Activities

1. (a) The Pell numbers are an infinite sequence of integers which correspond to the denominators of the closest rational approximations of $\sqrt{2}$. The Pell numbers are defined by the following recurrence relation (which looks very similar to the Fibonnacci sequence):

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{otherwise} \end{cases}$$

   Use this recurrence relation to write a recursive function, `pell`, which takes one parameter, n, and returns the n^{th} Pell number.

   (b) Write a separate function, named `(find-pell n)`, which uses your Pell function to find the largest Pell number which is less than n. You can try testing it by finding the largest Pell number less than 100.

   (c) The numerator for the rational approximation of $\sqrt{2}$ corresponding to a particular Pell number is half of the corresponding number in the sequence referred to as the *companion Pell numbers* (or Pell-Lucas numbers). The companion Pell numbers are defined by the recurrence relation:

$$Q_n = \begin{cases} 2 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 2Q_{n-1} + Q_{n-2} & \text{otherwise} \end{cases}$$

   Use this recurrence relation to write a function, named `(comp-pell n)`, which returns the n^{th} companion Pell number.

   (d) Finally write a function that uses the Pell number and companion Pell number functions, as described in Part a, to write a SCHEME function, named `(sqrt-2-approx n)`, to compute the n^{th} approximation for $\sqrt{2}$. You can test your new function to compute the approximation for $\sqrt{2}$ using the sixth Pell and companion Pell numbers.

2. Viète's formula is the following infinite product of nested radicals representing the mathematical constant π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

   Write a SCHEME function named `(viete n)` which approximates $\frac{2}{\pi}$ using Viète's formula.
   Note: You may want to use a helper function that takes an additonal parameter so that you can compute the numerator of each term easily (in a `let` form perhaps) as well as pass it to the recursive call.

3. It is an interesting fact the the square-root of any number may be expressed as a *continued fraction*. For example,

$$\sqrt{x} = 1 + \cfrac{x-1}{2 + \cfrac{x-1}{2 + \cfrac{x-1}{\ddots}}}$$

   Write a Scheme function called *new-sqrt* which takes two formal parameters x and n, where x is the number we wish to find the square root of and n is the number of continued fractions to compute recursively. Demonstrate that for large n, *new-sqrt* is very close to the builtin *sqrt* function.

4. **Nested Recursion**:The McCarthy 91 function is a recursive function, defined by the computer scientist John McCarthy as a test case for formal verification within computer science.

   The McCarthy 91 function is defined as

$$m91(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ m91(m91(n+11)), & \text{if } n \le 100 \end{cases}$$

   Define a SCHEME procedure, named `(m91 n)`, which evaluates to McCarthy's 91 function. Try a few values for n which are less than 100 and a few greater than 100.

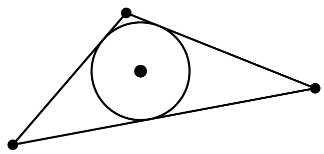

Figure 1: The circle inscribed in a triangle.

5. The radius R of the circle inscribed in a triangle with edge lengths A, B, and C is given by the formula

$$R = \sqrt{\frac{(S-A)(S-B)(S-C)}{S}} \qquad \text{where} \qquad S = \frac{A+B+C}{2}.$$

   Define a SCHEME function `iradius` which takes three parameters for the side lengths (perhaps call them A, B, and C) and returns the radius as given by the formula above. (You may use the built-in scheme function `sqrt` for this purpose. A `let` construct can save you a lot of typing.)

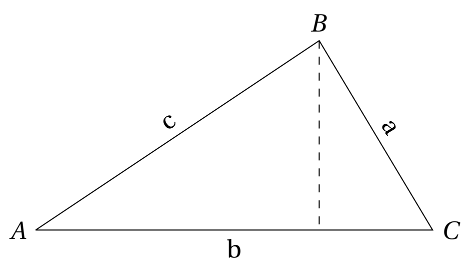

Figure 2: A triangle with sides a, b, and c.

6. In geometry, Heron's formula gives the area of a triangle when the length of all three sides, a, b, and c, are known.

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c}$$

   where

$$s = \frac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

   Define a SCHEME function, named `(heron a b c)`, which uses Heron's formula to compute the area of a triangle with sides with lengths a, b, and c. You must use `let` forms to define the variables s, s_a, s_b, and s_c.