

## PROBLEM 2

$$r(t) = u(t) e^{j 2\pi f t} + w(t)$$

where  $f$  is the frequency shift and  $w(t)$  is WGN with PSD No.

$$|u(t)|^2 = P > 0$$

$$\text{Matched filter } z = \frac{1}{T} \int_0^T u(t)^* r(t) dt$$

(a) MF response as  $z = x + v$  [x should have a sinc function]

$$\begin{aligned} z &= \frac{1}{T} \int_0^T u(t) [u(t) e^{j 2\pi f t} + w(t)] dt = \frac{1}{T} \int_0^T u(t) e^{j 2\pi f t} dt + \frac{1}{T} \int_0^T u(t) w(t) dt \\ &= \frac{P}{T} \int_0^T e^{j 2\pi f t} dt + \frac{1}{T} \int_0^T u(t) w(t) dt \\ &\quad \downarrow \\ &= \frac{P}{T} \frac{1}{2\pi i f} e^{j 2\pi f T} \Big|_0^T = \frac{P}{T} \frac{e^{j 2\pi f T} - 1}{2\pi i f} = \frac{P}{T} e^{j \pi f T} \frac{(e^{j \pi f T} - e^{-j \pi f T})}{2\pi i f} \\ &= P e^{j \pi f T} \frac{\sin(\pi f T)}{\pi f T} = P e^{j \pi f T} \operatorname{sinc}(\pi f T) \end{aligned}$$

$$\left\{ \begin{array}{l} x = P e^{j \pi f T} \operatorname{sinc}(\pi f T) \\ v = \frac{1}{T} \int_0^T u(t) w(t) dt \rightarrow \text{Gaussian RV } \sim \mathcal{CN}\left(0, \frac{N_o P}{T}\right) \end{array} \right.$$

$$(b) \text{SNR} = \frac{|x|^2}{E[|v|^2]} = \frac{P^2 \operatorname{sinc}^2(\pi f T)}{\frac{P N_o}{T}} = \frac{T P \operatorname{sinc}^2(\pi f T)}{N_o} \quad //$$

$$\begin{aligned} E[|V|^2] &= E[V V^*] = \frac{1}{T^2} E \left[ \int_0^T u(t) w(t) dt \int_0^T u(t)^* w(t)^* dt \right] = \\ &= \frac{1}{T^2} E \left[ \int_0^T \int_0^T u(t) w(t)^* \underbrace{w(t)^* w(t)}_{N_o} dt dt' \right] = \frac{1}{T^2} \int_0^T \int_0^T u(t) w(t)^* N_o \delta(t-t') dt dt' = \\ &= \frac{N_o}{T^2} \int_0^T |u(t)|^2 dt = \frac{N_o P T}{T^2} = P N_o \end{aligned}$$



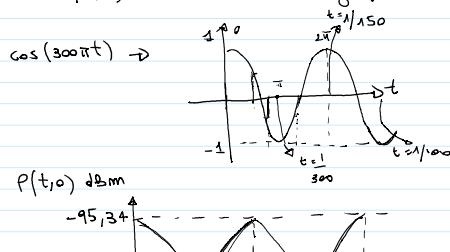
(d) 

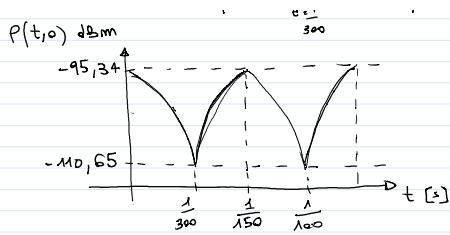
## PROBLEM 3 [SEE ALSO MATLAB CODE FOR THIS PROBLEM]

$$\begin{aligned} f_{d1} &= 100 \text{ Hz} & P_1 &= -100 \text{ dBm} \\ f_{d2} &= -50 \text{ Hz} & P_2 &= -103 \text{ dBm} \Rightarrow |h_m|^2 = 2|h_2|^2 \Rightarrow h_1 = \sqrt{2} h_2 \end{aligned}$$

