Relay Interactions Enable Remote Synchronization in Networks of Phase Oscillators

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***Abstract*—Remote synchronization describes a fascinating phenomenon where oscillators that are not directly connected via physical links evolve synchronously. This phenomenon is thought to be critical for distributed information processing in the mammalian brain, where long-range synchronization is empirically observed between neural populations belonging to spatially distant brain regions. Inspired by the growing belief that this phenomenon may be prompted by intermediate mediating brain regions, such as the *thalamus*, in this letter we derive a novel mechanism to achieve remote synchronization. This mechanism prescribes remotely synchronized oscillators to be stably connected to a cohesive relay in the network – a group of tightly connected oscillators mediating the distant ones. Remote synchronization unfolds whenever the stability of the subnetwork formed by relays and remotely synchronized oscillators is not affected by the rest of the oscillators. In accordance with our results, we find that remotely-synchronized cortico-thalamo-cortical circuits in the brain posses strong interconnection profiles. Finally, we demonstrate that the absence of cohesive relays prevents stable remote synchronization in a large class of cases, further validating our results.**

***Index Terms*—Biological systems, network analysis and control, Lyapunov methods, stability of nonlinear systems.**

I. Introduction

SYNCHRONIZATION is a universal phenomenon intimately related to the functioning of many natural and engineered systems [1]. In the brain, synchronization phenomena are thought to constitute the neural basis of cognition, memory, and large-scale information processing [2]–[4]. Empirical evidence demonstrates that brain regions that are not physically interconnected are capable of synchronizing, giving rise to what is known as *remote* synchronization [5]–[7]. In this context, the thalamus is believed to be an enabler of remote synchronization by functioning as a central hub that relays information to distant cortical regions [5]. However, the mechanism underlying this compelling phenomenon has not been fully characterized yet, and studies on remote synchronization remain few and sparse.

To investigate this phenomenon, we model neural activity as the network-wide product of interacting oscillators, where each oscillator represents a brain region [8]. Preliminary work has probed remote synchronization in phase-amplitude oscillators, producing seminal results [9]–[11]. Yet, there is compelling evidence that most of the information in brain-wide interactions can be explained by the phases of brain signals, not their amplitude [2], [12]. Hence, phase oscillators lend themselves as an ideal candidate for the modeling and analysis of remotely synchronizing brain regions.

In this work, we utilize heterogeneous Kuramoto oscillators to investigate the role that network topology and parameters play in the emergence of remote synchronization. Specifically, we derive conditions to ensure stable remote synchronization that prescribe the existence of a strongly connected set of oscillators acting as intermediate relay between remotely-synchronized nodes. Moreover, by analyzing human brain structural and functional data, we find evidence that the brain may enact a similar mechanism.

Related Work: Kuramoto-like models, known for their rich dynamics and fascinating behaviors [13], have been widely used for the study of neural synchronization phenomena [14]–[17]. Besides emerging in brain recordings, remote synchronization finds applications in climate research [18] and in secure communication technologies [19]. In the latter, concurrent remote synchronization of distant network nodes and asynchronous behavior of the intermediate ones allows for the secure distribution of critical information. Despite its importance, the characterization of remote synchronization in phase oscillators has remained elusive.

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Early attempts at the characterization of remote synchronization in phase oscillators have been made by employing phase shifts [20] and network symmetries (mathematically described by graph automorphisms) [21], [22]. Here, we present a different mechanism for remote synchronization, and show that network symmetries, although beneficial, are not necessary for the emergence of remote synchronization.

Paper Contribution: The main contribution of this letter is the derivation of a mechanism that guarantees the emergence of stable remote synchronization in networks of heterogeneous Kuramoto oscillators. We demonstrate that stability of remote synchronization is guaranteed whenever there exists a network partition where oscillators within the same group evolve cohesively, and at least one group consists of remotely-synchronized oscillators with strongly connected neighbors – a *cohesive relay*. We confirm the existence of a cohesive relay that enables remote synchronization in human brain data using a publicly available dataset. The mechanism proposed in this letter builds upon and extends previous results used to assess cluster synchronization [23], and complements previous work on relay synchronization [24].

Furthermore, we reveal that the absence of cohesive relays hinders the stability of the remote synchronization manifold in the class of networks comprising two groups of synchronized oscillators. This important result suggests that our condition may be (almost) necessary. Finally, since Kuramoto oscillators are hard to analyze with the Master Stability Function formalism, our findings shed light to the challenging analysis of exotic synchronization phenomena in this class of oscillators.

Mathematical Notation: $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{S}$ denote the real numbers, the nonnegative real numbers, the positive real numbers, and the unit circle, respectively. The set $\mathbb{T}^n = \mathbb{S} \times \cdots \times \mathbb{S}$ is the n-dimensional torus. We use $\mathbb{1}$ and e_i to represent the vector of all ones and the i-th canonical vector, respectively. The operation $A^\dagger$ denotes the Moore-Penrose pseudoinverse of the matrix A. An M-matrix is a real nonsingular matrix $A = [a_{ij}]$ such that $a_{ij} \leq 0$ for all $i \neq j$ and all leading principal minors are positive. Finally, $A \succ 0$ indicates that A is positive definite, and $D = \mathrm{diag}(c_1, \ldots, c_n)$ represents a diagonal matrix with (i, i)-th entry c_i, $i = 1, \ldots, n$.

II. Problem Setup and Preliminary Notions

In this letter, we characterize the stability properties of remotely-synchronized trajectories arising in networks of oscillators with Kuramoto dynamics. To this aim, let $\mathcal{G} = \{\mathcal{O}, \mathcal{E}\}$ be the connected and weighted graph representing the network of oscillators, with $\mathcal{O} = \{1, \ldots, n\}$ being the oscillator set, and $\mathcal{E}$ being the edge set. Let $A = [a_{ij}]$ be the sparse adjacency matrix of the network, where $a_{ij} > 0$ whenever $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The dynamics of the i-the oscillator is governed by

$$\dot{\theta}_i = \omega_i + \sum_{j \neq i} a_{ij} \sin(\theta_j - \theta_i), \tag{1}$$

where $\omega_i \in \mathbb{R}_{>0}$ and $\theta_i \in \mathbb{S}$ denote the natural frequency and the phase of the i-th oscillator, respectively, and a_{ij} represents the coupling strength of the undirected edge between oscillators i and j. We assume that there are no self-loops (i.e., $a_{ii} = 0$).

