Secure and Reliable Control Systems (ME 233)
HW1 Solutions
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1) For each of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 0 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 6 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix},$$

(a) Use Gaussian Elimination to determine the rank of the matrix.

Sol. Perform elementary row operations on each of the given matrices to get a row echelon form. Consider,

$$A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow{R_3=R_1-R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow{R_2=R_2/3, R_3=R_3/2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_2 - R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1=R_1-2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the $\text{Rank}(A_1) = 2$

$$A_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 6 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{R_3=R_3/2-R_1, R_2=R_2/2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{R_4=R_4-1} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - R_4} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{swap } R_2, R_4} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the $\text{Rank}(A_3) = 2$

(b)Find a basis for the range space of each matrix.

Sol. From (1a) row echelon form of $A_1 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. So, basis for the range space of A_1 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Similarly, row echelon form of $A_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, basis for the range space of A_3 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(a) Find a basis for the null space of each matrix.

Sol. To find a basis for the the null space of A_1, A_3, we need to solve the equation $A_1X = 0, A_3Y = 0$. Hence, a basis for the null space of A_1 is $\left\{ \begin{bmatrix} -0.5345 \\ 0.8018 \\ -2.2673 \end{bmatrix} \right\}$.

Similarly for A_3 we have a basis $\left\{ \begin{bmatrix} -0.7644 \\ 0.2715 \\ 0.5810 \\ -0.2715 \end{bmatrix}, \begin{bmatrix} -0.4837 \\ 0.6171 \\ -0.0667 \\ -0.6171 \end{bmatrix} \right\}$

2) Let $\mathcal{B} = v_1, v_2, \dots, v_n$ be a basis for a vector space $\mathcal{V}$. Show that every vector $v \in \mathcal{V}$ can be written as a unique linear combination of the vectors in $\mathcal{B}$.

Proof: We will prove by contradiction. Let, v can be expressed as two different linear combinations of vectors in $\mathcal{B}$. Hence, we have

$$v = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i$$

where $c_i, d_i \in \mathcal{F}$ (field of scalars may be $\mathbb{R}$ or $\mathbb{C}$) and $c_i \neq d_i$ atleast for some of the i's. Now, from the above representation of v we can rewrite,

$$\sum_{i=1}^{n} (c_i - d_i) v_i = 0 \overset{(a)}{\Longrightarrow} c_i - d_i = 0 \ \forall i$$

(a) follows from the definition of basis. As the vector's in a basis are linearly independent, linear combination of these vectors equals zero when the corresponding scalar coefficient of every vector (in the basis) equals 0. Hence, $c_i = d_i$ for all i which is a contradiction to the assumed hypothesis, and v cannot be expressed as different linear combinations of vectors in the given basis $\mathcal{B}$

3) For each of the following statements, either provide a short proof that it is true or provide a counterexample showing that it is false.

(a) Every vector space contains a zero vector

Sol). **True**, by the definition of vector space.

(b) A vector space may have more than one zero vector.

Sol). **False**. We shall prove by contradiction. Assume that z is another zero vector apart from our so called 0 vector. But from additive inverse property of vector space, $v + z = v$ for every vector v. Then in particular, $0 + z = 0$. Also, $0 + z = z$ (with respect to 0). Hence, we can only have one zero vector.

(c) The zero vector is a linear combination of any nonempty set of vectors.

Sol). **True**, $\mathbf{0} = 0v_1 + 0v_2 + \ldots + 0v_n$.

(d) If S is a set of linearly dependent vectors, then every vector in S is linear combination of other vectors in S.

Sol). **False**. Let $[3, 3], [6, 6], [1, 2]$ be linearly dependent vectors in the set S then the last vector cannot be written as linear combination of first two.

(e) Any set containing the zero vector is linearly dependent.

Sol) True. Let $\mathcal{A} = \{\mathbf{0}, v_1, v_2, \ldots, v_n\}$ be the set, then we have $c \cdot \mathbf{0} + 0 \cdot v_1 + 0 \cdot v_1 +, \ldots, +0 \cdot v_n = 0$, where $c \neq 0$. So, we have a linear combination of vectors in set $\mathcal{A}$ equal 0 with not all coefficients equal to zero. Hence, the set $\mathcal{A}$ is linearly dependent.

(f) Subsets of linearly dependent sets are linearly dependent.

Sol) False. Consider the set $\mathcal{B}$ with linearly independent vectors. It's obvious that $\mathcal{B} \subset \mathcal{B} \cup \{0\}$. But from (3e) any set with 0 vector is linearly dependent .

(g) Subsets of linearly independent sets are linearly independent.

