exist (that case should be interpreted as transitional dynamics).

Implication. There is no contradiction with the law of one price: P^T (tradable) is common; p_{it} (housing) is local. BGP existence requires common growth of p_{it} across cities equal to γ ; levels may differ by i .

9 Appendix D: Stationary Age Distribution on a BGP with $n \neq 0$

D.1 Demographic Accounting

Let $N(t)$ denote aggregate population and $f(a, t)$ the age density such that $N(t) = \int_0^A f(a, t) da$. On a BGP with constant population growth rate n , we have $N(t) = N_0 e^{nt}$.

The age density evolves according to the McKendrick–von Foerster equation. With deterministic death at age A (no mortality before A):

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial a} = 0, \quad a \in [0, A]. \quad (18)$$

D.2 Stationary Age Structure

Proposition 2 (BGP Age Distribution). *On a BGP with population growth rate n , the age density takes the form*

$$f(a, t) = e^{nt} \cdot \psi(a), \quad \psi(a) = \psi(0) e^{-na}.$$

The normalized cross-sectional age distribution is

$$\tilde{f}(a) \equiv \frac{f(a, t)}{N(t)} = \frac{n}{1 - e^{-nA}} e^{-na}, \quad a \in [0, A]. \quad (19)$$

Proof. Substitute the ansatz $f(a, t) = e^{nt} \psi(a)$ into the PDE:

$$ne^{nt} \psi(a) + e^{nt} \psi'(a) = 0 \implies \psi'(a) = -n\psi(a) \implies \psi(a) = \psi(0) e^{-na}.$$

Normalization requires $\int_0^A \tilde{f}(a) da = 1$:

$$\int_0^A C e^{-na} da = C \cdot \frac{1 - e^{-nA}}{n} = 1 \implies C = \frac{n}{1 - e^{-nA}}.$$

□

Remark 1 (Limiting Cases). As $n \rightarrow 0$, L'Hôpital's rule gives $\tilde{f}(a) \rightarrow 1/A$ (uniform). For $n > 0$ (growing population), young cohorts dominate; for $n < 0$ (shrinking population), old cohorts dominate:

$$\frac{\tilde{f}(A)}{\tilde{f}(0)} = e^{-nA} \begin{cases} < 1 & \text{if } n > 0, \\ = 1 & \text{if } n = 0, \\ > 1 & \text{if } n < 0. \end{cases}$$

D.3 Implication for the Detrended KFE

Under the share normalization $g_t(\Omega) = e^{-2\gamma t} \phi(\hat{b}, \hat{h}, \hat{\Omega})$, the stationary density ϕ must embed the e^{-na} age structure. Decompose:

$$\phi(\hat{b}, \hat{h}, i, a, n, a_n) = \tilde{f}(a) \cdot \phi|_a(\hat{b}, \hat{h}, i, n, a_n | a),$$

where $\phi|_a$ is the conditional distribution given age a .

Implementation. When propagating mass from age a to $a + \Delta a$ in a discrete-time approximation, apply the demographic correction factor:

$$\phi(\cdot, a + \Delta a) = e^{-n \Delta a} \cdot \phi^{\text{post-transition}}(\cdot, a), \quad (20)$$

where $\phi^{\text{post-transition}}$ is the density after within-period drift and discrete choice updates. This ensures the age marginal of ϕ matches (19).

Normalization. The total mass under the correct age weighting is

$$\int \phi d\hat{b} d\hat{h} da = 1, \quad \text{where the } a\text{-marginal satisfies } \int_0^A \tilde{f}(a) da = 1.$$

In discrete time with J periods of length Δa :

$$\sum_{j=1}^J \text{mass}(j) = 1, \quad \frac{\text{mass}(j)}{\sum_{j'} \text{mass}(j')} \approx \tilde{f}((j-1)\Delta a) \Delta a.$$

D.4 Numerical Verification

After solving the KFE, verify that the age marginals match theory:

$$\text{mass}(j) \propto e^{-n(j-1)\Delta a}, \quad j = 1, \dots, J.$$

Define the diagnostic:

$$\epsilon_{\text{age}} \equiv \max_j \left| \frac{\text{mass}(j)}{\sum_{j'} \text{mass}(j')} - \tilde{f}((j-1)\Delta a) \Delta a \right|.$$

A well-implemented KFE should achieve $\epsilon_{\text{age}} < 0.01$.

D.5 Timing Convention at Shock Ages

At shock ages $a \in \mathcal{A}^s$, households make discrete choices. The timing convention must be consistent between the HJB (backward) and KFE (forward).

Convention. At the beginning of age a , the sequence of events is:

- (i) **Fertility choice** (if $a \in [A_f^{\text{start}}, A_f^{\text{end}}]$ and $n = 0$): draw $\varepsilon^{n'}$ i.i.d. Type-I extreme value and choose parity $n' \in \{0, 1, 2, 3\}$.
- (ii) **Location choice**: draw $\varepsilon^{i'}$ i.i.d. Type-I extreme value and choose destination $i' \in \mathcal{I}$.
- (iii) **Tenure/size choice**: choose $h' \in \mathcal{H}$.
- (iv) **Within-period flow**: consumption c , rental h^R , wealth accumulation \dot{b} .
- (v) **Age advancement**: transition to $a + \Delta a$.

HJB implementation. Backward induction at age j :

$$\begin{aligned} V^H(b, h, i', j, n, a_n) &= \max_{h' \in \mathcal{H}} V^{\text{flow}}(\tilde{b}^H, h', i', j, n, a_n), \\ V^I(b, h, i, j, n, a_n) &= \frac{1}{\nu_\ell} \log \left(\sum_{i' \in \mathcal{I}} (\mathcal{E}_{i'} \mu_{ii'}) \exp(\nu_\ell V^{i'}(b, h, i, j, n, a_n)) \right), \\ V^{\text{fert}}(b, h, i, j, a_n) &= \frac{1}{\nu^n} \log \left(\sum_{n'=0}^3 \exp(\nu^n V^I(b, h, i, j, n', a_n)) \right). \end{aligned}$$

For agents with $n = 0$ at fertile ages, the value function is $V(b, h, i, j, 0, a_n) = V^{\text{fert}}(b, h, i, j, a_n)$.

KFE implementation. Forward iteration at age j :

(i) **Fertility:** For mass $\phi(b, h, i, j, 0, 1)$ (childless, no kids present), redistribute:

$$\phi(b, h, i, j, n', a'_n) += \pi^{n'}(b, h, i, j) \cdot \phi(b, h, i, j, 0, 1), \quad n' \in \{0, 1, 2, 3\},$$

where $a'_n = 0$ if $n' > 0$ (newborn children) and the child state updates accordingly.

(ii) **Location:** For each (n, a_n) , redistribute mass across destinations using $\pi^{i'}(b, h, i, j, n, a_n)$.

(iii) **Tenure:** Concentrate mass at optimal h' given post-relocation wealth.

(iv) **Drift:** Solve the KFE for within-period dynamics.

(v) **Advance:** $\phi(\cdot, j+1, \cdot) += \phi^{\text{post-drift}}(\cdot, j, \cdot)$.

Remark 2 (Timing consistency). *The fertility choice must occur before within-period drift in the KFE, matching the HJB where fertility affects the value function at age j . If fertility is applied after drift (i.e., to mass at $j+1$), there is a one-period delay in child costs and housing demand, creating an inconsistency between the value function and the cross-sectional distribution.*