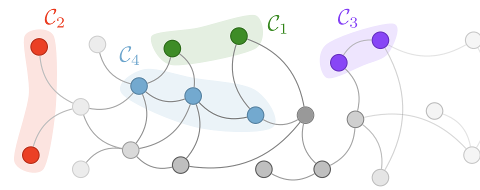


Fig. 1. A network with (remotely-)synchronized clusters of oscillators. The clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ comprise remotely-synchronized oscillators. Clusters $\mathcal{C}_3$ and $\mathcal{C}_4$ contain connected phase-synchronized oscillators. All the singletons $\mathcal{C}_5, \ldots, \mathcal{C}_p$ are depicted in shades of gray.

In what follows, we distinguish between phase synchronization and frequency synchronization of the oscillators.

Definition 1 [(Remote) Phase Synchronization]: We say that oscillators $i, j \in \mathcal{O}$ are *phase synchronized* if $\theta_i(t) = \theta_j(t)$ for all $t \geq 0$. Additionally, two phase-synchronized oscillators $i, j \in \mathcal{O}$ are *remotely-synchronized* if $a_{ij} = 0$.

Definition 2 [Frequency Synchronization (Manifold)]: We say that oscillators $i, j \in \mathcal{O}$ are *frequency-synchronized* if $\dot{\theta}_i(t) = \dot{\theta}_j(t)$ for all $t \geq 0$. Additionally, the frequency synchronization manifold for $\mathcal{G}$ is $\mathcal{M}_\mathcal{G} = \{\theta \in \mathbb{T}^n : \omega_i + \sum_k a_{ik}(\theta_k - \theta_i) = \omega_j + \sum_k a_{jk}(\theta_k - \theta_j), \forall i, j \in \mathcal{O}\}$.

Conditions for the oscillators in $\mathcal{G}$ to have an asymptotically stable frequency synchronization manifold $\mathcal{M}$ can be found in [13], [25], and demand that the coupling strengths dominate the heterogeneity of the natural frequencies.

Remote synchronization can be studied as a special case of *cluster synchronization*, where the oscillators can be partitioned into clusters (possibly, singletons) so that the oscillators in each cluster evolve identically. To formalize the treatment, consider a network partition $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$, with clusters satisfying $\mathcal{C}_k \cap \mathcal{C}_\ell = \emptyset$ for all $k, \ell \in \{1, \ldots, p\}$, $k \neq \ell$, and $\bigcup_{k=1}^{p} \mathcal{C}_k = \mathcal{O}$. The cluster synchronization manifold is

$$\mathcal{S}_\mathcal{C} = \{\theta \in \mathbb{T}^n : \theta_i = \theta_j \text{ for all } i, j \in \mathcal{C}_\ell, \;\; \ell = 1, \ldots, p\}.$$

To focus on remote synchronization, we assume that the first $m \geq 1$ clusters contain remotely-synchronized oscillators:

(A1) there exists $1 \leq m \leq p$ such that $a_{ij} = 0$ for all $i, j \in \mathcal{C}_\ell$ and $\ell \in \{1, \ldots, m\}$.

Fig. 1 illustrates a network partitioned into clusters.

To be stable, the cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ must be invariant. Sufficient conditions on the network weights and oscillators' natural frequencies for the invariance of $\mathcal{S}_\mathcal{C}$ have been derived elsewhere, and read as follows [23]:

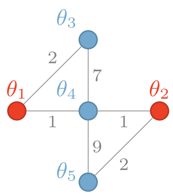
(C1) the natural frequencies satisfy $\omega_i = \omega_j$ for every $i, j \in \mathcal{C}_k$ and $k \in \{1, \ldots, m\}$;

(C2) The network weights satisfy $\sum_{k \in \mathcal{C}_\ell} a_{ik} - a_{jk} = 0$ for every $i, j \in \mathcal{C}_z$ and $z, \ell \in \{1, \ldots, m\}$, with $z \neq \ell$.

In the following, we assume that (C1) and (C2) are satisfied for the partition $\mathcal{C}$ being considered.

Notice that the invariance conditions (C1) and (C2) do not rely on the existence of network symmetries to support remotely-synchronized trajectories. This is contrast with previous studies [21], [22], where remote synchronization emerges from specific network configurations. Fig. 2 illustrates remote synchronization in the absence of network symmetries.

We conclude this section by stressing out that existing conditions for the stability of the cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ require each cluster in $\mathcal{C}$ to be connected



(a)

$$\tilde{A}\text{ and }A:\quad A = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 7 & 0 \\ 1 & 1 & 7 & 0 & 9 \\ 0 & 2 & 0 & 9 & 0 \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 7 & 0 \\ 1 & 1 & 7 & 0 & 9 \\ 2 & 0 & 0 & 9 & 0 \end{bmatrix}$$

(b)

Fig. 2. Fig. 2(a), where phases are color coded, depicts a network of identical oscillators ($\omega_i = \omega_j$ for all i, j) in which $\theta_1(t) = \theta_2(t)$, and $\theta_3(t) = \theta_4(t) = \theta_5(t)$, with conditions (C1) and (C2) being satisfied. In Fig. 2(b), the adjacency matrix A of the network in Fig. 2(a) changes to $\tilde{A} \neq A$ after a permutation of oscillators 1 and 2, thus implying that there is no network symmetry supporting synchronization of 1 and 2.

internally [23], [26], and do not cover the case of disconnected clusters. In the next section, we extend the results in [23] to account for the case of remote synchronization.

