ME223, Winter 2021 – Secure and Reliable Control Systems
HW1 Solutions (Linear Algebra)

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1. For each of the following matrices

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 0 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 6 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

(a) Use Gaussian Elimination to determine the rank of the matrix.

Solution. Perform elementary row operations on each of the given matrices to get a row echelon form. Consider

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow{r_3 = r_1 - r_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow[r_2 = r_2/3]{r_3 = r_1 - r_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_3 = r_2 - r_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

from which we can see that $\text{rank}(A_1) = 2$. Proceeding analogously, we find that

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 6 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

from which $\text{rank}(A_1) = 3$ and $\text{rank}(A_3) = 2$.

(b) Find a basis for the range space of each matrix.

Solution. Basis for the range space of A_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$

Basis for the range space of A_1 is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Basis for the range space of A_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Find a basis for the null space of each matrix.

Solution. The nullspace of A_1 is non-empty, since the matrix is square and not full rank. To compute the nullspace, we solve $A_1 x = 0$ for x. That is,

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that we can also use the row echelon form obtained by Gaussian elimination. The solution is $\mathcal{N} = \{[2 \ -3 \ 1]^\mathsf{T}\}$. For the matrix A_2, we have that $\mathcal{N} = \emptyset$. Finally, for the matrix A_3, we have that $\mathcal{N} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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2. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for a vector space $\mathcal{V}$. Show that every vector $v \in \mathcal{V}$ can be written as a unique linear combination of the vectors in $\mathcal{B}$. *(Note that uniqueness is the key here)*

Solution. We can prove the statement by contradiction. Suppose that v can be expressed as two different linear combinations of vectors in $\mathcal{B}$. Hence, we have

$$v = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i$$

where $c_i, d_i \in \mathcal{F}$ (the fields of scalars can be $\mathbb{R}$ or $\mathbb{C}$) for all $i \in \{1, \dots, n\}$, and $\exists i$ for which $c_i \neq d_i$. We can manipulate the above equality as follows:

$$\sum_{i=1}^{n} (c_i - d_i) v_i = 0 \overset{(\star)}{\implies} (c_i - d_i) = 0 \ \forall i \in \{1, \dots, n\},$$

where $(\star)$ follows from the definition of basis (i.e., nonzero vectors). Because the vectors in a basis are linearly independent, linear combinations of these vectors equal zero whenever the corresponding scalar coefficient of every vector (in the basis) equals 0. Therefore, $c_i = d_i$ for all $i \in \{1, \dots, n\}$, which is a contradiction to the hypothesis that v could be written as two distinct linear combinations of vectors in $\mathcal{B}$. This concludes on the uniqueness of the linear combination of vectors in $\mathcal{B}$ that defines v.

3. For each of the following statements, either provide a short proof that it is true or provide a counterexample showing that it is false.

(a) Every vector space contains a zero vector.

True, by the definition of vector space.

(b) A vector space may have more than one zero vector.

False. We can prove this by contradiction. Suppose that w is another zero vector apart from our so called $\mathbf{0}$ vector. But from the additive inverse property of a vector space, $v + w = v$ for every vector v. Then in particular, $\mathbf{0} + z = \mathbf{0}$. Also, $\mathbf{0} + z = z$ (with respect to $\mathbf{0}$). Therefore, we can only have one zero vector.

(c) The zero vector is a linear combination of any nonempty set of vectors.

True. $\mathbf{0} = 0v_1 + \cdots + 0v_n$

(d) If $\mathcal{S}$ is a set of linearly dependent vectors, then every vector in $\mathcal{S}$ is linear combination of other vectors in $\mathcal{S}$.

False. Let $\mathcal{S} = \{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\}$. The third vector CANNOT be written as a linear combination of the other two.

(e) Any set containing the zero vector is linearly dependent.

True. Let $\mathcal{S} = \{\mathbf{0}, v_1, \dots, v_k\}$. It holds that $\mathbf{0} = c\ \mathbf{0} + 0v_1 + \cdots + 0v_k$, for any $c \neq 0$. Thus, we have a linear combination of vectors in set $\mathcal{S}$ that yields $\mathbf{0}$ with at least one nonzero coefficient.

(f) Subsets of linearly dependent sets are linearly dependent.

