HW3 - Tommy Azzino

Wednesday, February 24, 2021 7:35 PM

PROBLEM 2

$$r(t) = u(t)\,e^{2\pi i f t} + w(t)$$

where f is the frequency shift and $w(t)$ is WGN with PSD N_0.

$|u(t)|^2 = P > 0$

Matched filter $z = \frac{1}{T}\int_0^T u(t)^* r(t)\,dt$

(a) MF response as $z = x + v$ [x should have a sinc function]

$$z = \frac{1}{T}\int_0^T u(t)\left[u(t)e^{2\pi i f t} + w(t)\right]dt = \frac{1}{T}\int_0^T u^2(t)e^{2\pi i f t}\,dt + \frac{1}{T}\int_0^T u(t)w(t)\,dt =$$

$$= \frac{P}{T}\int_0^T e^{2\pi i f t}\,dt + \frac{1}{T}\int_0^T u(t)w(t)\,dt$$

$$= \frac{P}{T}\,\frac{1}{2\pi i f}\,e^{2\pi i f t}\Big|_0^T = \frac{P}{T}\,\frac{e^{2\pi i f T}-1}{2\pi i f} = \frac{P}{T}\,e^{\pi i f T}\,\frac{\left(e^{\pi i f T} - e^{-\pi i f T}\right)}{2\pi i f}$$

$$= P\,e^{\pi i f T}\,\frac{\sin(\pi f T)}{\pi f T} = P\,e^{\pi i f T}\,\mathrm{sinc}(fT)$$

$$\begin{cases} x = P e^{\pi i f T}\,\mathrm{sinc}(fT) \\ v = \frac{1}{T}\int_0^T u(t)\,w(t)\,dt \longrightarrow \text{Gaussian RV } \sim \mathcal{CN}\left(0, \frac{N_0 P}{T}\right) \end{cases}$$

(b)

$$SNR = \frac{|x|^2}{E[|v|^2]} = \frac{P^2\,\mathrm{sinc}^2(fT)}{\frac{P \cdot N_0}{T}} = \frac{TP}{N_0}\,\mathrm{sinc}^2(fT)$$

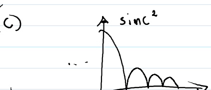
$$E[|v|^2] = E[v v^*] = \frac{1}{T^2}E\left[\int_0^T u(t)\,w(t)\,dt \int_0^T u(\tau)^* w(\tau)^*\,d\tau\right] =$$

$$= \frac{1}{T^2}E\left[\int_0^T\int_0^T u(t)u(\tau)^*\,\underbrace{w(t)\,w(\tau)^*}_{|w(t)|^2\delta(t-\tau)}\,dt\,d\tau\right] = \frac{1}{T^2}\int_0^T\int_0^T u(t)u(\tau)^* N_0\,\delta(t-\tau)\,dt\,d\tau =$$

($|w(t)|^2 = N_0$)

$$= \frac{N_0}{T^2}\int_0^T \underbrace{|u(t)|^2}_{P}\,d\tau = \frac{N_0 P \cancel{T}}{T^{\cancel{2}}} = \frac{P \cdot N_0}{T}$$

(c)



[MATLAB CODE]

(d) [MATLAB CODE]

PROBLEM 3

[SEE ALSO MATLAB CODE FOR THIS PROBLEM]

$f_{d1} = 100$ Hz $\quad P_1 = -100$ dBm

$f_{d2} = -50$ Hz $\quad P_2 = -103$ dBm $\quad \Rightarrow |h_1|^2 = 2|h_2|^2 \Rightarrow h_1 = \sqrt{2}h_2$

$$P(t,0) = \left|h_1 e^{j2\pi f_{d1} t} + h_2 e^{j2\pi f_{d2} t}\right|^2$$

$$= \left|\sqrt{2}h_2 e^{j2\pi 100 t} + h_2 e^{-j2\pi 50 t}\right|^2 = |h_2|^2\left|\sqrt{2}e^{j2\pi 100 t} + e^{-j2\pi 50 t}\right|^2$$