III. Stability of Remote Synchronization Through Relay Interactions

In this section, we present a condition to ensure that remote synchronization emerges in a network of heterogeneous Kuramoto oscillators. We focus on local stability because $\mathcal{S}_\mathcal{C}$ is, in general, not globally asymptotically stable. As it is not clear where on the manifold the system stabilizes, we are interested in any trajectory that converges to $\mathcal{S}_\mathcal{C}$. Our condition reveals that the existence of a strongly connected relay promotes the stability of remotely-synchronized trajectories.

A. A Sufficient Condition for the Stability of Remote Synchronization via Perturbation Theory and Relays

In order to derive our stability condition, we introduce the incremental dynamics of (1). Let $x_{ij} = \theta_j - \theta_i$, so that $\dot{x}_{ij} = \dot{\theta}_j - \dot{\theta}_i$. Stacking all the differences x_{ij} with $i < j$ yields a vector of phase differences $x = B^\mathsf{T}\theta$, with B being the oriented incidence matrix of $\mathcal{G}$.[1]

It is worth noting that $n-1$ phase differences $x_{\min}$ encode all phase trajectories, and that there always exists a full column-rank submatrix $B_{\min} \in \mathbb{R}^{n\times(n-1)}$ of B such that

$$x = B^\mathsf{T}(B_{\min}^\mathsf{T})^\dagger x_{\min},$$

where $x_{\min}$ is a set of $n-1$ phase differences that can be used to quantify synchronization among all oscillators. To see this, let $x_{\min} = B_{\min}^\mathsf{T}\theta$, with $B_{\min}$ being any full column-rank submatrix of B (e.g., the incidence matrix of a spanning tree of $\mathcal{G}$ [27]). Because $\ker(B^\mathsf{T}) = \ker(B_{\min}^\mathsf{T}) = \mathbb{1}$ by definition, it holds that $x = B^\mathsf{T}\theta = B^\mathsf{T}(\theta + c\mathbb{1}) = B^\mathsf{T}(B_{\min}^\mathsf{T})^\dagger x_{\min}$.

Let $\dot{x}_{\min}^{(k)} = g_k(x_{\min}^{(k)})$ denote the incremental dynamics of (1) restricted to a subnetwork $\mathcal{G}_k = \{\mathcal{O}_k, \mathcal{E}_k\}$, with $\mathcal{O}_k \subseteq \mathcal{O}$ and $\mathcal{E}_k = \{(i,j) \in \mathcal{E} : i, j \in \mathcal{O}_k\} \subseteq \mathcal{E}$, and let

$$\begin{aligned} J_k(\bar{x}_{\min}^{(k)}) &= \frac{\partial g_k}{\partial x_{\min}^{(k)}}(\bar{x}_{\min}^{(k)}) \\ &= -B_{\min,k}^\mathsf{T} B_k \operatorname{diag}(\{a_{ij}\cos(\bar{x}_{ij})\}_{(i,j)\in\mathcal{E}_k}) B_k^\mathsf{T}(B_{\min,k}^\mathsf{T})^\dagger \end{aligned} \tag{2}$$

[1] The $n \times |\mathcal{E}|$ oriented incidence matrix is defined entry-wise as $B_{k\ell} = -1$ if oscillator k is the source of the interconnection ℓ, $B_{k\ell} = 1$ if oscillator k is the sink of the interconnection ℓ, and $B_{k\ell} = 0$ otherwise.

be the Jacobian matrix computed at $\bar{x}_{\min}^{(k)}$ [23, Lemma 3.1], where each $\bar{x}_{ij}$ can be expressed as a function of $\bar{x}_{\min}^{(k)}$. The following theorem extends [23, Th. 3.2] to the case of disconnected clusters, thus providing a condition for the stability of the remote synchronization manifold.

Theorem 1 (Stable Remote Synchronization Through Relay Interactions and Weak Outer Couplings): Let $\mathcal{S}_\mathcal{C}$ be the cluster synchronization manifold associated with a partition $\mathcal{C} = \{\mathcal{C}_1, \dots \mathcal{C}_p\}$ of the network $\mathcal{G}$, with $\{\mathcal{C}_1, \dots, \mathcal{C}_m\}$ comprising disconnected oscillators. The cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ is locally asymptotically stable if there exists a partition $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ satisfying the following conditions:

1) $\mathcal{C}_\ell \subseteq \mathcal{F}_k$, for all $\ell \in \{1, \dots, p\}$ and some $k \in \{1, \dots, r\}$;
2) $\mathcal{G}_k = (\mathcal{F}_k, \mathcal{E}_k)$ is connected, for all $k \in \{1, \dots, r\}$,
3) There exists a locally asymptotically stable $\mathcal{M}_{\mathcal{G}_k}$ (see Definition 2) for the oscillators in the isolated subnetwork $\mathcal{G}_k = (\mathcal{F}_k, \mathcal{E}_k)$, for all $k \in \{1, \dots, r\}$;
4) the matrix $S \in \mathbb{R}^{r\times r}$, defined as

$$S = [s_{k\ell}] = \begin{cases} \lambda_{\max}^{-1}(P_k) - c^{(kk)} & \text{if } k = \ell, \\ -c^{(k\ell)} & \text{if } k \neq \ell, \end{cases} \tag{3}$$

is an M-matrix, where $P_k \succ 0$ is such that $J_k(0)P_k + P_kJ_k(0)^\mathsf{T} = -I$, with J_k as in (2), and, for any $i \in \mathcal{F}_k$,

$$c^{(k\ell)} = 2\max_r |\mathcal{F}_r| \cdot \begin{cases} \sum_{j\in\mathcal{F}_\ell} a_{ij}, & \text{if } \ell \neq k \\ \sum_{\ell\neq k}\sum_{j\in\mathcal{F}_\ell} a_{ij}, & \text{otherwise.} \end{cases} \tag{4}$$

