HW5 Tommy Azzino

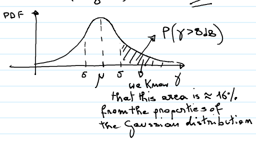
Saturday, March 27, 2021 6:23 PM

PROBLEM 1

(a) we have that $\gamma\,[dB] \sim \mathcal{N}(\mu, \sigma^2)$ where $\mu = 6dB$, $\sigma = 2dB$.

$P(\gamma > 8)$?

$$P(\gamma > 8) = P(\underset{8dB}{\gamma > \gamma_0}) = \int_{\gamma_0}^{\infty} \underbrace{f_\gamma(\gamma)}_{\text{pdf of this R.V.}}\, d\gamma = Q\left(\frac{\gamma_0 - \mu}{\sigma}\right) = 0{,}1587$$



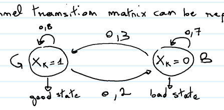
(b)

The probability the transmission will pass with MCS 2 is:

$$P(\gamma > 4 \mid \gamma \le 8) = \frac{P(\gamma > 4, \gamma \le 8)}{P(\gamma \le 8)} = \frac{P(\gamma > 4, \gamma \le 8)}{1 - P(\gamma > 8)} = \frac{P(\gamma > 4) - P(\gamma > 8)}{1 - P(\gamma > 8)} = 0{,}8114$$

PROBLEM 2

The channel transition matrix can be represented like this:



$$P = \begin{array}{c|cc} & G & B \\ \hline G & 0{,}8 & 0{,}2 \\ B & 0{,}3 & 0{,}7 \end{array}$$

$P(Y_k = 1 \mid X_k = 1) = 0{,}8 \Rightarrow P(Y_k = 0 \mid X_k = 1) = 0{,}2$

$P(Y_k = 1 \mid X_k = 0) = 0{,}4 \Rightarrow P(Y_k = 0 \mid X_k = 0) = 0{,}6$

(a)

$a_k(i) = P(X_k = i)$

$a_{k+1}(i)$ in terms of the values of $a_k(j)$?

$$\begin{cases} P(X_{k+1} = 1) = \underbrace{P(X_{k+1} = 1 \mid X_k = 1)}_{\text{Markov property}} P(X_k = 1) + \underbrace{P(X_{k+1} = 1 \mid X_k = 0)}_{\text{Markov property}} P(X_k = 0) \\ P(X_{k+1} = 0) = P(X_{k+1} = 0 \mid X_k = 1) P(X_k = 1) + P(X_{k+1} = 0 \mid X_k = 0) P(X_k = 0) \end{cases}$$

$$a_{k+1}(i) = \sum_{j=0}^{1} P(X_{k+1} = i \mid X_k = j) \underbrace{P(X_k = j)}_{a_k(j)} = \sum_{j=0}^{1} P(X_{k+1} = i \mid X_k = j)\, a_k(j)$$

In matrix form:

$$\begin{bmatrix} a_{k+1}(1) \\ a_{k+1}(0) \end{bmatrix} = P^T \begin{bmatrix} a_k(1) \\ a_k(0) \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0{,}2 \\ 0{,}3 & 0{,}7 \end{bmatrix}^T \begin{bmatrix} a_k(1) \\ a_k(0) \end{bmatrix} = \begin{bmatrix} 0{,}8\, a_k(1) + 0{,}3\, a_k(0) \\ 0{,}2\, a_k(1) + 0{,}7\, a_k(0) \end{bmatrix}$$

(b)

$$\alpha_k^\circ(i) = P(X_k = i \mid Y_0 = 0, \dots, Y_{k-1} = 0)$$

$$\alpha_{k+1}^\circ(i) = P(X_{k+1} = i \mid Y_0 = 0, \dots, Y_k = 0) = \frac{P(X_{k+1} = i, Y_k = 0 \mid Y_0 = 0, \dots, Y_{k-1} = 0)}{P(Y_k = 0 \mid Y_0 = 0, \dots, Y_{k-1} = 0)} =$$