False. Consider $\mathcal{S}$ from (d), which is a set of linearly dependent vectors. The subset $\bar{\mathcal{S}} = \{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\}$ is linearly independent.

(g) Subsets of linearly independent sets are linearly independent.

True. Let $\mathcal{S} = \{v_1, \dots, v_n\}$ be a set of linearly independent vectors. Consider $\mathcal{S}' \subset \mathcal{S}$ with $\mathcal{S}' = \{v_1, \dots, v_k\}$, with $k < n$. For the statement to be true, we need to show that $\sum_{i=i}^{k} c_i v_i = 0$ only if $c_i = \mathbf{0}$ for all $i \in \{1, \dots, k\}$. Consider

$$\mathbf{0} = \sum_{i=i}^{k} c_i v_i = \sum_{i=i}^{k} c_i v_i + \sum_{i=k+1}^{n} 0v_i.$$

The above manipulation expresses $\mathbf{0}$ as a linear combination of vectors in $\mathcal{S}$. However, by the definition of linearly independence of set $\mathcal{S}$, all the scalars should be zero. That is, $c_i = 0$ for all $i \in \{1, \dots, k\}$, which proves the claimed statement.

(h) The rank of a matrix is equal to the number of its nonzero columns.

False. Consider $A = \left[\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right]$, which is a rank-1 matrix. Clearly, the number of nonzero columns does not determine its rank.

(i) The product of two matrices always has rank equal to the smaller of the ranks of the two matrices.

False. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The multiplication AB yields

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is a rank-1 matrix. Therefore, $\text{rank}(A) = \text{rank}(B) = 2 \neq \text{rank}(AB) = 1$.

(j) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

True. Recall that the maximum number of linearly independent rows of A is equal to the maximum number of linearly independent columns of $A^\top$. Because $\text{rank}(A) = \text{rank}(A^\top)$, the claimed statement follows.

(k) Let c be a nonzero scalar and A be a matrix. Then $\text{rank}(cA) = \text{rank}(A)$.

True. Elementary row and column operations (such as the multiplication by a scalar) preserve the rank.

4. Consider the set of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

(a) Write the above system of equations using matrices and vectors.

Solution.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

(b) True or False: if $m < n$ the above system of equations is guaranteed to have a solution for $x_1, \ldots, x_n$.

Solution. False. It depends on whether the system is consistent or not. That is, it depends on the entries of the matrix A.

(c) True or False: if $m \geq n$ the above system of equations is guaranteed to have a solution for $x_1, \ldots, x_n$.

Solution. False. It depends on whether the system is consistent or not. That is, it depends on the entries of the matrix A.

(d) In terms of the range space of the matrix and using the vectors you defined in part (a), state the condition for the above system of equations to have a solution for $x_1, \ldots, x_n$.

Solution. The known vector b should be in the range space of A: $b \in \text{Im}(A)$.

(e) Suppose that $n > m$ and that the above set of equations has a solution. Show that the set of equations actually has an infinite number of solutions, and characterize all such solutions.

Solution. Notice that, because $n > m$, $\text{rank}(A) \leq m$. By the Rank-Nullity theorem, there exists a nullspace N of dimension at least $n - m$. Let $\bar{x}$ be a solution to the system of equations, so that $A\bar{x} = b$. Consider now $A(\bar{x} + cN) = A\bar{x} + cAN = b + 0 = b$, which proves that also $\bar{x} + cN$ is a solution for any $c \in \mathbb{R}$.

5. Suppose A is a $p \times n$ matrix and B is a $p \times m$ matrix, such that all columns of A are linearly independent. Furthermore, no nontrivial linear combination of columns in A is equal to a nontrivial linear combination of columns in B. Let N be a matrix whose rows form a basis for the left nullspace of B. Show that all columns of the matrix NA are linearly independent.

Solution. Let $v_1, \dots, v_n$ denote the columns of A, and $u_1, \dots, u_m$ denote the columns of B. By assumption, we have that

$$\sum_{i=1}^{n} c_i v_i \neq \sum_{i=1}^{m} d_i u_i \tag{1}$$

for scalars c_i, d_i not all zero. Because $NB = 0$, it holds that $N \sum_{i=1}^{m} d_i u_i = 0$. Plugging this into Eq. (1) (i.e., pre-multiplying by the left nullspace N) yields:

$$N \sum_{i=1}^{n} c_i v_i \neq N \sum_{i=1}^{m} d_i u_i = 0,$$

from which we have that $\sum_{i=1}^{n} c_i N v_i \neq 0$ for some scalars c_i not all zero. Notice that Nv_i denote exactly the columns of NA. Therefore, we have shown that arbitrary nontrivial scalars c_i generate nonzero combinations of columns of NA. The fact that $\sum_{i=1}^{n} c_i N v_i = 0$ only if $c_i = 0$ for all $i \in \{1, \dots, n\}$ implies that the columns of NA are linearly independent.

