* Eigen Decomposition.

if A is diagonalizable, $D = V^{-1}AV$

$A = V\underbrace{D}_{\text{diagonal}}V^{-1}$ diagonal $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

* arbitrary vector $\xrightarrow{?}$ linear combination of eigen vectors.

= what are coefficients of each eigen vector

$\downarrow$ how?

$A = V$ (eigen vectors) $\underline{x}$ $\quad \underline{x} = V^{-1}A$

$\begin{bmatrix} [v_1] & [v_2] \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

* linear transformation via eigen decomposition.

$T(\underline{x}) = A\underline{x}$, $\quad A = VDV^{-1}$

$A\underline{x} = VDV^{-1}\underline{x} = V(D(V^{-1}\underline{x}))$: $\quad \underline{x} \xrightarrow{T(x_1)} V^{-1}\underline{x} \xrightarrow{T(x_2)} D(V^{-1}\underline{x}) \xrightarrow{T(x_3)} V(D(V^{-1}\underline{x}))$

if $\underline{v}$ is eigen vector, $A\underline{v} = \lambda\underline{v}$, easy to calculate.

ex) $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$

not eigen vec. $\quad$ eigen vec $\underline{v_1}$ $\quad \underline{v_2}$

$\lambda_1\underline{v_1} \quad \lambda_2\underline{v_2}$

eigen vector의 선형결합으로 계산을 편리하게.

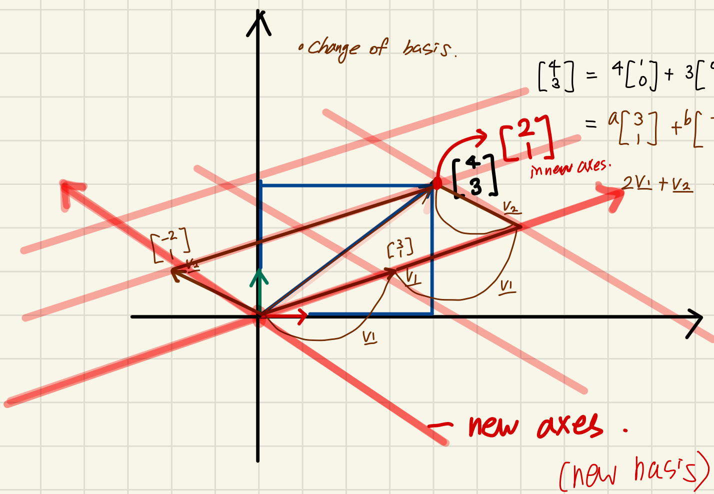
* Change of basis.

$$\mathrm{Span}\left\{\underbrace{\begin{bmatrix}-2\\1\end{bmatrix},\begin{bmatrix}-1\\5\end{bmatrix}}_{\text{basis}}\right\} = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

$\downarrow$

임의 벡터 $\begin{bmatrix}5\\7\end{bmatrix} = 5\cdot\begin{bmatrix}1\\0\end{bmatrix} + 7\begin{bmatrix}0\\1\end{bmatrix} = a\begin{bmatrix}-2\\1\end{bmatrix} + b\begin{bmatrix}-1\\5\end{bmatrix}.$

ex)



$\begin{bmatrix}4\\3\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$

$= a\begin{bmatrix}3\\1\end{bmatrix} + b\begin{bmatrix}-2\\1\end{bmatrix} = \overset{V}{\begin{bmatrix}3 & -2\\1 & 1\end{bmatrix}}\overset{\underline{x}}{\begin{bmatrix}a\\b\end{bmatrix}} = \begin{bmatrix}4\\3\end{bmatrix}$ / $\begin{bmatrix}a\\b\end{bmatrix} = V^{-1}\cdot\begin{bmatrix}4\\3\end{bmatrix}$

$2\underline{V_1} + \underline{V_2} \Rightarrow a=2 \quad b=1$ $\Rightarrow T(\underline{x}) : \begin{bmatrix}3 & -2\\1 & 1\end{bmatrix}$

$= \begin{bmatrix}6\\2\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}$ $\begin{bmatrix}a\\b\end{bmatrix} \to \begin{bmatrix}4\\3\end{bmatrix}$

↳ 통상적인 직교좌표계의 $\begin{bmatrix}4\\3\end{bmatrix}$ ⇒ basis를 $\begin{bmatrix}3\\1\end{bmatrix}$ $\begin{bmatrix}-2\\1\end{bmatrix}$로 바꿨을때의 $\begin{bmatrix}2\\1\end{bmatrix}$과 같다.

