Exercise 5

1. Clifford Gates and Paulis

a) To express any Pauli for 1 qubit up to a phase, we only need the X and Z. This is because $XX = I$ and $XZ = iY$ Therefore the generators for 1 qubit are X and Z.

For two qubits, we can use the generators $X \otimes I$, $I \otimes X$, $Z \otimes I$ and $I \otimes Z$:
$Y \otimes I = (iX \otimes I)(Z \otimes I) = iXZ \otimes II$
$I \otimes Y = (I \otimes iX)(I \otimes I) = iXZ \otimes II$ etc.

Because of $(a \otimes b)(c \otimes d) = ac \otimes bd$, we can say that for any N qubits, the set of generators to express any N-qubit Pauli is: $\bigcup_N X_n \cup \bigcup_N Z_n$, where $X_n = I^{\otimes n-1} \otimes X \otimes I^{\otimes N-n}$ and $Z_n = I^{\otimes n-1} \otimes Z \otimes I^{\otimes N-n}$.
A product of elements of these generators can be used to express any arbitrary N-qubit Pauli, up to a phase.

1b)
We need to be show, that if the relation holds for the generators described above, it also holds for a product of generators.
The relation $UPU^\dagger \sim P'$ holds for the generators, because the generators are guaranteed to be made up of Paulis and thus are n-qubit Paulis themselves.
Because $P_1P_2 = P_3$ a product of generators is also guaranteed to be a Pauli and therefore the relation also holds for any arbitrary n-qubit Pauli, as it can be expressed as such a product

2. Single-Qubit Clifford Gates

a) $XXX = X \sim X$
$XZX = iX \sim X$, therefore X is Clifford

$ZXZ = -iX$
$ZZZ = Z \sim Z$, therefore Z is Clifford

$YXY = -X \sim X$
$YZY = -Z \sim Z$, therefore Y is Clifford

b) $HXH = Z \sim Z$
$HYH = -Y \sim Y$
$HZH = X \sim X$
$HIH = I \sim I$, therefore H is Clifford

$SXS^\dagger = -Y \sim Y$
$SYS^\dagger = X \sim X$
$SZS^\dagger = Z \sim Z$
$SIS^\dagger = I \sim I$, therefore S is Clifford

$S^\dagger XS = Y \sim Y$
$S^\dagger YS = -X \sim X$
$S^\dagger ZS = Z \sim Z$
$S^\dagger IS = I \sim I$, therefore $S^\dagger$

c) $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$
$TYT^\dagger = \begin{pmatrix} 0 & -ie^{-i\frac{\pi}{4}} \\ ie^{i\frac{\pi}{4}} & 0 \end{pmatrix} \nsim P \; \forall P \in \{X, Y, Z, I\}$, therefore T is not Clifford.

3. Two-Qubit Clifford Gates

Because we found the generators, we only need to show that the relation holds for the generators, which in this case are: $X \otimes I$, $I \otimes X$, $Z \otimes I$ and $I \otimes Z$.

a) $C_x(X \otimes I)C_x = i(Y \otimes X) \sim Y \otimes X$
$C_x(I \otimes X)C_x = I \otimes X \sim I \otimes X$
$C_x(Z \otimes I)C_x = Z \otimes I) \sim Z \otimes I$
$C_x(I \otimes Z)C_x = Z \otimes Z \sim Z \otimes Z$, therefore C_x is Clifford

b) $C_H(X \otimes I)C_H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \nsim P \; \forall P \in P_2$

4. Three-Qubit Clifford Gates

a) Fredkin Gate:

$$CSwap = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

b) $CCNOT(I \otimes I \otimes Z)CCNOT^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \nsim P \; \forall P \in P_3$