

PROBLEM 1

$\hat{h}(m)$ is the new estimate of the channel

$$\hat{h}(m) = \frac{\sum_{k \in I} w_k \hat{h}_o(m+k)}{\sum_{k \in I} w_k}$$

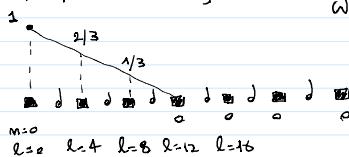
Suppose the use of triangular kernel
 $w_k = \max\left\{1 - \frac{|k|}{L}, 0\right\}$

\Rightarrow set of ref. symbol locations, $w_k = \max\left\{1 - \frac{|k|}{L}, 0\right\}$

$I = \{0, d, 2d, \dots, Md\} \rightarrow$ reference symbols are spaced every d positions

$$L=12, d=4, M=10 \Rightarrow I = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40\}$$

$$(a) \quad \hat{h}(0) = \frac{\sum_{k \in I} w_{-k} \hat{h}_o(k)}{\sum_{k \in I} w_{-k}}$$



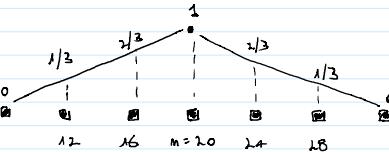
$$= \frac{w_0 \hat{h}_o(0) + w_4 \hat{h}_o(4) + w_8 \hat{h}_o(8)}{w_0 + w_4 + w_8}$$

[the remaining coeffs are 0]

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$$= \frac{\hat{h}_o(0) + \frac{1}{3} \hat{h}_o(4) + \frac{1}{3} \hat{h}_o(8)}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2} \hat{h}_o(0) + \frac{1}{3} \hat{h}_o(4) + \frac{1}{6} \hat{h}_o(8)$$

$$(b) \quad \hat{h}(20) = \frac{\sum_{k \in I} w_{20-k} \hat{h}_o(k)}{\sum_{k \in I} w_{20-k}}$$

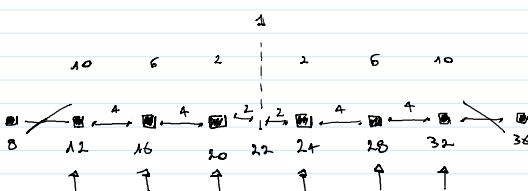


$$= \frac{\frac{1}{3} \hat{h}_o(12) + \frac{2}{3} \hat{h}_o(16) + \hat{h}_o(20) + \frac{2}{3} \hat{h}_o(24) + \frac{1}{3} \hat{h}_o(28)}{3}$$

$$= \frac{1}{9} \hat{h}_o(12) + \frac{2}{9} \hat{h}_o(16) + \frac{1}{3} \hat{h}_o(20) + \frac{2}{9} \hat{h}_o(24) + \frac{1}{9} \hat{h}_o(28)$$

$$(c) \quad [m=22]$$

$$\hat{h}(22) = \frac{\sum_{k \in I} w_{22-k} \hat{h}_o(k)}{\sum_{k \in I} w_{22-k}}$$



$$= \frac{\frac{1}{6} \hat{h}_o(12) + \frac{1}{2} \hat{h}_o(16) + \frac{5}{6} \hat{h}_o(20) + \frac{5}{6} \hat{h}_o(24) + \frac{1}{2} \hat{h}_o(28) + \frac{1}{6} \hat{h}_o(32)}{3}$$

$$= \frac{1}{18} \hat{h}_o(12) + \frac{1}{6} \hat{h}_o(16) + \frac{5}{18} \hat{h}_o(20) + \frac{5}{18} \hat{h}_o(24) + \frac{1}{6} \hat{h}_o(28) + \frac{1}{18} \hat{h}_o(32)$$

PROBLEM 3 [PROBLEM 2 IS AFTER PROBLEM 3]

