ME 121: Handout for discussion 5

Tommaso Menara *

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Abstract

In this discussion, we study the change of basis in 2 dimensions. Specifically, we learn basis rotations, and we analyze the orthogonal and non-orthogonal cases.

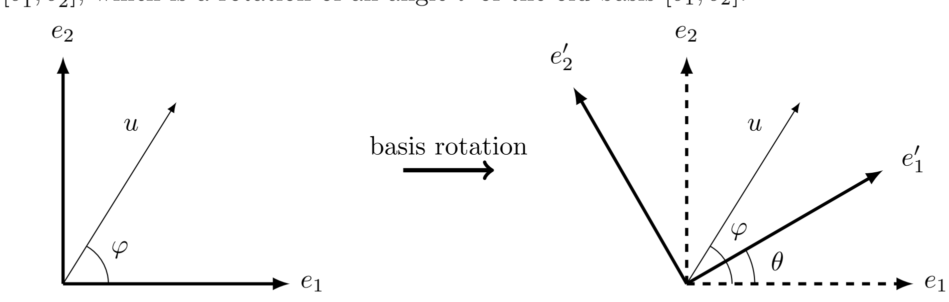
1 Change of Basis

A basis of a vector space V is a set of vectors in V that is linearly independent and spans V. An ordered basis is a list, rather than a set, meaning that the order of the vectors in an ordered basis matters.

A quote from G. Strang [1]: "*The standard basis vectors for $\mathbb{R}^n$ and $\mathbb{R}^m$ are the columns of I. That choice leads to a standard matrix, and $T(v) = Av$ in the normal way. But these spaces also have other bases, so the same T is represented by other matrices. A main theme of linear algebra is to choose the bases that give the best matrix for T* "

1.1 Basis Rotation

Typically, the vector u is expressed in an orthogonal basis with components e_1 and e_2. However, it is possible to write the vector u in a new orthogonal basis $[e_1', e_2']$, which is a rotation of an angle θ of the old basis $[e_1, e_2]$.



*Tommaso Menara is with the Department of Mechanical Engineering, University of California at Riverside, `tomenara@engr.ucr.edu`. All files are available at `www.tommasomenara.com`

The relationship between the two bases is expressed as a rotation:

$$\begin{aligned}\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} &= R(\theta) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \end{aligned} \tag{1}$$

Equivalently, the two components of the new rotated basis e_1' and e_2' read as:

$$\begin{aligned} e_1' &= \cos\theta\ \hat{\mathbf{\imath}} + \sin\theta\ \hat{\mathbf{\jmath}}, \\ e_2' &= -\sin\theta\ \hat{\mathbf{\imath}} + \cos\theta\ \hat{\mathbf{\jmath}}. \end{aligned}$$

The vector u can be written in the two different bases as:

$$[u]_{e_1,e_2} = \begin{bmatrix} |u|\cos\phi \\ |u|\sin\phi \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

or

$$[u]_{e_1',e_2'} = \begin{bmatrix} |u|\cos(\varphi-\theta) \\ |u|\sin(\varphi-\theta) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

How are $[x\ \ y]^\mathsf{T}$ and $[x'\ \ y']^\mathsf{T}$ related? Recall the trigonometric identities $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$. Then, we have

$$\begin{aligned} x' &= |u|(\cos\varphi\cos\theta + \sin\varphi\sin\theta) \\ &= (|u|\cos\varphi)\cos\theta + (|u|\sin\varphi)\sin\theta \\ &= x\cos\theta + y\sin\theta, \end{aligned}$$

$$\begin{aligned} y' &= |u|(\sin\varphi\cos\theta - \sin\theta\cos\varphi) \\ &= -(|u|\cos\varphi)\sin\theta + (|u|\sin\varphi)\cos\theta \\ &= -x\sin\theta + y\cos\theta, \end{aligned}$$

from which it follows that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This final equation corresponds to the rotation by an angle θ with the matrix $R(\theta)$ in (1).

