* linear equation.

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\to \begin{bmatrix} a_1 & a_2 & a_3 & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$\underline{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$\underline{a}^T\underline{x} = b$ (dot-product)

* linear system.

= 연립방정식

$$60x_1 + 5.5x_2 + 1\cdot x_3 = 66$$
$$65x_1 + 5.0x_2 + 0\cdot x_3 = 74$$
$$55x_1 + 6.0x_2 + 1\cdot x_3 = 78$$

$$\Rightarrow \begin{array}{c} \underline{a_1^T} \\ \underline{a_2^T} \\ \underline{a_3^T} \end{array} \overset{A}{\begin{bmatrix} \boxed{60 \quad 5.5 \quad 1} \\ \boxed{65 \quad 5.0 \quad 0} \\ \boxed{55 \quad 6.0 \quad 1} \end{bmatrix}} \overset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{\underline{b}}{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}$$

$$\underline{a_1} = \begin{bmatrix} 60 \\ 5.5 \\ 1 \end{bmatrix}, \quad \underline{a_2} = \begin{bmatrix} 65 \\ 5.0 \\ 0 \end{bmatrix} \quad \underline{a_3} = \begin{bmatrix} 55 \\ 6.0 \\ 1 \end{bmatrix} \Rightarrow A\underline{x} = \underline{b}, \quad \underline{a_1}^T\underline{x} = 66, \; a_2^T x = 74, \; a_3^T x = 78.$$

* identity matrix

is square matrix, whose diagonal entries are all "1".

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 .$$

* inverse matrix

squme matrix A, $A^{-1} \cdot A = A \cdot A^{-1} = I$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* why inlease matix is useful?

$$A \underline{x} = \underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b}$$

* determinant.

> invertible or not?

2x2 matrix → $\underbrace{ad-bc \neq 0}_{\det(A)}$.

↓

more than 2? : complicate.

↓

gaussian elimination.

> non - invertible

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$ ⇒ 항등식의 형태

or no solution.

* Rectangular matrix.

↓

least square "! 근사적으로 w와 해찾기.

* linear combination

Given vectors $\underline{v_1}, \underline{v_2}, \cdots \underline{v_p}$

Scalars $c_1, c_2, \cdots c_p$

$c_1 \times \begin{bmatrix} v_1 \end{bmatrix} + c_2 \begin{bmatrix} v_2 \end{bmatrix} + \cdots c_p \begin{bmatrix} v_p \end{bmatrix} : c_1 \underline{v_1} + c_2 \underline{v_2} + \cdots c_p \underline{v_p}$

= linear combination

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

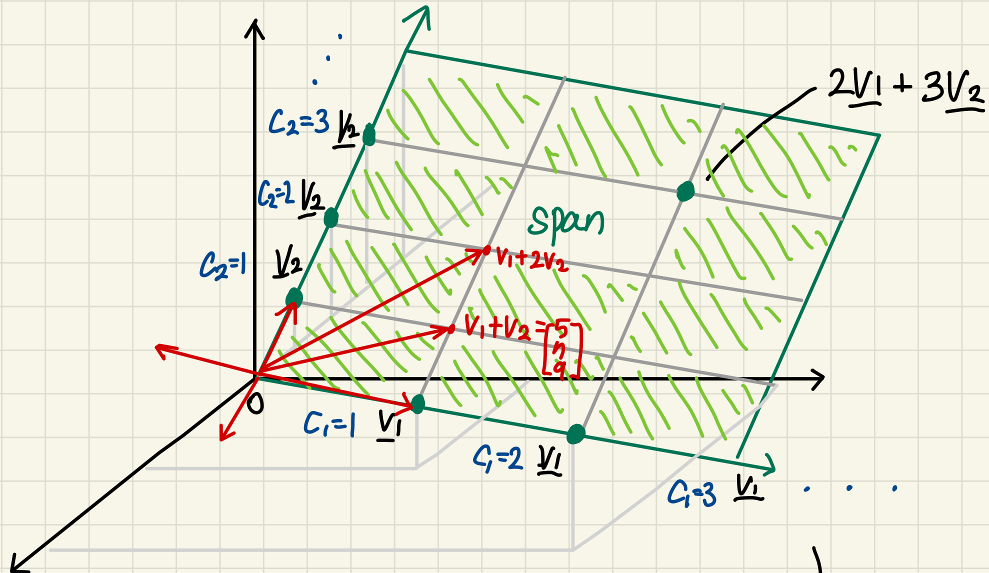
$$= \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

vector equation: $\underline{a_1} x_1 + \underline{a_2} x_2 + \underline{a_3} x_3 = \underline{b}$

* span.

$$\underbrace{c_1 \underset{\underline{v_1}}{\begin{bmatrix}1\\2\\3\end{bmatrix}} + c_2 \underset{\underline{v_2}}{\begin{bmatrix}4\\5\\6\end{bmatrix}}}_{\text{SPAN}}$$

SPAN.



