HW 7

Sunday, April 11, 2021 6:12 PM

PROBLEM 1

$\hat{h}_0$ is the raw estimate of the channel

$$\hat{h}(m) = \frac{\sum_{\ell \in I} w_\ell \, \hat{h}_0(m+\ell)}{\sum_{\ell \in I} w_\ell}$$

I → set of ref. symbol locations

Suppose the use of a triangular Kernel

$$w_\ell = \max\left\{1 - \frac{\ell}{L}, 0\right\}$$

UPDATED

$$\hat{h}(m) = \frac{\sum_{K \in I} \omega_{m-K} \, \hat{h}_0(K)}{\sum_{K \in I} \omega_{m-K}}$$

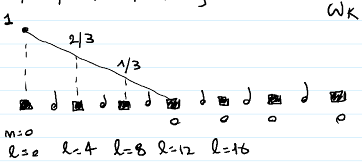
$$w_K = \max\left\{1 - \frac{|K|}{L}, 0\right\}$$

$I = \{0, d, 2d, \dots, Md\}$ → reference symbols are spaced every d positions

$L = 12$, $d = 4$, $M = 10$. $\Rightarrow I = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$

(a)

$$\hat{h}(0) = \frac{\sum_{K \in I} W_{-K} \, \hat{h}_0(K)}{\sum_{K \in I} W_{-K}}$$



$$= \frac{w_0 \hat{h}_0(0) + w_4 \hat{h}_0(4) + w_8 \hat{h}_0(8)}{w_0 + w_4 + w_8}$$

[the remaining coeffs are 0]

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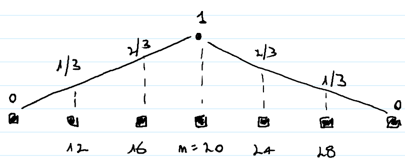
$$= \frac{\hat{h}_0(0) + \frac{2}{3}\hat{h}_0(4) + \frac{1}{3}\hat{h}_0(8)}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2}\hat{h}_0(0) + \frac{1}{3}\hat{h}_0(4) + \frac{1}{6}\hat{h}_0(8)$$

(b) $\boxed{m = 20}$

$$\hat{h}(20) = \frac{\sum_{K \in I} \omega_{20-K} \, \hat{h}_0(K)}{\sum_{K \in I} w_{20-K}}$$

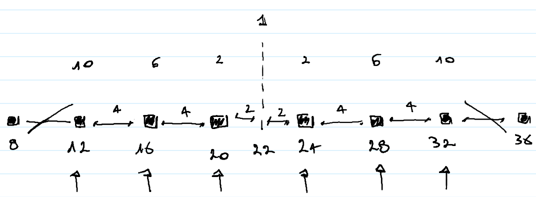


$$= \frac{\frac{1}{3}\hat{h}_0(12) + \frac{2}{3}\hat{h}_0(16) + \hat{h}_0(20) + \frac{2}{3}\hat{h}_0(24) + \frac{1}{3}\hat{h}_0(28)}{3}$$

$$= \frac{1}{9}\hat{h}_0(12) + \frac{2}{9}\hat{h}_0(16) + \frac{1}{3}\hat{h}_0(20) + \frac{2}{9}\hat{h}_0(24) + \frac{1}{9}\hat{h}_0(28)$$

(c) $\boxed{m = 22}$

$$\hat{h}(22) = \frac{\sum_{K \in I} \omega_{22-K} \, \hat{h}_0(K)}{\sum_{K \in I} \omega_{22-K}}$$



$$= \frac{\frac{1}{6}\hat{h}_0(12) + \frac{1}{2}\hat{h}_0(16) + \frac{5}{6}\hat{h}_0(20) + \frac{5}{6}\hat{h}_0(24) + \frac{1}{2}\hat{h}_0(28) + \frac{1}{6}\hat{h}_0(32)}{\underbrace{\frac{1}{6} + \frac{1}{2} + \frac{5}{6} + \frac{5}{6} + \frac{1}{2} + \frac{1}{6}}_{=3}}$$

$$= \frac{1}{18}\hat{h}_0(12) + \frac{1}{6}\hat{h}_0(16) + \frac{5}{18}\hat{h}_0(20) + \frac{5}{18}\hat{h}_0(24) + \frac{1}{6}\hat{h}_0(28) + \frac{1}{18}\hat{h}_0(32)$$

PROBLEM 3 [PROBLEM 2 IS AFTER PROBLEM 3]

Two BPSK symbols, $i = 0, 1$

$r_i = h x_i + w_i$ $\quad w_i \sim CN(0, N_0)$ $\quad x_i = \pm 1$

$h \sim CN(0, E_s)$ $\quad x_0 = 1 \rightarrow$ reference symbol.

