Exercise 3

1. Mutually unbiased bases

Show that the X, Y and Z bases are all unbiased with respect to each other: each state for one basis results in a completely random result for the other two bases.

Supplementary question: sho why the bases are unbiased supplementary one another.

2. Shifting certainty

The state of a single qubit is characterized by three numbers, $\langle \sigma^x \rangle$, $\langle \sigma^y \rangle$ and $\langle \sigma^z \rangle$, defined as

$$\langle \sigma^\alpha \rangle = p_0^\alpha - p_1^\alpha,$$

where p_0^z is the probability of the outcome 0 for a Z measurement, and so on.

For any single qubit superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$,

$\langle \sigma^x \rangle^2 + \langle \sigma^y \rangle^2 + \langle \sigma^z \rangle^2 = 1$

Verify this for the following states.

- (a) $|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$
- (b) $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$

Supplementary question: Verify this for the most generic qubit. Use these results to show why the sum is a constant.

3. A useful matrix

Find the 2×2 matrix M such that

$$p_0^z - p_1^z = \langle\psi|M|\psi\rangle \;\; \forall|\psi\rangle$$

Supplementary question: Find another matrix that works and if you can't show why this can't be done.