Exercise 4

Part 1

a) State a is orthogonal if $\langle\bar{0}|\bar{1}\rangle = 0$. So we need to find a matrix $|\bar{1}\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, where $(\cos\theta \quad \sin\theta)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = a_0\cos\theta + a_1\sin\theta = 0$. One solution is when $|\bar{1}\rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$.

b) $|\bar{+}\rangle = \left(\frac{|\bar{0}\rangle + |\bar{1}\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta + \sin\theta \\ sin\theta - \cos\theta \end{pmatrix}$

$|\bar{-}\rangle = \left(\frac{|\bar{0}\rangle - |\bar{1}\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta - \sin\theta \\ sin\theta + \cos\theta \end{pmatrix}$

Part 2

a) $XX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\times 0 + 1\times 1 & 0\times 1 + 1\times 0 \\ 0\times 1 + 1\times 0 & 1\times 1 + 0\times 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$YY = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0\times 0 + (-i)i & 0\times -i + (-i)\times 0 \\ i\times 0 + 0\times i & i\times(-i) + 0\times 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$ZZ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 0\times 0 & 1\times 0 + 0\times(-1) \\ 1\times 0 + 0\times(-1) & 0\times 0 + (-1)\times(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

b)

$XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -YX$

$YZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -ZY$

$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -ZX$

c) $XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ \sim Z$

$YZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -iX \sim X$

$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = iY \sim Y$

d)

X:

$det(X - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$

For $\lambda = 1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, so $a_0 = a_1$. Because of normalization condition the eigenstate is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = 1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, so $a_0 = a_1$. Because of normalization condition the eigenstate is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = -1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = -\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$, so $a_0 = a_1$. Because of normalization condition the eigenstate is either $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Y:

$det(Y - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda_{1,2} = \pm 1$

For $\lambda = 1$: $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} -ia_1 \\ ia_0 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

Because of normalization condition: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

For $\lambda = -1$: $\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \begin{pmatrix}-ia_1\\ia_0\end{pmatrix} = -\begin{pmatrix}a_0\\a_1\end{pmatrix}$

Because of normalization condition: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

Z:

$det(Z-\lambda I) = (1-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda_{1,2} = \pm 1$

For $\lambda = 1$: $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \begin{pmatrix}a_0\\-a_1\end{pmatrix} = \begin{pmatrix}a_0\\a_1\end{pmatrix}$ ⇒ Eigenstate: $\begin{pmatrix}1\\0\end{pmatrix}$

For $\lambda = -1$: $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \begin{pmatrix}a_0\\-a_1\end{pmatrix} = -\begin{pmatrix}a_0\\a_1\end{pmatrix}$ ⇒ Eigenstate: $\begin{pmatrix}0\\1\end{pmatrix}$

Part 3

a) To find the eigenvalues and eigenstates, the following must hold: $det(H-\lambda I) = 0$, where λ is the eigenvalue.

$det\left(\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = (\frac{1}{\sqrt{2}}-\lambda)(-\frac{1}{\sqrt{2}}-\lambda) - (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = 0$

$\lambda^2 - 1 = (\lambda+1)(\lambda-1) = 0 \Rightarrow \lambda_{1,2} = \pm 1$

For $\lambda = 1$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}a_0+a_1\\a_0-a_1\end{pmatrix} = \begin{pmatrix}a_0\\a_1\end{pmatrix}$

System of equations:

1: $\frac{1}{\sqrt{2}}a_0 + \frac{1}{\sqrt{2}}a_1 - a_0 = 0$

2: $\frac{1}{\sqrt{2}}a_0 - \frac{1}{\sqrt{2}}a_1 - a_1 = 0$

$\frac{1}{\sqrt{2}}a_0 + \frac{1}{\sqrt{2}}a_1 - a_0 = \frac{1}{\sqrt{2}}a_0 - \frac{1}{\sqrt{2}}a_1 - a_1 \Rightarrow \frac{1}{\sqrt{2}}a_1 - a_0 = -\frac{1}{\sqrt{2}}a_1 - a_1 \Rightarrow a_0 = \sqrt{2}a_1 + a_1$