6. Left- and right-inverses of matrices:
(a) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$. In this case, B is called a right-inverse for A.

Solution. Consider the matrix $AA^\top$ and recall the following lemma (from any Linear Algebra textbook):

Lemma 0.1 *Let A be an $m \times n$ matrix with rank m, then $AA^\top$ is full rank and* $\operatorname{rank}(AA^\top) = m$*. If matrix A is with rank n, then $A^\top A$ is full rank and* $\operatorname{rank}(A^\top A) = n$*.*

Proof of Lemma 0.1: Consider A with $\operatorname{rank}(A) = m$. This implies that $v^\top A = 0$ only if $v = 0$ (since all the rows are linearly independent). Let v be such that $v^\top AA^\top = 0$. Then, it also holds that $v^\top AA^\top v = 0$. But $v^\top AA^\top v = (v^\top A)(v^\top A)^\top = 0 \Rightarrow v^\top A = 0$, which follows from the fact that the dot product of a vector with itself equals 0 when the vector itself is zero. However, we already know that $v^\top A = 0$ only if $v = 0$, which implies that $v^\top AA^\top = 0$ only if $v = 0$, proving that $AA^\top$ is full rank and $\operatorname{rank}(AA^\top) = m$. The proof for the case of A with $\operatorname{rank}(A) = n$ proceeds analogously. ■

From Lemma 1, the matrix $AA^\top$ is full rank and its inverse exists. Therefore, $(AA^\top)(AA^\top)^{-1} = I_m$. By letting $B = A^\top(AA^\top)^{-1}$, we have that $AB = I_m$. This proves the existence of a right inverse for A.

(b) Let A be an $m \times n$ matrix with rank n. Prove that there exists an $n \times m$ matrix B such that $BA = I_n$. In this case, B is called a left-inverse for A.

Solution. By Lemma 0.1 the inverse $(A^\top A)^{-1}$ exists. Then, we proceed analogously as in (a), and write $(A^\top A)^{-1}(A^\top A) = I_n$. By letting $B = (A^\top A)^{-1}A^\top$, we have that $BA = I_n$. This proves the existence of a left inverse for A.

7. Eigenvalues and eigenvectors.
(a) Prove that the eigenvalues of an upper-triangular matrix are just the diagonal elements of that matrix.

Solution. Consider a general upper-tringular matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{bmatrix}$$

The eigenvalues of U are the roots of the characteristic polynomial $\det(U - \lambda I) = 0$. The following properties are easy to check:

- $U - \lambda I$ is also an upper-triangular matrix
- the determinant satisfies $\det(U - \lambda I) = \prod_{i=1}^{n}(a_{ii} - \lambda)$

Therefore, the eigenvalues of U equal its diagonal entries.

(b) Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & -5 & 1 \\ -3 & 3 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

Solution. The eigenvalues of A are computed from $\det(A - \lambda I) = 0$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -5 & 1 \\ -3 & 3-\lambda & -3 \\ -1 & -1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 6) + 5(3\lambda) + (6 - \lambda) \\ &= -\lambda(\lambda - 6)(\lambda + 3), \end{aligned}$$

from which we obtain the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = -3$. It is not surprising to see that one eigenvalue is zero, since $\text{rank}(A) = 2$ due to the first and the last columns being identical (i.e., linearly dependent).

To compute the eigenvectors associated to these eigenvalues, we need to solve $Av_i = \lambda_i v_i$, which yields

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) For the above matrix, use the Gershgorin Disc Theorem to sketch the regions where the eigenvalues will be contained, and verify that your answer to the point above satisfies the theorem.

Solution. Using Gershgorin Disc Theorem the regions containing eigenvalues are shown in Fig. 1.