Sol) True. Let $\mathcal{S} = \{v_1, v_2, \ldots, v_n\}$ be set of linearly independent vectors and $\mathcal{S}' = \{v_1, v_2, \ldots, v_k\}$ be a subset of $\mathcal{S}$ and obviously $k < n$. We claim that if $\sum\limits_{i=1}^{k} c_i v_i = 0$ for some scalars c_i, then $c_i = 0$ for all $1 \leq i \leq k$. Consider the following,

$$\begin{aligned} 0 &= \sum_{i=1}^{k} c_i v_i \\ &= \sum_{i=1}^{k} c_i v_i + \sum_{i=k+1}^{n} 0 \cdot v_i \end{aligned}$$

Thus by the above manipulation, we have expressed the linear combination of vectors in S' as linear combination of vectors in $\mathcal{S}$ which equals 0. But, by the definition of linearly independence of set $\mathcal{S}$ all the scalars should be zero. In particular, $c_i = 0$ for all i's. Hence, the claim is proved.

(h) The rank of a matrix is equal to the number of its nonzero columns.

Sol). **False**. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. This particular matrix has 2 non-zero columns but it's rank is 1.

(i) The product of two matrices always has rank equal to the smaller of the ranks of the two matrices.

Sol). **False**. Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then we have $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\text{Rank}(A) = \text{Rank}(B) = 2 \neq \text{Rank}(AB)(= 1)$

(j) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

Sol). **True**. We note that the maximum number of linearly independent rows of A equals to the maximum number of linearly independent columns of A^T, which is precisely rank of A^T. But $\text{Rank}(A) = \text{Rank}(A^T)$.

(k) Let c be a nonzero scalar and A be a matrix. Then $\text{Rank}(cA) = \text{A}$

Sol). **True**. Elementary operations preserves the rank.

4) Consider the set of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_x + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_x + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_x + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

(a) Write the above system of equations using matrices and vectors.

Sol).

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{B}$$

(b) Write the above system of equations using matrices and vectors.

Sol). **False**. It depends on system being consistent or inconsistent, which depends on the elements of A.

(c) Write the above system of equations using matrices and vectors.

Sol). **False**. It depends on system being consistent or inconsistent, which depends on the elements of A.

(d) In terms of the range space of the matrix and using the vectors you defined in part a), state the condition for the above system of equations to have a solution for $x_1, \ldots, x_n$.

Sol). B should be in the range space of A compactly $B \in \mathcal{R}(A)$

(e) Suppose that $n > m$ and that the above set of equations has a solution. Show that the set of equations actually has an infinite number of solutions, and characterize all such solutions.

Sol). Let X be a solution satisfying $AX = B$. Given that $n > m$ the matrix should admit nontrivial null space because $\text{Rank}(A) \leq \min(m, n) = m$ (look into Rank-Nullity theorem). Also, we note that there are infinite vectors in a nontrivial null space ($AV = 0 \implies A(cV) = 0$, where c is a scalar). Hence, by assertions it is clear that, if X is a solution to $AX = B$ then $X + cV$ is also a solution, since, $A(X + cV) = AX + A(cV) = B + 0 = B$.

5) Suppose A is $p \times n$ matrix and B is a $p \times m$, such that all columns of A are linearly independent, and furthermore, no nontrivial linear combination of columns in A is equal to nontrivial linear combination of columns in B. Let N be a matrix whose rows form a basis for the left nullspace of B. Show that all columns of the matrix NA are linearly independent.

Sol. Let columns of A be $v_1, v_2, \ldots, v_n$ and that of B be $\mu_1, \mu_2, \ldots, \mu_m$. Given that,

$$\sum_{i=1}^{n} c_i v_i \neq \sum_{i=1}^{m} d_i \mu_i \tag{1}$$

for all nontrivial scalars c_i, d_i. Also, $NB = 0 \implies N \sum_{i=1}^{m} d_i \mu_i = 0$. From (1) we have,

$$N \sum_{i=1}^{n} c_i v_i \neq N \sum_{i=1}^{m} d_i \mu_i = 0$$

Hence, $N \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} c_i N v_i \neq 0$ for not all $c_i = 0$. But Nv_i are precisely columns of NA. Hence, we have showed that linear combination of columns of NA not equals 0 for arbitrary scalars. It equals 0 when all $c_i = 0$ implying that columns of NA are linearly independent.

Lemma 1: Let A be an $m \times n$ matrix with rank m, then AA^T is full rank and equals m. If matrix A is with rank n, then $A^T A$ is full rank and equals n.

