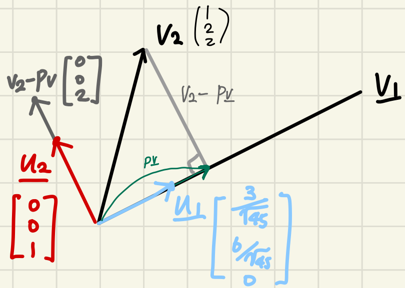
* Gram-Schmidt orthogonalization.

$$\underline{v_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \qquad \underline{v_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

↓

$$\underline{u_1} = \begin{bmatrix} \frac{3}{\sqrt{45}} \\ \frac{6}{\sqrt{45}} \\ 0 \end{bmatrix}$$



$$v_2 \cdot u_1 = \frac{15}{\sqrt{45}} \text{ (길이)}.$$

projection vector v_2 $= \frac{15}{\sqrt{45}} \begin{pmatrix} \frac{3}{\sqrt{45}} \\ \frac{6}{\sqrt{45}} \\ 0 \end{pmatrix}$

$\underline{u_1}$ (방향)

$$v_2 - \underline{pv} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

↓

$$\text{unit} = \underline{u_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

⇒ independent $\underline{v_1}$, $\underline{v_2}$ → $\underline{u_1}$, $\underline{u_2}$

orthonormal vectors.

same span.

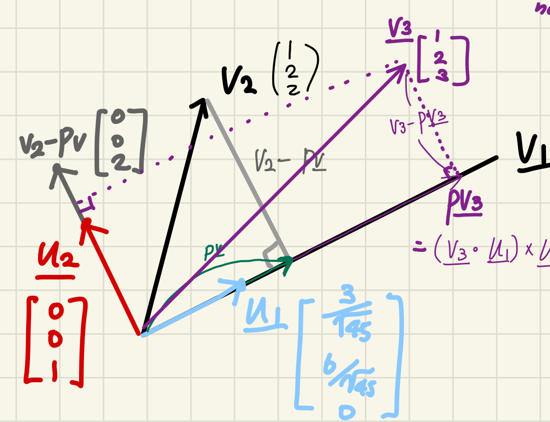
* v_3 $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\xrightarrow{\perp}$ $\underline{u_1}$ ①, $\xrightarrow{\perp}$ $\underline{u_2}$ ②

v_3 − ① − ②

= (new orthogonal)

normalize

= new orthonormal.



$= (\underline{v_3} \cdot \underline{u_1}) \times \underline{u_1}$

$\rightarrow$ original input matrix 복원.

$$\begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix} \Rightarrow \underset{\text{normalize}}{=} \begin{bmatrix} \frac{3}{\sqrt{45}} & 0 \\ \frac{6}{\sqrt{45}} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} & \boxed{4} \end{bmatrix}$$

$$\rightarrow \left[\boxed{1}\begin{bmatrix} \\ \\ \end{bmatrix} + \boxed{2}\begin{bmatrix} \\ \\ \end{bmatrix}, \quad \boxed{3}\begin{bmatrix} \\ \\ \end{bmatrix} + \boxed{4}\begin{bmatrix} \\ \\ \end{bmatrix} \right]$$

$$\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{45}} \\ \frac{6}{\sqrt{45}} \\ 0 \end{bmatrix} \times \sqrt{45} + \boxed{0}\begin{bmatrix} \\ \\ \end{bmatrix} \quad \therefore \boxed{1} = \sqrt{45} \quad \boxed{2} = 0$$

how do we made $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

$$= \text{normalize}\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\frac{15}{\sqrt{45}}\right) \begin{bmatrix} \frac{3}{\sqrt{45}} \\ \frac{6}{\sqrt{45}} \\ 0 \end{bmatrix} \right)$$

normalize with 2.