$$\begin{aligned} P(t, 0) &= |h_1 e^{j 2\pi f_{d1} t} + h_2 e^{j 2\pi f_{d2} t}|^2 \\ &= |h_2 e^{j 2\pi 100t} + h_2 e^{-j 2\pi 50t}|^2 = |h_2|^2 |h_2 e^{j 2\pi 100t} + e^{-j 2\pi 50t}|^2 \\ &= |h_2|^2 (h_2 e^{j 2\pi 100t} + e^{-j 2\pi 50t})(h_2 e^{-j 2\pi 100t} + e^{j 2\pi 50t}) \\ &= |h_2|^2 (2 + \sqrt{2} e^{j 2\pi 150t} + \sqrt{2} e^{-j 2\pi 150t} + 1) \\ &= |h_2|^2 (3 + 2\sqrt{2} \cos(2\pi 150t)) \end{aligned}$$

$$\Rightarrow P(t, 0) = -103 \text{ dBm} + 10 \log_{10}(3 + 2\sqrt{2} \cos(2\pi 150t))$$





(b) The average received power is:

$$P_{\text{avg}} = -103 + 10 \log_{10}(3) = -98.23 \text{ dBm}$$

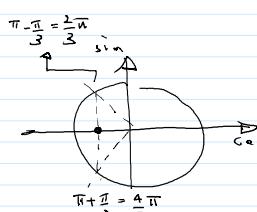
(c) The time it takes for going from the maximum to the minimum is:

$$\Delta t = \frac{1}{300} - 0 = \frac{1}{300} \approx 3.33 \text{ ms}$$

(d)  $10 \log_{10} (3 + 2\sqrt{2} \cos(2\pi 150t)) > 2$

$$3 + 2\sqrt{2} \cos(2\pi 150t) > 10^{0.2}$$

$$\cos(2\pi 150t) > \frac{10^{0.2} - 3}{2\sqrt{2}} = -\frac{1}{2}$$



The fraction of time the power is greater than -101 dBm is:

$$f = \frac{\frac{4\pi}{3}}{2\pi} = \frac{2}{3} \approx 67\% \text{ of the time}$$

#### PROBLEM 4

(a)  $\rho(\tau)$  is the relative change in  $h(t)$  over a time  $\tau$

$$\rho(\tau) = \frac{E |h(t) - h(t+\tau)|^2}{E |h(t)|^2} \quad R(\tau) = E [h(t)h^*(t-\tau)]$$

$$\begin{aligned} \rho(\tau) &= \frac{E [(h(t) - h(t+\tau))(h(t) - h(t+\tau))^*]}{E [h(t)h^*(t)]} \\ &= \frac{E [h(t)h^*(t) - h(t)h^*(t+\tau) - h(t+\tau)h^*(t) + h(t+\tau)h^*(t+\tau)]}{E [h(t)h^*(t)]} \\ &= \frac{R(0) - R(-\tau) - R^*(-\tau) + R(0)}{R(0)} = 2 - \underbrace{\frac{R(-\tau) + R^*(-\tau)}{R(0)}}_{2\operatorname{Re}[R(-\tau)]} \\ &= 2 - \frac{2\operatorname{Re}[R(-\tau)]}{R(0)} \end{aligned}$$

(b)  $h(t)$  follows a Jakes' spectrum with uniform angular distribution.

$R(\tau) = R(0) J_0(2\pi f_{\max} \tau)$ , where  $J_0(\cdot)$  is Bessel function of the first kind.

Plot  $R(\tau)/R(0)$  vs  $\tau f_{\max}$ , for  $\tau f_{\max} \in [0, 5]$

[MATLAB CODE)

(c)

$$R(-\tau) = R(0) J_0(-2\bar{u} f_{\max} \tau)$$

$$\rho(\tau) = \frac{2 - 2 R(0) J_0(-2\bar{u} f_{\max} \tau)}{R(0)}$$

$$= 2 - 2 J_0(-2\bar{u} f_{\max} \tau) = 2 [1 - J_0(-2\bar{u} f_{\max} \tau)]$$

$$\rho(\tau) = 10\% = 0,1 \Rightarrow 0,1 = 2 [1 - J_0(-2\bar{u} f_{\max} \tau)] \Rightarrow 0,05 = 1 - J_0(-2\bar{u} f_{\max} \tau)$$

$$\Rightarrow J_0(-2\bar{u} f_{\max} \tau) = J_0(2\bar{u} f_{\max} \tau) = 0,95 \Rightarrow \tau = 357,5357$$