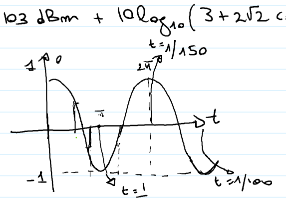
$$= |h_2|^2\left(\sqrt{2}e^{j2\pi 100 t} + e^{-j2\pi 50 t}\right)\left(\sqrt{2}e^{-j2\pi 100 t} + e^{+j2\pi 50 t}\right) =$$

$$= |h_2|^2\left(2 + \sqrt{2}e^{j2\pi 150 t} + \sqrt{2}e^{-j2\pi 150 t} + 1\right) =$$

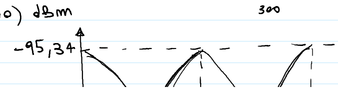
$$= |h_2|^2\left(3 + 2\sqrt{2}\cos(2\pi 150 t)\right)$$

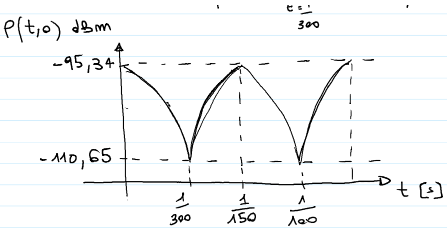
$$\Rightarrow P(t,0) = -103\text{ dBm} + 10\log_{10}\left(3 + 2\sqrt{2}\cos(2\pi 150 t)\right)$$

$\cos(300\pi t) \rightarrow$



$P(t,0)$ dBm





(b) The average received power is:

$$P_{AVG} = -103 + 10\log_{10}(3) = -98,23 \text{ dBm}$$

(c) The time it takes for going from the maximum to the minimum is:

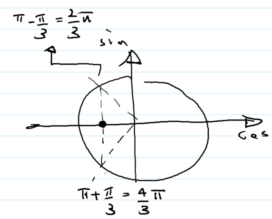
$$\Delta t = \frac{1}{300} - 0 = \frac{1}{300} \approx 3,33 \text{ ms}$$

(d)

$$10\log_{10}\left(3 + 2\sqrt{2}\cos\left(2\pi 150 t\right)\right) > 2$$

$$3 + 2\sqrt{2}\cos\left(2\pi 150 t\right) > 10^{0,2}$$

$$\cos\left(2\pi 150 t\right) > \frac{10^{0,2} - 3}{2\sqrt{2}} = -\frac{1}{2}$$



The fraction of time the power is greater than -101 dBm is:

$$f = \frac{\frac{4\cancel{\pi}}{3}}{2\cancel{\pi}} = \frac{2}{3} \approx 67\% \text{ of the time}$$

PROBLEM 4

(a) $\rho(\tau)$ is the relative change in $h(t)$ over a time τ

$$\rho(\tau) = \frac{E|h(t) - h(t+\tau)|^2}{E|h(t)|^2} \qquad R(\tau) = E[h(t)h^*(t-\tau)]$$

$$\rho(\tau) = \frac{E[(h(t) - h(t+\tau))(h(t) - h(t+\tau))^*]}{E[h(t)h^*(t)]}$$

$$= \frac{E[h(t)h^*(t) - h(t)h^*(t+\tau) - h(t+\tau)h^*(t) + h(t+\tau)h^*(t+\tau)]}{E[h(t)h^*(t)]}$$

$$= \frac{R(0) - R(-\tau) - R^*(-\tau) + R(0)}{R(0)} = 2 - \left[\frac{\overbrace{R(-\tau) + R^*(-\tau)}^{2Re[R(-\tau)]}}{R(0)}\right]$$

$$= 2 - \frac{2Re[R(-\tau)]}{R(0)}$$

(b) $h(t)$ follows a Jakes' spectrum with uniform angular distribution.

$R(\tau) = R(0) J_0(2\pi f_{max}\tau)$, where $J_0(\cdot)$ is Bessel function of the first Kind.

Plot $R(\tau)/R(0)$ vs τf_{max}, for $\tau f_{max} \in [0,5]$

[MATLAB CODE]

(c)

$$R(-\tau) = R(0) J_0(-2\pi f_{max} \tau)$$

$$\rho(\tau) = 2 - \frac{2 \cancel{R(0)} J_0(-2\pi f_{max}\tau)}{\cancel{R(0)}}$$

$$= 2 - 2 J_0(-2\pi f_{max}\tau) = 2\left[1 - J_0(-2\pi f_{max}\tau)\right]$$

$$\rho(\tau) = 10\% = 0{,}1 \Rightarrow 0{,}1 = 2\left(1 - J_0(-2\pi f_{max}\tau)\right) \Rightarrow 0{,}05 = 1 - J_0(-2\pi f_{max}\tau)$$

$$\Rightarrow J_0(-2\pi f_{max}\tau) = J_0(2\pi f_{max}\tau) = 0{,}95 \Rightarrow \tau = 357{,}5357$$