$$= \frac{\sum_{j=0}^{1} P(X_{k+1} = i, Y_k = 0 \mid Y_0 = 0, \dots, Y_{k-1} = 0, X_k = j)\, P(X_k = j \mid Y_0 = 0, \dots, Y_{k-1} = 0)}{\sum_{j=0}^{1} P(Y_k = 0 \mid Y_0 = 0, \dots, Y_{k-1} = 0 \mid X_k = j)\, P(X_k = j \mid Y_0 = 0, \dots, Y_{k-1} = 0)} \to \alpha_k^\circ(j)$$

$$= \frac{\sum_{j=0}^{1} P(X_{k+1} = i \mid X_k = j) P(Y_k = 0 \mid X_k = j)\, \alpha_k^\circ(j)}{\sum_{j=0}^{1} P(Y_k = 0 \mid X_k = j)\, \alpha_k^\circ(j)} \Rightarrow \begin{cases} \alpha_{k+1}^\circ(1) = \dfrac{0{,}8 \cdot 0{,}2\, \alpha_k^\circ(1) + 0{,}3 \cdot 0{,}6\, \alpha_k^\circ(0)}{0{,}2\, \alpha_k^\circ(1) + 0{,}6\, \alpha_k^\circ(0)} \\ \alpha_{k+1}^\circ(0) = \dfrac{0{,}2 \cdot 0{,}2\, \alpha_k^\circ(1) + 0{,}7 \cdot 0{,}6\, \alpha_k^\circ(0)}{0{,}2\, \alpha_k^\circ(1) + 0{,}6\, \alpha_k^\circ(0)} \end{cases}$$

$$\begin{cases} \alpha_{k+1}^\circ(1) = \dfrac{0{,}16\, \alpha_k^\circ(1) + 0{,}18\, \overbrace{\alpha_k^\circ(0)}^{(1 - \alpha_k^\circ(1))}}{0{,}2\, \alpha_k^\circ(1) + 0{,}6\, \alpha_k^\circ(0)} = \dfrac{0{,}18 - 0{,}02\, \alpha_k^\circ(1)}{0{,}6 - 0{,}4\, \alpha_k^\circ(1)} \\ \alpha_{k+1}^\circ(0) = \dfrac{0{,}04\, \alpha_k^\circ(1) + 0{,}42\, \alpha_k^\circ(0)}{0{,}2\, \underbrace{\alpha_k^\circ(1)}_{(1 - \alpha_k^\circ(1))} + 0{,}6\, \alpha_k^\circ(0)} = \dfrac{0{,}04 + 0{,}38\, \alpha_k^\circ(0)}{0{,}2 + 0{,}4\, \alpha_k^\circ(0)} \end{cases}$$

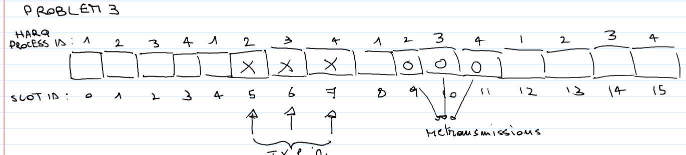
c)

$$P(T = k) = P(Y_k = 1, Y_0 = 0, \dots, Y_{k-1} = 0) = P(Y_k = 1 \mid Y_0 = 0, \dots, Y_{k-1} = 0) P(Y_0 = 0, \dots, Y_{k-1} = 0) = \underbrace{\sum_{j=0}^{1} P(Y_k = 1 \mid X_k = j)\, \alpha_k^\circ(j)}_{0{,}8\, \overset{1 - \alpha_k^\circ(0)}{\alpha_k^\circ(1)} + 0{,}4\, \alpha_k^\circ(0) = 0{,}8 - 0{,}4\, \alpha_k^\circ(0)} \cdot P(Y_0 = 0, \dots, Y_{k-1} = 0)$$

$$P(Y_0 = 0, \dots, Y_{k-1} = 0) = P(T > k-1) = 1 - P(T \le k-1) = 1 - \sum_{j=0}^{k-1} P(T = j)$$

$$P(T = k) = \left[1 - \sum_{j=0}^{k-1} P(T = j)\right]\left[0{,}8 - 0{,}4\, \alpha_k^\circ(0)\right]$$

PROBLEM 3



(a)

PDU Index	First Slot PDU decoded index
0	0
1	1
2	2
3	3
4	4
5	9

0	0
1	1
2	2
3	3
4	4
5	9
6	10
7	11
8	8
9	12

(b) Decoded PDUs are delivered in order to the upper layer.

