Detection and characterization of extrasolar planets

T. T. Koskinen

March 23, 2017

1 Hunting for honey bees

What does it take to observe Jupiter orbiting the Sun at 0.05 AU from a distance of only 1 light year? The area of the solar disk is $A_{\mathrm{Sun}} = 1.52 \times 10^{16}$ m^2, with the solar radius of $R_{Sun} = 695700$ km, and the area of Jupiter's disk is $A_J = \pi R_{\mathrm{Jeq}} R_{\mathrm{Jp}} = 1.5 \times 10^{16}$ m^2 where Jupiter's equatorial and polar radii are $R_{\mathrm{Jeq}} = 71492$ km and $R_{\mathrm{Jp}} = 66854$ km, respectively. This corresponds to a regular honey bee with dimensions of 0.01 x 0.01 m (area of $A_{\mathrm{hb}} = 10^{-4}$ m^2) flying around a rectangular camping lantern with dimensions of 0.125 x 0.8 m (area of $A_{\mathrm{L}} = 0.01$ m^2) at a distance of 0.6 m or 2 feet. On light year in this system corresponds to a distance of 772 km or 480 miles ($ly = 9.4607 \times 10^{15}$ m) with a distance conversion factor of $\sqrt{A_{\mathrm{hb}}/A_J} = 8.16 \times 10^{-11}$ from real world to the lantern-bee system. Observing the Hot Jupiter system from a distance of 1 light year is equivalent to trying to see the honey bee in Montreal, Canada from Washington, DC circling around the lantern, which is hundreds of thousands of times brighter than a typical camping lantern. Note that in reality Hot Jupiters are much further - for example, the first known Hot Jupiter 51 Peg b is 50 light years away.

2 Radial velocity semi-amplitude

In this section, we derive the relationship that ties the observed radial velocity semi-amplitude K_1 of the host star to the period, eccentricity and minimum mass ($M_p \sin i$) of an orbiting planet. Throughout, we assume that there is only one planet in the system. The center of mass of the star-planet system is located at:

$$\mathbf{R}_{\mathrm{CM}} = \frac{M_s \mathbf{r}_1 + M_p \mathbf{r}_2}{M_s + M_p} \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors connecting the star and the planet, respectively, to the origin of the coordinate system. Placing the origin at the center of mass, we have at all times

$$M_s r_1 = M_p r_2 = M_p (r - r_1) \rightarrow r_1 = \frac{M_p}{M_s + M_p} r. \tag{2}$$

The orbit for the star traces a small ellipse with a semi-major axis a_1, given by

$$r_1 = \frac{a_1(1-e^2)}{1+e\cos\theta} \tag{3}$$

where θ is the true anomaly between the periastron and the position along the orbit, measured with respect to the center of mass. By virtue of equation (2), we have

$$a_1 = \frac{M_p}{M_s + M_p} a$$

and

$$r_1 = \frac{M_p}{M_s + M_p} \frac{a(1-e^2)}{1+e\cos\theta}. \tag{4}$$

If the x axis points from the CM to periastron, the x, y coordinates of the position vector $\mathbf{r}_1$ are

$$\mathbf{r}_1 = \begin{pmatrix} r_1 \cos\theta \\ r_1 \sin\theta \end{pmatrix} \tag{5}$$

and thus

$$\dot{\mathbf{r}}_1 = \begin{pmatrix} \dot{r}_1 \cos\theta - r_1 \dot{\theta} \sin\theta \\ \dot{r}_1 \sin\theta + r_1 \dot{\theta} \cos\theta \end{pmatrix}. \tag{6}$$

Now, differentiate equation (3) to get:

$$\dot{r}_1 = \frac{a_1 e (1-e^2) \dot{\theta} \sin\theta}{(1+e\cos\theta)^2} = \frac{e r_1^2 \dot{\theta} \sin\theta}{a_1(1-e^2)}. \tag{7}$$

Substituting this to the x component of $\dot{\mathbf{r}}_1$ yields

$$\begin{aligned} \dot{r}_{1x} &= \frac{e r_1^2 \dot{\theta} \sin\theta \cos\theta}{a_1(1-e^2)} - r_1 \dot{\theta} \sin\theta \\ &= \frac{r_1^2 \dot{\theta}}{a_1(1-e^2)} \left[e \sin\theta \cos\theta - \frac{a_1 \sin\theta (1-e^2)}{r_1} \right] = -\frac{r_1^2 \dot{\theta}}{a_1(1-e^2)} \sin\theta. \end{aligned} \tag{8}$$

Following through with a similar derivation for the y component, we have

$$\dot{\mathbf{r}}_1 = \frac{h_1}{M_s a_1 (1-e^2)} \begin{pmatrix} -\sin\theta \\ \cos\theta + e \end{pmatrix} \tag{9}$$

where $h_1 = M_s r_1^2 \dot{\theta}$ is the constant angular momentum of the star.