Proof: We will use perturbation theory of dynamical systems [28, Ch. 9] to prove the stability of $\mathcal{S}_\mathcal{C}$. Here, the nominal systems are the isolated sets $\mathcal{F}_1, \dots, \mathcal{F}_r$, whose stability is perturbed by the phase trajectories of the oscillators belonging to interconnected sets. We first show that, because the oscillators in $\mathcal{G}_1, \dots, \mathcal{G}_r$ have stable frequency synchronization manifolds, there exist r quadratic Lyapunov functions for the linearized incremental dynamics of the isolated sets $\mathcal{F}_1, \dots, \mathcal{F}_r$. Notice that the frequency-synchronized trajectories of the oscillators in $\mathcal{F}_1, \dots, \mathcal{F}_r$ uniquely identify equilibria $\bar{x}_{\min}^{(k)}$, $k \in \{1, \dots, r\}$, of their respective incremental dynamics on $\mathcal{G}_1, \dots, \mathcal{G}_r$. By applying a change of coordinates $y^{(k)} = x_{\min}^{(k)} - \bar{x}_{\min}^{(k)}$ such that the linearized incremental dynamics are centered at the origin, we can define r Lyapunov functions that read as $V_k(y^{(k)}) = y^{(k)\,\mathsf{T}}P_ky^{(k)}$, with $P_k \succ 0$ such that $J_k(0)P_k + P_kJ_k(0)^\mathsf{T} = -I$, and satisfy $V_k(0) = 0$, for all $k \in \{1, \dots, r\}$. Next, we define $\tilde{y}_{\min} = [y^{(1)}, \dots, y^{(r)}]^\mathsf{T}$ as the minimum incremental variables for the entire partition $\mathcal{C}$ of $\mathcal{G}$, so that $\tilde{y}_{\min} = 0$ implies cluster synchronization. Let the Lyapunov candidate for the incremental dynamics $\dot{\tilde{y}}_{\min}$ be

$$V(\tilde{y}_{\min}) = \sum_{k=1}^{r} d_kV_k(\tilde{y}^{(k)}), \quad d_k > 0. \tag{5}$$

By the invariance condition (C2) and [23, Lemma 3.1], we can apply perturbation theory of dynamical systems [28, Ch. 9.5] to obtain that the derivative of (5) satisfies $\dot{V}(\tilde{y}_{\min}) \leq (DS + S^\mathsf{T}D)\|\tilde{y}_{\min}\|$, where $D = \operatorname{diag}(d_1, \dots, d_r)$, and S is as in (3). Finally, [28, Lemma 9.7 and Th. 9.2] define the constants $c^{(k\ell)}$ as in (4), the matrix S as in (3), and conclude on the origin of $\dot{\tilde{y}}_{\min}$ being locally stable if S is an M-matrix, thus proving the stability of $\mathcal{S}_\mathcal{C}$. ■

Theorem 1 introduces an additional partition of the oscillators besides $\mathcal{C}$, where the sets $\mathcal{F}_1, \dots, \mathcal{F}_r$ contain one or

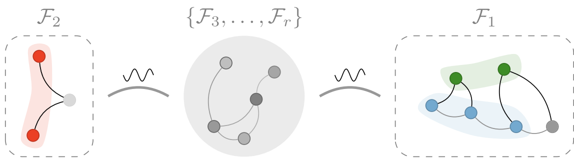


Fig. 3. Remote synchronization through stable relay interconnections. This figure illustrates a partition $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_r\}$ of the network in Fig. 1 satisfying the conditions in Theorem 1. Requiring that the matrix in (3) is an M-matrix guarantees that the stable frequency-synchronized trajectories in $\mathcal{F}_1$ and $\mathcal{F}_2$ are not perturbed by the rest of the oscillators.

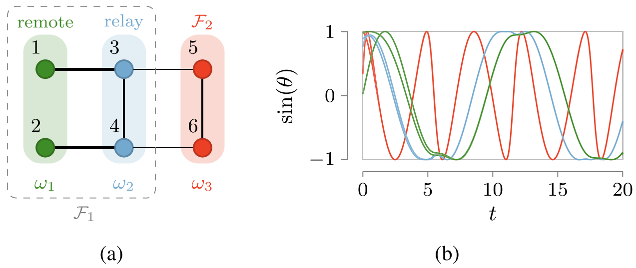


Fig. 4. Fig. 4(a) illustrates a network partitioned into $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ that displays stable remote synchronization by satisfying Theorem 1 with $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2\}$, where $\mathcal{F}_1 = \{\mathcal{C}_1, \mathcal{C}_2\}$. The cluster $\mathcal{C}$ (in blue) being a cohesive relay for the remote cluster $\mathcal{C}_1$ (in green). In this example $\mathcal{F}_2 = \mathcal{C}_3$ (in red). We fix the network weights as $a_{13} = a_{24} = a_{34} = a_{56} = 10$ and $a_{35} = a_{46} = 1$, and the natural frequencies as $\omega_1 = 0$, $\omega_2 = 0.5$, $\omega_3 = 2.1$. It can be shown that the dynamics $\dot{x}^{(1)}_{\min} = [\dot{x}_{13}\ \dot{x}_{24}\ \dot{x}_{34}]^{\mathsf{T}}$ for $\mathcal{F}_1$ (in grey, dashed) has a stable equilibrium $\bar{x}^{(1)}_{\min} = [0\ 0.025\ 0]^{\mathsf{T}}$. The matrix $S = \left[\begin{smallmatrix} 7.7139 & -4 \\ -4 & 36 \end{smallmatrix}\right]$ in (3) is an M-matrix. Fig. 4(b) illustrates the phases evolution starting from random initial conditions close to $\mathcal{S}_{\mathcal{C}}$. The phases, which are color coded according to Fig. 4(a), converge to $\mathcal{S}_{\mathcal{C}}$, and the oscillators in $\mathcal{C}_1$ are remotely synchronized.

more clusters each, and whose oscillators are required to be coupled strongly enough to achieve frequency synchronization when isolated from the other sets. That is, while the partition $\mathcal{C}$ encodes which oscillators are phase- and which are remotely-synchronized, $\mathcal{F}$ encodes the frequency synchronization behavior of the oscillators.