PDU Index	PDU Arrival time at the Higher layer [Slot Index]
0	3
1	4
2	5
3	6
4	7
5	12
6	13
7	14
8	14
9	15

→ PDU 8 was received correctly before PDU 7. However, PDU 8 is delivered to the higher layer only when PDU 7 is received correctly.

PROBLEM 6

(a) • $r_K = w_K \quad w_K \sim CN(0,N) \Rightarrow r_K \sim CN(0,N)$

$$\hat{N} = \frac{1}{K}\sum_{K=1}^{K} |r_K|^2$$

$$r_K = r_K^R + j r_K^I \quad \begin{cases} r_K^R \sim N\left(0, \frac{N}{2}\right) \\ r_K^I \sim N\left(0, \frac{N}{2}\right) \end{cases} \quad \frac{N}{2} \to \sigma^2 \qquad |r_K|^2 = \left(\sqrt{r_K^{R\,2} + r_K^{I\,2}}\right)^2 = r_K^{R\,2} + r_K^{I\,2}$$

$$\hat{N} = \frac{1}{K}\sum_{K=1}^{K} r_K^{R\,2} + r_K^{I\,2} = \frac{1}{K}\sum_{K=1}^{K} r_K^{R\,2} + \frac{1}{K}\sum_{K=1}^{K} r_K^{I\,2} = \frac{1}{K}\left[\sum_{K=1}^{K} r_K^{R\,2} + \sum_{K=1}^{K} r_K^{I\,2}\right]$$

Now, we know that the chi-square distribution with K degrees of freedom is the distribution of the sum of the squares of K independent standard normal variables.

$$S_K^R = r_K^R\sqrt{\frac{N}{2}} \; ; \quad S_K^I = r_K^I\sqrt{\frac{N}{2}}$$

$$\hat{N} = \frac{1}{K}\left[\sum_{K=1}^{K} \frac{S_K^{R\,2}}{\frac{2}{N}} + \sum_{K=1}^{K} \frac{S_K^{I\,2}}{\frac{2}{N}}\right] = \frac{N}{2K}\underbrace{\left[\sum_{K=1}^{K} S_K^{R\,2} + \sum_{K=1}^{K} S_K^{I\,2}\right]} \sim \frac{N}{2K}\chi^2(2K)$$

Chi-square symbol

sum of 2 independent identical (iid) chi-square distributions is chi-square with K+K = 2K degrees of freedom (assuming independency)

$\hat{N}$ is a scaled chi-square distribution with 2K degrees of freedom

• $r_K = h_K x_K + w_K \quad h_K \sim CN(0, E_s) \quad w_K \sim CN(0,N)$

$$\hat{S} = \frac{1}{M}\sum_{K=1}^{M} |r_K|^2 \;, \quad |x_K| = 1$$

$$h_K = h_K^R + j h_K^I \quad h_K^R \sim N\left(0, \frac{E_s}{2}\right) \quad h_K^I \sim N\left(0, \frac{E_s}{2}\right)$$

$$|r_K| = |h_K||\underset{=1}{x_K}| + |w_K| = |h_K| + |w_K| \Rightarrow r_K \sim CN(0, E_s + N)$$ → assuming independency

$$r_K^R \sim N\left(0, \frac{E_s+N}{2}\right) \quad r_K^I \sim N\left(0, \frac{E_s+N}{2}\right) \Rightarrow S_K^R = r_K^R\sqrt{\frac{E_s+N}{2}} \quad S_K^I = r_K^I\sqrt{\frac{E_s+N_0}{2}}$$

$$\hat{S} = \frac{E_s+N}{2M}\left[\sum_{K=1}^{M} S_K^{R\,2} + \sum_{K=1}^{M} S_K^{I\,2}\right] \underset{\text{iid}}{\sim} \frac{E_s+N}{2M}\chi^2(2M)$$

