misc

<u>Schwarz</u> $[0,1] \xrightarrow{f} \mathbb{R}$ continuous, $f(x+h)+f(x-h)-2f(x)=o(h^2)$ for all $x \in (0,1)$ implies $f(x)=ax+b$ is affine.

<u>proof</u> wlog $f(0)=f(1)=0$. let $g_\varepsilon(t)=f(t)-\varepsilon t(1-t)$. then $\frac{g(t+h)+g(t-h)-2g(t)}{h^2} \to 2\varepsilon$ for $t \in (0,1)$ meaning g_ε has maximum at 0 or 1. hence $g_\varepsilon \leq 0$ for all ε and $f \leq 0$. similarly $f \geq 0$. ∎

<u>claim</u> $a_n = o(n)$ (weakly) increasing sequence of positive integers $\implies \dfrac{n}{a_n}$ contains all positive integers.

<u>proof</u> as ka_n has the same properties, it suffices to show $a_n = n$ has a solution. at $n=1$ we have $\geq$, and at some point $<$. we cannot however have a flip from $>$ to $<$ as $a_n > n \implies a_{n+1} \geq n+1$. ∎

<u>claim</u> a bounded harmonic function $g : \mathbb{R}^n \to \mathbb{R}$ is constant.

<u>proof</u> fix two points p, q. then $f(p)-f(q) = \dfrac{\int_{B_R(p)\Delta B_r(q)} \pm f}{\mathrm{Vol} B_R}$, but $\mathrm{Vol}(B_R(p)\Delta B_R(q)) = o(\mathrm{Vol} B_R)$ is negligble. ∎

<u>claim</u> a bounded harmonic function $g : \mathbb{Z}^n \to \mathbb{R}$ is constant.

<u>proof</u> suppose not. let $r(x) = \Delta_{e_1} g(x) = g(x+e_1)-g(x)$. then $r(x)+r(x+e_1)+\cdots+r(x+ke_1)$ is uniformly bounded. wlog $s = \sup r(x) > 0$. fix x_ε with $r(x_\varepsilon) > s-\varepsilon$. since $r(x) \leq s$ and $\frac{1}{2d}\sum_{x\sim y} r(x) = r(y)$, we get $g(x_\varepsilon+e_1) > s-2d\varepsilon$. taking k big and ε small yields arbitrarily large $r(x_\varepsilon)+r(x_\varepsilon+e_1)+\cdots+r(x_\varepsilon+ke_1) > (k+1)s-\varepsilon(1+2d+\cdots+(2d)^k)$. ∎

<u>claim</u> assume $\mathbb{Z} = \bigsqcup_{j\in[k]} a_j + d_j\mathbb{Z}$ is a nontrivial ($k \geq 2$) partition of the integers into arithmetic progressions. then the differences d_j are not all distinct.

<u>proof</u> wlog we have $\mathbb{Z}_{\geq 0} = \bigsqcup a_j + d_j \mathbb{Z}_{\geq 0}$. now $\dfrac{1}{1-x} = \sum x^n = \sum_j \sum_m x^{a_j+d_j m} = \sum_j \dfrac{x^{a_j}}{1-x^{d_j}}$. since there is no pole at $e^{2\pi i/\max d_j}$, the maximal d_j has to be the difference of at least two progressions. ∎