Two possibilities $x^{(1)} = (1,1)$ or $x^{(2)} = (1,-1)$

(a)

$\rightarrow \underline{x}^{(1)} = (1,1)$ $\begin{cases} r_0 = h \cdot 1 + w_0 = h + w_0 \\ r_1 = h \cdot 1 + w_1 = h + w_1 \end{cases}$ $\quad \underline{r} = h\underline{x}^{(1)} + \underline{w}$

$$E(\underline{r} \mid \underline{x}^{(1)}) = \begin{bmatrix} E[r_0 \mid x_0 = 1] \\ E[r_1 \mid x_1 = 1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} //$$

$$Var(\underline{r} \mid \underline{x}^{(1)}) = \overbrace{Var(h)}^{E_s} \underbrace{\underline{x}^{(1)} \underline{x}^{(1)*}}_{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} + N_0 I_2 = E_s \underline{x}^{(1)} \underline{x}^{(1)*} + N_0 I_2 = \begin{pmatrix} E_s + N_0 & E_s \\ E_s & E_s + N_0 \end{pmatrix} //$$

$\rightarrow \underline{x}^{(2)} = (1,-1)$ $\begin{cases} r_0 = h\,1 + w_0 = h + w_0 \\ r_1 = h(-1) + w_1 = -h + w_1 \end{cases}$

$$E[\underline{r} \mid \underline{x}^{(2)}] = \begin{bmatrix} E[r_0 \mid x_0 = 1] \\ E[r_1 \mid x_1 = -1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} //$$

$$Var(\underline{r} \mid \underline{x}^{(2)}) = Var(h) \underbrace{\underline{x}^{(2)} \underline{x}^{(2)*}}_{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} + N_0 I_2 = E_s \underline{x}^{(2)} \underline{x}^{(2)*} + N_0 I_2 = \begin{pmatrix} E_s + N_0 & -E_s \\ -E_s & E_s + N_0 \end{pmatrix} //$$

multivariate normal distribution

$$f_{\underline{x}}(x_1, \ldots, x_K) = \frac{\exp\left(-(\underline{x} - \underline{\mu})^H \Sigma^{-1} (\underline{x} - \underline{\mu})\right)}{\pi^K \det(\Sigma)}$$

Σ is the covariance matrix

(b)

$$LLR = \log\left[\frac{p(\underline{r} \mid \underline{x} = (1,1))}{p(\underline{r} \mid \underline{x} = (1,-1))}\right]$$

$$\left[Var(\underline{r} \mid \underline{x}^{(1)})\right]^{-1} = \frac{1}{N_0}\left[I_2 - \frac{\gamma}{1 + \gamma \|\underline{x}^{(1)}\|^2} \underline{x}^{(1)} \underline{x}^{(1)*}\right]$$

$\gamma = E_s / N_0$

$$Var(\underline{r} \mid \underline{x}^{(1)}) = E_s \underline{x}^{(1)} \underline{x}^{(1)*} + N_0 I_2$$

$$= N_0 \left(I_2 + \overbrace{\frac{E_s}{N_0}}^{\gamma} \underline{x}^{(1)} \underline{x}^{(1)*}\right)$$

$$p(\underline{r} \mid \underline{x} = (1,1)) = \frac{1}{\pi^2 (2E_s N_0 + N_0^2)} \exp\left(-\underline{r}^* \frac{1}{N_0}\left(I_2 - \frac{E_s/N_0}{1 + 2E_s/N_0}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)\underline{r}\right)$$

$$\left[Var(\underline{r} \mid \underline{x}^{(2)})\right]^{-1} = \frac{1}{N_0}\left[I_2 - \frac{\gamma}{1 + \gamma \|\underline{x}^{(2)}\|^2} \underline{x}^{(2)} \underline{x}^{(2)*}\right]$$

$$p(\underline{r} \mid \underline{x} = (1,-1)) = \frac{1}{\pi^2 (2E_s N_0 + N_0^2)} \exp\left(-\underline{r}^* \frac{1}{N_0}\left(I_2 - \frac{E_s/N_0}{1 + 2E_s/N_0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\underline{r}\right)$$