∘ Suppose $A\underline{V_1} = -1\underline{V_1}$, $A\underline{V_2} = 2\underline{V_2}$

$T(\underline{x}) = A\underline{x} = VDV^{-1}\underline{x} = V(D(V^{-1}\underline{x}))$

let $\underline{y} = V^{-1}\underline{x}$, $V\underline{y} = \underline{x}$

$$\underline{y} = \begin{bmatrix}a\\b\end{bmatrix} = V^{-1}\underline{x}$$

$$= [\mathrm{eigen}\,\underline{V_1},\ \mathrm{eigen}\,\underline{V_2}]^{-1}\cdot\begin{bmatrix}4\\3\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\end{bmatrix} \ (a=2, b=1)$$

* Element-wise Scaling

$$\text{diagonal}\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} = 1x_1 + 2x_2 + \cdots 5x_5$$

element wise

$$T(\underline{x}) = V(D(\underbrace{V^{-1}\underline{x}}_{\underline{y}})) = V\boxed{(D\underline{y})}$$

let $D\underline{y} = \underline{z}$, $\quad D\underline{y} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \overset{\underline{y}}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

λ_1 λ_2

element wise

$\because D = \text{diagonal}$.

$$\underline{y} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

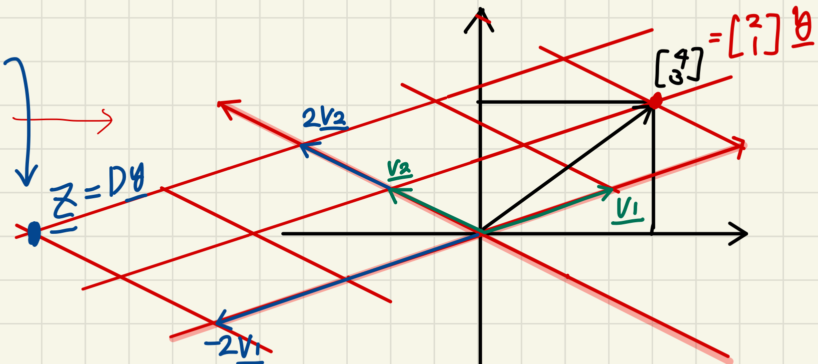
coefficient of linear combinations of eigen vector.

* Back to original basis.

$$V(D\underline{y}^{\underline{z}})$$

직교좌표계로 다시 변환하는 과정.

$$= [\underline{V_1} \quad \underline{V_2}]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underline{v_1}z_1 + \underline{v_2}z_2$$



$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \underline{z}.$$

* Overview of transformation using eigen decomposition.

$$
\begin{tikzcd}
\underline{x} \arrow[r, "A"] \arrow[d, "V^{-1}"'] & T(\underline{x}) = A\underline{x} = V(D(V^{-1}\underline{x})) \\
\underline{y} \arrow[r, "D"] & \underline{z} \arrow[u, "V"']
\end{tikzcd}
$$

* Linear trans via A^k

$$A = VDV^{-1}, \quad A^k = \underbrace{(VD\cancel{V^{-1}})(\cancel{V}D\cancel{V^{-1}})\cancel{\cdots}(\cancel{V}DV^{-1})}_{k}$$

$$= VD^kV^{-1}$$

$$D^k = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{bmatrix}^k$$

$$= \begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \lambda_k^k \end{bmatrix}$$

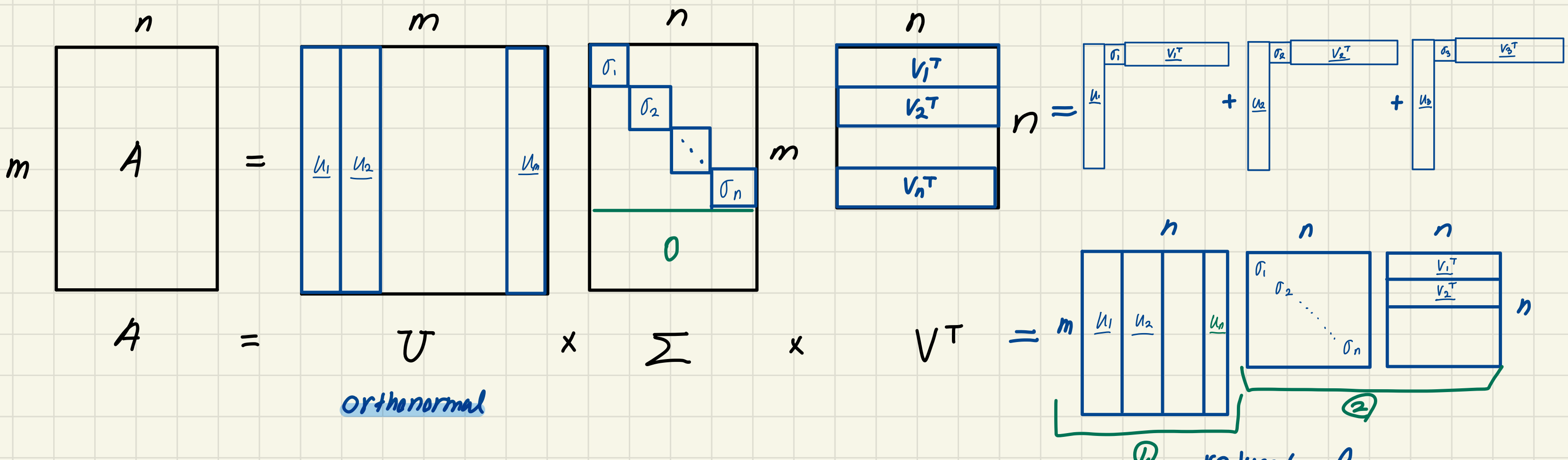
* Singular vector decomposition.

