ME 121: Handout for discussion 9

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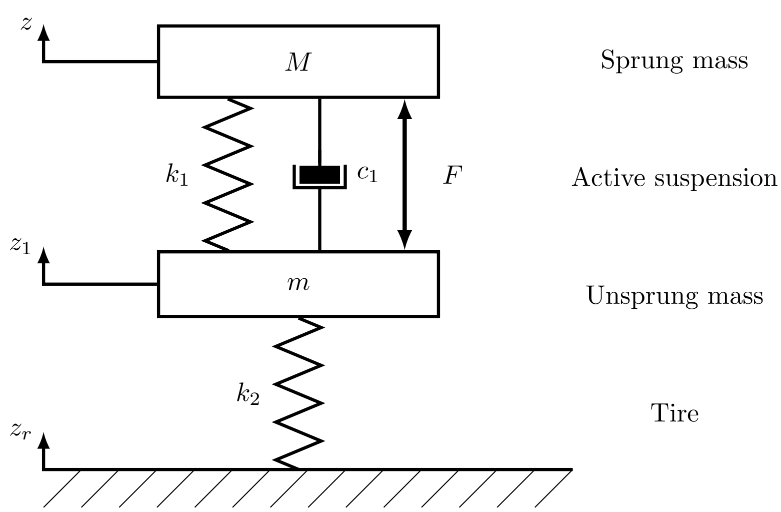
Abstract

In this discussion, we study pole placement techniques for controllers and observers, and apply these concepts to the quarter-car model developed at the beginning of the class. Further, we design a linear quadratic regulator (LQR) for this model.

1 Pole Placement for the Quarter-Car Model

Let us recall the quarter-car model with active suspension that we have derived in Discussion 2.



The state variables are the masses displacement and their velocity: $z = x_1$, $\dot{z} = x_2$, $z_1 = x_3$, and $\dot{z}_1 = x_4$. Hence, the state equation for the above model is:

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$$
\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{M} & -\frac{c_1}{M} & \frac{k_1}{M} & \frac{c_1}{M} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & \frac{c_1}{m} & -\frac{k_1+k_2}{m} & -\frac{c_1}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \\ -\frac{1}{m} \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m} \end{bmatrix} \dot{z}_r
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} F \\ \dot{z}_r \end{bmatrix}
$$

1.1 Full State Feedback

Let us design a full state-feedback controller for the system. Assuming for now that all the states can be measured (this assumption is probably not true but is sufficient for this problem). Thus, $C = I$ since we assume to have access to all the states. The controller K can only control the force input F but not the road disturbance, so we only need the first column of B, which we call B_1.

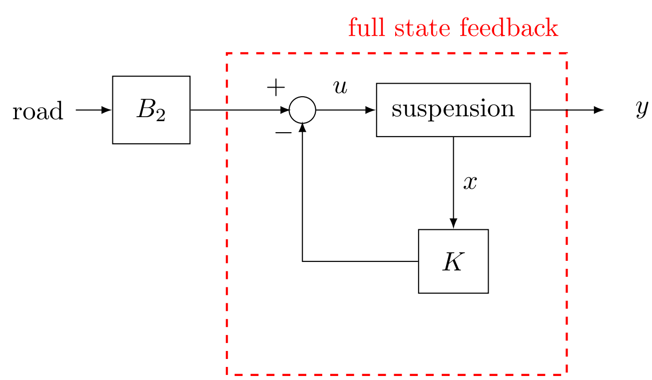
The poles of the system transfer function are the roots of the characteristic equation given by $|sI - A| = 0$. Full state feedback is utilized by commanding the input vector u. Consider an input proportional (in the matrix sense) to the state vector,

$$
u = -Kx.
$$

Substituting into the state space equations above yields

$$
\begin{aligned}
\dot{x} &= (A - B_1 K)x \\
y &= (C - DK)x.
\end{aligned}
$$

The poles of the full state feedback system are given by the characteristic equation of the matrix $A - B_1K$, $\det[sI - (A - B_1K)] = 0$. Comparing the terms of this equation with those of the desired characteristic equation yields the values of the feedback matrix K that forces the closed-loop eigenvalues to the pole locations specified by the desired characteristic equation.