Remark 1 (Cohesive Relays Support Stable Remote Synchronization): Conditions (i) and (ii) in Theorem 1 require that any set $\mathcal{F}_k$ containing a cluster of remotely-synchronized oscillators $\mathcal{C}_\ell$, $\ell \in \{1, \ldots, m\}$ must also contain one or more connected clusters from $\{\mathcal{C}_{m+1}, \ldots, \mathcal{C}_p\}$, which in turn act as a relay for the remote oscillators in $\mathcal{F}_k$. Such relay is also *cohesive*, as all its oscillators behave cohesively due to its internal weights being large enough to guarantee frequency synchronization and the M-matrix condition. Taken together, these requirements reveal that stable remote synchronization is ensured by the existence of cohesive relays, which provide sufficient "inertia" to preserve remote synchronization despite the perturbations from other oscillator sets.

Fig. 3 illustrates remotely-synchronized oscillators supported by cohesive relays in a partition $\mathcal{F}$ for the network in Fig. 1. Finally, Fig. 4 presents an example for Theorem 1.

B. Remotely-Synchronized Brain Regions Possess Strong Relay Interconnections

While Theorem 1 does not provide a method to choose the partition $\mathcal{F}$ and, thus, which oscillators constitute relays,

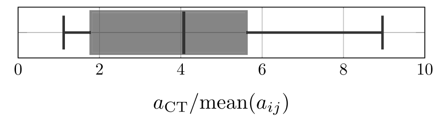


Fig. 5. Distribution of cortico-thalamic interconnections of remotely synchronized cortical regions in anatomical brain networks of $N = 20$ subjects. For each subject, we plot the cortico-thalamic interconnection weight of remotely-synchronized cortical regions a_{CT} divided by the subject-specific average interconnection weight mean(a_{ij}).

empirical studies have identified candidate relay regions in the brain to be the backbone of a set $\mathcal{F}_k \subset \mathcal{F}$. We interrogate the publicly available `NKI Rockland` dataset [29] to verify whether there exists a relay whose connections with brain regions that display remote synchronization are much stronger than the average connection strength across all brain regions. More in detail, we analyze cortico-thalamo-cortical circuits [5], where disconnected cortical regions synchronize via interconnections with the thalamus.

In the considered dataset, the anatomical organization of the brain is encoded by adjacency matrices $W_i \in \mathbb{R}^{188\times188}_{\geq 0}$. We analyze the first $N = 20$ subjects in the dataset, and find that each subject possesses pairs of disconnected cortical regions whose activity is highly synchronous (Pearson correlation coefficients ≥ 0.9). For each subject, we select the pair displaying the largest mean coupling strengths with the thalamus. We find that, across all 20 subjects, the mean cortico-thalamic interconnection of remotely synchronized regions is 4.079 ± 0.303 SEM (Standard Error of the Mean) times the mean network weight. Fig. 5 illustrates the distribution of cortico-thalamic interconnection weights divided by the average interconnection weight of each subject. Remarkably, even the smallest among these values is larger than one. Because a partition $\mathcal{F}$ cannot be uniquely identified from the available data, we cannot check whether Theorem 1 is satisfied. Nevertheless, the above finding is in line with the theoretical requirement of strong relay interconnections, and suggests that remote synchronization in the brain may be supported by strong interactions with a cohesive relay.

IV. Unstable Remote Synchronization in the Absence of Cohesive Relays

In this section, we provide evidence that the absence of cohesive relays hinders the emergence of stable remote synchronization in a large class of networks. To do so, we first assess whether frequency-synchronized sets that include remotely-synchronized oscillators are necessary to enable stable remote synchronization of a single cluster of disconnected oscillators. We focus on the case of networks partitioned into two clusters ($\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2\}$), where $\dot{\theta}_i \neq \dot{\theta}_j$ for all $i \in \mathcal{C}_1$ and $j \in \mathcal{C}_2$. We set to zero all intra-cluster couplings in $\mathcal{C}_1$, so that $\mathcal{C}_2$ is a (non-cohesive) relay for the remote oscillators in $\mathcal{C}_1$. In this configuration, there does not exist a partition $\mathcal{F}$ satisfying Theorem 1.

The next theorem shows that this specific configuration yields at best *marginal stability* of $\mathcal{S}_{\mathcal{C}}$ – as phase trajectories that start in the vicinity of $\mathcal{S}_{\mathcal{C}}$ never converge to it as $t \to \infty$. To present our result, we define $\bar{\omega} = \omega_2 - \omega_1$, with

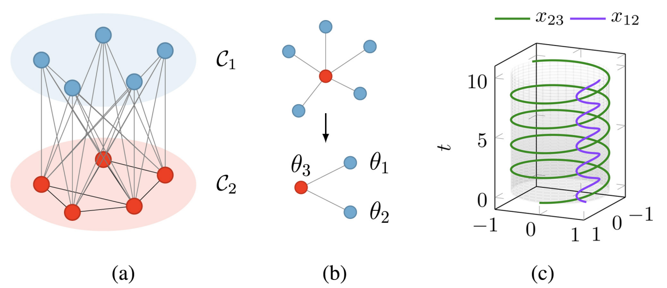


Fig. 6. Fig. 6(a) illustrates a 2-cluster network where the remote oscillators in $\mathcal{C}_1$ are connected to all the oscillators in $\mathcal{C}_2$, thus satisfying the condition in Theorem 2. Fig. 6(b) illustrates that, due to the synchronization of all the oscillators in $\mathcal{C}_2$ (in red) whenever the oscillators in $\mathcal{C}_1$ share common neighbors, the network in Fig. 6(a) can be reduced to a star network in order to analyze its stability properties. When studying a specific perturbation of $\theta(0) \in \mathcal{S}_\mathcal{C}$ so that only the oscillators in $\mathcal{C}_1$ are outside of $\mathcal{S}_\mathcal{C}$ at $t = 0$, the star network in the top panel can be further reduced to a 3-node star. Fig. 6(c) depicts the periodic trajectories (in Cartesian coordinates) $x_{12} = \theta_2 - \theta_1$ and $x_{23} = \theta_3 - \theta_2$ of the 3-node star in Fig. 6(b) starting from $\theta_1(0) = -1$ and $\theta_2(0) = \theta_3(0) = 0$, which satisfy Theorem 2.