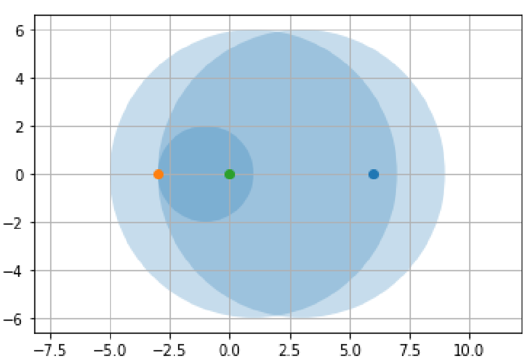

Figure 1: *Gershgorin Disks for A (shaded regions). The colored dots are the actual eigenvalues, which lies in the disks union.*

(d) For the above matrix, find a matrix T so that $T^{-1}AT = \Lambda$, where Λ is a diagonal matrix.

Solution. A matrix T that diagonalizes A can be chosen as the matrix of the eigenvectors of A (regardless of their order). Hence, the matrix

$$T = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A via the transformation $T^{-1}AT$.

(e) Prove or disprove: If $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$, where k is any nonnegative integer. If you are proving that it is true, provide a proof for a general square matrix A (not just the specific matrix above).

Solution. Let A be any general square matrix. We want to verify the relation $A^n = T\Lambda^n T^{-1}$, where n is a nonnegative integer. We start with the case $n=1$: $A^1 = T\Lambda^1 T^{-1}$. Then, for $n=2$, we have

$$A^2 = AA = (T\Lambda T^{-1})(T\Lambda T^{-1}) = T\Lambda I \Lambda T^{-1} = T\Lambda^2 T^{-1}.$$

The same holds for $n = 3, \ldots, k-1$ until, for $n = k$:

$$A^k = A^{k-1} \cdot A = (T\Lambda T^{-1})(T\Lambda T^{-1}) \cdots (T\Lambda T^{-1}) = (T\Lambda^{k-1} T^{-1})(T\Lambda T^{-1}) = T\Lambda^{k-1} I \Lambda T^{-1} = T\Lambda^k T^{-1},$$

which follows by applying the principle of complete induction.

(f) Prove or disprove: An invertible matrix can have an eigenvalue at zero.

Solution. We disprove this statement by contradiction. Suppose the matrix A has an eigenvalue $\lambda = 0$, and let w the (nonzero) left eigenvector associated with $\lambda = 0$. Then, we have that $w^\mathsf{T} A A^{-1} = w^\mathsf{T} I$, from which it follows that $(w^\mathsf{T} A)A^{-1} = w^\mathsf{T} I \Rightarrow (w^\mathsf{T} 0)A^{-1} = 0 = w^\mathsf{T} I$, which is a contradiction, as w is nonzero by definition.

8. Let

$$A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} \beta \\ 0 \\ 1 \end{bmatrix},$$

where $\beta \in \mathbb{R}$ is some constant.

(a) For what values of β is the pair (A, B) controllable?

Solution. Recall that the pair (A, B) is controllable if and only $\operatorname{rank}(\mathcal{C}_{n-\operatorname{rank}(B)}) = n$, where $n = 3$ is the dimension of the system. Then, we need to check the rank of the controllability matrix $\mathcal{C}_2$:

$$\mathcal{C}_2 = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} \beta & \beta - 1 & 3 + \beta \\ 0 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

By visual inspection, one can see that $\operatorname{rank}(\mathcal{C}_2) = 2$ whenever $\beta = 0$. To see this, notice that if $\beta = 0$, the first and second row of $\mathcal{C}_2$ are linearly dependent. Thus, controllability is guaranteed for all $\beta \neq 0$, as $\operatorname{rank}(\mathcal{C}_2) = 3$. You can also convince yourself by computing the determinant of $\mathcal{C}_2$, which yields $\det(\mathcal{C}_2) = 6\beta$, which is nonzero for all $\beta = 0$.

(b) Is the system stable?

Solution. The stability of the system is dictated by the eigenvalues of the matrix A. The simple command `eig(A)` in Matlab reveals that $\lambda = \{-2, -2, 1\}$. If we are dealing with a discrete-time system, the system is stable if and only if $|\lambda_i| < 1$, which is not the case. If we are dealing with a continuous-time system, then the system is stable if and only if $\operatorname{real}(\lambda_i) < 0$, which is again not the case.