Two BPSK symbols, $i = 0, 1$

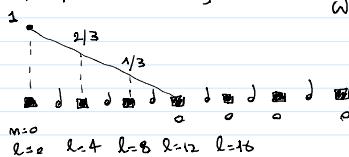
$$r_i = h x_i + w_i \quad w_i \sim \mathcal{CN}(0, N_0) \quad x_i = \pm 1$$

UPDATED

$$\hat{h}(m) = \frac{\sum_{k \in I} w_{m-k} \hat{h}_o(k)}{\sum_{k \in I} w_{m-k}}$$

$$w_k = \max\left\{1 - \frac{|k|}{L}, 0\right\}$$

$$(a) \quad \hat{h}(0) = \frac{\sum_{k \in I} w_{-k} \hat{h}_o(k)}{\sum_{k \in I} w_{-k}}$$



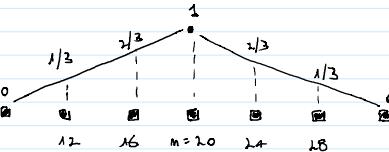
$$= \frac{w_0 \hat{h}_o(0) + w_4 \hat{h}_o(4) + w_8 \hat{h}_o(8)}{w_0 + w_4 + w_8}$$

[the remaining coeffs are 0]

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$$= \frac{\hat{h}_o(0) + \frac{1}{3} \hat{h}_o(4) + \frac{1}{3} \hat{h}_o(8)}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2} \hat{h}_o(0) + \frac{1}{3} \hat{h}_o(4) + \frac{1}{6} \hat{h}_o(8)$$

$$(b) \quad \hat{h}(20) = \frac{\sum_{k \in I} w_{20-k} \hat{h}_o(k)}{\sum_{k \in I} w_{20-k}}$$

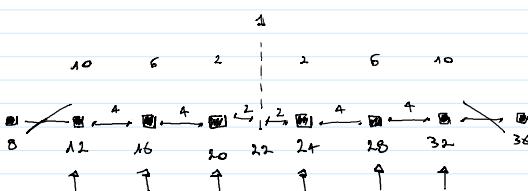


$$= \frac{\frac{1}{3} \hat{h}_o(12) + \frac{2}{3} \hat{h}_o(16) + \hat{h}_o(20) + \frac{2}{3} \hat{h}_o(24) + \frac{1}{3} \hat{h}_o(28)}{3}$$

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$$(c) \quad [m=22]$$

$$\hat{h}(22) = \frac{\sum_{k \in I} w_{22-k} \hat{h}_o(k)}{\sum_{k \in I} w_{22-k}}$$



$$= \frac{\frac{1}{6} \hat{h}_o(12) + \frac{1}{2} \hat{h}_o(16) + \frac{5}{6} \hat{h}_o(20) + \frac{5}{6} \hat{h}_o(24) + \frac{1}{2} \hat{h}_o(28) + \frac{1}{6} \hat{h}_o(32)}{3}$$

$$= \frac{1}{18} \hat{h}_o(12) + \frac{1}{6} \hat{h}_o(16) + \frac{5}{18} \hat{h}_o(20) + \frac{5}{18} \hat{h}_o(24) + \frac{1}{6} \hat{h}_o(28) + \frac{1}{18} \hat{h}_o(32)$$

$$h \sim CN(0, E_s) \quad x_0 = 1 \rightarrow \text{reference symbol.}$$

Two possibilities $\underline{x}^{(1)} = (1, 1)$ or $\underline{x}^{(2)} = (1, -1)$

$$\xrightarrow{(a)} \underline{x}^{(1)} = (1, 1) \quad \begin{cases} r_0 = h \cdot 1 + w_0 = h + w_0 \\ r_1 = h \cdot 1 + w_1 = h + w_1 \end{cases} \quad \underline{r} = h \underline{x}^{(1)} + \underline{w}$$

$$E(\underline{r} | \underline{x}^{(1)}) = \begin{bmatrix} E[r_0 | x_0=1] \\ E[r_1 | x_1=1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} //$$