* 유한한 벡터의 선형결합으로
만들어 낼 수 있는 공간

3차원공간
2개의 벡터
↓
2차원의 span.

* Vector equation with span.

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

미지수의 개수 = 주어진 벡터의 개수. (span)

span.

방정식의 개수 = 벡터가 속한 전체 차원

original dimension.

$$a_1 \underline{x_1} + a_2 \underline{x_2} + a_3 \underline{x_3} = \underline{b}$$

↳ solution exists when $b \in \mathrm{Span}\{a_1, a_2, a_3\}$

* multiplications of matrix as linear combination of vectors.

$$\underbrace{\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix}}_{\text{vector}} \cdot \underbrace{x_1}_{\text{weight}} + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

- Multi-colums.

$$\overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = [\underline{x} \ \underline{y}]$$

$$\Rightarrow \underline{x} = \underline{a_1} \cdot b_{11} + \underline{a_2} \cdot b_{21} + \underline{a_3} \cdot b_{31} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underline{y} = \underline{a_1} \cdot b_{12} + \underline{a_2} \cdot b_{22} + \underline{a_3} \cdot b_{23} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

* Matrix multiplications as Row Combinations.

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \boxed{1 \quad 1 \quad 1} \\ \boxed{1 \quad 0 \quad -1} \\ \boxed{0 \quad 1 \quad 1} \end{bmatrix} = \begin{matrix} 1 \times [1 \ 1 \ 0] \\ + 2 \times [1 \ 0 \ -1] \\ + 3 \times [0 \ 1 \ 1] \end{matrix}$$

$$(A \cdot \underline{x})^T \qquad \underline{x}^T \cdot A^T$$

- multiple rows

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \underline{x}^T \\ \underline{y}^T \end{bmatrix}$$

$$\underline{x}^T = 1 \cdot [1 \ 1 \ 0] + 2 \cdot [1 \ 0 \ 1] + 3 \cdot [1 \ -1 \ 1]$$

$$\underline{y}^T = 1 \cdot [1 \ 1 \ 0] + 0 \cdot [1 \ 0 \ 1] + -1 [1 \ -1 \ 1]$$

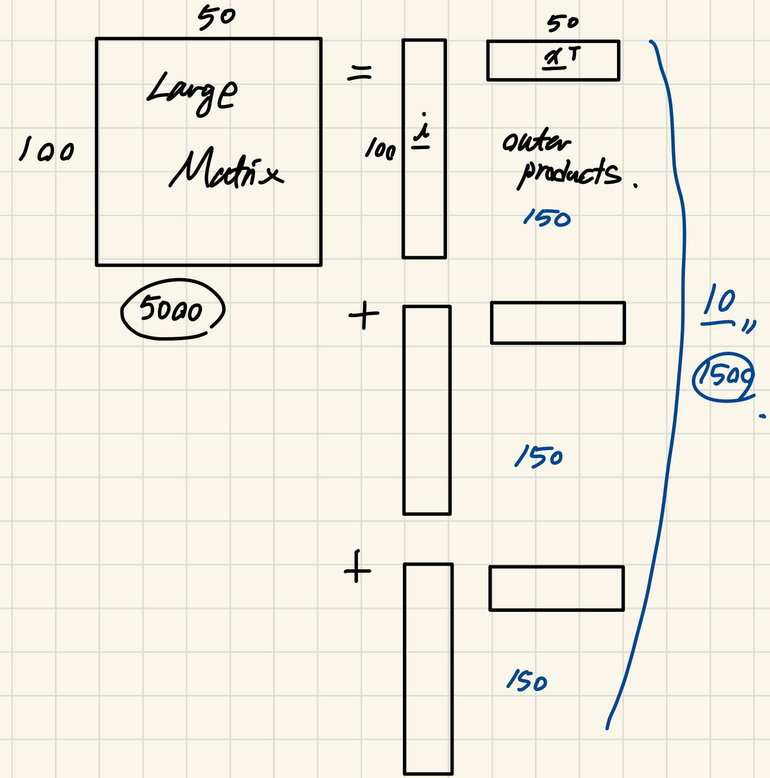
* Matrix multiplications as Sum of (rank-1) Outer product.