Proof: Given A is having full row rank m implies that the $x^T A = 0$ only when $x^T = 0$ (since $x^T A$ is linear combination of rows of A which are linearly independent). Let x be such that $x^TAA^T = 0$. When then have $x^TAA^Tx = 0$. But $x^TAA^Tx = (x^TA)(x^TA)^T \implies x^TA = 0$ (since the dot product of a vector with itself, x^Tin this case, equals 0 when the vector itself is zero). But we already know that $x^TA = 0$ happens only when $x^T = 0$. Hence, it follows that $x^TAA^T = 0$ happens when $x^T = 0$ indicating that AA^T is having full rank $= m$. Similar proof works for A with rank n.

6) Left- and right-inverses of matrices:

(a) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

Sol). Consider the matrix AA^T. From Lemma 1, the matrix AA^T is of full rank and inverse exist. So, $(AA^T)(AA^T)^{-1} = I_m$. Set $B = A^T(AA^T)^{-1}$ and we have $AB = I_m$. This proves that there exist a right inverse for A.

(b) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

Sol). Consider the matrix A^TA. From Lemma 1, the matrix A^TA is of full rank and inverse exist. So, $(A^TA)^{-1}(A^TA) = I_m$. Set $B = (A^TA)^{-1}A^T$ and we have $BA = I_n$. This proves that there exist a left inverse for A.

7) Eigenvalues and eigenvectors.

(a) Prove that the eigenvalues of an upper-triangular matrix are just the diagonal elements of that matrix.

Sol. A square matrix is said to be *Upper Triangular* if all the elements below main diagonal are zero. For instance,

$$U = \begin{bmatrix} \mathrm{d}_1 & \mathrm{x} & \mathrm{x} & \dots & \mathrm{x} \\ 0 & \mathrm{d}_2 & \mathrm{x} & \dots & \mathrm{x} \\ 0 & 0 & \mathrm{d}_3 & \dots & \mathrm{x} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{d}_n \end{bmatrix}_{nxn}$$

Note that all x′s need not be equal, they might be different and are represented in this form for brevity. The eigenvalues of U are nothing but the roots of the characteristic polynomial $\det(U - \lambda I) = 0$. The following properties are easy to check

- $U - \lambda I$ is also an upper triangular matrix
- Determinant of an upper triangular matrix equals product of its main diagonal entries.

Hence, by above properties $\det(U - \lambda I) = \prod_{i=1}^{n}(\mathrm{d}_i - \lambda)$. The roots are above polynomial in λ are precisely the eigenvalues which equals main diagonal entries of the matrix U. This concludes the proof.

(b) Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 1 & -5 & 1 \\ -3 & 3 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

Sol. The eigenvalues of matrix A are roots of its characteristic polynomial $\det(A - \lambda I)$. So, we begin by expanding the determinant of $A - \lambda I$,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -5 & 1 \\ -3 & 3-\lambda & -3 \\ -1 & -1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 6) + 5(3\lambda) + (6 - \lambda) \\ &= -\lambda^3 + 3\lambda^2 + 18\lambda \\ &= -\lambda(\lambda - 6)(\lambda + 3) \end{aligned}$$

The roots of above polynomial are $\lambda = \{0, 6, -3\}$, which are precisely our eigenvalues. It's not surprising to see one eigenvalue is 0, since $\text{Rank}(A) = 2$ (you can look into the relations governing rank's and eigenvalues in any Linear Algebra book).

To find the eigenvector for corresponding eigenvalues $\lambda = \{0, 6, -3\}$, plug each λ into the equation $Av = \lambda v$ and solve for vector v. Hence, we have the following eigenvectors,

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ +\frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

(c) For the above matrix, use the Gershgorin Disc Theorem to sketch the regions where the eigenvalues will be contained, and verify that your answer to the point above satisfiers theorem.

Sol. Using Gershgorin Disc Theorem (look for definition in any Linear Algebra book) the regions containing eigenvalues are shown in the Fig 1.

(d) For the above matrix, find a matrix T so that $T^{-1}AT = \Lambda$, where Λ is a diagonal matrix.

Sol. We need to find a matrix T that diagonalizes (condition for diagonalization can be found in any standard linear Algebra book) A. For our purpose we pick T which contains eigenvectors of A as it column's(can order in anyway). Hence,

$$T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

will diagonalize the matrix A.