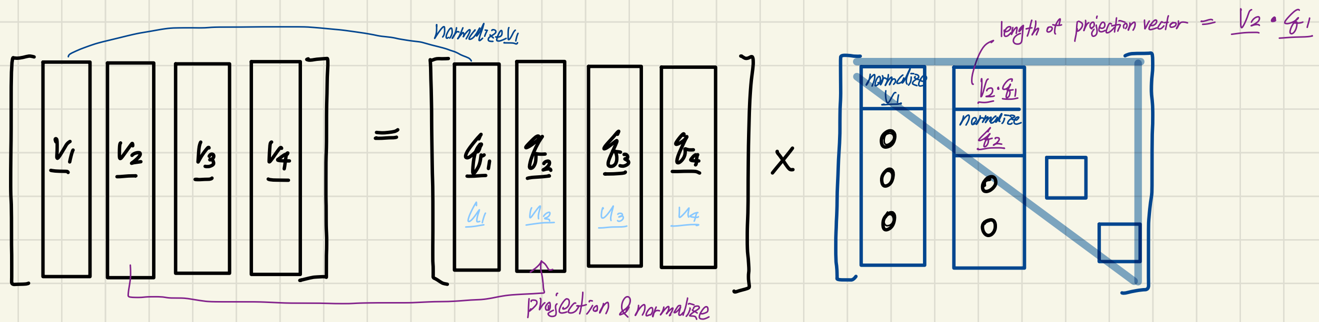
$$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{15}{\sqrt{45}} \times \begin{bmatrix} \frac{3}{\sqrt{45}} \\ \frac{6}{\sqrt{45}} \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{3} = \frac{15}{\sqrt{45}} \quad \boxed{4} = 2.$$

projection vec의 길이

normalize 정도.

⟹ * QR factorization.



Vectors into Orthonormal

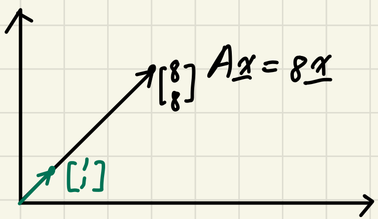
$$A \quad = \quad Q \quad \times \quad R$$

linearly independent Matrix   orthonormal matrix   triangular matrix.

* Eigen value & vector.

$A\underline{x} = \lambda \underline{x}$. , A is a square matrix.

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



* Computational advantage of eigenvector

$$\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$(2\times1 + 6\times1)$, $\quad 8\times1$,

$(5\times1 + 3\times1)$ $\quad 8\times1$.

* equation

$$(A - \lambda I)\underline{x} = \underline{0}$$

$$\boxed{A}\,\boxed{\underline{x}} - \boxed{\lambda}\,\boxed{\underline{x}} = 0$$

$$\underbrace{\left(\boxed{A} - \boxed{\lambda}\,\boxed{I}\right)}_{\text{linearly dependent}}\boxed{\underline{x}} = 0.$$

linearly dependent $\Rightarrow$ non-trivial solution.

$$\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} - 8I \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}. = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \underline{x} = \underline{0}.$$

$\underline{v_1}$ $\underline{v_2}$

dependent column.

$\therefore \underline{x} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any non zero scalar c,

* Null space

$$A\underline{x} = 0 \rightarrow \{\underline{x}\} = Nul(A)$$

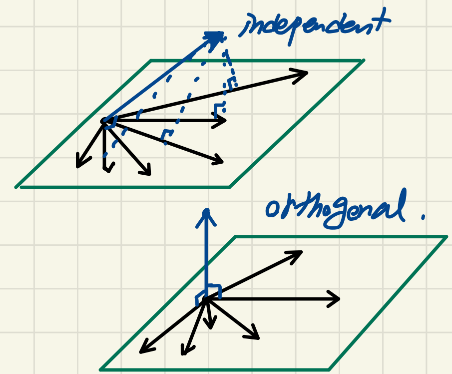
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ A's column $\underline{a_1}$, $\underline{a_2}$ is linearly independent.

