

### Exercise A

Let  $y$  be an inverse Gaussian distribution, with support  $[0, \infty)$  and density

$$p(y; \xi, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{1/2} y^{-3/2} e^{-\frac{\sqrt{\lambda\xi}}{2} \left(\frac{\lambda}{y} + \xi y\right)}$$

Then

$$p(y; \xi, \lambda) = \exp\left\{-\frac{1}{2}\xi y - \frac{1}{2}\frac{\lambda}{y} + \left(\lambda\xi\right)^{1/2} + \frac{1}{2}\log(\lambda) - \frac{3}{2}\log(y) - \frac{1}{2}\log(2\pi)\right\}$$

Set  $\boxed{\theta = -\frac{1}{2}\frac{\xi}{\lambda}}$  so that  $-\frac{\xi}{2} = \theta\lambda$  and  $\xi\lambda = -2\theta\lambda^2$   
 $\hookrightarrow$  hence,  $\theta < 0$ .

This gives:

$$p(y; \theta, \lambda) = \exp\left\{\theta\lambda y - \lambda(-2\theta)^{1/2} - \frac{1}{2}\frac{\lambda}{y} + \frac{1}{2}\log(\lambda) - \frac{3}{2}\log(y) - \frac{1}{2}\log(2\pi)\right\}$$

$= c(\lambda, y)$

Now, set  $\boxed{\phi = \frac{1}{\lambda}}$ , giving

*Caution here, we need a minus (fixed below)*

$$p(y; \theta, \phi) = \exp\left\{\frac{\theta y}{\phi} \oplus (-2\theta)^{1/2} + c(\phi, y)\right\}.$$

Thus, we have

$\theta = -\frac{1}{2}\frac{\xi}{\lambda}$  canonical parameter, with  $\theta < 0$ ;  $\phi = \frac{1}{\lambda}$  dispersion.

$$\boxed{b(\theta) = -(-2\theta)^{1/2}; \quad a(\phi) = \phi}$$

$$c(\phi, y) = -\frac{1}{2\phi y} - \frac{1}{2}\log(\phi) - \frac{3}{2}\log(y) - \frac{1}{2}\log(2\pi)$$

## Mean function

$$b'(\theta) = \frac{\partial}{\partial \theta} (-2\theta)^{1/2} = \cancel{1} \cdot \frac{1}{2(-2\theta)^{1/2}} \cdot \cancel{-2} = \frac{1}{(-2\theta)^{1/2}} = \mu(\theta)$$

$$\Rightarrow \mu^2 = \frac{1}{-2\theta} \Leftrightarrow -2\theta = \frac{1}{\mu^2} \Leftrightarrow \theta = \frac{1}{-2\mu^2}$$

$$\boxed{\theta(\mu) = \frac{1}{-2\mu^2}} \quad \text{Canonical link}$$

## Variance function

$$b''(\theta) = \frac{\partial}{\partial \theta} (-2\theta)^{-1/2} = -\frac{1}{2} (-2\theta)^{-3/2} \cdot (-2) = \frac{1}{(-2\theta)^{3/2}}$$

$$\Rightarrow v(\mu) = \frac{1}{[-2\theta(\mu)]^{3/2}} = \frac{1}{\left(\frac{\cancel{-2}}{\cancel{-2}\mu^2}\right)^{3/2}} = \mu^3$$

$$\boxed{v(\mu) = \mu^3, \quad \text{var}(y) = \phi \mu^3}$$