ω_1, ω_2 being the natural frequencies of the oscillators in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Finally, let $\bar{a} = \sum_{k \in \mathcal{C}_2} a_{ik} + \sum_{k \in \mathcal{C}_1} a_{jk}$, for any $i \in \mathcal{C}_1$ and $j \in \mathcal{C}_2$.

Theorem 2 (Instability of $\mathcal{S}_\mathcal{C}$ if Common Neighbors Are Not Frequency-Synchronized): Consider the network $\mathcal{G}$ partitioned as $\mathcal{P} = \{\mathcal{C}_1, \mathcal{C}_2\}$, with $a_{ij} = 0$ for all $i, j \in \mathcal{C}_1$ and $a_{ij} = a_{ik} > 0$ for any $i \in \mathcal{C}_1$ and all $j, k \in \mathcal{C}_2$. If $\bar{\omega} > \bar{a}$, then the cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ is not asymptotically stable. Furthermore, for any initial condition $\theta(0) = \tilde{\theta}(0) + \alpha e_i$, with $\tilde{\theta}(0) \in \mathcal{S}_\mathcal{C}$, $\alpha \in \mathbb{T}$, and $i \in \mathcal{C}_1$, the solution to (1) is periodic and does not belong to $\mathcal{S}_\mathcal{C}$.

Proof: We prove that there exist an infinite number of time instants $t_1, t_2, \ldots$ such that $\theta(0) = \theta(t_1) = \theta(t_2) = \ldots$. Owing to [23, Lemma 3.4], the condition $\bar{\omega} > \bar{a}$ implies that $\dot{\theta}_i \neq \dot{\theta}_j$ for all $i \in \mathcal{C}_1$ and $j \in \mathcal{C}_2$. Notice that any remote oscillator can equivalently be seen as a singleton cluster, and that the phases of such clusters cancel out in the dynamics of $\dot{x}_{ij}^{(2)}$ for all $i, j \in \mathcal{C}_2$. Hence, due to (C1) and the fact that $\mathcal{C}_2$ is internally connected, it holds that $|x_{ij}| \to 0$ for all $i, j \in \mathcal{C}_2$ [23]. In this case, we observe that any network akin to Fig. 6(a) can be conveniently analyzed as a star network, where $\mathcal{C}_2$ is considered as the oscillator at the center of the star because its oscillators synchronize (see Fig. 6(b)).

Let us reorder the oscillators so that the last one is the star center, and consider initial conditions $\theta(0) = c\mathbb{1} + [\alpha \ 0 \ \ldots \ 0]^\mathsf{T}$; that is, all oscillators start at the same value $c \in \mathbb{T}$ and $x_{1k} = \alpha$, for all $k = 2, 3, \ldots, |\mathcal{C}_1|$. This choice of $\theta(0)$ allows us to further reduce the star network to a 3-oscillator star, where synchronized oscillators $2, \ldots, |\mathcal{C}_1|$ become a single oscillator, as illustrated in Fig. 6(b). To see this, recall that the synchronized trajectories of $2, \ldots, |\mathcal{C}_1|$ are invariant due to (C1), (C2). Finally, for simplicity, let us set $a_{ij} = 1$ for all a_{ij}.[2] We are left with studying a 3-oscillator star obeying equations $\dot{\theta}_1 = \sin(\theta_3 - \theta_1)$, $\dot{\theta}_2 = \sin(\theta_3 - \theta_2)$, $\dot{\theta}_3 = \omega_3 - \sin(\theta_3 - \theta_1) - \sin(\theta_3 - \theta_2)$.

We are now ready to demonstrate that there exist $t_1, t_2, \ldots$ such that $\theta(0) = \theta(t_1) = \theta(t_2) = \ldots$ We first define $\gamma = \frac{1}{2}(\theta_2 - \theta_1)$ and $\varphi = \omega_3 - \frac{\theta_1 + \theta_2}{2}$. On the unit circle, $\gamma(t)$ represents the evolution of half of the difference between the two outer oscillators, while $\varphi(t)$ represents the evolution of the difference between $\theta_3(t)$ and the center of the difference between the outer oscillators. Since $\theta_3 - \theta_1 = \gamma + \varphi$ and $\theta_3 - \theta_2 = \gamma - \varphi$, the time evolution of γ and φ can be written as $\dot{\gamma} = \frac{1}{2}\left[\sin(\varphi - \gamma) - \sin(\varphi + \gamma)\right]$, $\dot{\varphi} = \gamma_3 - \frac{3}{2}\left[\varphi - \gamma) + \sin(\varphi + \gamma)\right]$. Notice that $\dot{\gamma} = 0$ for $(\gamma, \varphi) = (\frac{\alpha}{2}, k\frac{\pi}{2})$ with k odd. Next, we get rid of the dependence of γ from time by considering $\frac{\dot{\gamma}}{\dot{\varphi}} = \frac{\partial \gamma}{\partial t}\frac{\partial t}{\partial \varphi} = \frac{\partial \gamma}{\partial \varphi}$. By recalling that φ is monotonically increasing (as $\bar{\omega} > \bar{a}$ implies that $\dot{\theta}_3 \neq \dot{\theta}_1$), we can study $\gamma(\varphi)$ and $\frac{\partial \gamma}{\partial \varphi}(\varphi)$ as a function of φ. In particular, showing that $\gamma(-\frac{\pi}{2}) = \gamma(\frac{3\pi}{2})$ is equivalent to show that x_{12} does not converge to zero. To do so, recall that $\gamma(\frac{3\pi}{2}) = \gamma(-\frac{\pi}{2}) + \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau$.