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\underline{a}} \times \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{\underline{b}^T} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

outerproduct

low rank factorization

- Large Matrix를 근사적으로 표현



- Sum of outer-product

$$\begin{bmatrix} \underline{a} & \underline{c} \end{bmatrix} \cdot \begin{bmatrix} \underline{b}^T \\ \underline{d}^T \end{bmatrix} = \underline{a}\underline{b}^T + \underline{c}\underline{d}^T$$

↓

- Word-embedding
- principle component analysis
- singular vector decomposition.

⋮

* Linear Independence

> Uniqueness of solution $A\underline{x} = \underline{b}$

think of linear combination of vectors.

$$\underline{a_1}x_1 + \underline{a_2}x_2 + \underline{a_3}x_3 = \underline{b}$$

x_1, x_2, x_3 is unique when

$\underline{a_1}$, $\underline{a_2}$, $\underline{a_3}$ are linearly independent.

• formal definition.

Consider $x_1\underline{v_1} + x_2\underline{v_2} + \cdots x_p\underline{v_p} = \underline{0}$

trivial solution : $\underline{x} = \underline{0}$.

if $v_1 \sim v_p$ are independent,

$\underline{x} = \underline{0}$ is the only solution.

elif $v_1 \sim v_p$ are dependent,

at least one x_i is non zero.

* Subspace vs span.

("닫혀있다"의 개념.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \right\}$$

$$0 \cdot [\;] + 0 [\;] = \in$$ 닫혀있다.

linear combination

- A subspace H is defined as a subset of $\mathbb{R}^n$ closed under linear combination.

($\underline{u_1}, \underline{u_2} \in H$, $c, d \Rightarrow c\underline{u_1} + d\underline{u_2} \in H$.

✓ span $\{\underline{v_1}, \cdots \underline{v_p}\}$ is always a subspace. why?

$$\text{span} \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{v_2} \right\}$$

ex) $2 \cdot \underline{v_1} + 1 \cdot \underline{v_2} = \begin{matrix} 2 \\ 4 \\ 6 \end{matrix} \;\; \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \;\; \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix} \underline{v_3}.$

$-1 v_1 + 1 v_2 = \begin{matrix} -1 \\ -2 \\ -3 \end{matrix} \;\; \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \;\; \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \underline{v_4}.$

↓ $-1 \cdot \underline{v_3} + 2 \cdot v_4 = -1(2v_1 + v_2) + 2(-v_1 + v_2)$

$-4\underline{v_1} + \underline{v_2}$ ".

결국 $\underline{v_1}$, $\underline{v_2}$ 의 선형결합꼴.

* Basis of a Subspace.

- fully spans the given subspace H.
- linearly independent.

기저벡터의 선형결합으로 span 내의 vec을
표현할때 coefficient 값이 unique (independent).

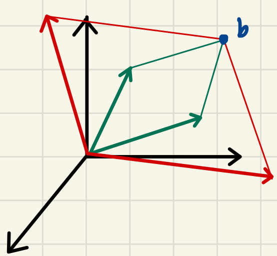
* Rank of matrix.

- rank A = dim colum space (A).

= column 들이 span하는 subspace의 dim

= basis 의 개수.

* non uniqueness of Basis.



<unique>
dimension of subspace

= number of basis.

* Linear Transformation.

= matrix as a function!

- $x_1 = 1$, $x_2 = 2$

$4x_1 + 5x_2 \to \boxed{T : x \mapsto 3x} \to 42$

$x_1 = 1 \to \boxed{T : x \mapsto 3x} \to 3$

$x_2 = 2 \to \boxed{T : x \mapsto 3x} \to 6$

$42 = \Rightarrow 4 \cdot T(x_1) + 5 \cdot T(x_2) = 42.$

if $3x+2$ → different (bias term include)

not linear trans

$T(\underline{x}) = A\underline{x}$ for all $\underline{x} \in \mathbb{R}^n$,

$$A = [\, T(\underline{e_1}) \;\; T(\underline{e_2}) \cdots T(\underline{e_n}) \,]$$

(e_j = jth Column of Identity matrix in $\mathbb{R}^{n \times n}$).

* matrix of Linear Trans.

