

ME 121: Handout for discussion 5

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Abstract

In this discussion, we study the change of basis in 2 dimensions. Specifically, we learn basis rotations, and we analyze the orthogonal and non-orthogonal cases.

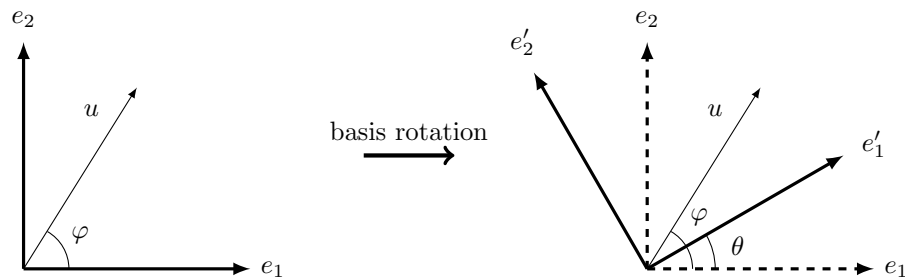
1 Change of Basis

A basis of a vector space V is a set of vectors in V that is linearly independent and spans V . An ordered basis is a list, rather than a set, meaning that the order of the vectors in an ordered basis matters.

A quote from G. Strang [1]: “*The standard basis vectors for \mathbb{R}^n and \mathbb{R}^m are the columns of I . That choice leads to a standard matrix, and $T(v) = Av$ in the normal way. But these spaces also have other bases, so the same T is represented by other matrices. A main theme of linear algebra is to choose the bases that give the best matrix for T* ”

1.1 Basis Rotation

Typically, the vector u is expressed in an orthogonal basis with components e_1 and e_2 . However, it is possible to write the vector u in a new orthogonal basis $[e'_1, e'_2]$, which is a rotation of an angle θ of the old basis $[e_1, e_2]$.



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The relationship between the two bases is expressed as a rotation:

$$\begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (1)$$

Equivalently, the two components of the new rotated basis e'_1 and e'_2 read as:

$$\begin{aligned} e'_1 &= \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}, \\ e'_2 &= -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}. \end{aligned}$$

The vector u can be written in the two different bases as:

$$[u]_{e_1, e_2} = \begin{bmatrix} |u| \cos \phi \\ |u| \sin \phi \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

or

$$[u]_{e'_1, e'_2} = \begin{bmatrix} |u| \cos(\phi - \theta) \\ |u| \sin(\phi - \theta) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

How are $[x \ y]^T$ and $[x' \ y']^T$ related? Recall the trigonometric identities $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ and $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$. Then, we have

$$\begin{aligned} x' &= |u|(\cos \phi \cos \theta + \sin \phi \sin \theta) \\ &= (|u| \cos \phi) \cos \theta + (|u| \sin \phi) \sin \theta \\ &= x \cos \theta + y \sin \theta, \end{aligned}$$

$$\begin{aligned} y' &= |u|(\sin \phi \cos \theta - \sin \theta \cos \phi) \\ &= -(|u| \cos \phi) \sin \theta + (|u| \sin \phi) \cos \theta \\ &= -x \sin \theta + y \cos \theta, \end{aligned}$$

from which it follows that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This final equation corresponds to the clockwise rotation by an angle θ , as in (1). Notice that, intuitively, rotating clockwise the components of u is equivalent to a counter-clockwise rotation of its old basis.

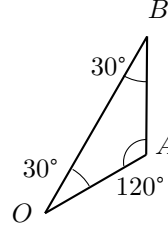
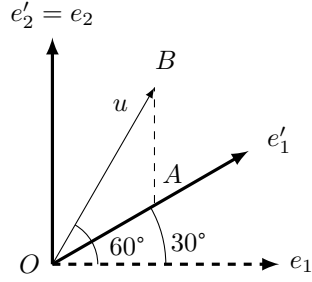
1.2 Different example with non-orthogonal basis: e_1 is rotated by 30° but e_2 is held fixed

The components of the new basis can be written as

$$\begin{aligned} e'_1 &= \cos 30^\circ \hat{e}_1 + \sin 30^\circ \hat{e}_2 \\ e'_2 &= \hat{e}_2, \end{aligned}$$

which can be expressed in matrix form as

$$\begin{bmatrix} e'_1 \\ e'_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$



Recall the following trigonometric identity:

$$\frac{OB}{\sin 120} = \frac{OA}{\sin 30},$$

which yields

$$OA = \frac{\sin 30}{\sin 120} OB,$$

and by fixing $|u| = OB = 1$, $OA = AB = \frac{1}{\sqrt{3}}$.

Hence, being $B = [\frac{1}{2} \ \frac{\sqrt{3}}{2}]^T$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

We can now write the expression for u in the new basis:

$$\begin{aligned} u &= x'e'_1 + y'e'_2 \\ &= x' \left(\frac{\sqrt{3}}{2} e_1 + \frac{1}{2} e_2 \right) + y' e_2 \\ &= \underbrace{x' \frac{\sqrt{3}}{2}}_x e_1 + \underbrace{\left(x' \frac{1}{2} + y' \right)}_y e_2 \end{aligned}$$

and by plugging the coordinates of B in the expressions for x and y in the

previous equation, we derive the expressions for x' and y' :

$$\begin{aligned}\frac{1}{2} &= x' \frac{\sqrt{3}}{2} \Rightarrow x' = \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2} &= y' = \frac{1}{\sqrt{3}}.\end{aligned}$$

In matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}}_{A^T} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Finally, from the previous equation, we obtain the general expression for the change of basis by left-multiplying by the inverse of A^T :

$$\mathbf{x} = A^T \mathbf{x}' \Rightarrow \mathbf{x}' = (A^T)^{-1} \mathbf{x}.$$

1.3 General derivation of the change of basis

In this section, we quickly recall the derivation of some useful equations involved in the change of basis. Consider the basis $[e'_1 \ e'_2]$ with components

$$\begin{aligned}e'_1 &= a_{11}e_1 + a_{12}e_2 \\ e'_2 &= a_{21}e_1 + a_{22}e_2.\end{aligned}$$

Then, a vector u can be expressed in the basis $[e'_1 \ e'_2]$ as:

$$\begin{aligned}u &= x'e'_1 + y'e'_2 \\ &= x'(a_{11}e_1 + a_{12}e_2) + y'(a_{21}e_1 + a_{22}e_2) \\ &= \underbrace{(x'a_{11} + y'a_{21})}_{x} e_1 + \underbrace{(x'a_{12} + y'a_{22})}_{y} e_2.\end{aligned}$$

In matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{A^T} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

In summary:

$$e' = Ae, \quad \mathbf{x} = A^T \mathbf{x}', \quad \mathbf{x}' = (A^T)^{-1} \mathbf{x}.$$

In 2-dimensional cases, if A defines a transformation of basis vectors, then the components of vectors are transformed by $(A^T)^{-1}$ under that change of basis. Notice that, in the particular case of orthogonal transformation matrices such as the rotation matrix, it holds that $A^T = A^{-1}$, or $(A^T)^{-1} = A$.

References

- [1] G. Strang. *Introduction to linear algebra*, volume 3. Wellesley-Cambridge Press Wellesley, MA, 1993.