ME 121: Handout for discussion 3

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Abstract

In this discussion, we study second-order systems and their time responses. This systems have great variation in the types of responses. To gain insight into second-order systems, we make use of the Spring-Mass-Damper system that we have developed in the previous class. Finally, we quickly review Bode diagrams.

1 Intro to Second-Order Systems

The order of a differential equation is the highest degree of derivative present in that equation. A system whose input-output equation is a second order differential equation is called Second-Order System. The time domain analyzes the functioning of the system on basis of time. This analysis is only effective when nature of input plus mathematical model of the control system is known. Second-order systems are described by the following differential equation:

$$\frac{\partial^2}{\partial t^2}x + 2\zeta\omega_n\frac{\partial}{\partial t}x + \omega_n^2 x = f(t), \tag{1}$$

where ω_n is the natural frequency of the system and ζ is the damping ratio.

For the sake of simplicity, we first assume that (i) $\zeta > 0$ so that the system is stable, and (ii) $f(t) = 0$ so that the system is not perturbed by exogenous inputs. Then, we rewrite Equation (2) as:

$$\frac{1}{\omega_n^2}\frac{\partial^2}{\partial t^2}x + \frac{2\zeta}{\omega_n}\frac{\partial}{\partial t}x + x = 0. \tag{2}$$

The natural frequency $\omega_n > 0$ determines the time-scale of the response.[1] Thus, the damping ratio ζ determines the type of natural response that the system will display. Namely:

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[1] We assume $\omega_n > 0$ without loss of generality.

(i) If $0 < \zeta < 1$, then such a second-order system is **underdamped**, the poles have imaginary components, and the natural response contains some amount of oscillatory component. Lower values of ζ correspond with relatively more oscillatory responses, i.e., are more lightly damped.

(ii) If $\zeta = 1$, then such a second-order system is **critically damped**, and the poles are coincident on the negative real axis at a location $-\omega_n$.

(iii) If $\zeta > 1$, then such a second-order system is **overdamped**, and the poles are at distinct locations on the negative real axis. This case can also be thought of as two independent first-order systems.

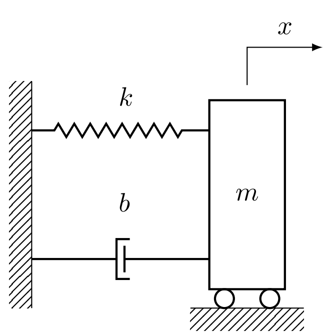
There are a number of factors that make second order systems important. They are simple and exhibit oscillations and overshoot. Higher order systems are based on second order systems. In the case of mechanical second order systems, energy is stored in the form of inertia whereas in case of electrical systems, energy can be stored in a capacitor or inductor. There are two components of any system's time response, which are:

- Transient response
- Steady state response

Typically, meaningful test signals are: Impulse, Step, Ramp, Parabolic. The function `stepinfo` computes the step-response characteristics of the system we are interested in.

1.1 Example: Mass-Spring-Damper

The second-order system which we will study in this section is shown below:



Recall the governing equation:

$$-b\dot{x} - kx = m\ddot{x} \quad \Leftrightarrow \quad m\ddot{x} + b\dot{x} + kx = 0.$$

The pole locations are conveniently parameterized in terms of the damping ratio ζ, and natural frequency ω_n, where

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{b}{2\sqrt{km}}.$$

The natural frequency ω_n is the frequency at which the system would oscillate if the damping b were zero. The damping ratio is the ratio of the actual damping b to the critical damping $b_c = 2\sqrt{km}$.

> **EXERCISE:** The solutions to the characteristic equation of the Mass-Spring-Damper system can read as follows:
>
> - $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, where $\lambda_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$,
> - $x(t) = e^{-\zeta\omega_n t} C \cos(\omega_d t - \phi)$, where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$,
> - $x(t) = (c_1 + c_2 t)e^{\lambda t}$.
>
> Assign each of the above solutions to its corresponding case (i), (ii), or (iii) in Section 1.

