Exercise F

Case i) Poisson with canonical link

Suppose that $y_i \sim \text{Poisson}(\mu_i)$, with $\log(\mu_i) = \beta_1 + \beta_2 x_i$, in cui $x_i \{0,1\}$.

The likelihood equations are:

$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} \mu_i \qquad \text{and} \qquad \sum_{i=1}^{m} y_i x_i = \sum_{i=1}^{m} \mu_i x_i$$

where $x_i = 1$, with $i = 1, \dots, m_A$, so $\sum_{i=1}^{m_A} y_i = \sum_{i=1}^{m_A} \mu_i$

However, note that $\mu_i = \exp(\beta_1 + \beta_2 x_i)$, therefore $\mu_i = \mu_A = \exp(\beta_1 + \beta_2)$ for $i = 1, \dots, m_A$. Consequently

$$\frac{1}{m_A} \sum_{i=1}^{m_A} \mu_A = \frac{1}{m_A} \sum_{i=1}^{m_A} y_i \implies \hat{\mu}_i = \hat{\mu}_A = \bar{y}_A, \text{ for } i = 1, \dots, m_A.$$

Moreover

$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} \mu_i \iff \sum_{i=m_A+1}^{m} y_i + \cancel{\overbrace{\sum_{i=1}^{m_A} y_i}^{m_A \bar{y}_A}} = \sum_{i=m_A+1}^{m} \mu_i + \cancel{\overbrace{\sum_{i=1}^{m_A} \mu_i}^{m_A \bar{y}_A}}$$

$$\iff \frac{1}{m_B} \sum_{i=m_A+1}^{m} y_i = \frac{1}{m_B} \sum_{i=m_A+1}^{m} \mu_i$$

with $\mu_i = \exp(\beta_1 + \beta_2 \cdot 0) = \exp(\beta_1) = \mu_B$, $i = m_A+1, \dots, m$, from which we obtain that $\hat{\mu}_B = \bar{y}_B$.

Case ii) Poisson with any link function

The likelihood equations in this case are

$$\sum_{i=1}^{m} \frac{y_i - \mu_i}{\mu_i} \cdot \frac{1}{g'(\mu_i)} = 0, \quad \text{and} \quad \sum_{i=1}^{m} \frac{y_i - \mu_i}{\mu_i} \cdot \frac{1}{g'(\mu_i)} x_i = 0.$$

where $x_i \in \{0,1\}$, so that $\mu_i = \mu_A = g^{-1}(\beta_1 + \beta_2)$ for $i = 1, \dots, m_A$ and $\mu_i = \mu_B = g^{-1}(\beta_1)$ for $i = m_A+1, \dots, m$. Hence, the two equations become

$$\cancel{\sum_{i=1}^{m_A} \frac{y_i - \mu_A}{\mu_A} \cdot \frac{1}{g'(\mu_A)}} + \sum_{i=m_A+1}^{m} \frac{y_i - \mu_B}{\mu_B} \cdot \frac{1}{g'(\mu_B)} = 0, \quad \text{and} \quad \sum_{i=1}^{m_A} \frac{y_i - \mu_A}{\mu_A} \cdot \frac{1}{g'(\mu_A)} = 0$$

it implies that can be canceled.

Focusing on the second equation:

$$\sum_{i=1}^{m_A} \frac{y_i - \mu_A}{\cancel{\mu_A}} \cdot \frac{1}{\cancel{g'(\mu_A)}} = 0 \iff \sum_{i=1}^{m_A} y_i = \sum_{i=1}^{m_A} \mu_A \implies \hat{\mu}_A = \frac{1}{m_A} \sum_{i=1}^{m_A} y_i = \bar{y}_A$$

Similarly, from the first equation:

$$\sum_{i=m_A+1}^{m} \frac{y_i - \mu_B}{\cancel{\mu_B}} \cdot \frac{1}{\cancel{g'(\mu_B)}} = 0 \implies \sum_{i=m_A+1}^{m} y_i = \sum_{i=m_A+1}^{m} \mu_B \implies \hat{\mu}_B = \bar{y}_B.$$

Case iii, general GLM

The proof is almost identical to case ii. The estimation equations are

$$\sum_{i=1}^{m} w_i \frac{(y_i-\mu_i)}{v(\mu_i)} \cdot \frac{1}{g(\mu_i)} = 0 \quad \text{and} \quad \sum_{i=1}^{m} w_i \frac{(y_i-\mu_i)}{v(\mu_i)} \cdot \frac{1}{g(\mu_i)} x_i = 0.$$

where $x_i \in \{0,1\}$, so that $\mu_i = \mu_A = g^{-1}(\beta_1+\beta_2)$ for $i = 1,\dots,m_A$ and $\mu_i = \mu_B = g^{-1}(\beta_1)$ for $i = m_A+1,\dots,m$. Hence, the two equations become

$$\cancel{\sum_{i=1}^{m_A} w_i \frac{(y_i-\mu_A)}{v(\mu_A)} \cdot \frac{1}{g'(\mu_A)}} + \sum_{i=m_A+1}^{m} w_i \frac{(y_i-\mu_B)}{v(\mu_B)} \cdot \frac{1}{g'(\mu_B)} = 0; \quad \sum_{i=1}^{m_A} w_i \frac{(y_i-\mu_A)}{v(\mu_A)} \cdot \frac{1}{g'(\mu_A)} = 0$$

It implies that can be cancelled

Focusing on the second equation:

$$\sum_{i=1}^{m_A} w_i \frac{(y_i-\mu_A)}{v(\cancel{\mu_A})} \cdot \frac{1}{g'(\cancel{\mu_A})} = 0 \iff \sum_{i=1}^{m_A} w_i y_i = \mu_A \sum_{i=1}^{m_A} w_i \Rightarrow \hat{\mu}_A = \frac{\sum_{i=1}^{m_A} w_i y_i}{\sum_{i=1}^{m_A} w_i} = \bar{y}_{\omega,A}$$

Similarly, you get from the first equation that

$$\sum_{i=m_A+1}^{m} w_i \frac{(y_i-\mu_B)}{v(\cancel{\mu_B})} \cdot \frac{1}{g'(\cancel{\mu_B})} = 0 \iff \sum_{i=m_A+1}^{m} w_i y_i = \mu_B \sum_{i=m_A+1}^{m} w_i \Rightarrow \hat{\mu}_B = \bar{y}_{\omega,A}$$