ME 121: Handout for discussion 6

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Abstract

In this discussion, we recall the definitions of eigenvalues and eigenvectors. Further, we solve linear systems by means of different methods such as the matrix exponential.

1 Eigenvalues and Eigenvectors

Let A be a linear transformation represented by a matrix A. If there is a vector $v \neq 0$ in R^n such that

$$Av = \lambda v$$

for some scalar λ, then λ is called the eigenvalue of A with corresponding (right) eigenvector v. Letting A be a $k \times k$ square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix}$$

with eigenvalue λ, then the corresponding eigenvectors satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix},$$

which corresponds to the homogeneous system

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} - \lambda & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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The latter equation can be rewritten in compact form as

$$(A - \lambda I)v = 0, \tag{1}$$

where I denotes the identity matrix. As shown in Cramer's rule, a linear system of equations has nontrivial solutions if and only if the determinant vanishes, so the solutions of equation (1) are given by

$$\det(A - \lambda I) = 0.$$

This equation is known as the characteristic equation of A.

2 Solution to a System of Linear Differential Equations

Consider the system of differential equations

$$\begin{cases} \dot{x} = -5x + y, \\ \dot{y} = x - 5y. \end{cases}$$

In matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix} \mathbf{x} = A\mathbf{x}. \tag{2}$$

By solving the characteristic equation $\det(A - \lambda I) = 0$, we obtain that the eigenvalues of A are $\lambda_1 = -6$ and $\lambda_2 = -4$. Further, the eigenvectors of A can be computed from $(A - \lambda_1 I)v_1 = 0$ and $(A - \lambda_2 I)v_2 = 0$, whose solutions are

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution of the system (2) is represented in terms of the matrix exponential as

$$\mathbf{x}(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants, and the matrix exponential is defined as

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

Recall the following useful equality, which holds for the case of A diagonalizable[1]:

$$e^{At} = V e^{Dt} V^{-1},$$

[1] That is, in the case of distinct eigenvalues. If the matrix is not diagonalizable, one approach is to use the Jordan form to obtain the matrix exponential.

where V is a diagonalizing matrix of eigenvectors, and $D = \text{diag}(\lambda_1, \lambda_2)$ is the diagonal matrix of the eigenvalues of A (see Appendix for the derivation of this result). We can now compute the matrix exponential for the system (2):

$$\begin{aligned} e^{At} &= Ve^{Dt}V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-6t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-6t} + e^{-4t} & -e^{-6t} + e^{-4t} \\ -e^{-6t} + e^{-4t} & e^{-6t} + e^{-4t} \end{bmatrix}. \end{aligned}$$

Finally, if initial conditions are given, i.e. $x(t_0) = x_0$ and $y(t_0) = y_0$, the solution to the initial value problem is

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}(0) = e^{A(t-t_0)} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

3 An Alternative Method to Solve Linear Systems

Consider the system of differential equations

$$\begin{cases} \dot{x} = x + 3y, \\ \dot{y} = x - y. \end{cases}$$

In matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \mathbf{x} = A\mathbf{x}. \tag{3}$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -2$, and their respective eigenvectors read

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

To solve the system (3), we let

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \tag{4}$$

whose time derivative is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{5}$$

By plugging (4) and (5) into the system (3), we obtain

$$\lambda e^{\lambda t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} e^{\lambda t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \Rightarrow \quad \lambda \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

from which, by moving all terms to the left-hand side,

$$(\lambda I - A) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 0.$$

Finally, by solving the latter equation for x_0 and y_0, for the cases $\lambda = \lambda_1$ and $\lambda = \lambda_2$, and by (4), the general solution to (3) reads as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants.

If $x(0)$ and $y(0)$ are given, we can solve the initial conditions problem and compute the constants c_1 and c_2. For instance, let $\mathbf{x}(0) = [1\ 2]^\mathsf{T}$. Then,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 e^{0} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{0} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

from which

$$\begin{cases} 1 = 3c_1 - c_2, \\ 2 = c_1 + c_2. \end{cases}$$

The above system of equations yields $c_1 = \frac{3}{4}$ and $c_2 = \frac{5}{4}$. Thus, the solution to the initial condition problem is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{3}{4} e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{5}{4} e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} e^{2t} - \frac{5}{4} e^{-2t} \\ \frac{3}{4} e^{2t} + \frac{5}{4} e^{-2t} \end{bmatrix}.$$

EXERCISE: Solve the linear system (3) with the matrix exponential and the same initial conditions $x(0) = 1$ and $y(0) = 2$. You should obtain the same solution as with the alternative method.

Appendix

The exponential of a matrix A can also be determined when A is diagonalizable. That is, whenever we know a matrix P such that $P^{-1}AP$ is a diagonal matrix D. Then, $A = PDP^{-1}$, and using $(PDP^{-1})^k = PD^kP^{-1}$, we obtain

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\ &= PIP^{-1} + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \dots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots\right)P^{-1} = Pe^DP^{-1}. \end{aligned}$$