lamp problems

problem[1] n lamps in a row. initial random position. evolution : a lamp will be lit at time $t+1$ iff its two neighbors shined the same at time t. an edge lamp at time $t+1$ takes its neighbors state at time t.[2] determine the n for which no matter the starting states, the lamps will eventually all be lit forever.

solution[3] iff $n+1$ is a power of two.

restatement over F_2, where (on,off)$=(0,1)$, the evolution is summing (xoring) the neighbors. i.e. is linear and given by $A \in F_2^{n\times n}$, $A_{ij} = \delta_{|i-j|=1}$. the question reads : for which n is A nilpotent. we call $x \in F_2^n$ s.t. $\exists k : A^k x = \vec{0}$ terminating.

observation $\ker A = \{(1,0,1,0,\dots,0,1,0,1), \vec{0}\}$ if n odd and $\{\vec{0}\}$ if n even. in particular A invertible for n even.

observation given a length n vector $\vec{x}$, the length $2n+1$ symmetric vector $(\overleftarrow{x}, 0, \vec{x})$ [4] evolves to remain symmetric with 0 as its central term, with each side evolving as if it were unaffected by the other terms, just as a length n vector would.

observation as even n fail, it suffices to show n works $\iff$ $2n+1$ works.

$\Longleftarrow$ fix $\vec{x} \in F_2^n$. as any vector of length $2n+1$ terminates, $(\overleftarrow{x}, 0, \vec{x})$ terminates, meaning x does. $\square$

$\Longrightarrow$ fix $\vec{v} \in F_2^{2n+1}$. then $\vec{v} + \overleftarrow{v}$ is symmetric with 0 as the center term. hence terminating. i.e. $A^k(\vec{v} + \overleftarrow{v}) = \vec{0}$ and in other words $A^k \vec{v}$ is symmetric [as A and $\leftharpoonup$ commute]. hence $A^{k+1}\vec{v}$ is symmetric with 0 at its center, and so terminates. $\square$

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[2] e.g. (on,off,on,on)$\to$ (off,on,off,on)$\to$(on,on,on,off)$\to$(on,on,off,on)...

[3] this is my personal solution, there are others.

[4] where $\overleftarrow{x} = (x_n, \dots, x_1)$ if $x = (x_1, \dots, x_n)$ and a vector $(t_1, \dots, t_n)$ is symmetric if $t_i = t_j$ for $i + j = n+1$.