projections

<u>von Neumann, Halperin</u> let $M_1, \dots, M_k \leq H$ be closed subspaces. then $(\mathrm{Proj}_{M_1} \dots \mathrm{Proj}_{M_k})^n \xrightarrow{p} \mathrm{Proj}_{\cap M_j}$ pointwise.

<u>proof (Netyanun, Solomon)</u> write $T = \mathrm{Proj}_{M_1} \dots \mathrm{Proj}_{M_k}$, $M = \bigcap M_j$. we'll proceed in steps.

step i : $H = M \oplus \overline{\mathrm{img}(I-T)}$.

<u>proof</u> we have $M = \{\text{fixed pts of } T\} = \ker(I-T)$. indeed, $\subseteq$ is obvious and the other direction follows as each projection decreases the length of x if it isn't fixed. similarly $M = \ker(I - T^*) = \mathrm{img}(I-T)^{\perp}$ as T^* is T in reveresed order. □

step ii : $\|(I-T)y\|^2 \leq k\left(\|y\|^2 - \|Ty\|^2\right)$.

<u>proof</u> we have $\|(I-T)y\| \leq \sum \|S_j y - S_{j+1} y\|$ where $S_j = P_1 \dots P_j$. now $\|S_j y - S_{j+1} y\|^2 = \|S_j y\|^2 - \|S_{j+1} y\|^2$ by Pythagoras, so it remains to note that $\left(\sum_{j=1}^{k} a_j\right)^2 \leq k \sum_{j=1}^{k} a_j^2$ by Cauchy-Schwarz. □

step iii : $T^n x - T^{n+1} x \to 0$.

<u>proof (Kakutani)</u> $\|T^n x\|$ is decreasing and hence convergent. so we're done by letting $y = T^n x$ in step ii. □

the result is trivial if $x \in M$. step iii gives the result for $x \in \mathrm{img}(I-T) \subseteq M^{\perp}$, which one extends to $\overline{\mathrm{img}(I-T)} = M^{\perp}$. ■

the following is a linear analog of Banach's fixed point theorem

<u>von Neumann</u> let $\|A\| \leq 1$ be a contraction operator on H and $M = \{\text{fixed pts of } A\}$. then $A^n x \xrightarrow{a} \mathrm{Proj}_M(x)$.

<u>lemma</u> $M^{\perp} = \overline{\mathrm{img}(I-A)}$.

<u>proof</u> it suffices to show $M = \{\text{fixed pts of } A^*\} = \ker(I - A^*)$. but as $\|A\| = \|A^*\| \leq 1$, we need to show only one inclusion. for unit x we have $Ax = x \implies 1 = \langle Ax, x\rangle = \langle x, A^* x\rangle$ and $A^* x = x$ by Cauchy Schwarz. □

<u>proof</u> the result is trivial if $x \in M$. also if $x = (A-I)y \in \mathrm{img}(I-A)$ then $\frac{1}{n+1}\sum_{k=0}^{n} A^k x = \frac{A^{n+1}y - y}{n+1} \to 0$. one easily extends this to $x \in \overline{\mathrm{img}(I-A)} = M^{\perp}$ and is done by linearity. ■