$$\text{Var}(\underline{r} | \underline{x}^{(1)}) = \underbrace{\text{Var}(h) \underline{x}^{(1)} \underline{x}^{(1)*}}_{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} + N_o I_2 = E_s \underline{x}^{(1)} \underline{x}^{(1)*} + N_o I_2 = \begin{pmatrix} E_s + N_o & E_s \\ E_s & E_s + N_o \end{pmatrix} //$$

$$\rightarrow \underline{x}^{(2)} = (1, -1) \quad \begin{cases} r_0 = h \cdot 1 + w_0 = h + w_0 \\ r_1 = h \cdot (-1) + w_1 = -h + w_1 \end{cases}$$

$$E(\underline{r} | \underline{x}^{(2)}) = \begin{bmatrix} E[r_0 | x_0=1] \\ E[r_1 | x_1=-1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} //$$

$$\text{Var}(\underline{r} | \underline{x}^{(2)}) = \text{Var}(h) \underbrace{\underline{x}^{(2)} \underline{x}^{(2)*}}_{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} + N_o I_2 = E_s \underline{x}^{(2)} \underline{x}^{(2)*} + N_o I_2 = \begin{pmatrix} E_s + N_o & -E_s \\ -E_s & E_s + N_o \end{pmatrix} //$$

(b) $LLR = \log \left[\frac{P(r | \underline{x} = (1, 1))}{P(r | \underline{x} = (1, -1))} \right]$

multivariate normal distribution

$$f_{\underline{X}}(\underline{x}_1, \dots, \underline{x}_K) = \frac{\exp(-(\underline{x} - \underline{\mu})^H \Sigma^{-1} (\underline{x} - \underline{\mu}))}{\pi^K \det(\Sigma)}$$

Σ is the covariance matrix

$$\left[\text{Var}(r | \underline{x}^{(1)}) \right]^{-1} = \frac{1}{N_o} \left[I_2 - \frac{\gamma}{1 + \gamma \| \underline{x}^{(1)} \|^2} \underline{x}^{(1)} \underline{x}^{(1)*} \right] \quad \gamma = E_s / N_o$$

$$\text{Var}(r | \underline{x}^{(1)}) = E_s \underline{x}^{(1)} \underline{x}^{(1)*} + N_o I_2$$

$$= N_o \left(I_2 + \frac{E_s}{N_o} \underline{x}^{(1)} \underline{x}^{(1)*} \right)$$

$$P(r | \underline{x} = (1, 1)) = \frac{1}{\pi^2 (2E_s N_o + N_o^2)} \exp \left(-r^* \frac{1}{N_o} \left(I_2 - \frac{E_s / N_o}{1 + 2E_s / N_o} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) r \right)$$

$$\left[\text{Var}(r | \underline{x}^{(2)}) \right]^{-1} = \frac{1}{N_o} \left[I_2 - \frac{\gamma}{1 + \gamma \| \underline{x}^{(2)} \|^2} \underline{x}^{(2)} \underline{x}^{(2)*} \right]$$

$$P(r | \underline{x} = (1, -1)) = \frac{1}{\pi^2 (2E_s N_o + N_o^2)} \exp \left(-r^* \frac{1}{N_o} \left(I_2 - \frac{E_s / N_o}{1 + 2E_s / N_o} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) r \right)$$

$$LLR = \log \left[\frac{P(r|X^{(1)})}{P(r|X^{(2)})} \right] = \left[\underline{r}^* \frac{1}{N_0} \left(I_2 - \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \underline{r} - \underline{r}^* \frac{1}{N_0} \left(I_2 - \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \underline{r} \right]$$

$$= \underline{r}^* \left(\frac{1}{N_0} \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \right) \underline{r} = \frac{E_s/N_0}{2E_s+N_0} \underline{r}^* \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \underline{r} = \frac{E_s/N_0}{2E_s+N_0} 4 \operatorname{Re}\{r_0 r_1^*\}$$

$$\frac{1}{N_0} \frac{E_s}{N_0} \frac{N_0}{2E_s+N_0} = \frac{E_s}{N_0(2E_s+N_0)}$$