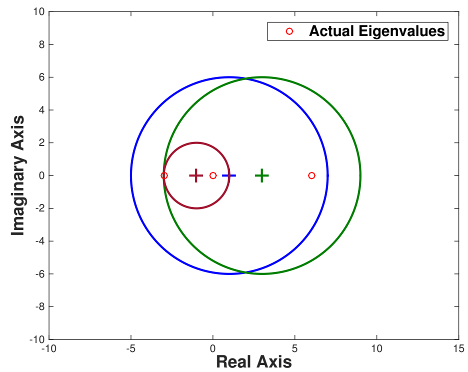

Figure 1: Gershgorin Disc for A, $'+'$ indicates center of each circle (with same color) and $\circ$ are actual eigenvalues which lies in the regions (obtained using the theorem)

(e) Prove or disprove: If $A = T\Lambda T^{-1}$, then $A^k = T\Lambda^k T^{-1}$, where k is any nonnegative integer.

Sol. Let A be any general square matrix. We want to verify the relation $A^n = T\Lambda^n T^{-1}$, where n is nonegative integer. For,

$$\begin{aligned}
&n = 1,\;\; A = T\Lambda T^{-1}\\
&n = 2,\;\; A^2 = AA = (T\Lambda T^{-1})(T\Lambda T^{-1}) = T\Lambda \underbrace{T^{-1}T}_{I} \Lambda T^{-1} = T\Lambda^2 T^{-1}\\
&n = 3,\;\; A^3 = A^2 A = (T\Lambda^2 T^{-1})(T\Lambda T^{-1}) = T^{-1} = T\Lambda^3 T^{-1}\\
&\;\;\vdots\\
&n = k,\;\; A^k \overset{(a)}{=} A^{k-1}A = (T\Lambda^{k-1}T^{-1})(T\Lambda T^{-1}) = T^{-1} = T\Lambda^k T^{-1}
\end{aligned}$$

where (a) follows by applying the principle of complete induction. This concludes the proof.

(f) Prove or disprove: An invertible matrix can have an eigenvalue at zero.

Sol. We disprove this statement by contradiction. Let us assume that an invertible matrix say, A has an eigenvalue (λ) at 0. But, by the definition of eigenvalue we have $Av = \lambda v = 0v = 0$. This shows that there are some non-negative vector's which maps A to 0 implying that the mapping Av is not surjective further implying A will not have an inverse. We arrived at a contradiction indicating that assumed hypothesis is wrong and hence A will not have an inverse.

Alternatively, $Av = 0$ implies that A has non-empty kernel (also called null-space) and from "Rank-Nullity Theorem" Rank$(A) + ker(A) = n$. We see that Rank$(A) < n$ (since ker$(A) \neq 0$). As matrix A is not having full rank, inverse fails to exist.

Alternatively (by Prof. Fabio), for the invertible matrix, say A, $AA^{-1} = I$. Let w^T be a non-zero left eigenvector corresponding to the eigenvalue $\lambda = 0$. Then we have, $w^T AA^{-1} = w^T I$. It follows that $w^T A = \lambda w^T = 0$, implying $w^T AA^{-1} = 0 = w^T I$, which is a contradiction since w^T is non-zero by assumption.

8) Let

$$A = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} \beta \\ 0 \\ 1 \end{bmatrix},$$

where $\beta \in \mathbb{R}$ is some constant.

(a) For what values of β is the pair (A, B) controllable
(b) Is the system stable?

Sol.

(a) Recall Theorem 3.7 about the *Controllability of discrete-time systems*, which says that the system described below

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

is controllable if and only $\text{Rank}(\mathcal{C}_{n-\text{Rank}(B)}) = n$, where n is the dimension of state. Let's begin by computing *controllability matrix* $(n = 3, \text{Rank}(B) = 1)$

$$\text{Rank}(\mathcal{C}_{n-\text{Rank}(B)}) = \mathcal{C}_2 = \begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 3+b & b-1 & b \\ 3 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

From the above controllability matrix it is obvious that $\text{Rank}(\mathcal{C}_2) = 3$ when $b \neq 0$. If $b = 0$, we see that first and second rows of $\mathcal{C}_n$ are linearly dependent and rank < 3 (rank $= 2$). To convince yourself convert the problem into the following equivalent problem, $\mathcal{C}_n = 3 \iff \det(A) \neq 0$.

$$\det(A) = -6b$$

The above expression not equals 0 when $b \neq 0$ only. Hence, the range of values of b for which the system is controllable are $\mathbb{R}\backslash\{0\}$.

(b) To determine stability of the system we need information about A, since to analyze stability we set $u = 0$ and B becomes irrelevant. From Theorem 3.5, discrete-time linear system is stable if all the eigenvalues of A have magnitude less than 1. To compute eigenvalues set

$$\det(A - \lambda I) = 0.$$

Use command 'eig(A)' in MatLab to compute eigenvalues, which turn out to be $\lambda = -2, -1, 1$. As magnitude of all the eigenvalues of A are not strictly smaller than one, we conclude that system is not stable.