It holds that the integral in the above equation is zero, as it can be shown that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau. \tag{6}$$

Specifically, we prove by mathematical induction that, for any $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\frac{\partial \gamma}{\partial \varphi}(\varphi) = \frac{\partial \gamma}{\partial \varphi}(\pi - \varphi)$. This can be done by letting $\varphi_1 = \frac{\pi}{2}$, and considering for the base step of the induction $\varphi_1^+ = \varphi + \delta$ and $\varphi_1^- = \varphi - \delta$. In the limit for $\delta \to 0$, it holds that $\frac{\partial \gamma}{\partial \varphi}(\varphi_1^+) = \frac{\partial \gamma}{\partial \varphi}(\varphi_1^-)$. The inductive step can be proven analogously, thus concluding on the symmetry of $\frac{\partial \gamma}{\partial \varphi}$ in the unit circle with respect to the axis $(-\frac{\pi}{2}, \frac{\pi}{2})$, which implies the equivalence in (6) since we have that

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\tau) \mathrm{d}\tau + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial \gamma}{\partial \varphi}(\pi - \beta) \mathrm{d}\beta = 0.\end{aligned}$$

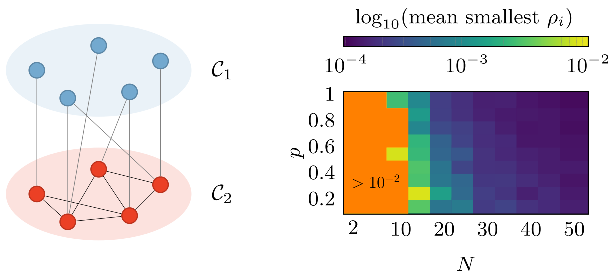
This concludes the proof that x_{12} circles back to $x_{12}(0)$ an infinite amount of times, thus implying that the remote synchronization manifold is not asymptotically stable. ∎

Theorem 2 shows that if every oscillator $i \in \mathcal{C}_1$ receives the same "input" $\sum_{j \in \mathcal{C}_2} a_{ij} \sin(\theta_j - \theta_i)$ from all the oscillators in $\mathcal{C}_2$, as illustrated in 6(a), the cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ is at best marginally stable. Additionally, Theorem 2 demonstrates the existence of a family of initial conditions, which can be arbitrarily close to $\mathcal{S}_\mathcal{C}$, that yields periodic trajectories not belonging to $\mathcal{S}_\mathcal{C}$. Fig. 6(c) illustrates periodic trajectories as in Theorem 2.

We confirm by means of Floquet stability theory [30] that network topologies not as accurately crafted as the ones in Theorem 2 yield an unstable $\mathcal{S}_\mathcal{C}$. We remark that it is possible to employ Floquet stability theory because the phase differences between the oscillators in $\mathcal{C}_1$ and the ones in $\mathcal{C}_2$ are periodic whenever the two clusters are not frequency-synchronized (i.e., $\bar{\omega} > \bar{a}$) [23, Lemma 3.4].

We generated 10^4 2-cluster networks with varying size and connectivity profiles, where $a_{ij} = 0$ for all $i, j \in \mathcal{C}_1$, and $\mathcal{C}_2$ is a random connected topology (Erdös-Rényi graphs with

[2]The reasoning below can be extended to any weights $a_{ij} \in \mathbb{R}_{\geq 0}$.



(a) (b)

Fig. 7. Fig. 7(a) illustrate a 2-cluster network with random inter-cluster coupling and no common neighbors. Fig. 7(b) depicts consistently unstable Floquet characteristic multipliers of a 2-cluster network for varying topologies and cluster sizes. We generated, for each cluster size $|\mathcal{C}_i| = N$ and probability p of interconnection between two nodes, 100 Erdös-Rényi graph topologies with weights drawn uniformly from $[0, 1]$. Each entry of the matrix plot represents the mean unstable characteristic multiplier ρ_i averaged over 100 different network instances satisfying: (i) clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of N oscillators each, (ii) the conditions for the invariance of $\mathcal{S}_\mathrm{R}$, (iii) $a_{ij} = 0$ for all $i, j \in \mathcal{C}_1$.

edge probability $p \in [0.1, 1]$). Fig. 7 summarizes the stability analysis of $\mathcal{S}_\mathcal{C}$ in all these networks.

While not excluding that remote synchronization may arise in specific instances where multiple phase-unlocked clusters interact, our results suggest that the sufficient mechanism proposed in Section III may be (almost) necessary for the stability of $\mathcal{S}_\mathcal{C}$.

V. Conclusion

We have studied remote synchronization in networks of heterogeneous Kuramoto oscillators. Motivated by the hypothesis that remote synchronization in the brain is promoted by intermediate brain regions that relay information between disconnected ones, we have proposed a novel mechanism to achieve stable remote synchronization. Our main result prescribes the existence of cohesive relays, which are required to be strongly coupled to remote oscillators and weakly perturbed by the other ones. We have also analyzed brain data and found that regions that remotely-synchronize are strongly connected to a common relay. Finally, we have demonstrated that the case of 2 clusters where no cohesive relay exists does not admit stable remote synchronization. The latter result suggests that our sufficient conditions may also be necessary. We leave the validation of this conjecture as a topic for future research.

References

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[2] F. Varela, J. P. Lachaux, E. Rodriguez, and J. Martinerie, "The BrainWeb: Phase synchronization and large-scale integration," *Nat. Rev. Neurosci.*, vol. 2, no. 4, pp. 229–239, 2001.

[3] J. Fell and N. Axmacher, "The role of phase synchronization in memory processes," *Nat. Rev. Neurosci.*, vol. 12, no. 2, pp. 105–118, 2011.