The time it takes for the channel to change 10% is:  $\approx 358 \mu s$

$\approx$

PROBLEM 5

(a)

$$p(\tau_l) = \frac{1}{\lambda} e^{-\tau_l/\lambda}$$

$$R(\Delta f) = E[H(t, f) H^*(t, f - \Delta f)]$$

$$= E\left[\frac{1}{\sqrt{L}} \sum_{l=1}^L g_l e^{2\pi i(f t - \tau_l f)} \overline{\frac{1}{\sqrt{L}} \sum_{l=1}^L g_l^* e^{-2\pi i(f t - \tau_l f - \Delta f)}}\right]$$

$$= E\left[\frac{1}{L} \sum_{l=1}^L |g_l|^2 e^{-2\pi i \tau_l \Delta f}\right] = \frac{1}{L} \sum_{l=1}^L E[|g_l|^2] E[e^{-2\pi i \tau_l \Delta f}] =$$

$$= \frac{1}{L} \sum_{l=1}^L G \cdot E[e^{-2\pi i \tau_l \Delta f}] \quad (*)$$

$$E[e^{-2\pi i \tau_l \Delta f}] = E[e^{i \tau_l \xi_l}] = \frac{1}{1 - i \lambda \xi_l} =$$

$$\xi_l = -2\bar{u} \Delta f$$

↑ characteristic function of an exponential R.V.

$$= \frac{1}{1 - i \lambda (-2\bar{u} \Delta f)} = \frac{1}{1 + 2\pi i \lambda \Delta f}$$

$$(*) R(\Delta f) = \frac{1}{L} \sum_{l=1}^L \frac{G}{1 + 2\pi i \lambda \Delta f} = \frac{G}{1 + 2\pi i \lambda \Delta f}$$

(b)

$$\rightarrow E |H(t, f) - H(t, f + w)|^2 = E [(H(t, f) - H(t, f + w))(H^*(t, f) - H^*(t, f + w))]$$

$$= E [H(t, f) H^*(t, f) - H(t, f) H^*(t, f + w) - H(t, f + w) H^*(t, f) + H(t, f + w) H^*(t, f + w)]$$

$$= R(0) - R(w) - R^*(w) + R(0) = 2R(0) - 2 \operatorname{Re}[R(w)]$$

$$\rightarrow E |H(t, f)|^2 = E [H(t, f) H^*(t, f)] = R(0)$$

.. . . . .

$$\rightarrow E |H(t, f)|^2 = E [H(t, f) H^*(t, f)] = R(0)$$

$$\rightarrow R(\Delta f) = \frac{G}{1 + 2\pi n \Delta f \lambda} \frac{1 - 2\pi n \Delta f \lambda}{(1 - 2\pi n \Delta f \lambda)} = \frac{G(1 - 2\pi n \Delta f \lambda)}{1 + 4\pi^2 \Delta f^2 \lambda^2}$$

$$Re[R(\Delta f)] = \frac{G}{1 + 4\pi^2 \Delta f^2 \lambda^2}$$

$$\rightarrow 2R(0) - 2Re[R(w)] \leq \frac{1}{2}R(0)$$

$$\frac{3}{2}R(0) - 2Re[R(w)] \leq 0$$

$$\frac{3}{2}G - \frac{2G}{1 + 4\pi^2 w^2 \lambda^2} \leq 0 \Rightarrow 3G - \frac{4G}{1 + 4\pi^2 w^2 \lambda^2} \leq 0$$

$$3G + 12G\pi^2 w^2 \lambda^2 - 4G \leq 0 \Rightarrow 12G\pi^2 w^2 \lambda^2 \leq G$$