∴ there's no non-triv solution.

$$\begin{bmatrix} x \\ y \end{bmatrix} \perp \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$



$$\Rightarrow A = \begin{bmatrix} \underline{a_1}^T \\ \vdots \\ \underline{a_m}^T \end{bmatrix}, \quad \underline{a_1}^T\underline{x} = 0, \ \underline{a_2}^T\underline{x} = 0, \cdots \underline{a_m}^T\underline{x} = 0.$$

* Orthogonal complement

여집합.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \dim(\text{Col } n + \text{Row } n^c.)$$

$\text{Col}(A) \in \mathbb{R}^3$

$\text{Row}(A) \in \mathbb{R}^2 = \text{Span of } \{[1\ 2], [3\ 4]\}$

$\mathbb{R}^3 \to 3 = \dim \text{Row}(A) + \dim \text{Nul}(A).$

$$\left[\begin{array}{l} \text{Nul}(A) = \text{Row}(A)^\perp \\ \text{Nul}(A^T) = \text{Col}(A)^\perp \end{array}\right.$$

* Null space vs eigen vectors.

$(A - \lambda I)\underline{x} = \underline{0}.$

eigen vectors $\in \text{Nul}(A - \lambda I)$

$$\lambda = 3.$$

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 3 \\ 3 & 6 & 12 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \underbrace{3\ \ 6\ \ 9}_{\text{dependent}} \end{bmatrix} \begin{matrix} \perp \\ \perp \\ \perp \end{matrix} \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} = \underline{0}.$$

$\Leftrightarrow \det(A - \lambda I) = 0$

(characteristic equation.

* characteristic equation ⇒ eigen vector.

$$\det\begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix} \neq 0.$$

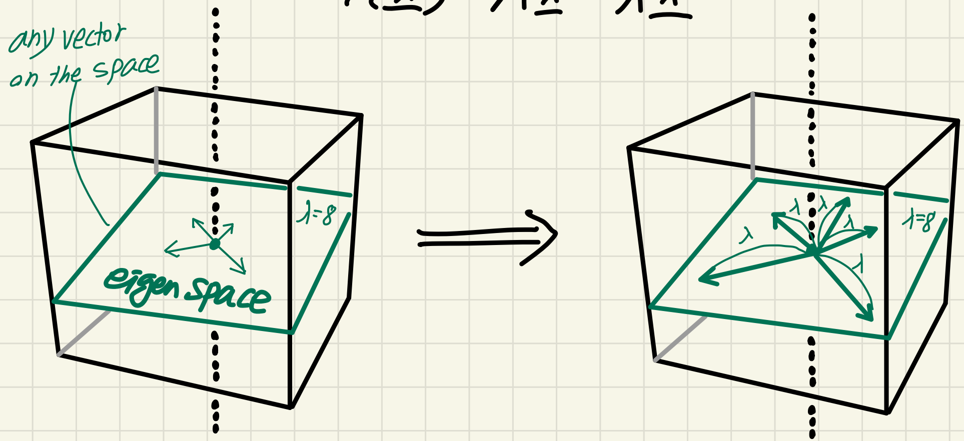
$$\det\begin{pmatrix} 2-\lambda & 6 \\ 5 & 3-\lambda \end{pmatrix} = 0$$

$$\rightarrow (2-\lambda)(3-\lambda) = 30.$$

$$\lambda = 8, -3.$$

$$\left[\begin{array}{l} (A-8I)\underline{x} = 0 \\ \\ (A+3I)\underline{x} = 0. \end{array}\right.$$

$$T(\underline{x}) = A\underline{x} = \lambda\underline{x}$$



multiplication by A.

* Diagonalization.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D = V^{-1} A V$$

diagonalizable matrix A.

- finding V and D.

$$D = V^{-1}AV \Rightarrow VD = AV$$

$$V = [\, \underline{V_1} \;\; \underline{V_2} \; \cdots \; \underline{V_n} \,]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\begin{array}{ccccccccc} AV & = & [\, A\underline{V_1} & A\underline{V_2} & A\underline{V_3} & \cdots & A\underline{V_n} \,] \\ \parallel & & \parallel & \parallel & \parallel & & \parallel \\ VD & = & [\, \lambda_1 \underline{V_1} & \lambda_2 \underline{V_2} & \lambda_3 \underline{V_3} & \cdots & \lambda_n \underline{V_n} \,] \end{array}$$

$\therefore$ $A\underline{V_1} = \lambda \underline{V_1}$, $A\underline{V_2} = \lambda \underline{V_2}$ $\cdots$ $A\underline{V_n} = \lambda_n \underline{V_n}$

eigen vector = $\underline{V_K}$, egien value = λ_K