Exercise A

Let y be an inverse Gaussian distribution, with support $[0,\infty)$ and density

$$f(y;\xi,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{1/2} y^{-3/2} e^{\sqrt{\lambda\xi}}\, e^{-\frac{1}{2}\left(\frac{\lambda}{y}+\xi y\right)}$$

Then

$$f(y;\xi,\lambda)=\exp\left\{-\frac{1}{2}\xi y-\frac{1}{2}\frac{\lambda}{y}+(\lambda\,\xi)^{1/2}+\frac{1}{2}\log(\lambda)-\frac{3}{2}\log(y)-\frac{1}{2}\log(2\pi)\right\}$$

Set $\boxed{\theta=-\frac{1}{2}\frac{\xi}{\lambda}}$ so that $-\frac{\xi}{2}=\theta\lambda$ and $\xi\lambda=-2\theta\lambda^2$

↳ hence, $\theta<0$.

This gives:

$$f(y;\theta,\lambda)=\exp\left\{\theta\lambda y-\lambda(-2\theta)^{1/2}\underbrace{-\frac{1}{2}\frac{\lambda}{y}+\frac{1}{2}\log(\lambda)-\frac{3}{2}\log(y)-\frac{1}{2}\log(2\pi)}\right\}$$

Now, set $\boxed{\phi=\frac{1}{\lambda}}$, giving

Careful here, we need a minus (fixed below)

$$f(y;\theta,\phi)=\exp\left\{\frac{\theta y+(-2\theta)^{1/2}}{\phi}+c(\phi,y)\right\}.$$

Thus, we have

$\theta=-\frac{1}{2}\frac{\xi}{\lambda}$ canonical parameter, with $\theta<0$; $\phi=\frac{1}{\lambda}$ dispersion.

$$\boxed{b(\theta)=-(-2\theta)^{1/2};}\qquad a(\phi)=\phi$$

$$c(\phi,y)=-\frac{1}{2\phi y}-\frac{1}{2}\log(\phi)-\frac{3}{2}\log(y)-\frac{1}{2}\log(2\pi)$$

Mean function

$$b'(\theta) = \frac{\partial}{\partial \theta} - (-2\theta)^{1/2} = \cancel{-} \frac{1}{\cancel{2}(-2\theta)^{1/2}} \cdot \cancel{-2} = \frac{1}{(-2\theta)^{1/2}} = \mu(\theta)$$

$$\Rightarrow \mu^2 = \frac{1}{-2\theta} \iff -2\theta = \frac{1}{\mu^2} \iff \theta = \frac{1}{-2\mu^2}$$

$$\boxed{\theta(\mu) = \frac{1}{-2\mu^2}}$$

Canonical link

Variance function

$$b''(\theta) = \frac{\partial}{\partial \theta} (-2\theta)^{-1/2} = -\frac{1}{2}(-2\theta)^{-3/2} \cdot (-2) = \frac{1}{(-2\theta)^{3/2}}$$

$$\Rightarrow V(\mu) = \frac{1}{[-2\theta(\mu)]^{3/2}} = \frac{1}{\left(\frac{\cancel{-2}}{\cancel{-2}\mu^2}\right)^{3/2}} = \mu^3$$

$$\boxed{V(\mu) = \mu^3, \quad \text{var}(y) = \phi\mu^3}$$