weak convergence

question $a_n : \sum a_n b_n$ converges $\forall b \in \ell_2 \overset{?}{\implies} a \in \ell_2$.

weak convergence $x_n \xrightarrow{w} x : \langle x_n, y\rangle \to \langle x, y\rangle$ for all y.

observations i. weak limits are unique and linear. ii. $x_n \longrightarrow x \implies x_n \xrightarrow{w} x$. iii. $x_n \xrightarrow{w} x \implies x_n \longrightarrow x$ if $\dim H$ finite.

observation $(u_n)^\infty$ orthonormal $\implies u_n \xrightarrow{w} 0$ by Bessel.

exercise i. $x_n \xrightarrow{w} x \implies \|x\| \le \liminf \|x_n\|$. ii. $x_n \xrightarrow{w} x$ and $\|x_n\| \longrightarrow \|x\| \implies x_n \longrightarrow x$.

weak Cauchy : $\langle x_n, y\rangle$ converges for all y.

claim weak Cauchy $\implies$ bounded.

proof given x_n, let $C_n = \{y : \forall k |\langle x_k, y\rangle| \le n\}$. then C_n closed, $\bigcup C_n = H$. by Baire, $\mathrm{Ball}_r y_0 \subseteq C_{n_0}$ has nonempty interior. if x_m is nonzero, we get $\langle x_m, \frac{r x_m}{2\|x_m\|}\rangle$ is in absolute value at most $2n_0$, hence $\|x_m\| \le 4n_0/r$ is bounded. ■

claim x_n bounded, $\langle x_n, z\rangle$ converges for all z in a dense subset $\implies x_n$ weak Cauchy.

proof let $\|x_n\| \le M$. fix y, ε. find $\|y - z_0\| \le \frac{\varepsilon}{4M}$. so $\forall n, m \ge N_0$ we have $|\langle x_n - x_m, z_0\rangle| \le \varepsilon/2 \implies |\langle x_n - x_m, y\rangle| \le \varepsilon$. ■

claim weak Cauchy implies weak convergence.

proof $\lim\langle y, x_n\rangle$ is a well defined linear functional. it is bounded because x_n is. by Riesz, $\lim\langle y, x_n\rangle = \langle y, x\rangle$. ■

exercise conclude a positive answer to the above question.

claim $x_n \in H$ bounded $\implies \exists x_{n_k}$ weakly convergent.

proof assuming H separable : fix y_n dense. as $\langle y_1, x_n\rangle$ bounded, there is a convergent subsequence given by $x_{1,n}$. continue with $\langle y_2, x_{1,n}\rangle$ etc, we have $x_{m,n}$. let $x'_n = x_{n,n}$ denote the diagonal subsequence. so $\langle y_k, x'_n\rangle$ converges as its eventually a subsequence of $\langle y_j, x_{j,n}\rangle$. by the above claims, x'_n weakly convergent. □

exercise finish the nonseparable case using $H_0 = \mathrm{Clos}(\mathrm{Span}\{x_n\})$.

[Banach-Saks] $x_n \xrightarrow{w} x$ implies $\exists x_{n_k} \xrightarrow{a} x$, i.e. $\dfrac{x_{n_1} + \cdots + x_{n_k}}{k} \to x$.

proof wlog $x = 0$, $\|x_n\| \le M$. as $\langle x_n, y\rangle \longrightarrow 0$ for all y, we pick $x'_1 = x_1$ and inductively x'_n s.t. $|\langle x'_j, x'_n\rangle| \le \frac{1}{n-1}$ for $j = 1, \ldots, n-1$. we get $\left\|\sum_{j=1}^k x'_j\right\|^2 \le kM^2 + 2(\frac{1}{1} + \frac{2}{2} + \cdots + \frac{k-1}{k-1}) = o(k^2)$, i.e. $x'_k \xrightarrow{a} 0$. ■

corollary a closed convex set is closed under weak limits.

exercise C closed, bounded, convex, $C \xrightarrow{f} \mathbb{R}$ convex, bounded from below with $x_n \to x \implies f(x) \le \liminf f(x_n)$. then f assumes its minimum.

exercise two linear operators S, T on H with $\langle Tx, y\rangle = \langle x, Sy\rangle$ implies S, T bounded.

exercise in ℓ_2 we have $x_k \xrightarrow{w} x$ iff x_n bounded and $x_k \xrightarrow{p} x$ pointwise.