1.2 LQR design

In optimal control theory, the goal is not only to control a system, but to do so by optimizing a certain metric (or cost). Typically, it would be desirable to control a system with the minimum amount of energy possible. The case where the cost is described by a quadratic function is called the LQ problem. The LQR (linear quadratic regulator) algorithm reduces the amount of work done by the control systems engineer to optimize the controller. The LQR algorithm is basically an automated way of finding an appropriate state-feedback controller.

Consider the linear continuous-time system $\dot{x} = Ax + Bu$ with a cost functional defined as

$$J = \int_0^\infty (x^\mathsf{T} Q x + u^\mathsf{T} R u)\ \mathrm{d}t.$$

The feedback control law that minimizes the value of the cost takes the form $u = -Kx$ where K is given by

$$K = R^{-1}(B^\mathsf{T} P)$$

and P is found by solving the continuous time algebraic Riccati equation

$$A^\mathsf{T} P + PA - (PB)R^{-1}(B^\mathsf{T} P) + Q = 0,$$

where the matrices Q and R must satisfy $Q = Q^\mathsf{T} \succ 0$, $R = R^\mathsf{T} \succ 0$. The larger these values in the matrices Q and R are, the more you penalize the quantities they are related to. More in detail, by choosing a large (resp., small) value for R, we try to stabilize the system by using a large (resp., small) amount of energy. This is because the weights in R penalize the control input u. Similarly, choosing large values for Q means that we try to stabilize the system with the least possible changes in the states and small Q implies less concern about the changes in the states.

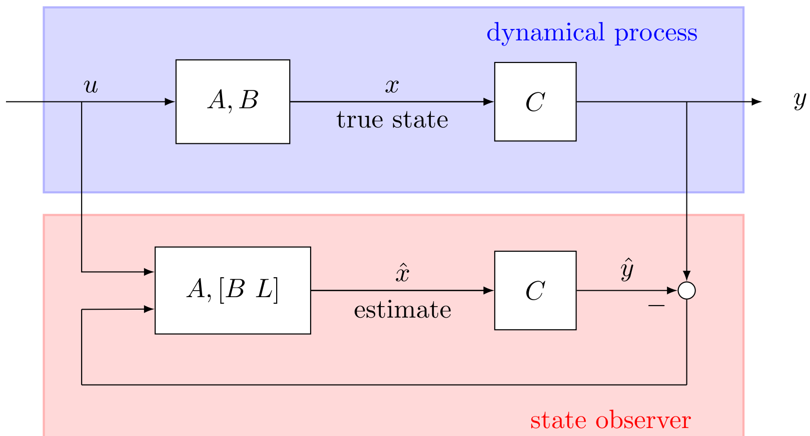
2 Design of an Observer for the Quarter-Car Model

In order to compute the full state feedback, we need to be able to measure the full state, which is not always possible in practice. Therefore, in cases where the full state is not accessible, we can design a state observer. The state observer let us estimate the states that we do not have access to. We can construct a state estimate $\hat{x}$ such that the law $u = -K\hat{x}$ retains similar pole assignment and closed-loop properties to the full state feedback $u = -Kx$. We can achieve this by designing a state estimator (or observer) of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du).$$

The estimator poles are the eigenvalues of $A - LC$, which can be arbitrarily assigned by proper selection of the estimator gain matrix L, provided that (C, A)

is observable. Generally, the estimator dynamics should be faster than the controller dynamics (eigenvalues of $A - BK$).



The dynamics of $\hat{x}$ derives from the theory developed by Luenberger. The main idea is to correct the estimation equation with a feedback from the estimation error $y(t) - \hat{y}(t)$. The output of the estimated state dynamics reads $\hat{y} = C\hat{x} + Du$, hence we have

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x} - Du).
\end{aligned}$$

The dynamics of the state estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ reads as

$$\begin{aligned}
\dot{\tilde{x}} &= Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x} - Du) \\
\dot{\tilde{x}} &= (A - LC)\tilde{x}.
\end{aligned}$$

If the matrix $A - LC$ is stable, the error dynamics converge to zero exponentially fast the the state estimate $\hat{x}$ converges to the true state x.

Finally, replacing x by its estimate $\hat{x}$ in $u = -Kx$ yields the dynamic output-feedback compensator:

$$\begin{aligned}
\dot{\hat{x}} &= [A - LC - (B - LD)K]\hat{x} + Ly \\
u &= -K\hat{x}
\end{aligned}$$

Note that the resulting closed-loop dynamics are

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}.$$