Problems: Small-Scale Fading
ECE-GY 6023. Wireless Communications

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1. *Doppler shift and frequency offset:*

   (a) Suppose that a vehicle is traveling at 100 km/h directly towards a base station. If the carrier frequency is 2.5 GHz, what is the maximum Doppler?

   (b) Frequency shifts can also occur due to differences in the local oscillators (LOs) at the TX and RX. Suppose that the LO error is 1 part per million (ppm) meaning that the frequency error is 1 millionth of the carrier. What is the frequency shift?

2. *Doppler and match filter*: Suppose that we receive a signal

$$r(t) = u(t)e^{2\pi i f t} + w(t),$$

   where $u(t)$ is a transmitted signal, f is a frequency shift and $w(t)$ is WGN with PSD N_0. Assume $|u(t)|^2 = P$ for all t for some received power level $P > 0$. We then compute a matched filter,

$$z = \frac{1}{T}\int_0^T u(t)^* r(t) dt.$$

   (a) Write the MF response as $z = x + v$ where x is due to the signal and v is due to the noise. Your expression for x should have a sinc function in it.

   (b) Write an expression for the SNR defined as

$$\mathsf{SNR} = \frac{|x|^2}{\mathbb{E}|v|^2}.$$

   The expression should be in terms of the integration time T, SNR P/N_0 and frequency offset f.

   (c) When there is no frequency offset (i.e. $f = 0$), the SNR increases linearly with the integration time. However, when there is an uncompensated frequency offset, there is an optimal integration time beyond which the SNR begins to drop. Find the maximum SNR and optimal integration time.

   The maximization will not have a closed-form answer. So, we will use MATLAB. Specifically, plot the SNR as a function of T when $P/N_0 = 1$ and $f = 1$. Then, write an expression to translate your answer to other values of P/N_0 and f.

   (d) Suppose that the frequency offset is $f = 100$ Hz, the received power is $P =$ -100 dBm and the noise power density is $N_0 =$ -140 dBm/Hz. What is the optimal integration time T and maximum SNR?

3. *Two path channel:* A received signal has two paths:

   - Path 1: power -100 dBm and Doppler shift 100 Hz,
   - Path 2: power -103 dBm and Doppler shift -50 Hz.

   (a) Draw the signal power as a function of time.

   (b) What is the average receive power in dBm?

   (c) What is the time it takes to go from the maximum to minimum power?

   (d) What is the fraction of time the power is greater than -101 dBm?

4. *Two path channel:* Suppose that a narrowband complex channel $h(t)$ be modeled as a wide-sense stationary random process.

   (a) Let $\rho(\tau)$ be the relative change in $h(t)$ over a time τ.

   $$\rho(\tau) = \frac{\mathbb{E}|h(t) - h(t+\tau)|^2}{\mathbb{E}|h(t)|^2}$$

   Write $\rho(\tau)$ in terms of the autocorrelation function $R(\tau) = \mathbb{E}h(t)h^*(t-\tau)$.

   (b) If $h(t)$ follows a Jakes' spectrum with a uniform angular distribution, the autocorrelation is given by ,

   $$R(\tau) = R(0)J_0(2\pi f_{max}\tau),$$

   where f_{max} is the maximum Doppler shift and $J_0(\cdot)$ is the Bessel of the first kind. Plot $R(\tau)/R(0)$ vs. τf_{max} for $\tau f_{max} \in [0, 5]$.

   (c) Using the autocorrelation function in the previous part, if $f_{max} = 200\,\text{Hz}$, what is the time it takes the channel to change by 10%?

5. *Auto-correlation.* Consider a multipath fading channel of the form

$$y(t) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i f_\ell} x(t - \tau_\ell), \quad f_\ell = f_{max}\cos(\theta_\ell),$$

where L is the number of paths, and for each path ℓ, g_ℓ is its complex gain, θ_ℓ the AoA, f_ℓ the Doppler shift and τ_ℓ the delay. This channel has a time-varying frequency response

$$H(t, f) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i (f_\ell t - \tau_\ell f)}, \quad f_\ell = f_{max}\cos\theta_\ell. \tag{1}$$

Suppose that we model the path parameters statistically. That is, we model $g_\ell, \theta_\ell, \tau_\ell$ as random variables that are all independent from one another and across different values of ℓ. Assume that the distributions of the random variables are identical for different path indices ℓ. In addition, assume that g_ℓ are zero mean, with average power gain $\mathbb{E}|g_\ell|^2 = G$ for some value $G > 0$.

(a) Suppose that the delays are exponentially distributed with pdf,

$$p(\tau_\ell) = \frac{1}{\lambda} e^{-\tau_\ell/\lambda},$$

for some λ representing the delay spread. What is the autocorrelation function in frequency

$$R(\delta f) = \mathbb{E}\left[H(t,f)H^*(t,f-\Delta f)\right].$$

(b) Suppose we define the 3 dB coherence bandwidth as the frequency W where the frequency response changes by less than 3 dB. That is,

$$\mathbb{E}|H(t,f) - H(t,f+W)|^2 \leq \beta \mathbb{E}|H(t,f)|^2, \quad \beta = 0.5.$$

Under the above exponential delay model, if the delay spread is $\lambda = 100\,\text{ns}$, what is the 3 dB bandwidth?