Exercise C

The general likelihood equations for a GLM are:

$$\sum_{i=1}^{m} \omega_i \frac{(y_i - \mu_i)}{v(\mu_i)} \cdot \frac{1}{g'(\mu_i)} x_{ir} = 0, \qquad r = 1, \dots, p.$$

i. In the Beetles data, we have $x_{i1} = 1$ and $x_{i2} = x_i$ (log-dose). Moreover, the weights are equal to $\omega_i = m_i$, $v(\mu_i) = \mu_i(1-\mu_i)$ and the link function is the logit.

Hence:

$$g(\mu_i) = \log\left(\frac{\mu_i}{1-\mu_i}\right) \Rightarrow g'(\mu_i) = \frac{1-\mu_i}{\mu_i} \cdot \underbrace{\frac{\partial}{\partial \mu_i} \frac{\mu_i}{1-\mu_i}}_{= \frac{1}{(1-\mu_i)^2}}$$

$$= \frac{\cancel{1-\mu_i}}{\mu_i} \cdot \frac{1}{(1-\mu_i)^{\cancel{2}}} = \frac{1}{\mu_i(1-\mu_i)}.$$

From this, we recover that $v(\mu_i) g'(\mu_i) = \frac{\mu_i(1-\mu_i)}{\mu_i(1-\mu_i)} = 1$, a well-known property of the canonical link. Hence, we get

$$\sum_{i=1}^{m} m_i (y_i - \mu_i) \cdot 1 = 0 \quad \text{and} \quad \sum_{i=1}^{m} m_i (y_i - \mu_i) x_i = 0,$$

where $\mu_i = g^{-1}(\beta_1 + \beta_2 x_i) = \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta x_i}}$, obtaining

$$\sum_{i=1}^{m} m_i \left(y_i - \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta_2 x_i}} \right) = 0, \quad \sum_{i=1}^{m} m_i \left(y_i - \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta_2 x_i}} \right) x_i = 0.$$

Or, if you prefer

$$\sum_{i=1}^{m} m_i y_i = \sum_{i=1}^{m} m_i \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta_2 x_i}}; \quad \sum_{i=1}^{m} m_i y_i x_i = \sum_{i=1}^{m} m_i \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta_2 x_i}} \cdot x_i = 0.$$

ii In Poisson regression with AIDS data we have that $x_{i1} = 1$, $x_{i2} = t_i$ and constant weights $w_i = t_i$, with variance function $v(\mu_i) = \mu_i$. The canonical link is $g(\mu_i) = \log(\mu_i)$, obtaining.

$$g'(\mu_i) = \frac{1}{\mu_i}, \quad \text{therefore} \quad v(\mu_i)\, g'(\mu_i) = \frac{\mu_i}{\mu_i} = 1.$$

Hence the likelihood equations are

$$\sum_{i=1}^{m} (y_i - \mu_i) = 0 \quad \text{and} \quad \sum_{i=1}^{m} (y_i - \mu_i) t_i = 0, \quad \text{with } \mu_i = e^{\beta_1 + \beta_2 x_i}.$$

We therefore obtain

$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} e^{\beta_1 + \beta_2 x_i} \quad \text{and} \quad \sum_{i=1}^{m} y_i t_i = \sum_{i=1}^{m} t_i e^{\beta_1 + \beta_2 x_i}$$

iii In a Binomial regression model with link Cauchy and considering again the Beetles data, we have

$$\sum_{i=1}^{m} m_i \frac{(y_i - \mu_i)}{\mu_i(1-\mu_i)} \cdot \frac{1}{g'(\mu_i)} = 0 \quad \text{and} \quad \sum_{i=1}^{m} m_i \frac{y_i - \mu_i}{\mu_i(1-\mu_i)} \frac{1}{g'(\mu_i)} x_i = 0.$$

The link function is $g(\mu_i) = \tan(\tilde{\pi}(\mu_i - 1/2))$, so that

$$g'(\mu_i) = \frac{\tilde{\pi}}{\sin^2(\tilde{\pi}\mu_i)}; \quad g^{-1}(\eta_i) = \frac{1}{2} + \frac{\operatorname{atan}(\eta_i)}{\tilde{\pi}}.$$

Substituting, we get

$$\sum_{i=1}^{m} \frac{m_i y_i \sin^2(\tilde{\pi}\mu_i)}{\tilde{\pi}\mu_i(1-\mu_i)} = \sum_{i=1}^{m} \frac{m_i \mu_i \sin^2(\tilde{\pi}\mu_i)}{\tilde{\pi}\mu_i(1-\mu_i)}; \quad \sum_{i=1}^{m} \frac{x_i m_i y_i \sin^2(\tilde{\pi}\mu_i)}{\tilde{\pi}\mu_i(1-\mu_i)} = \sum_{i=1}^{m} \frac{x_i m_i \mu_i \sin^2(\tilde{\pi}\mu_i)}{\tilde{\pi}\mu_i(1-\mu_i)}$$

where $\mu_i = \frac{1}{2} + \frac{\operatorname{atan}(\beta_1 + \beta_2 x_i)}{\tilde{\pi}}.$