misc

<u>Schwarz</u> $[0,1] \xrightarrow{f} \mathbb{R}$ continuous, $f(x+h)+f(x-h)-2f(x) = o(h^2)$ for all $x \in (0,1)$ implies $f(x) = ax+b$ is affine.

<u>proof</u> wlog $f(0) = f(1) = 0$. let $g_\varepsilon(t) = f(t) - \varepsilon t(1-t)$. then $\frac{g(t+h)+g(t-h)-2g(t)}{h^2} \to 2\varepsilon$ for $t \in (0,1)$ meaning g_ε has maximum at 0 or 1. hence $g_\varepsilon \le 0$ for all ε and $f \le 0$. similarly $f \ge 0$. ■

<u>claim</u> $a_n = o(n)$ (weakly) increasing sequence of positive integers $\implies \dfrac{n}{a_n}$ contains all positive integers.

<u>proof</u> as ka_n has the same properties, it suffices to show $a_n = n$ has a solution. at $n = 1$ we have $\ge$, and at some point $<$. we cannot however have a flip from $>$ to $<$ as $a_n > n \implies a_{n+1} \ge n+1$. ■

<u>claim</u> a bounded harmonic function $g : \mathbb{R}^n \to \mathbb{R}$ is constant.

<u>proof</u> fix two points p, q. then $f(p) - f(q) = \dfrac{\int_{B_R(p)\Delta B_r(q)} \pm f}{\mathrm{Vol} B_R}$, but $\mathrm{Vol}(B_R(p)\Delta B_R(q)) = o(\mathrm{Vol} B_R)$ is negligble. ■

<u>claim</u> a bounded harmonic function $g : \mathbb{Z}^n \to \mathbb{R}$ is constant.

<u>proof</u> suppose not. let $r(x) = \Delta_{e_1} g(x) = g(x+e_1) - g(x)$. then $r(x) + r(x+e_1) + \cdots + r(x+ke_1)$ is uniformly bounded. wlog $s = \sup r(x) > 0$. fix x_ε with $r(x_\varepsilon) > s - \varepsilon$. since $r(x) \le s$ and $\frac{1}{2d}\sum_{x\sim y} r(x) = r(y)$, we get $g(x_\varepsilon + e_1) > s - 2d\varepsilon$. taking k big and ε small yields arbitrarily large $r(x_\varepsilon) + r(x_\varepsilon + e_1) + \cdots + r(x_\varepsilon + ke_1) > (k+1)s - \varepsilon(1 + 2d + \cdots + (2d)^k)$. ■