Therefore $\hat{S}$ is a scaled chi-square distribution with 2M degrees of freedom

b) From wikipedia: A F-distributed random variable with parameters d_1 and d_2 arises from the ratio of two appropriately scaled chi-square random variables $\left[X = \frac{U_1/d_1}{U_2/d_2}\right]$

$$\frac{\hat{S}}{\hat{N}} = \frac{\frac{E_s+N}{2M}\left[\sum_{K=1}^{M} S_K^{R\,2} + \sum_{K=1}^{M} S_K^{I\,2}\right]}{\frac{N}{2K}\left[\sum_{K=1}^{K} S_K^{R\,2} + \sum_{K=1}^{K} S_K^{I\,2}\right]} = \frac{(E_s+N)K}{N\,M}\,\frac{\chi^2(2M)}{\chi^2(2K)} = \frac{(E_s+N)\cancel{K}}{N\cancel{M}}\,\frac{\cancel{2M}\,1}{\cancel{2K}}\left[\frac{\chi^2(2M)/2M}{\chi^2(2K)/2K}\right]$$

$\sim F(2M, 2K)$ assuming independency

$$\frac{\hat{S}}{\hat{N}} \sim \left(\frac{E_s+N}{N}\right)F(2M, 2K)$$

$2K \quad \ldots$

$$\frac{\hat{S}}{\hat{N}} \sim \left(\frac{E_s+N}{N}\right) F(2M, 2K) \quad \to \text{F-distribution}$$

$\sim F(2M, 2K)$
assuming independency of the two chi-square distributions

c)

$$\hat{\gamma} = \max\left\{0, \frac{\hat{S}}{\hat{N}} - 1\right\} \qquad \gamma_{true} = \frac{E_s}{N} = 3\,dB \quad \gamma_{tol} = 0{,}5\,dB \quad , \quad K = M$$

$$\hat{\gamma} = \max\left\{0, \frac{\hat{S}}{\hat{N}} - 1\right\} = \begin{cases} 0 & \text{if } \frac{\hat{S}}{\hat{N}} < 1 \Rightarrow \gamma_{err} = |\gamma_{true} - 0| = \gamma_{true} > \gamma_{tol} \text{ with prob } 1 \\ \frac{\hat{S}}{\hat{N}} & \text{if } \frac{\hat{S}}{\hat{N}} > 1 \Rightarrow \gamma_{err} = |\hat{\gamma} - \gamma_{true}| \end{cases}$$

$$P(\gamma_{err} < \gamma_{tol}) = P(|\hat{\gamma} - \gamma_{true}| < \gamma_{tol}) = P(-\gamma_{tol} < \hat{\gamma} - \gamma_{true} < \gamma_{tol})$$

$$= P\left(-\gamma_{tol} < \frac{\hat{S}}{\hat{N}} - 1 - \gamma_{true} < \gamma_{tol}\right) = P\left(1 + \gamma_{true} - \gamma_{tol} < \frac{\hat{S}}{\hat{N}} < \gamma_{tol} + \gamma_{true} + 1\right)$$

$\sim \frac{E_s+N}{N} F(2K, 2K)$

$$= P\left(\frac{1 + \gamma_{true} - \gamma_{tol}}{\frac{E_s+N}{N}} < \left(\frac{\hat{S}}{\hat{N}}\right)^* < \frac{\gamma_{tol} + \gamma_{true} + 1}{\frac{E_s+N}{N}}\right)$$

$\left(\frac{\hat{S}}{\hat{N}}\right)^* \sim F(2K, 2K)$

$$= P\left(\frac{1 + \gamma_{true} - \gamma_{tol}}{1 + \gamma_{true}} < \left(\frac{\hat{S}}{\hat{N}}\right)^* < \frac{\gamma_{tol} + \gamma_{true} + 1}{\gamma_{true} + 1}\right)$$

$$= \mathrm{FCDF}\left(\frac{\gamma_{tol}}{\gamma_{true}+1} + 1\right) - \mathrm{FCDF}\left(1 - \frac{\gamma_{tol}}{\gamma_{true}+1}\right) \qquad \text{[SEE MATLAB CODE]}$$