$$LLR = \log\left[\frac{p(\underline{r}|\underline{x}^{(1)})}{p(\underline{r}|\underline{x}^{(2)})}\right] = \left[\underline{r}^* \frac{1}{N_0}\left(I_2 - \frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right)\underline{r} - \underline{r}^* \frac{1}{N_0}\left(I_2 - \frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)\underline{r}\right]$$

$$= \underline{r}^*\left(\frac{1}{N_0}\frac{E_s/N_0}{1+\frac{2E_s}{N_0}}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\right)\underline{r} = \frac{E_s/N_0}{2E_s+N_0}\underline{r}^*\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\underline{r} = \frac{E_s/N_0}{2E_s+N_0}4\,\mathrm{Re}\{r_0 r_1^*\}$$

$$\frac{1}{\cancel{N_0}}\frac{E_s}{N_0}\frac{\cancel{N_0}}{2E_s+N_0} = \frac{E_s}{N_0(2E_s+N_0)} \qquad 2[r_0 r_1^* + r_1 r_0^*] = 2[(a+ib)(c-id)+(c+id)(a-ib)]$$

$$ac - \cancel{iad} + \cancel{icb} + bd + ca - \cancel{icb} + \cancel{ida} + bd$$

$$\frac{1}{1+\frac{2E_s}{N_0}} = \frac{1}{\frac{2E_s+N_0}{N_0}} = \frac{N_0}{2E_s+N_0} \qquad [r_0^*\ r_1^*]\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix} = [2r_1^*\quad 2r_0^*]\begin{bmatrix}r_0\\ r_1\end{bmatrix} = 2r_0r_1^* + 2r_1r_0^*$$

$$2ac + 2bd \quad [\mathrm{Re}\{(a+ib)(c-id)\} = ac+bd]$$

$$+\frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$$

$$\underline{r}^*A\underline{r} - \underline{r}^*B\underline{r} = r^*(A-B)\underline{r} \Rightarrow A - B = \frac{1}{N_0}\left(\cancel{I_2} - \frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \cancel{I_2} + \frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)$$

$$= \frac{1}{N_0}\frac{E_s/N_0}{1+2E_s/N_0}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}$$

PROBLEM 2

$$\hat{h}(m) = \frac{1}{2}\left[\hat{h}_0(m-L) + \hat{h}_0(m+L)\right] \qquad \hat{h}_0(K) = h(K) + V(K), \quad V(K) \sim \mathcal{CN}(0, N_v)$$

$$R(K) = E[h(m)h(m+K)^*] \leftarrow \text{WSS}$$

(a) MSE?

$$\varepsilon = E|\hat{h}(m) - h(m)|^2 \qquad \hat{h}(m) = \frac{1}{2}[h(m-L) + v(m-L) + h(m+L) + v(m+L)]$$

$$= E[(\hat{h}(m) - h(m))(\hat{h}(m) - h(m))^*] = E[\underbrace{\hat{h}(m)\hat{h}^*(m)}_{(1)} - \underbrace{\hat{h}(m)h(m)^*}_{(2)} - \underbrace{h(m)\hat{h}(m)^*}_{(3)} + \underbrace{h(m)h(m)^*}_{R(0)}]$$

$$(1)\ E[\hat{h}(m)\hat{h}^*(m)] = \frac{1}{4}E[h(m-L) + v(m-L) + h(m+L) + v(m+L)][h^*(m-L) + v^*(m-L) + h^*(m+L) + v^*(m+L)]$$

$$= \frac{1}{4}[R(0) + R(2L) + N_v + 0 + R(-2L) + R(0) + 0 + N_v] = \frac{1}{2}R(0) + \frac{1}{4}[R(2L) + R(2L)^*] + \frac{1}{2}N_v$$

assuming noise is iid

$$(2)\ E[\hat{h}(m)h^*(m)] = E\left[\frac{1}{2}(h(m-L) + v(m-L) + h(m+L) + v(m+L))h^*(m)\right]$$

$$= \frac{1}{2}R(L) + \frac{1}{2}R(-L) = \frac{1}{2}[R(L) + R(L)^*]$$

$$(3)\ E[h(m)\hat{h}(m)^*] = E\left[\frac{1}{2}h(m)(h^*(m-L) + v^*(m-L) + h^*(m+L) + v^*(m+L))\right]$$

$$= \frac{1}{2}R(-L) + \frac{1}{2}R(L) = \frac{1}{2}[R(L) + R(L)^*]$$

$$\varepsilon = E|\hat{h}(m) - h(m)|^2 = \frac{3}{2}R(0) + \frac{1}{4}[\overbrace{R(2L) + R(2L)^*}^{2\mathrm{Re}\{R(2L)\}}] - [\overbrace{R(L) + R(L)^*}^{2\mathrm{Re}\{R(L)\}}] + \frac{1}{2}N_v$$

$$= \frac{3}{2}R(0) + \frac{1}{2}\mathrm{Re}\{R(2L)\} - 2\mathrm{Re}\{R(L)\} + \frac{1}{2}N_v$$

