ME 121: Handout for discussion 7

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Abstract

In this discussion, we solve Problem 5 of the Midterm. These questions are a good recap of useful topics in linear algebra.

1 Problem 5 Midterm

State with reasons whether the following are true or false. You can provide counterexamples for false statements.

(i) The eigenvectors for the matrix $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ are linearly dependent.

(ii) The matrix $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ is orthogonal.

(iii) The transformation that computes the determinant of a matrix is linear.

(iv) The columns of a 4×5 matrix are linearly dependent.

(v) One of the eigenvalues of a reflection transformation is zero.

(vi) One of the eigenvalues of a projection transformation is zero.

(vii) One of the eigenvalues of any orthogonal transformation must be zero.

(viii) If A and B are matrices such that $AB = 0$, then either $A = 0$ or $B = 0$.

(ix) The null space of an $m \times n$ matrix contains vectors in $\mathbb{R}^m$.

(x) If a finite set S of non-zero vectors spans a vector space V, then some subset of S is a basis of V.

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2 Solution to Problem 5

(i) False: the eigenvalues of the matrix are 0 and -1. Since there are two different eigenvalues, the corresponding eigenvectors are independent.
The eigenvalues are computed from $\det(A-\lambda I) = 0$, which yields $\lambda^2+\lambda = 0$, whose solutions are $\lambda_1 = 0$ and $\lambda_2 = -1$. The eigenvectors associated with these eigenvalues are $v_1 = [-2\ 1]^\mathsf{T}$ and $v_2 = [-1\ 1]^\mathsf{T}$, respectively, which are clearly independent.

(ii) False: A orthogonal matrix must have each column with unit norm (length 1). Different columns must be orthogonal, i.e., their dot product must be zero. Alternately, you can check whether $A^\mathsf{T}A = AA^\mathsf{T} = I$. In this case, the latter equality does not hold. Finally, another method to check whether the matrix is orthogonal, is to verify if its determinant is ± 1.

(iii) False: Let us show that this is false with an example in which the property $\mathcal{L}(A+B) = \mathcal{L}(A) + \mathcal{L}(B)$ does not hold. Consider the matrices $A = I_{2\times 2}$ and $B = \mathrm{diag}(1,0)$. Then, $\det(A) = 1$ and $\det(B) = 0$, but $\det(A+B) = 2 \neq 1 = \det(A) + \det(B)$. Hence, the determinant is not a linear transformation.

(iv) True: The columns of a 4×5 matrix are 5, but the rank of the matrix is at most the $\min(\#rows, \#columns)$, which is 4 in this case. Being the rank the maximum number of linearly independent rows (or columns), one column can be expressed as a linear combination of the other 4. A trivial example is $\begin{bmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$. Clearly, the column $[a\ b\ c\ d]^\mathsf{T}$ can be expressed as a linear combination of the previous 4 columns.

(v) False: The linear transformation matrix for a reflection across the line $y = mx$ is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

The eigenvalues and eigenvectors of a linear transformation gives you information about the scaling factors and the directions of the transformation itself. A zero eigenvalue means that there is a non-zero vector that is being transformed to zero. This cannot be the case for a reflection transformation.

(vi) True: The vector perpendicular to the direction along which the transform projects is an eigenvector corresponding to zero eigenvalue. Moreover, all projection matrices are positive semi-definite. Specifically, their eigenvalues are either 0 or 1. Thus, the vector perpendicular to the direction along which the transformation projects satisfies $Av = 0$, which is equivalent to the eigenproblem for the zero eigenvalue. As an example, consider the projection from $\mathbb{R}^3$ to $\mathbb{R}^2$: $P = \text{diag}(1, 1, 0)$. The vector orthogonal to the direction of the projection is $[0\ 0\ a]^{\mathsf{T}}$ for any a, which is also an eigenvector of P associated to the zero eigenvalue.

(vii) False: An orthogonal transformation preserves length. Hence, if one eigenvalue is zero, that means that a non-zero vector is being reduced to zero. But the zero vector and only the zero vector has zero length. Alternatively, since the determinant of a matrix equals the product of all its eigenvalues, and the determinant of an orthogonal matrix is always unitary, the orthogonal transformation cannot have a zero eigenvalue. Finally, one could also argue as follows. Taking the norm of the eigenproblem $Av = \lambda v$ yields $\|Av\| = \|\lambda v\| = |\lambda|^2 \|v\|$. The left-hand side becomes:

$$\begin{aligned}
\|Av\| &= (Av)^{\mathsf{T}}(Av) && \text{by definition of length,}\\
&= v^{\mathsf{T}} A^{\mathsf{T}} Av && \text{because } A \text{ is a real matrix,}\\
&= v^{\mathsf{T}} v && \text{because } A^{\mathsf{T}}A = I \text{ for orthogonal matrices,}\\
&= \|v\| && \text{by definition of length.}
\end{aligned}$$

It follows that $\|v\| = |\lambda|^2 \|v\|$. Since v is an eigenvector, $v \neq 0$, and we can cancel $\|v\|$ on both sides of the equation, obtaining $1 = |\lambda|^2$. Finally, because any length is nonnegative, we are left with $|\lambda| = 1$.

(viii) False. An easy counterexample to such a claim is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Notice how $A \neq 0$, $B \neq 0$ and $AB = 0$.

(ix) False: the vectors belong to $\mathbb{R}^n$. In fact, if A is an $m \times n$ matrix, the definition of kernel reads $N(A) = \{\mathbf{x} \in \mathbb{R}^n \ :\ A\mathbf{x} = 0\}$.

(x) True: Since the set already spans V, by picking a subset that is also linearly independent, we are choosing a basis for V.