Exercise 3

1. Mutually unbiased bases

I couldn't find a way to prove this generally, so I just calculated all the probabilities of getting a 0 or a 1 for each of the other bases for each basis state. If they are all $\frac{1}{2}$ then they are unbiased, because there's a 50/50 chance of getting either a 0 or 1. Technically, I didn't have to calculate P_1^α for each base, because if P_0^α is known it can be calculated because of the normalization condition of the amplitudes.

Z-Bases:
For $|0\rangle$:
$P_0^x = |\langle -|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^x = |\langle +|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_0^y = |\langle L|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^y = |\langle R|0\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$
For $|1\rangle$:
$P_0^x = |\langle -|1\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^x = |\langle +|1\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_0^y = |\langle L|1\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^y = |\langle R|1\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$

X-Bases
For $|-\rangle$:
$P_0^Z = |\langle 0|-\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Z = |\langle 1|-\rangle|^2 = \frac{1}{2}$
$P_0^Y = |\langle L|-\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Y = |\langle R|-\rangle|^2 = \frac{1}{2}$

For $|+\rangle$:
$P_0^Z = |\langle 0|+\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Z = |\langle 1|+\rangle|^2 = \frac{1}{2}$
$P_0^Y = |\langle L|+\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Y = |\langle R|+\rangle|^2 = \frac{1}{2}$

Y-Bases
For $|L\rangle$:
$P_0^Z = |\langle 0|L\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-i}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Z = |\langle 1|L\rangle|^2 = \frac{1}{2}$
$P_0^x = |\langle -|L\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-i}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^X = |\langle +|L\rangle|^2 = \frac{1}{2}$

For $|R\rangle$:
$P_0^Z = |\langle 0|R\rangle|^2 = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^Z = |\langle 1|R\rangle|^2 = \frac{1}{2}$
$P_0^x = |\langle -|R\rangle|^2 = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$
$P_1^X = |\langle +|R\rangle|^2 = \frac{1}{2}$

2. Shifting certainty

$\langle\sigma^x\rangle^2 + \langle\sigma^y\rangle^2 + \langle\sigma^z\rangle^2 = 1$, where $\sigma^\alpha = p_0^\alpha - p_1^\alpha$. So:
$(p_0^x - p_1^x)^2 + (p_0^y - p_1^y)^2 + (p_0^z - p_1^z)^2 = 1$
$(|\langle -|\psi\rangle|^2 - |\langle +|\psi\rangle|^2)^2 + (|\langle L|\psi\rangle|^2 - |\langle R|\psi\rangle|^2)^2 + (|\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2)^2 = 1$

$$\left(\left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2-\left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2\right)^2+\left(\left|\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2-\left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2\right)^2+\left(\left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2-\left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2\right)^2=1$$

$(|\frac{a_0-a_1}{\sqrt{2}}|^2-|\frac{a_0+a_1}{\sqrt{2}}|^2)^2+(|\frac{a_0+ia_1}{\sqrt{2}}|^2-|\frac{a_0-ia_1}{\sqrt{2}}|^2)^2+(|a_0|^2-|a_1|^2)^2=1$

$(\frac{|a_0-a_1|^2-|a_0+a_1|^2}{|\sqrt{2}|^2})^2+(\frac{|a_0+ia_1|^2-|a_0-ia_1|^2}{|\sqrt{2}|^2})^2+(|a_0|^2-|a_1|^2)^2=1$

$\frac{1}{4}(|a_0-a_1|^2-|a_0+a_1|^2)^2+\frac{1}{4}(|a_0+ia_1|^2-|a_0-ia_1|^2)^2+(|a_0|^2-|a_1|^2)^2=1$

a) $a_0=\cos\theta$ and $a_1=\sin\theta$

$(\frac{|\cos\theta-\sin\theta|^2-|\cos\theta+\sin\theta|^2}{2})^2+(\frac{|\cos\theta+i\sin\theta|^2-|\cos\theta-i\sin\theta|^2}{2})^2+(\cos^2\theta-\sin^2\theta)^2=1$

$\frac{(\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta-\cos^2\theta-2\cos\theta\sin\theta-\sin^2\theta)^2}{4}+\frac{1-1}{4}+(\cos^2\theta-\sin^2\theta)^2=1$