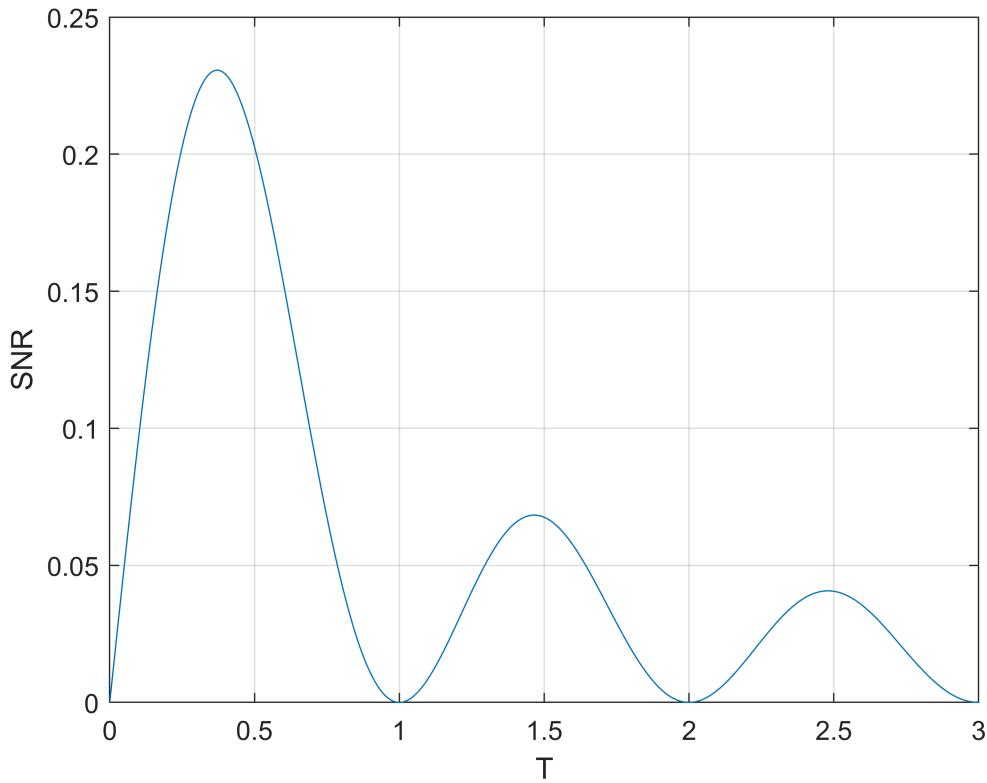
$$w^2 \leq \frac{G}{12G\pi^2 \lambda^2} = \frac{1}{12\pi^2 \lambda^2} \Rightarrow w = \frac{1}{\sqrt{12\pi^2 \lambda^2}} = 918,88 \text{ kHz}$$

$$|w| \leq 918,88 \text{ kHz}$$

Problem 2

(c)

```
npoints = 10000;
t_values = linspace(0,3,npontos);
p_N0 = 1;
f_offset = 1;
f_snr = @(t,f_off,p_0) sinc(f_off*t).^2.*t*p_0;
snr = f_snr(t_values,f_offset,p_N0);
plot(t_values,snr); grid on;
xlabel("T"); ylabel("SNR");
```



```
[snr_max, idx_max] = max(snr);
Tmax = t_values(idx_max);
fprintf(1, 'max snr = %.2f\n', snr_max);
```