↓ Rectangular matrix $A = \mathbb{R}^{m \times n}$.

$A = U\Sigma V^T$ $\quad\quad$ $A^T = V\Sigma^T U^T$

> Basic form.

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$



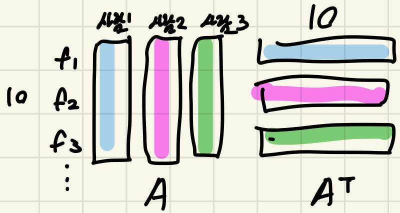
reduced - form

↳ original A = linear comb of ①, ②

= ①'s columns = basis of A.

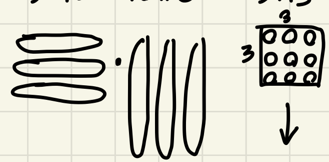
* Eigen decomposition in ML.

∨ feature by data-item.



inner product

① $A^TA = 3 \times 10 \cdot 10 \times 3 = 3 \times 3$

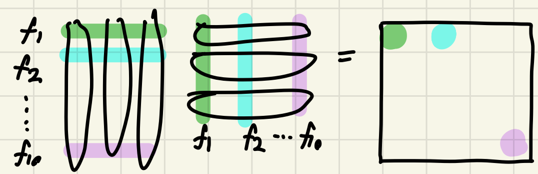


각 사람별 pairwise
similarity matrix (correlation).

vector similarity

outer product.

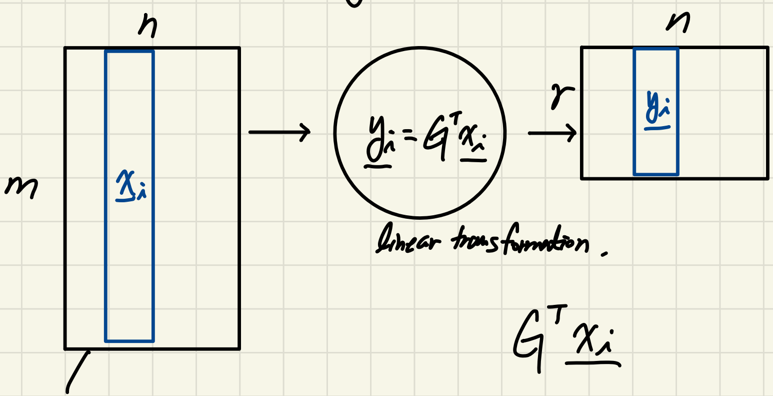
② $AA^T = 10 \times 3 \cdot 3 \times 10 = 10 \times 10$.



feature 간 similarity

- covariance matrix in principal component analysis (PCA)
- gram matrix in style transfer.

* Dimension - reducing transformation.



feature by data-item

* 차원을 축소하더라도
= 여전히) similarity를 가장 잘
보존하는 matrix.

$G^T \underline{x_i}$

$$X = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i \underline{u_i}\,\underline{v_i}^T$$

$$\hat{G} = U_r = [\underline{u_1}\ \underline{u_2}\ \cdots\ \underline{u_r}].$$

$\Uparrow$

find linear transformation $G^T \underline{x_i}$

$G \in \mathbb{R}^{m \times r}$ are orthonormal, best peserves the pairwise similarity

$$Y^T Y = (G^T X)^T G^T X = X^T G G^T X \leftarrow \hat{G} = \arg\min_{G} \| S - X^T G G^T X \|_F \text{ subject to } G^T G = I_k.$$