Exercise 5

1. Clifford Gates and Paulis

(a) For n qubits, there are 4^n possible tensor products of Paulis (one of which is the n-qubit identity). Show that (up to a phase) these can be expressed as a product of $2n$ n-qubit Paulis.

(b) If U is a Clifford gate, the following property holds

$$UPU^\dagger \sim P' \quad \forall P,$$

where P and P' are Paulis and $\sim$ denotes equality up to a factor of ± 1 or $\pm i$. If this relation holds for the $2n$ Pauli generators of part (a), show that it also holds for all n-qubit Paulis.

2. Single-Qubit Clifford Gates

(a) Show that the Paulis are Cliffords themselves.

(b) Show that H, S and $S^\dagger$ are Clifford gates.

(c) Show that $T = S^{1/2}$ is not a Clifford gate.

3. Two-Qubit Clifford Gates

For more than one qubit, Clifford gates map between tensor products of Pauli operators.

For two qubits

$$U\left(P \otimes Q\right)U^\dagger \sim P' \otimes Q' \quad \forall P, Q$$

where P, Q, P' and Q' are all Paulis and $\sim$ denotes equality up to a factor of ± 1 or $\pm i$.

(a) Show that the controlled-NOT is a Clifford gate.

(b) Show that the controlled-Hadamard is not a Clifford gate.

4. Three-Qubit Clifford Gates

(a) Provide an example of a three-qubit Clifford gate, and show that it is indeed a Clifford. This should be a truly three qubit gate, and therefore not one that can be expressed purely as a tensor product of single- and two-qubit gates.

(b) Show that the Toffoli gate is not Clifford.

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