- Suppose T is a linear trans from $\mathbb{R}^2$ to $\mathbb{R}^3$.

$$T\left(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1 \text{ of } 2\times2}\right) = \begin{bmatrix}2\\-1\\1\end{bmatrix}, \quad T\left(\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2 \text{ of } 2\times2}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

find a formula $T(\underline{x})$, $\mathbb{R}^2$

$$\Rightarrow \underline{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1 \cdot \begin{bmatrix}1\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\end{bmatrix}.$$

$$\Rightarrow T(\underline{x}) = T\left(x_1 \cdot \begin{bmatrix}1\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= x_1 \cdot \underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}_{\begin{bmatrix}2\\-1\\1\end{bmatrix}} + x_2 \cdot \underbrace{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}_{\begin{bmatrix}0\\1\\2\end{bmatrix}}.$$

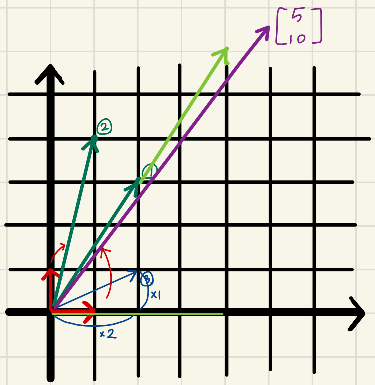
$$= \begin{bmatrix}2 & 0\\-1 & 1\\1 & 2\end{bmatrix} \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$T(e_1)$ $T(e_2)$

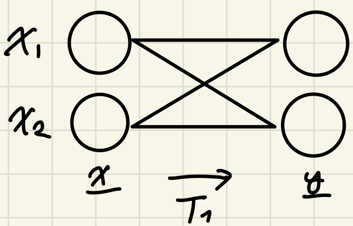
* linear transformation with Neural Network

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$\mathbb{R}^2$



fully connected layer



if $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ ①

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}$ ②

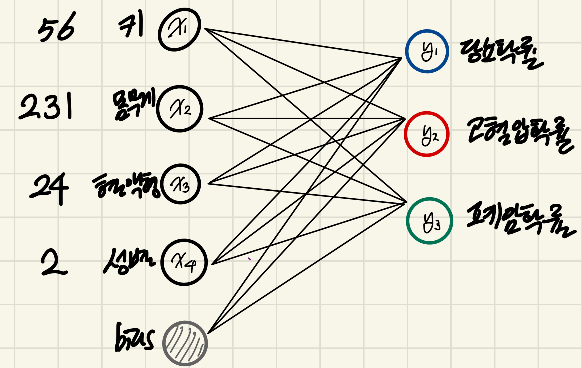
$$\begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \begin{matrix}2\\1\end{matrix}$$

$$③ \begin{bmatrix}2\\1\end{bmatrix} = 2\cdot\begin{bmatrix}1\\0\end{bmatrix} + 1\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

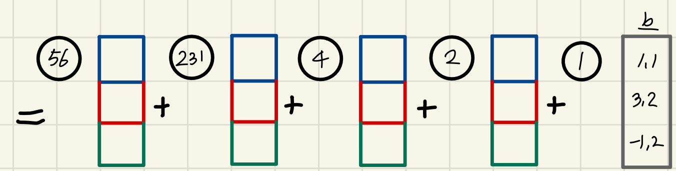
$$2\cdot\begin{bmatrix}2\\3\end{bmatrix} + 1\cdot\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix}$$

• Affine layer in NN.

fully connected layer with bias term.



$$\Rightarrow \underset{W}{\begin{bmatrix}0.2 & -0.5 & 0.1 & 2\\1.5 & 1.3 & 2.1 & 1\\-0.2 & 0.3 & 0.7 & -1.3\end{bmatrix}} \cdot \underset{\underline{x}}{\begin{bmatrix}56\\231\\4\\2\end{bmatrix}} + \underset{\underline{b}}{\begin{bmatrix}1.1\\3.2\\-1.2\end{bmatrix}} = \underset{\underline{y}}{\begin{bmatrix}-46.8\\439.9\\71.1\end{bmatrix}}$$



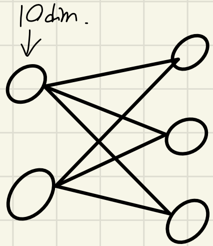
* onto and one-to-one

onto: 공역 = 치역

one-to-one: 일대일 (하나의 x당 하나의 y짝)

$T: \mathbb{R}^3 \to \mathbb{R}^2$, 공역∈치역. (전사)

think of neural net



output

640×480×3

Decoder.

$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

input dim < output dim

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$: one-to-one (X).

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\8\end{bmatrix}$$

$$T: \underline{y} = A\underline{x} = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}.$$

$$\begin{bmatrix}1 & 3 & 5\\2 & 4 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix} \Rightarrow A\underline{x} = \underline{b}$$

$b \in \mathrm{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\8\end{bmatrix}\right\}$

$\underline{x}$ unique? : $\begin{bmatrix}1 & 3 & 5\\2 & 4 & 8\end{bmatrix}$ ⟨ dependent? // independent?