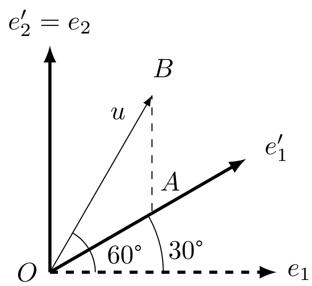
1.2 Different example with non-orthogonal basis: e_1 is rotated by 30° but e_2 is held fixed

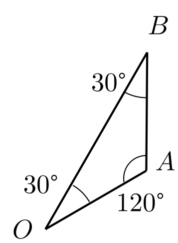
The components of the new basis can be written as

$$\begin{aligned} e_1' &= \cos 30\ \hat{e}_1 + \sin 30\ \hat{e}_2 \\ e_2' &= \hat{e}_2, \end{aligned}$$

which can be expressed in matrix form as

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$





Recall the following trigonometric identity:

$$\frac{OB}{\sin 120} = \frac{OA}{\sin 30},$$

which yields

$$OA = \frac{\sin 30}{\sin 120} OB,$$

and by fixing $|u| = OB = 1$, $OA = AB = \frac{1}{\sqrt{3}}$.

Hence, being $B = [\frac{1}{2}\ \frac{\sqrt{3}}{2}]^\mathsf{T}$, we have

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \\ \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}. \end{aligned}$$

We can now write the expression for u in the new basis:

$$\begin{aligned} u &= x' e_1' + y' e_2' \\ &= x' \left(\frac{\sqrt{3}}{2} e_1 + \frac{1}{2} e_2 \right) + y' e_2 \\ &= \underbrace{x' \frac{\sqrt{3}}{2}}_{x} e_1 + \underbrace{\left(x' \frac{1}{2} + y' \right)}_{y} e_2 \end{aligned}$$

and by plugging the coordinates of B in the expressions for x and y in the previous equation, we derive the expressions for x' and y':

$$\begin{aligned} \frac{1}{2} = x' \frac{\sqrt{3}}{2} \quad &\Rightarrow \quad x' = \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2} &= y' = \frac{1}{\sqrt{3}}. \end{aligned}$$

In matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}}_{A^\mathsf{T}} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Finally, from the previous equation, we obtain the general expression for the change of basis by left-multiplying by the inverse of A^T:

$$\mathbf{x} = A^\mathsf{T} \mathbf{x}' \quad \Rightarrow \quad \mathbf{x}' = \left(A^\mathsf{T} \right)^{-1} \mathbf{x}.$$

1.3 General derivation of the change of basis

In this section, we quickly recall the derivation of some useful equations involved in the change of basis. Consider the basis $[e_1' \; e_2']$ with components

$$\begin{aligned} e_1' &= a_{11} e_1 + a_{12} e_2 \\ e_2' &= a_{21} e_1 + a_{22} e_2. \end{aligned}$$

Then, a vector u can be expressed in the basis $[e_1' \; e_2']$ as:

$$\begin{aligned} u &= x' e_1' + y' e_2' \\ &= x' (a_{11} e_1 + a_{12} e_2) + y' (a_{21} e_1 + a_{22} e_2) \\ &= \underbrace{(x' a_{11} + y' a_{21})}_{x} e_1 + \underbrace{(x' a_{12} + y' a_{22})}_{y} e_2. \end{aligned}$$

In matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{A^\mathsf{T}} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

In summary:

$$e' = Ae, \qquad \mathbf{x} = A^\mathsf{T}\mathbf{x}', \qquad \mathbf{x}' = \left(A^\mathsf{T}\right)^{-1}\mathbf{x}.$$

In 2-dimensional cases, if A defines a transformation of basis vectors, then the components of vectors are transformed by $\left(A^\mathsf{T}\right)^{-1}$ under that change of basis. Notice that, in the particular case of orthogonal transformation matrices such as the rotation matrix $R(\theta)$ in (1), it holds that $A^\mathsf{T} = A^{-1}$, or $\left(A^\mathsf{T}\right)^{-1} = A$.

References

[1] G. Strang. *Introduction to linear algebra*, volume 3. Wellesley-Cambridge Press Wellesley, MA, 1993.