$$\frac{1}{1+2E_s/N_0} = \frac{1}{\frac{2E_s+N_0}{N_0}} = \frac{N_0}{2E_s+N_0}$$

$$2[r_0 r_1^* + r_1 r_0^*] = 2[(a+ib)(c-id) + (c+id)(a-id)]$$

$$[r_0^* r_1^*] \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = [2r_1^* 2r_0^*] \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

$$ac - iad + ieb + bd + ca - ieb + ida + bd \\ 2ac + 2bd \underbrace{[\operatorname{Re}(a+ib)(c-id)]}_{ac+bd} = ac + bd$$

$$+ \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{r}^* A \underline{r} - \underline{r}^* B \underline{r} = \underline{r}^*(A-B)\underline{r} \Rightarrow A-B = \frac{1}{N_0} \left(\overbrace{I_2 - \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}^{\cancel{I_2}} - \cancel{I_2} + \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{N_0} \frac{E_s/N_0}{1+2E_s/N_0} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

PROBLEM 2

$$\hat{h}(m) = \frac{1}{2} [\hat{h}_o(m-L) + \hat{h}_o(m+L)]$$

$$\hat{h}_o(k) = h(k) + v(k), v(k) \sim CN(0, N_v)$$

$$R(k) = E[\hat{h}(m) \hat{h}(m+k)^*] \leftarrow WSS$$

(a) MSE?

$$\varepsilon = E |\hat{h}(m) - h(m)|^2 \quad \hat{h}(m) = \frac{1}{2} [h(m-L) + v(m-L) + h(m+L) + v(m+L)]$$

$$= E [(\hat{h}(m) - h(m)) (\hat{h}(m) - h(m))^*] = E \left[\hat{h}(m) \hat{h}^*(m) - \hat{h}(m) h(m)^* - h(m) \hat{h}^*(m) + h(m) h(m)^* \right]$$

$$(1) E[\hat{h}(m) \hat{h}^*(m)] = \frac{1}{4} E[h(m-L) + v(m-L) + h(m+L) + v(m+L)] [h^*(m-L) + v^*(m-L) + h^*(m+L) + v^*(m+L)]$$

$$= \frac{1}{4} [R(o) + R(2L) + N_v + 0 + R(-2L) + R(o) + 0 + N_v] = \frac{1}{2} R(o) + \frac{1}{4} [R(2L) + R(2L)^*] + \frac{1}{2} N_v$$

assuming noise is i.i.d

$$(2) E[\hat{h}(m) h^*(m)] = E \left[\frac{1}{2} [h(m-L) + v(m-L) + h(m+L) + v(m+L)] h^*(m) \right]$$

$$= \frac{1}{2} R(L) + \frac{1}{2} R(-L) = \frac{1}{2} [R(L) + R(L)^*]$$

$$(3) E[\hat{h}(m) \hat{h}^*(m)] = E \left[\frac{1}{2} h(m) (h^*(m-L) + v^*(m-L) + h^*(m+L) + v^*(m+L)) \right]$$

$$= \frac{1}{2} R(-L) + \frac{1}{2} R(L) = \frac{1}{2} [R(L) + R(L)^*]$$

$$\varepsilon = E |\hat{h}(m) - h(m)|^2 = \frac{3}{2} R(o) + \frac{1}{4} \left[\overbrace{R(2L) + R(2L)^*}^{2 \operatorname{Re}\{R(2L)\}} - \overbrace{[R(L) + R(L)^*]}^{2 \operatorname{Re}\{R(L)\}} + \frac{1}{2} N_v \right]$$

$$= \frac{3}{2} R(o) + \frac{1}{2} \operatorname{Re}\{R(2L)\} - 2 \operatorname{Re}\{R(L)\} + \frac{1}{2} N_v //$$

$$(b) [\text{SEE MATLAB}] R(k) = E_s J_0(2\pi f_{\max} k T)$$

$$(c) [\text{SEE MATLAB}]$$

The minimum overhead requested is $\sim \frac{1}{1840}$ { See MATLAB for the precise result }