max snr = 0.23

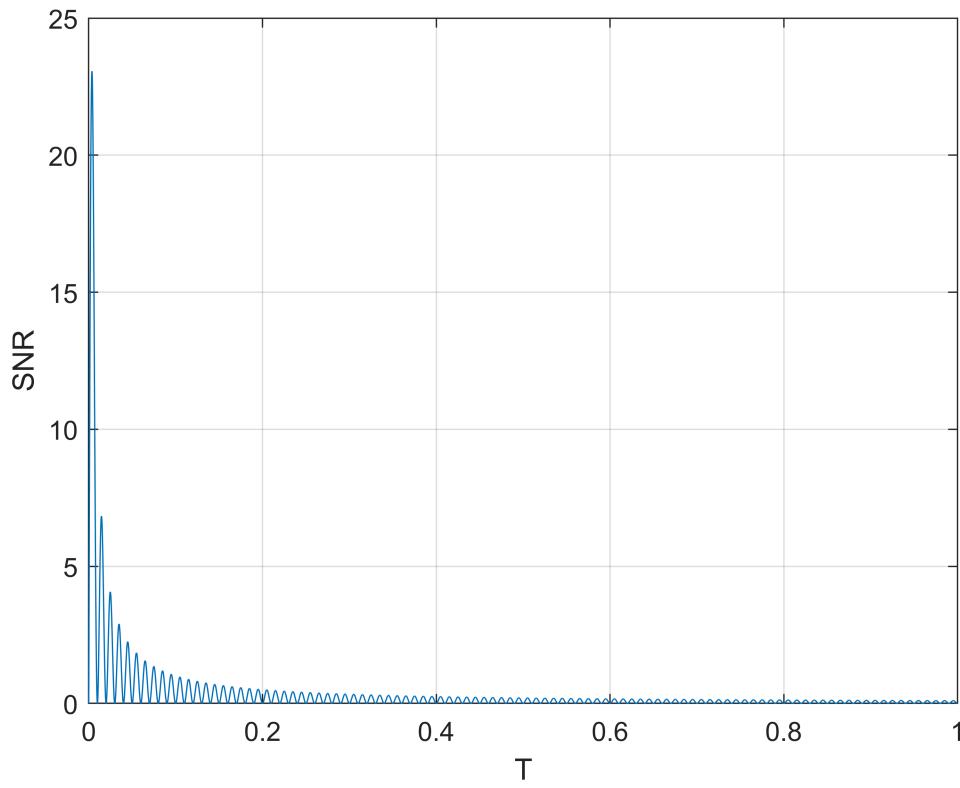
```
fprintf(1, 'optimal integration time = %f\n', Tmax);
```

```
optimal integration time = 0.371137
```

```
% Expression to evaluate other values for P/N0 and (offset) f;  
f_snr_max = @(f_offset,p_N0) sinc(f_offset*Tmax)^2*Tmax*p_N0;
```

(d)

```
t_values = linspace(0,1,npolyt);  
P = -100; % dBm  
Plin = db2pow(P-30); % convert to dBW, then to linear  
N0 = -140;  
N0lin = db2pow(N0-30);  
P_overN0 = Plin/N0lin;  
new_offset = 100;  
snr_2 = f_snr(t_values,new_offset,P_overN0);  
plot(t_values,snr_2); grid on;  
xlabel("T"); ylabel("SNR");
```



```
[snr_max2, idx_max2] = max(snr_2);  
Tmax2 = t_values(idx_max2);  
fprintf(1, 'max snr = %.2f\n', snr_max2);
```

```
max snr = 23.06
```

```

fprintf(1, 'max snr [dB] = %.2f\n', pow2db(snr_max2));

max snr [dB] = 13.63

fprintf(1, 'optimal integration time [ms] = %f\n', Tmax2*1e3);

optimal integration time [ms] = 3.700370

```

### Problem 3

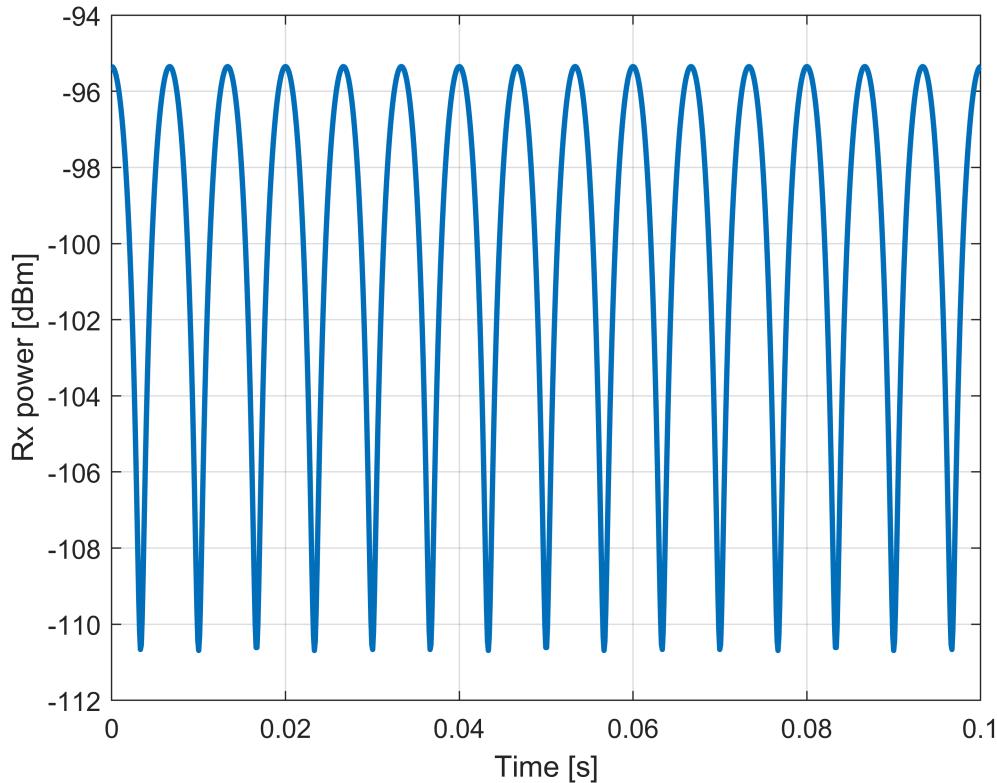
```

RxP = [-100, -103];
RxPlin = db2pow(RxP-30);
dopp = [100, -50];
t = linspace(0,0.1,1000)';
phase = 2*pi*t*dopp;
gaindB = [0,-3];

hpow = db2pow(gaindB);
h = sqrt(hpow);

H = exp(1i*phase).*h;
Pow_t = -100+ 10*log10(abs(sum(H,2)).^2);
figure;
plot(t,Pow_t,'LineWidth',2); grid on;
xlabel("Time [s]"); ylabel("Rx power [dBm]")