MATLAB

PROBLEM 7

$$r_k = h x_k + w_k \ , \quad w_k \sim CN(0, N) \qquad |h x_k|^2 = E_s$$

$$\hat{h} = \frac{\sum_{k=1}^{K} x_k^* r_k}{\sum_{k=1}^{K} |x_k|^2} \ , \qquad \hat{N} = \frac{\alpha}{K} \sum_{k=1}^{K} |r_k - \hat{h} x_k|^2$$

a) Find α in order to have an unbiased estimator for $\hat{N}$ (i.e. $E[\hat{N}] = N$)

$$\hat{h} = \frac{\sum_{k=1}^{K} \overbrace{x_k^* (h x_k}^{|x_k|^2 h} + w_k)}{\sum_{k=1}^{K} |x_k|^2} = h \frac{\sum_{k=1}^{K} |x_k|^2}{\sum_{k=1}^{K} |x_k|^2} + \frac{\sum_{k=1}^{K} x_k^* w_k}{\sum_{k=1}^{K} |x_k|^2} = h + \frac{\sum_{k=1}^{K} x_k^* w_k}{\sum_{k=1}^{K} |x_k|^2}$$

$$E[\hat{N}] = E\left[\frac{\alpha}{K} \sum_{k=1}^{K} \left| h x_k + w_k - \left(h + \frac{\sum_{k=1}^{K} x_k^* w_k}{\sum_{k=1}^{K} |x_k|^2}\right) x_k \right|^2\right] \qquad |x_k|^2 = \frac{E_s}{h^2}$$

$$= \frac{\alpha}{K} E\left[\sum_{k=1}^{K} \left| w_k - x_k \frac{\sum_{k=1}^{K} x_k^* w_k}{\sum_{k=1}^{K} |x_k|^2} \right|^2\right]$$

$$= \frac{\alpha}{K} E\left[\sum_{K=1} \left| w_K - x_K \frac{\sum_{K=1} x_K^* w_K}{\sum_{K=1} |x_K|^2} \right|^2 \right]$$

$$= \frac{\alpha}{k} \sum_{K=1}^{K} E\left[|w_K|^2 \left(1 - \frac{1}{K}\right)^2 \right] + \frac{E_s}{h^2} \frac{E\left[\sum |x_\ell^* w_K|^2 \right]}{K^2 \frac{E_s^2}{h^4}} \Rightarrow |x_K|^2 = \frac{E_s}{h^2}$$

$$= \frac{\alpha}{K} N \left(\frac{(K-1)^2}{K^2} + \frac{K-1}{K^2} \right) = \frac{\alpha}{K} N \left[\frac{K^2 - 2K + \cancel{1} + K - \cancel{1}}{K^2} \right] = \frac{\alpha}{\cancel{K}} N \frac{\cancel{K}(K-1)}{K} = \frac{\alpha}{N} \frac{(K-1)}{K}$$

$\alpha = \frac{K}{K-1}$ gives the unbiased estimator

(b) Unbiased estimate of E_s

$$\hat{E}_s = \frac{1}{K} \sum_{K=1}^{K} |r_K|^2 - \hat{N} \Rightarrow E\left[\hat{E}_s\right] = \frac{1}{K} \sum_{K=1}^{K} E\left[|r_K|^2\right] - N = \frac{\cancel{K}}{\cancel{K}}(E_s + N) - N = E_s$$

unbiased

$E\left[\hat{E}_s\right] = E_s$

PROBLEM 9

$$R(\gamma) = \min \{ \rho_{max}, \alpha \log_2(1 + \gamma) \}$$

All symbols in transmission K experience same SNR $\gamma_K = K = 1, \dots, K$

(a) For a given SNR target (γ_{target}) for the codeword

The condition on each γ_K is $\sum_{j=1}^{K} \gamma_K \geq \gamma_{target}$ for $K = 1, \dots, K$

if the packet passes after K transmission, the rate with chase combining will be

$$R_{chase}(\gamma) = \min \left\{ \frac{\rho_{max}}{K}, \frac{\alpha}{K} \log_2 (1 + \gamma_{target}) \right\}$$

(b) The condition on each γ_K is $\frac{1}{K} \sum_{K=1}^{K} M_K \geq \log_2(1 + \gamma_{target})$

$$\frac{1}{K} \sum_{K=1}^{K} \log_2(1 + \gamma_K) \geq \log_2(1 + \gamma_{target})$$

$$\frac{1}{K}\sum_{K=1}^{K}\log_2(1+\gamma_K) \geq \log_2(1+\gamma_{target})$$

The rate with IR will be:

$$R_{IR}(\gamma) = \min\left\{ \rho_{max},\ \alpha\log_2(1+\gamma_{target}) \right\}$$

(c) $\boxed{\text{SEE MATLAB}}$