By using equation (2), we can write the angular momentum of the star as:

$$h_1 = \frac{M_s M_p^2}{(M_s + M_p)^2} r^2 \dot{\theta} = \frac{M_p}{M_s + M_p} \mu_{sp} r^2 \dot{\theta} = \frac{M_p}{M_s + M_p} h \tag{10}$$

where the reduced mass is:

$$\mu_{sp} = \frac{M_s M_p}{M_s + M_p}. \tag{11}$$

Now, the radial equation of motion for the combined planet-star system is

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G(M_s + M_p)}{r^2}. \tag{12}$$

We will make the substitution $r = 1/u$ and recognize that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = ku^2 \tag{13}$$

where $k = h/\mu_{sp}$. Thus we have

$$\dot{r} = -\frac{1}{u^2} \frac{\mathrm{d}u}{\mathrm{d}\theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} = -k \frac{\mathrm{d}u}{\mathrm{d}\theta} \tag{14}$$

and

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}\theta} \frac{\mathrm{d}\theta}{\mathrm{d}t} = -k^2 u^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}. \tag{15}$$

Using this in equation (12) yields the equation of motion for u:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{G(M_s + M_p)}{k^2}, \tag{16}$$

which has a solution

$$u = \frac{G(M_s + M_p)}{k^2} (1 + e \cos\theta) \to r = \frac{k^2}{G(M_s + M_p)(1 + e \cos\theta)}. \tag{17}$$

Going back to the dependence of r on θ for an ellipse (e.g., equation 3), we deduce by comparison that

$$k = \frac{h}{\mu_{sp}} = \sqrt{G(M_s + M_p) a (1 - e^2)} \tag{18}$$

and thus

$$h_1 = \sqrt{\frac{G M_p^4 M_s^2 a (1 - e^2)}{(M_s + M_p)^3}}. \tag{19}$$

Substituting this equation into equation (9) and using equation (2) to replace a_1 gives

$$\dot{\mathbf{r}}_1 = \sqrt{\frac{GM_p^2}{(M_s+M_p)a(1-e^2)}} \begin{pmatrix} -\sin\theta \\ \cos\theta+e \end{pmatrix}. \tag{20}$$

We now have to project velocity to the line of sight to the Earth. We use the same x, y coordinate system centered at the CM as before and define the z axis appropriately. We define the inclination i as the angle between the z axis and the line of sight. We also define the longitude of periapsis ω as the angle between the x axis and the direction perpendicular to a projection of the line of sight onto the orbital (x,y) line. In this system, the unit vector along the line of sight is

$$\mathbf{k} = \begin{pmatrix} \sin\omega\sin i \\ \cos\omega\sin i \\ \cos i \end{pmatrix} \tag{21}$$

and the projection of the stellar radial velocity vector onto the line of sight is

$$v_{r,1} = \dot{\mathbf{r}}_\mathbf{1} \cdot \mathbf{k} = \sqrt{\frac{G}{(M_s+M_p)a(1-e^2)}} M_p \sin i\,[\cos(\omega+\theta)+e\cos\omega] \tag{22}$$

where we used the trigonometric identity

$$\cos(\omega+\theta) = \cos\omega\cos\theta - \sin\theta\sin\omega. \tag{23}$$

Finally, the maximum and minimum projected radial velocities are

$$v_{r,\max} = \sqrt{\frac{G}{(1-e^2)}}(M_p\sin i)(M_s+M_p)^{-1/2}a^{-1/2}\,[1+e\cos\omega] \tag{24}$$

$$v_{r,\min} = \sqrt{\frac{G}{(1-e^2)}}(M_p\sin i)(M_s+M_p)^{-1/2}a^{-1/2}\,[e\cos\omega-1] \tag{25}$$

and thus, finally, the radial velocity semi-amplitude is

$$K_1 = \frac{1}{2}\left(v_{r,\max} - v_{r,\min}\right) = \sqrt{\frac{G}{(1-e^2)}}(M_p\sin i)(M_s+M_p)^{-1/2}a^{-1/2}. \tag{26}$$

Kepler's third law is used to obtain the semi-major axis from the measured period:

$$P^2 = \frac{4\pi}{G(M_s+M_p)}a^3. \tag{27}$$

If the orbit is circular with an inclination of 90°, we can write

$$K_1 \approx \sqrt{\frac{G}{M_s r}}M_p \to M_p = \sqrt{\frac{M_s r}{G}}K_1, \tag{28}$$

which we could have simply derived by combining

$$\begin{aligned} V_p &= \sqrt{\frac{GM_s}{r}} \quad (29) \\ M_pV_p &= M_sK_1. \quad (30) \end{aligned}$$