```



```

fprintf(1, 'max rx power [dBm] = %.2f\n', max(Pow_t));
max rx power [dBm] = -95.35

fprintf(1, 'min rx power [dBm] = %.2f\n', min(Pow_t));
min rx power [dBm] = -110.69

[min_v,idx] = min(Pow_t);
t_min = t(Pow_t < min_v+1);
fprintf(1, 'tmin [s] = %f\n', t_min(1));

tmin [s] = 0.003203

fprintf(1, 'avg power [dBm] = %.2f\n', mean(Pow_t));
avg power [dBm] = -100.00

frac = sum(Pow_t >= -101)/length(Pow_t);
fprintf(1, 'frac power > -101 dBm = %f\n', frac*100);

frac power > -101 dBm = 66.400000

```

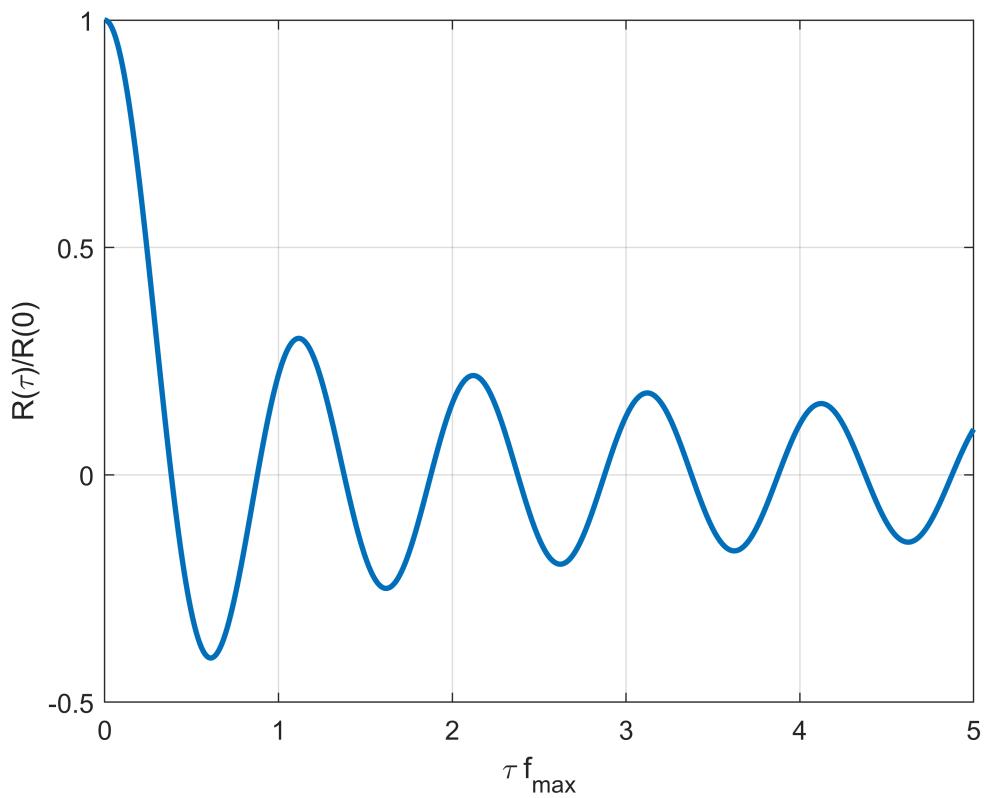
#### Problem 4

(b) and (c )

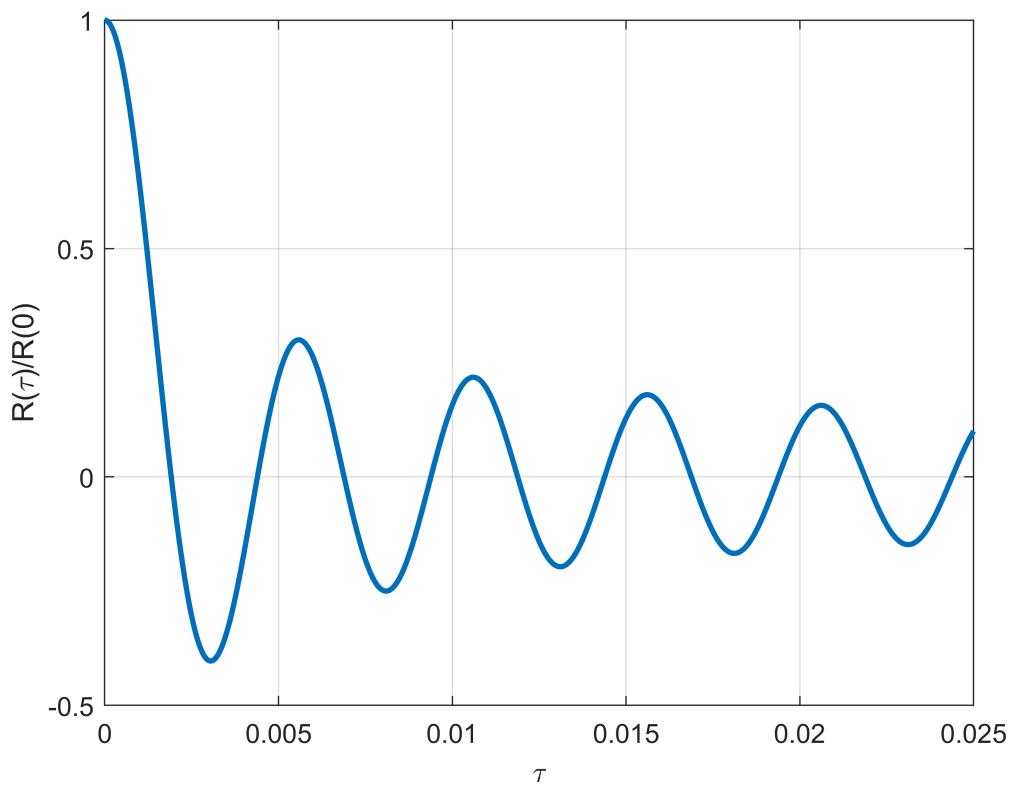
```

npoints = 10000;
tau_fmax = linspace(0,5,npolynts);
R_tau_over_0 = besselj(0,tau_fmax*2*pi);
figure;
plot(tau_fmax, R_tau_over_0, "LineWidth", 2);
grid on; xlabel('\tau f_{max}'); ylabel('R(\tau)/R(0)');

```



```
figure;
fmax = 200;
J0 = besselj(0,-tau_fmax*2*pi);
plot(tau_fmax/fmax, J0, "LineWidth",2);
grid on; xlabel('\tau'); ylabel('R(\tau)/R(0)');
```



```
tau = tau_fmax/fmax;
tau_v = tau(J0 >= 0.95);
fprintf(1, 'time it takes for the channel to change 10 perc [us] = %f\n', tau_v(end)*1e6);
```

```
time it takes for the channel to change 10 perc [us] = 357.535754
```