* Gauss - Jordan Elimination.

= 연립일차방정식의 풀이법!

$$\begin{pmatrix}1&2\\2&5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}$$

$\downarrow$ 확장행렬

$$\left(\begin{array}{cc|c}1&2&4\\2&5&9\end{array}\right)\Rightarrow\left(\begin{array}{cc|c}2&4&8\\2&5&9\end{array}\right)\uparrow\ominus$$

$$\Rightarrow\left(\begin{array}{cc|c}1&2&4\\0&1&1\end{array}\right)\Rightarrow\left(\begin{array}{cc|c}1&2&4\\0&2&2\end{array}\right)\downarrow\ominus$$

$$\Rightarrow\left(\begin{array}{cc|c}1&0&2\\0&1&1\end{array}\right)\quad\therefore\ x=2,\ y=1.$$

항등

$$\because\ I_2\times\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}.$$

* inverse matrix with gauss elimination.

$$A\times A^{-1}=I$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1&x_2\\y_1&y_2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\Rightarrow\left[\begin{array}{l}\left(\begin{array}{cc|c}a&b&1\\c&d&0\end{array}\right),\\ \left(\begin{array}{cc|c}a&b&0\\c&d&1\end{array}\right).\end{array}\right.$$

$$=\left(\begin{array}{cc|cc}a&b&1&0\\c&d&0&1\end{array}\right)\Rightarrow\left(\begin{array}{cc|cc}a&b&1&0\\0&d-\frac{bc}{a}&-\frac{c}{a}&1\end{array}\right)\times\frac{a}{ad-bc}.$$

$$\Rightarrow\left(\begin{array}{cc|cc}a&b&1&0\\0&1&\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{array}\right)$$

$$\Rightarrow\underbrace{\left(\begin{array}{cc|cc}1&0&\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\0&1&\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{array}\right)}_{A^{-1}}.$$

* determinant

(determinant) $\Rightarrow$ (2x2) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow \det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

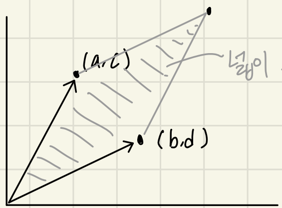
→ 각각의 minor 행렬과 곱해주는 형태.

$$\begin{vmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = + a_{11} \times (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) - \boxed{a_{12}} \times (a_{21} \cdot a_{33} - a_{23} \cdot a_{31}) + a_{13} \times (a_{21} \cdot a_{32} - a_{22} \cdot a_{31})$$

$-$

$+$

$\vdots$



넓이 = $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

① $\det(A) = 0 \Leftrightarrow A$ is singular.

② A is rank-deficient $\Leftrightarrow \det(A) = 0$

③ for diagonal matrix, $\det(A)$ = 대각성분

④ triangular matrix, $\det(A)$ = 〃

⑤ $\det(AB) = \det(A) \cdot \det(B)$.

⑥ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

⑦ $\det(A) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \lambda_n$.

* trace ?

정사각행렬에 대해서만 정의.

$$tr(A) = \sum_{i=1}^{n} a_{ii} \quad \text{(대각성분)}$$

$$= \text{scalar}$$

- $tr(A+B) = tr(A) + tr(B)$.
- $tr(A^T) = tr(A)$
- $tr(AB) = tr(BA)$
- $tr(a^T b) = tr(b\, a^T)$

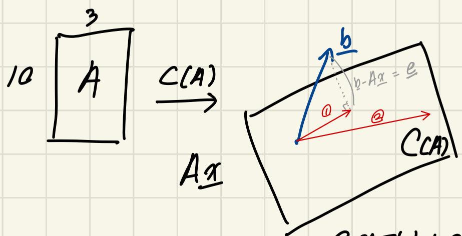
* $tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$.

cyclic property.

ABCD
BCDA
CDAB
DABC

* Least squares (최소자승법)

<problem> : A의 column space에서 span 할 수 없는 vec b



$A\underline{x} = \underline{b}$ 의 해는 존재X

$\Downarrow$

최대한 가까운 $\underline{x}$ 값을 찾자.

e의 길이가 가장 작은 $\underline{x}$.

$= \|e\|^2_{2norm}$ 을 줄이자. = 수직해야한다.

= $A\underline{x}$ 와 $\underline{b}-A\underline{x}$ 의 dot product $=0$인 $\hat{x}$.

$$(b-A\hat{x})^T \cdot A\hat{x} = 0$$

$$\underbrace{(b^TA-\hat{x}^TA^TA)}_{=0.}\cdot\hat{x} = 0$$

$$(b^TA)^T = (\hat{x}^TA^TA)^T$$

$$\Leftrightarrow A^Tb = \underbrace{\overset{(3\times10)\cdot(10\times3)}{A^TA}}_{\text{normal equation.}}\hat{x}$$

$A^TA = 3\times3$, $rank(A^TA) = rank(A)$.

$\therefore A^TA$ is invertible.

$$\Rightarrow \hat{x} = (A^TA)^{-1}A^T\cdot b \longrightarrow A\hat{x} = A\cdot(A^TA)^{-1}\cdot A^T\cdot \underline{b}$$

$\underline{x}$: 알고싶은 녀석.

사용되는 예시.

$\underline{z}$ (measurement) $= A\underline{x} + \underline{w}$ //

* eigen value & eigen vector

$\overset{m\times n}{A}\,\overset{n\times 1}{\underline{v}}$ $\overset{m\times 1}{=}$ $\lambda \times \overset{n\times 1}{\underline{v}}$ $\therefore m=n.$

- $A\underline{v} = \lambda \underline{v}$ (A는 정사각행렬)

λ = eigen value, $\underline{v}$ = eigen vector

$\underline{v} \to \boxed{A} \to A\underline{v}$

function

= linear transformation.