[4] J. F. Hipp, D. J. Hawellek, M. Corbetta, M. Siegel, and A. K. Engel, "Large-scale cortical correlation structure of spontaneous oscillatory activity," *Nat. Neurosci.*, vol. 15, no. 6, pp. 884–890, 2012.

[5] R. Vicente, L. L. Gollo, C. R. Mirasso, I. Fischer, and G. Pipa, "Dynamical relaying can yield zero time lag neuronal synchrony despite long conduction delays," *Proc. Nat. Acad. Sci.*, vol. 105, no. 44, pp. 17157–17162, 2008.

[6] L. L. Gollo, C. R. Mirasso, and A. E. P. Villa, "Dynamic control for synchronization of separated cortical areas through thalamic relay," *NeuroImage*, vol. 52, no. 3, pp. 947–955, 2010.

[7] G. Arnulfo *et al.*, "Long-range phase synchronization of high-frequency oscillations in human cortex," *Nat. Commun.*, vol. 11, no. 1, p. 5363, 2020.

[8] D. S. Bassett and O. Sporns, "Network neuroscience," *Nat. Neurosci.*, vol. 20, no. 3, pp. 353–364, 2017.

[9] A. Bergner *et al.*, "Remote synchronization in star networks," *Phys. Rev. E*, vol. 85, no. 2, 2012, Art. no. 026208.

[10] L. V. Gambuzza, A. Cardillo, A. Fiasconaro, L. Fortuna, J. Gómez-Gardenes, and M. Frasca, "Analysis of remote synchronization in complex networks," *Chaos Interdiscipl. J. Nonlinear Sci.*, vol. 23, no. 4, 2013, Art. no. 043103.

[11] J. Emenheiser, A. Salova, J. Snyder, J. P. Crutchfield, and R. M. D'Souza, "Network and phase symmetries reveal that amplitude dynamics stabilize decoupled oscillator clusters," 2020. [Online]. Available: arXiv:2010.09131.

[12] A. Ponce-Alvarez, G. Deco, P. Hagmann, G. L. Romani, D. Mantini, and M. Corbetta, "Resting-state temporal synchronization networks emerge from connectivity topology and heterogeneity," *PLoS Comput. Biol.*, vol. 11, no. 2, pp. 1–23, 2015.

[13] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[14] D. Cumin and C. P. Unsworth, "Generalising the Kuramoto model for the study of neuronal synchronisation in the brain," *Physica D Nonlinear Phenomena*, vol. 226, no. 2, pp. 181–196, 2007.

[15] A. Politi and M. Rosenblum, "Equivalence of phase-oscillator and integrate-and-fire models," *Phys. Rev. E*, vol. 91, no. 4, 2015, Art. no. 042916.

[16] T. Menara, G. Baggio, D. S. Bassett, and F. Pasqualetti, "A framework to control functional connectivity in the human brain," in *Proc. IEEE Conf. Decis. Control*, Nice, France, Dec. 2019, pp. 4697–4704.

[17] Y. Qin, M. Cao, B. D. O. Anderson, D. S. Bassett, and F. Pasqualetti, "Mediated remote synchronization: The number of mediators matters," *IEEE Control Syst. Lett.*, vol. 5, no. 3, pp. 767–772, Jul. 2021.

[18] D. Maraun and J. Kurths, "Epochs of phase coherence between El Niño/Southern Oscillation and Indian monsoon," *Geophys. Res. Lett.*, vol. 32, no. 15, 2005, Art. no. L15709.

[19] L. Zhang, A. E. Motter, and T. Nishikawa, "Incoherence-mediated remote synchronization," *Phys. Rev. Lett.*, vol. 118, Apr. 2017, Art. no. 174102.

[20] Y. Qin, Y. Kawano, and M. Cao, "Stability of remote synchronization in star networks of Kuramoto oscillators," in *Proc. IEEE Conf. Decis. Control*, Miami Beach, FL, USA, Dec. 2018, pp. 5209–5214.

[21] V. Nicosia, M. Valencia, M. Chavez, A. Díaz-Guilera, and V. Latora, "Remote synchronization reveals network symmetries and functional modules," *Phys. Rev. Lett.*, vol. 110, Apr. 2013, Art. no. 174102.

[22] V. Vuksanović and P. Hövel, "Functional connectivity of distant cortical regions: Role of remote synchronization and symmetry in interactions," *NeuroImage*, vol. 97, pp. 1–8, Aug. 2014.

[23] T. Menara, G. Baggio, D. S. Bassett, and F. Pasqualetti, "Stability conditions for cluster synchronization in networks of heterogeneous Kuramoto oscillators," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 1, pp. 302–314, Mar. 2020.

[24] L. V. Gambuzza, M. Frasca, L. Fortuna, and S. Boccaletti, "Inhomogeneity induces relay synchronization in complex networks," *Phys. Rev. E*, vol. 93, Apr. 2016, Art. no. 042203.

[25] S. Jafarpour and F. Bullo, "Synchronization of Kuramoto oscillators via cutset projections," *IEEE Trans. Autom. Control*, vol. 64, no. 7, pp. 2830–2844, Jul. 2019.

[26] T. Menara, G. Baggio, D. S. Bassett, and F. Pasqualetti, "Exact and approximate stability conditions for cluster synchronization of Kuramoto oscillators," in *Proc. Amer. Control Conf.*, Philadelphia, PA, USA, Jul. 2019, pp. 205–210.

[27] C. Godsil and G. F. Royle, *Algebraic Graph Theory* (Graduate Texts in Mathematics). New York, NY, USA: Springer, 2001.

[28] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice Hall, 2002.

[29] K. B. Nooner *et al.*, "The NKI-Rockland sample: A model for accelerating the pace of discovery science in psychiatry," *Front. Neurosci.*, vol. 6, p. 152, Oct. 2012.

[30] W. J. Rugh, *Linear System Theory* (Information and System Sciences). Upper Saddle River, NJ, USA: Prentice Hall, 1993.