ME 121: Handout for discussion 2

Tommaso Menara *

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Abstract

In this discussion we learn how to model a Mass-Spring-Damper System in Matlab and how to use the functions `ss`, `tf`, and `ss2tf`.

1 Dynamic Systems

Dynamic systems evolve over time according to some fixed governing rule. In most cases, physical systems can be represented by first-order differential equations:

$$\dot{x} = \frac{\partial x}{\partial t} = f(x(t), u(t), t). \tag{1}$$

In the above equation, x is a vector representing the state of the system, u is a vector representing the external (or exogenous) inputs, and $f(\cdot)$ is a (possibly nonlinear) function representing the time derivative (rate of change) of the state x. At any future time t_1, we can compute the state $x(t_1)$ if we know the state at initial time t_0, $x(t_0)$, and the series of inputs u, by integrating equation (1).

If the function $f(\cdot)$ has fixed parameters, then the system is said to be time-invariant and equation (1) can be simplified by removing the time dependency of $f(\cdot)$ as:

$$\dot{x} = f(x(t), u(t)). \tag{2}$$

The above equation is still nonlinear. By performing a linearization about an equilibrium point (e.g., the working point of an electro-mechanical system), we can study the system behavior in a small operating range around such equilibrium. Then, the system can be written in state-space form as:

$$\dot{x} = Ax + Bu,$$

which is the matrix equation of the linearized system (2) about an equilibrium, where A is the system matrix that represents interactions between the states and B is the input matrix that represents how the external inputs enter the system.

*Tommaso Menara is with the Department of Mechanical Engineering, University of California at Riverside, `tomenara@engr.ucr.edu`, `www.tommasomenara.com`

1.1 State-Space representation

For continuous linear time-invariant (LTI) systems, the state-space representation reads:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3}$$

where C is the output matrix and D is the feedforward matrix. The output equation (3) is often necessary because not all state variables can be directly measured or of interest. The output matrix, C, is used to specify which state variables (or combinations thereof) are available for use by the controller. Also, it is often the case that the outputs do not directly depend on the inputs (only through the state variables), in which case D is the zero matrix.

1.2 Transfer Function Representation

Using the Laplace transform, it is possible to convert a system's time-domain representation into a frequency-domain input/output representation, known as the transfer function. By doing so, we transform the governing differential equation into an algebraic equation, which is often easier to analyze. The Laplace transform of a time domain function, $f(t)$, is defined below:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt,$$

where the parameter $s = \sigma + j\omega$ is a complex frequency variable.

The transfer function from input $U(s)$ to output $Y(s)$ is as follows:[1]

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0},$$

where $Y(s)$ and $U(s)$ are the Laplace Transforms of $y(t)$ and $u(t)$, respectively. It is useful to factor the numerator and denominator of the transfer function into what is termed zero-pole-gain form:

$$G(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2)...(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)...(s - p_{n-1})(s - p_n)}.$$

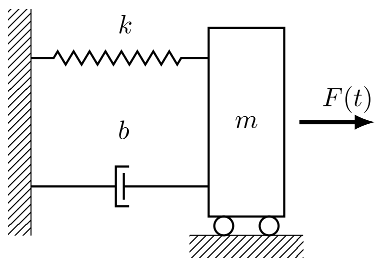
The zeros of the transfer function, $z_1, \ldots, z_m$, are the roots of the numerator polynomial (i.e. the values of s such that $N(s) = 0$). The poles of the transfer function, $p_1, \ldots, p_n$, are the roots of the denominator polynomial (i.e. the values of s such that $D(s) = 0$). Both the zeros and poles may be complex valued (have both real and imaginary parts). The system Gain is $K = b_m/a_n$.

[1] Recall that the Laplace transform of the n-th derivative of a function is: $\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - ... - f^{(n-1)}(0)$.

Finally, to transform directly the state-space representation into the transfer function representation, one can use:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D.$$

2 Example: Mass-Spring-Damper



To model this system, we need to draw the free body diagram (FBD) of this system and use of Newton's second and third laws of motion:[2]

$$\Sigma \mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{x}}{dt^2}.$$

In the x-direction we have:

$$\Sigma F_x = F(t) - b\dot{x} - kx = m\ddot{x}, \tag{4}$$

while there are no forces acting in the y-direction. Equation (4), known as the governing equation, completely characterizes the dynamic state of the system.

We choose the position and velocity as our state variables:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

The state equation in this case is:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t).$$

The Laplace transform for this system assuming zero initial conditions is

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

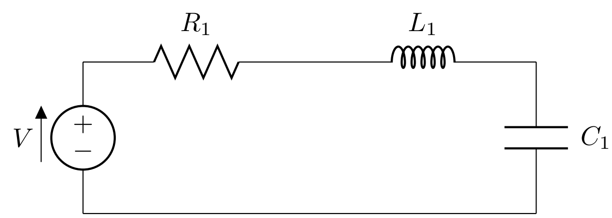
and, therefore, the transfer function from force input to displacement output is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}.$$

[2]Newton's second law states that the sum of the forces acting on a body equals the product of its mass and acceleration. Newton's third law states that if two bodies are in contact, then they experience the same magnitude contact force, just acting in opposite directions.

3 Exercise: RLC circuit

Try to model the following RLC circuit in Matlab. Find and code its state-space and transfer function representations.



Hint1: use Kirchoff Current Law to derive the following governing equation:

$$V(t) - Ri - L\frac{di}{dt} - \frac{1}{C}\int i\mathrm{d}t = 0.$$

Then, choose the charge on the capacitor and current through the circuit (inductor) as the state variables.

$$\mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix},$$

where $q = \int i\mathrm{d}t$.
Hint2: In the output equation, only the current i is observed.