방향이 바뀌지 않는 벡터와 값을 찾자.

(정사각행렬 A이 선형변환 후에도)

$A\underline{v} - \lambda\underline{v} \Rightarrow \underbrace{(A - \lambda\cdot I)}_{=0}\underline{v} = 0$

$\therefore$ $\det(A-\lambda I) = 0$ 을 만족하는 λ.

$N(A-\lambda I)$에 존재하는 $\underline{v}$

중 basis를 eigen vec으로 대표 삼는다.

(independents 의 집합).

①
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)^2 - 1 = 0. \quad \lambda = 1, 3.$$

eigen value

eigen vector

- $\lambda = 1 \to \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\cdot \underline{v} = 0 \quad (N(A-\lambda I)) \quad \therefore \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3 \to \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\cdot \underline{v} = 0, \quad \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

* eigen decomposition.

A : 2x2, eigen value : 2개, (λ_1, λ_2)
eigen vector : 2개, $(\underline{v}_1, \underline{v}_2)$

$A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$

$$\underbrace{A[\underline{v}_1, \underline{v}_2]}_{AV} = [\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2]$$

$$= \underbrace{[\underline{v}_1, \underline{v}_2]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}$$

$$\therefore A = V \Lambda V^{-1}$$

(eigen decomposition)

diagonalizable matrix = independent eigen vector n개.
(nxn) matrix

① A^k calculate $= V \cdot \Lambda \cancel{V^{-1}} \cdot \cancel{V} \Lambda V^{-1} \cdot \cdots$

② $A^{-1} = (V\Lambda V^{-1})^{-1} = V\Lambda^{-1} V^{-1}$

③ $\det(A) = \det(V\Lambda V^{-1})$

$= \cancel{\det(V)} \cdot \det(\Lambda) \cdot \cancel{\det(V^{-1})}$

$\det(V^{-1}) = \frac{1}{\det(V)}$

$= \lambda_1 \cdot \lambda_2 \cdots = \prod^{n} \lambda_n$

④ rank-deficient $\Leftrightarrow \det(A) = 0$
$\Leftrightarrow$ 0인 eigenvalue가 하나이상 존재.

⑤ A^T's eigen value = A's eigenvalue.

⑥ if A is orthogonal, $\lambda_i = \pm 1$

*⑦ Diagonalizable matrix 의 non-zero eigen value 의 수 = rank(A)

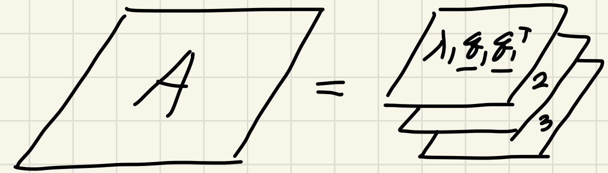
Symmetric matrix $\rightarrow$ diagonalizable

$(A^T = A) \longrightarrow A = V \Lambda V^{-1}$

$A^T = V^{-T} \Lambda V^T, \quad A = A^T$

$\therefore \quad V = V^{-T}, \quad V^{-1} = V^T$

$$\Rightarrow A = \underset{\text{orthogonal.}}{Q} \Lambda Q^T$$



$$A = \begin{bmatrix} \underline{q_1} & \underline{q_2} & \underline{q_3} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} \underline{q_1^T} \\ \underline{q_2^T} \\ \underline{q_3^T} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \underline{q_1}, & \lambda_2 \underline{q_2} & \lambda_3 \underline{q_3} \end{bmatrix}$$

$$= \lambda_1 \underbrace{\underline{q_1}\,\underline{q_1}^T}_{\text{rank1}} + \lambda_2 \cdot \underbrace{\underline{q_2}\,\underline{q_2}^T}_{\text{rank1}} + \lambda_3 \underbrace{\underline{q_3}\,\underline{q_3}^T}_{\text{rank1}}$$

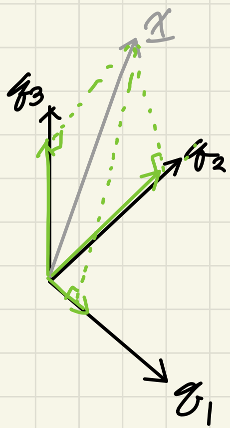
$$\lambda_1 \times \boxed{\text{rank1}} \oplus \lambda_2 \times \boxed{\text{rank1}} \oplus \lambda_3 \times \boxed{\text{rank1}}$$

$A = A^T$, $\quad A = \lambda_1 \cdot \underline{q}_1 \, \underline{q}_1^T + \lambda_2 \cdot \underline{q}_2 \cdot \underline{q}_2^T + \lambda_3 \cdot \underline{q}_3 \cdot \underline{q}_3^T$, $\quad 3 \times 3$ / $q_1 \perp q_2 \perp q_3$.

dot product

$$\underline{x} \longrightarrow \boxed{A} \longrightarrow A\underline{x} = (\lambda_1 \cdot \underbrace{\underline{q}_1 \boxed{\underline{q}_1^T) \cdot \underline{x}}}_{\text{projection된 vector}} + (\lambda_2 \cdot \underline{q}_2 \boxed{\underline{q}_2^T) \cdot \underline{x}} + (\lambda_3 \, \underline{q}_3 \boxed{\underline{q}_3^T) \underline{x}}$$

projection된 vector.



ex) $\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \underset{q_1^T \underline{x}}{\boxed{1}} \times \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{q_1} + \underset{q_2^T \underline{x}}{\boxed{2}} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{q_2} + \underset{q_3^T x_3}{\boxed{3}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{q_3}$

λ_1, λ_2, λ_3 만큼 늘려주기. $= A\underline{x}$