The time it takes for the channel to change 10% is: $\approx 358\,\mu s$

PROBLEM 5

(a)

$$p(\tau_\ell) = \frac{1}{\lambda} e^{-\tau_\ell/\lambda}$$

$$R(\Delta f) = E\left[H(t,f) H^*(t, f-\Delta f)\right]$$

$$= E\left[\frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell e^{2\pi i (f_\ell t - \tau_\ell f)} \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} g_\ell^* e^{\overbrace{-2\pi i (f_\ell t - \tau_\ell (f-\Delta f))}^{-2\pi i (f_\ell t - \tau_\ell f) - 2\pi i \tau_\ell \Delta f}}\right]$$

$$= E\left[\frac{1}{L} \sum_{\ell=1}^{L} |g_\ell|^2 e^{-2\pi i \tau_\ell \Delta f}\right] = \frac{1}{L} \sum_{\ell=1}^{L} E[|g_\ell|^2] E\left[e^{-2\pi i \tau_\ell \Delta f}\right] =$$

$$= \frac{1}{L} \sum_{\ell=1}^{L} G_i E\left[e^{-2\pi i \tau_\ell \Delta f}\right] \quad (*)$$

$$E\left[e^{-2\pi i \tau_\ell \Delta f}\right] = E\left[e^{i \tau_\ell \xi}\right] = \frac{1}{1 - i\lambda\xi} =$$

$\xi = -2\pi\Delta f$

characteristic function of an exponential R.V.

$$= \frac{1}{1 - i\lambda(-2\pi\Delta f)} = \frac{1}{1 + 2\pi i \lambda \Delta f}$$

$$(*) \quad R(\Delta f) = \frac{1}{L} \sum_{\ell=1}^{L} \frac{G}{1 + 2\pi i \Delta f \lambda} = \frac{G}{1 + 2\pi i \Delta f \lambda}$$

(b)

$$\rightarrow E\left|H(t,f) - H(t,f+W)\right|^2 = E\left[\left(H(t,f) - H(t,f+W)\right)\left(H^*(t,f) - H^*(t,f+W)\right)\right]$$

$$= E\left[H(t,f)H^*(t,f) - H(t,f)H^*(t,f+W) - H(t,f+W)H^*(t,f) + H(t,f+W)H^*(t,f+W)\right]$$

$$= R(0) - R(W) - R^*(W) + R(0) = 2R(0) - 2\,\mathrm{Re}\left[R(W)\right]$$

$$\rightarrow E\left|H(t,f)\right|^2 = E\left[H(t,f)H^*(t,f)\right] = R(0)$$

$\rightarrow E|H(t,f)|^2 = E[H(t,f)H^*(t,f)] = R(0)$

$$\rightarrow R(\Delta f) = \frac{G}{1+2\pi i\Delta f\lambda}\,\frac{1-2\pi i\Delta f\lambda}{(1-2\pi i\Delta f\lambda)} = \frac{G(1-2\pi i\Delta f\lambda)}{1+4\pi^2\Delta f^2\lambda^2}$$

$$Re[R(\Delta f)] = \frac{G}{1+4\pi^2\Delta f^2\lambda^2}$$

$$\rightarrow 2R(0) - 2Re[R(W)] \leq \frac{1}{2}R(0)$$

$$\frac{3}{2}R(0) - 2Re[R(W)] \leq 0$$

$$\frac{3}{2}G - \frac{2G}{1+4\pi^2W^2\lambda^2} \leq 0 \Rightarrow 3G - \frac{4G}{1+4\pi^2W^2\lambda^2} \leq 0$$

$$3G + 12G\pi^2W^2\lambda^2 - 4G \leq 0 \Rightarrow 12G\pi^2W^2\lambda^2 \leq G$$

$$W^2 \leq \frac{\cancel{G}}{12\cancel{G}\pi^2\lambda^2} = \frac{1}{12\pi^2\lambda^2} \Rightarrow W = \frac{1}{\sqrt{12\pi^2\lambda^2}} = 918,88\ \text{KHz}$$

$$|W| \leq 918,88\ \text{KHz}$$