We are now ready to study the responses of the Mass-Spring-Damper system in Matlab. Building on the file available from the previous discussion, plot a figure that illustrates the undamped, underdamped, critically damped and overdamped cases:

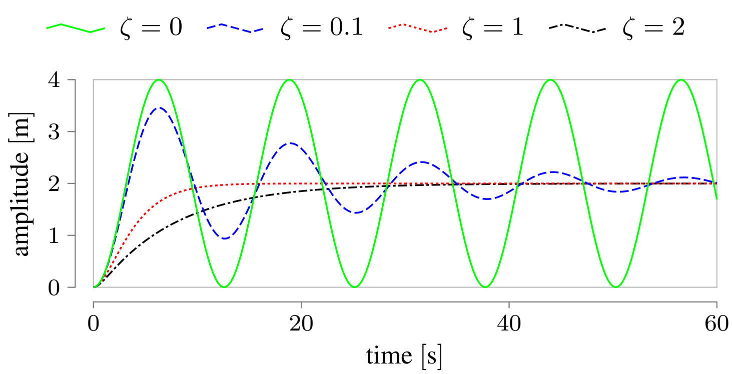


Figure 1: Step responses of the Mass-Spring-Damper system with parameters $m = 2$ [kg], $k = 0.5$ [N/m], and $b = \{0, 0.2, 2, 4\}$[Ns/m].

Notice how the case $\zeta = 0$ takes $1/\omega_n = 2\pi/\sqrt{0.5 \cdot 2} \approx 12.5$ seconds to complete one oscillation.

1.2 Matlab commands

- `step(sys)` calculates the step response of a dynamic system. For the state-space case, zero initial state is assumed.
- `stepinfo(sys)` computes the step-response characteristics for a dynamic system model sys. The function returns the characteristics in a structure containing the fields:

- `RiseTime`: Time it takes for the response to rise from 10% to 90% of the steady-state response.
- `SettlingTime`: Time it takes for the error $|y(t) - y_{\text{final}}|$ between the response $y(t)$ and the steady-state response y_{final} to fall to within 2% of y_{final}.
- `SettlingMin`: Minimum value of $y(t)$ once the response has risen.
- `SettlingMax`: Maximum value of $y(t)$ once the response has risen.
- `Overshoot`: Percentage overshoot, relative to y_{final}.
- `Undershoot`: Percentage undershoot.
- `Peak`: Peak absolute value of $y(t)$.
- `PeakTime`: Time at which the peak value occurs.

2 Bode Diagrams

The Bode plots of the systems studied in Fig. 1 are the following.

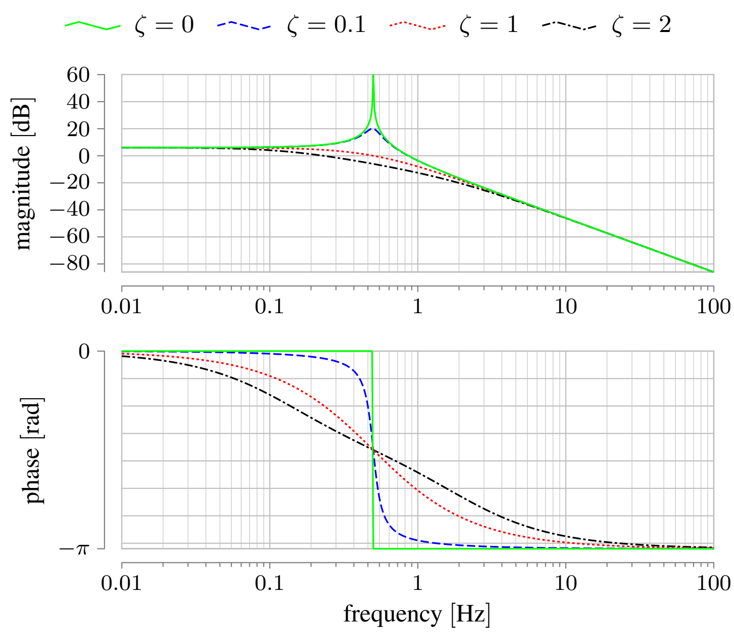


Figure 2: Bode plot of the Mass-Spring-Damper system with parameters $m = 2$ [kg], $k = 0.5$ [N/m], and $b = \{0, 0.2, 2, 4\}$[Ns/m].

Code the transfer function of the system and plot the Bode diagrams. Compute the system's static gain and explain the differences around the natural frequency $\omega_n = \sqrt{k/m} = 0.5$ in terms of the location of the poles.