(b) [SEE MATLAB] $R(K) = E_s \boxed{J_0(2\pi f_{max} KT)}$

(c) [SEE MATLAB]

The minimum overhead requested is $\sim \frac{1}{1840}$ {see MATLAB for the precise result}

Wireless Communications EL-GY 6023

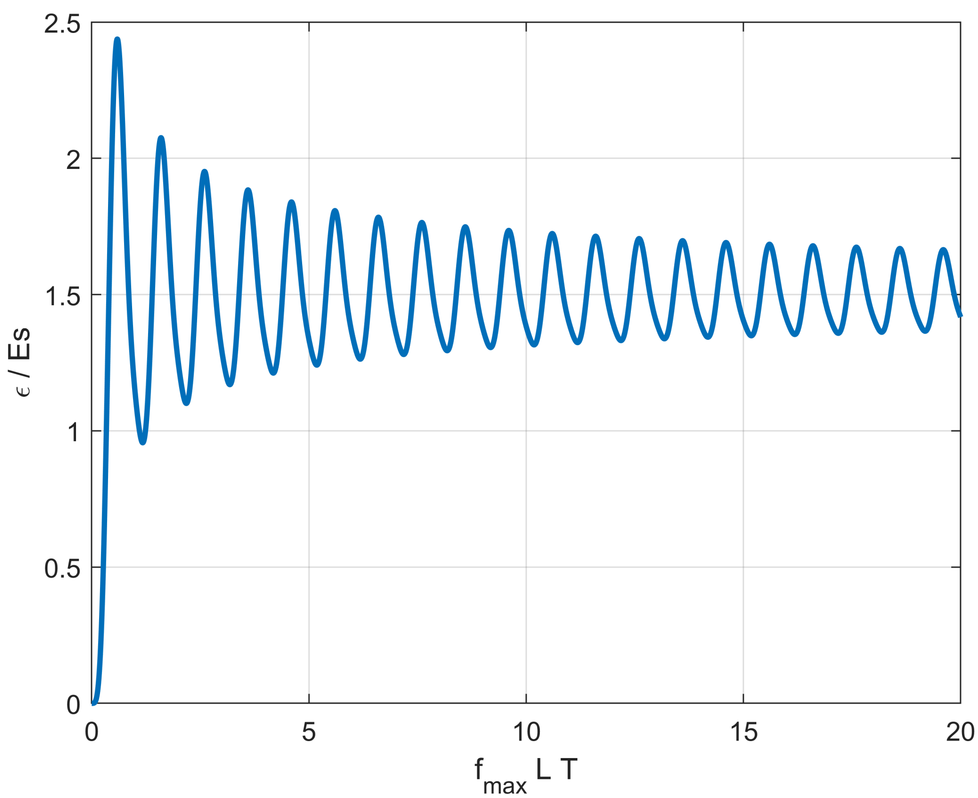
Homework 7 - Tommy Azzino (ta1731)

Problem 2

(b)

```
npoints = 10000;
fmax_L_T = linspace(0,20,npoints);
mse = 1.5*besselj(0, zeros(1, length(fmax_L_T))) + ...
    0.5*real(besselj(0, fmax_L_T*4*pi)) - 2*real(besselj(0, fmax_L_T*2*pi));

figure;
plot(fmax_L_T, mse, "LineWidth", 2);
grid on; xlabel('f_{max} L T'); ylabel('\epsilon / Es');
```



(c)

```
% one reference symbol every 2L samples
fmax = 100;
T = 1e-6;
epsilon_over_Es = 1/db2pow(20);

fmax_L_T_target = fmax_L_T(mse < epsilon_over_Es);
L = fmax_L_T_target(end)/(fmax*T);
```

```
min_overhead = 1/(2*L);
fprintf(1, 'The minimum overhead is: %f\n', min_overhead);
```

```
The minimum overhead is: 0.000543
```