$\frac{(-4\cos\theta\sin\theta)^2}{4}+(\frac{0}{2})^2+(\cos^2\theta-\sin^2\theta)^2=1$

$\frac{16\cos^2\theta\sin^2\theta}{4}+(\cos^2\theta-\sin^2\theta)^2=1$

$4\cos^2\theta\sin^2\theta+\cos^4\theta-2\cos^2\theta\sin^2\theta+\sin^4\theta=1$

$\cos^4\theta+2\cos^2\theta\sin^2\theta+\sin^4\theta=1$

$(\cos^2\theta+\sin^2\theta)^2=1^2=1$ Correct

b) $a_0=\frac{1}{\sqrt{2}}$ and $a_1=\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)$

$\frac{1}{4}\left(|\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2$

$\frac{1}{4}((\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos(\phi))^2+(\frac{1}{\sqrt{2}}\sin(\phi))^2-((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos(\phi))^2+(\frac{1}{\sqrt{2}}\sin(\phi))^2))^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi$

$\frac{1}{4}(\frac{1}{2}-\cos(\phi)+\frac{1}{2}\cos^2(\phi)+\frac{1}{2}\sin^2(\phi)-((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos(\phi))^2+(\frac{1}{\sqrt{2}}\sin(\phi))^2))^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos$

$\frac{1}{4}(\frac{1}{2}-\cos(\phi)+\frac{1}{2}\cos^2(\phi)+\frac{1}{2}\sin^2(\phi)-(\frac{1}{2}+\cos(\phi)+\frac{1}{2}\cos^2(\phi)+\frac{1}{2}\sin^2(\phi)))^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos$

$\frac{1}{4}(\frac{1}{2}-\cos(\phi)+\frac{1}{2}\cos^2(\phi)+\frac{1}{2}\sin^2(\phi)-\frac{1}{2}-\cos(\phi)-\frac{1}{2}\cos^2(\phi)-\frac{1}{2}\sin^2(\phi)))^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos$

$\frac{1}{4}(-2\cos(\phi))^2+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2=1$

$\cos^2(\phi)+\frac{1}{4}\left(|\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(i\cos\phi-\sin\phi)|^2-|\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}(i\cos\phi-\sin\phi)|^2\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2=1$

$\cos^2(\phi)+\frac{1}{4}\left(((\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sin\phi)^2+(\frac{1}{\sqrt{2}}\cos\phi)^2)-((\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\sin\phi)^2+(\frac{1}{\sqrt{2}}\cos\phi)^2)\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2=1$

$\cos^2(\phi)+\frac{1}{4}\left(\frac{1}{2}+\sin\phi+\frac{1}{2}\sin^2\phi+\frac{1}{2}\cos^2\phi-(\frac{1}{2}-\sin\phi+\frac{1}{2}\sin^2\phi+\frac{1}{2}\cos^2\phi)\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2=1$

$\cos^2(\phi)+\frac{1}{4}\left(\frac{1}{2}+\sin\phi+\frac{1}{2}\sin^2\phi+\frac{1}{2}\cos^2\phi-\frac{1}{2}+\sin\phi-\frac{1}{2}\sin^2\phi-\frac{1}{2}\cos^2\phi\right)^2+(|\frac{1}{\sqrt{2}}|^2-|\frac{1}{\sqrt{2}}(\cos\phi+i\sin\phi)|^2)^2=1$

$\cos^2(\phi)+\frac{1}{4}(2\sin\phi)^2+(\frac{1}{2}-\frac{1}{2})^2=1$

$\cos^2\phi+\sin^2\phi=1$ Correct

3. A useful matrix

$p_0^z-p_1^z=\langle\psi|M|\psi\rangle$

$\left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2-\left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix}\right|^2=|a_0|^2-|a_1|^2=a_0^*a_0-a_1^*a_1=\begin{pmatrix}a_0^* & a_1^*\end{pmatrix}M\begin{pmatrix}a_0\\a_1\end{pmatrix}$

For this to be true, the following must hold: $M|\psi\rangle=M\begin{pmatrix}a_0\\a_1\end{pmatrix}=\begin{pmatrix}a_0\\-a_1\end{pmatrix}$

An example where this is true is when $M=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}=\sigma_3$