3 Transit Depth

Consider a spherical polar coordinate system with the polar axis along the line from the center of a transiting planet host star to Earth. In the absence of interstellar absorption, the flux $F_{s\nu}$ at frequency ν from the star at distance d incident on Earth can be written as:

$$F_{s\nu}(d) = 2\pi\left(\frac{R_s}{d}\right)^2 \int_0^1 I_\nu(\mu)\mu \mathrm{d}\mu \quad (31)$$

where R_s is the radius of the star, $I(\mu)$ is intensity and $\mu = \cos\theta$. If intensity is constant on the stellar disk, we have

$$F_{s\nu}(d) = \pi I_\nu \left(\frac{R_s}{d}\right)^2 = F_{s\nu}(R_s)\left(\frac{R_s}{d}\right)^2 \quad (32)$$

where $F_{s\nu}(R_s)$ is the flux at the stellar surface. The transmitted stellar flux during transit is:

$$F_{t\nu}(d) = 2\pi\left(\frac{R_s}{d}\right)^2 \int_0^1 I_\nu(\mu)\exp\left[-\tau_\nu(\mu)\right]\mu \mathrm{d}\mu \quad (33)$$

where $\tau_\nu(\mu)$ is the optical depth along the line of sight corresponding to μ. Assuming an orbital inclination of $i = 90^\circ$, constant intensity on the stellar disk, and an optically thick planet at $R_s \sin\theta \le R_p$ or

$$\mu_p \ge \sqrt{1 - \left(\frac{R_p}{R_s}\right)^2} \quad (34)$$

gives

$$F_{t\nu}(d) = 2\pi I_\nu \left(\frac{R_s}{d}\right)^2 \int_0^{\mu_p} \mu \mathrm{d}\mu = F_{s\nu}(R_s)\left(\frac{R_s}{d}\right)^2 \left[1 - \left(\frac{R_p}{R_s}\right)^2\right]. \quad (35)$$

Relative flux in transit is then simply given by

$$\frac{F_{t\nu}(d)}{F_{s\nu}(d)} = 1 - \left(\frac{R_p}{R_s}\right)^2 \quad (36)$$

and thus the transit depth is

$$\Delta_\nu = 1 - \frac{F_{t\nu}(d)}{F_{s\nu}(d)} = \left(\frac{R_p}{R_s}\right)^2. \tag{37}$$

This is just the ratio of the area of the disk of the planet to the area of the disk of the host star. Equations (31) and (33) are more general and can be used to calculate transit depths for the planetary atmosphere in the presence variations in stellar intensity such as limb darkening or active regions.

4 Pressure scale height and the hypsometric equation

The equation of hydrostatic equilibrium is:

$$\frac{\partial p}{\partial r} = -\rho(r)g(r) \tag{38}$$

where p is pressure, ρ is mass density and $g(r)$ is gravitational acceleration. Thus we have

$$\mathrm{d}\ln p = -\frac{\mathrm{d}r}{H(r)} \tag{39}$$

where the pressure scale height is

$$H(r) = \frac{kT(r)}{m(r)g(r)} \tag{40}$$

where $T(r)$ is temperature, $g(r)$ is gravity and $m(r)$ is mean molecular weight. Assuming that the scale height is constant with radius (altitude), we get pressure

$$p(r) = p(r_0)\exp\left(-\frac{r-r_0}{H}\right) \tag{41}$$

and the hypsometric equation for pressure level radii:

$$r - r_0 = H\ln\left(\frac{p_0}{p}\right). \tag{42}$$

From the last equation we see that the extent of the atmosphere and thus the variations in transit depth at different wavelengths depend on the scale height of the atmosphere. Giant planets with high temperature and low gravity produce the largest atmospheric signatures.

5 Secondary eclipse

In the case of uniform emission from the stellar and planetary disks, the secondary eclipse depth is simply given by

$$\Sigma_\nu = \frac{F_{p\nu}}{F_{s\nu}} \left(\frac{R_p}{R_s} \right)^2 \tag{43}$$

where F_p and F_s are the surface fluxes emitted by the planet and the star, respectively.

6 Exercise

Calculate the transit depth of the atmosphere of Trappist-1e in spectral lines for which the line of sight becomes optically thick at the 1 mbar level in the atmosphere, assuming a temperature of 273 K and a rough overall composition of 0.78 bar of N_2 and 0.22 bar of O_2 for the atmosphere. The mass and radius of Trappist-1e are 0.62 M_E and 0.918 R_E, respectively, giving a surface gravity of $g = 6.1$ m s^{-2}. Here we used $R_E = 6371$ km and $M_E = 5.972 \times 10^{24}$ kg. The mean molecular weight is

$$m \approx 0.78 \times 28 + 0.22 \times 32 = 28.9 \text{ amu} \tag{44}$$

and thus the pressure scale height is

$$H = \frac{kT}{mg} = 12.9 \text{ km}.$$

The hypsometric equation from above is

$$r - r_0 = H \ln \left(\frac{p_0}{p} \right) = 6.91H = 89 \text{ km}.$$

The transit depth can now be approximated from

$$\Delta = \left(\frac{r}{R_s} \right)^2 \approx \left(\frac{R_p + 89 \text{ km}}{R_s} \right)^2 = 0.532\%,$$

which is 1.03 times the transit depth of 0.516% that I obtain by using the published values for the radius and mass of the planet (Gillon et al., 2017). This means that the differential transit depth is of the order of 1.6×10^{-4}.

References

Gillon, M., et al., 2017. Seven temperate terrestrial planets around the nearby ultracool dwarf star TRAPPIST-1. Nature, 542, 456–460.