Insert into 2: $\frac{1}{\sqrt{2}}(\sqrt{2}a_1 + a_1) - \frac{1}{\sqrt{2}}a_1 - a_1 = 0 \Rightarrow 0 = 0 \Rightarrow a_1 = 1$

Insert $a_1 = 1$ into 2: $\frac{1}{\sqrt{2}}a_0 = \frac{1}{\sqrt{2}} + 1 \Rightarrow a_0 = 1 + \sqrt{2}$

So eigenstate for $\lambda = 1$ is $\begin{pmatrix}1+\sqrt{2}\\1\end{pmatrix}$

For $\lambda = -1$:

$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}a_0+a_1\\a_0-a_1\end{pmatrix} = -\begin{pmatrix}a_0\\a_1\end{pmatrix}$

System of equations:

1: $\frac{1}{\sqrt{2}}a_0 + \frac{1}{\sqrt{2}}a_1 + a_0 = 0$

2: $\frac{1}{\sqrt{2}}a_0 - \frac{1}{\sqrt{2}}a_1 + a_1 = 0$

System of equations:

$\frac{1}{\sqrt{2}}a_0 - \frac{1}{\sqrt{2}}a_1 + a_1 = \frac{1}{\sqrt{2}}a_0 + \frac{1}{\sqrt{2}}a_1 + a_0 \Rightarrow -\frac{1}{\sqrt{2}}a_1 + a_1 = \frac{1}{\sqrt{2}}a_1 + a_0 \Rightarrow a_0 = a_1 - \sqrt{2}a_1$

Insert in to 2:

$\frac{1}{\sqrt{2}}(a_1 - \sqrt{2}a_1) - \frac{1}{\sqrt{2}}a_1 + a_1 = 0 \Rightarrow 0 = 0 \Rightarrow a_1 = 1$

$a_0 = 1-\sqrt{2}$

Eigenstate for $\lambda = -1$: $\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$

b) $HH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1\times 1+1\times 1 & 1\times 1+1\times(-1) \\ 1\times 1+(-1)\times 1 & 1\times 1+(-1)\times(-1) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ c)

For $P_1 = X$: $HXH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z \sim Z$

Für $P_1 = Y$

$HYH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y \sim Y$ Für $P_1 = Z$

$HZH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = X \sim X$

Part 4

For Z_1, we can say that $Z_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes Z = \begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}$.

Where the commutation with all elements from the other set (X_0, Y_0, Z_0) is true.

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}$$

$\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}$ only true if $a == -d$ and $b == c$

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & -d \end{pmatrix}$$

also only true if $b = c = 0$, so only possible solution is $Z_1 = Z \otimes Z$

For X_1 we can say that:

$$X_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes X = \begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix}$$

Commutation:

$\begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix}(X \otimes X) = (X \otimes X)\begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix}$ only true if $a = d$ and $b = c$

$$\begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix} (Z \otimes Z) = (Z \otimes Z) \begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix} (Y \otimes Z) = (Y \otimes Z) \begin{pmatrix} 0 & a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & d \\ c & 0 & d & 0 \end{pmatrix}$$

only true if $b = -c$ and $a = d$. Therefore, $b = c = -c = 0$ and therefore $X_1 = I \otimes X$

For Y_1 we can say that $Y_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes Y = \begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix}$

Because of Commutation:

$$\begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix} (I \otimes Z) = (I \otimes Z) \begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix} \text{ Only true if } a = d = 0$$

$$\begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix} (Y \otimes Z) = (Y \otimes Z) \begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix}$$

Only true if $c = -b$ and $a = -d$, but that is already guaranteed.

$$\begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix} (X \otimes X) = (X \otimes X) \begin{pmatrix} 0 & -ai & 0 & -bi \\ ai & 0 & bi & 0 \\ 0 & -ci & 0 & -di \\ ci & 0 & di & 0 \end{pmatrix}$$

Therefore $Y_1 = Y \otimes Y$

Because $P_1 P_2 \sim P_3$. $Y_1 X_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -i(Y \otimes Z) \sim Y \otimes Z = Z_1$