ME 121: Handout for discussion 8

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Abstract

In this discussion, we review the Laplace transform.

1 The Laplace transform and its properties

The Laplace transform is an integral transform that takes a function of a real variable t (time) to a function of a complex variable s (complex frequency). The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

where s is a complex number frequency parameter $s = \sigma + \mathrm{i}\,\omega$, with real numbers σ and ω. An alternate notation for the Laplace transform of f is $\mathcal{L}\{f\}$.

The Laplace transform has a long list of properties. The following table contains some of the most useful properties in the time and frequency domain.

	Time domain	Frequency domain
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
Derivative	$f'(t)$	$sF(s) - f(0^-)$
Second derivative	$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$
General derivative	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0^-)$
Time-domain integration	$\int_0^t f(\tau)\,\mathrm{d}\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
Frequency shifting	$e^{at}f(t)$	$F(s-a)$
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$	$F(s) \cdot G(s)$

where u is the Heaviside step function and $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.

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To convince ourselves that the above properties hold, it could be of interest to see the derivation of at least the derivative formula. By applying the definition of Laplace transform, we have $G(s) = \int_0^\infty e^{-st} f'(t)\mathrm{d}t$. Integration by parts yields:

$$G(s) = e^{-st} f(t)|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\mathrm{d}t = \lim_{t\to\infty} f(t)e^{-st} - f(0) + sF(s),$$

from which we can recover the formula $G(s) = sF(s) - f(0)$.

Useful results in control theory are the initial and final value theorems, which make use of the Laplace transform to obtain initial and final values of functions in the time domain:

- **Initial value theorem:**

$$f(0^+) = \lim_{s\to\infty} sF(s).$$

- **Final value theorem:**

$$f(\infty) = \lim_{s\to 0} sF(s),$$

  if all poles of $sF(s)$ are in the left half-plane. The final value theorem is useful because it gives the long-term behavior of a function without having to perform difficult calculations.

We conclude this section with a review of some selected Laplace transforms.

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Frequency domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence
unit impulse	$\delta(t)$	1	all s
unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
ramp	$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
exponential decay	$te^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}(s) > -\alpha$
sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$

A few example of derivations of the above Laplace transforms:

- Unit step: $F(s) = \int_0^\infty e^{-st}\mathrm{d}t = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$, provided we can say that $e^{-st} \to 0$ as $t \to \infty$, which is true for $\mathrm{Re}(s) > 0$.

- Cosine: we first rewrite $f(t) = \cos(\omega t)$ as $f(t) = \frac{e^{\mathrm{i}\omega t}+e^{-\mathrm{i}\omega t}}{2}$, then we apply the definition of Laplace transform.

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}\frac{e^{\mathrm{i}\omega t}+e^{-\mathrm{i}\omega t}}{2}\mathrm{d}t = \frac{1}{2}\int_0^\infty e^{(-s+\mathrm{i}\omega)t}\mathrm{d}t + \frac{1}{2}\int_0^\infty e^{(-s-\mathrm{i}\omega)t}\mathrm{d}t \\ &= \frac{1}{2(s-\mathrm{i}\,\omega)} + \frac{1}{2(s+\mathrm{i}\,\omega)} = \frac{s}{s^2+\omega^2} \end{aligned}$$