HW5 Tommy Azzino

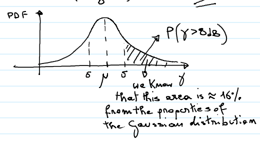
Saturday, March 27, 2021 6:23 PM

PROBLEM 1

(a) we have that $\gamma\,[dB] \sim \mathcal{N}(\mu, \sigma^2)$ where $\mu = 6\,dB$, $\sigma = 2\,dB$.

$P(\gamma > 8)$?

$$P(\gamma > 8) = P(\underset{8\,dB}{\gamma > \gamma_0}) = \int_{\gamma_0}^{\infty} \underbrace{f_\gamma(\gamma)}_{\text{pdf of this R.V.}}\, d\gamma = Q\left(\frac{\gamma_0 - \mu}{\sigma}\right) = 0{,}1587$$



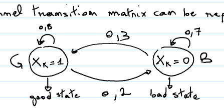
(b)

The probability the transmission will pass with MCS 2 is:

$$P(\gamma > 4 \mid \gamma \le 8) = \frac{P(\gamma > 4, \gamma \le 8)}{P(\gamma \le 8)} = \frac{P(\gamma > 4, \gamma \le 8)}{1 - P(\gamma > 8)} = \frac{P(\gamma > 4) - P(\gamma > 8)}{1 - P(\gamma > 8)} = 0{,}8114$$

PROBLEM 2

The channel transition matrix can be represented like this:



$$P = \begin{array}{c c} & \begin{array}{cc} G & B \end{array} \\ \begin{array}{c} G \\ B \end{array} & \begin{bmatrix} 0{,}8 & 0{,}2 \\ 0{,}3 & 0{,}7 \end{bmatrix} \end{array}$$

$P(Y_K = 1 \mid X_K = 1) = 0{,}8 \Rightarrow P(Y_K = 0 \mid X_K = 1) = 0{,}2$

$P(Y_K = 1 \mid X_K = 0) = 0{,}4 \Rightarrow P(Y_K = 0 \mid X_K = 0) = 0{,}6$

(a)

$a_K(i) = P(X_K = i)$

$a_{K+1}(i)$ in terms of the values of $a_K(j)$?

$$\begin{cases} P(X_{K+1} = 1) = \underbrace{P(X_{K+1} = 1 \mid X_K = 1)}_{\text{Markov property}} P(X_K = 1) + \underbrace{P(X_{K+1} = 1 \mid X_K = 0)}_{\text{Markov property}} P(X_K = 0) \\ P(X_{K+1} = 0) = P(X_{K+1} = 0 \mid X_K = 1) P(X_K = 1) + P(X_{K+1} = 0 \mid X_K = 0) P(X_K = 0) \end{cases}$$

$$a_{K+1}(i) = \sum_{j=0}^{1} P(X_{K+1} = i \mid X_K = j) \underbrace{P(X_K = j)}_{a_K(j)} = \sum_{j=0}^{1} P(X_{K+1} = i \mid X_K = j)\, a_K(j)$$

In matrix form:

$$\begin{bmatrix} a_{K+1}(1) \\ a_{K+1}(0) \end{bmatrix} = P^T \begin{bmatrix} a_K(1) \\ a_K(0) \end{bmatrix} = \begin{bmatrix} 0{,}8 & 0{,}2 \\ 0{,}3 & 0{,}7 \end{bmatrix}^T \begin{bmatrix} a_K(1) \\ a_K(0) \end{bmatrix} = \begin{bmatrix} 0{,}8\, a_K(1) + 0{,}3\, a_K(0) \\ 0{,}2\, a_K(1) + 0{,}7\, a_K(0) \end{bmatrix}$$

(b)

$$\alpha_K^{o}(i) = P(X_K = i \mid Y_0 = 0, \ldots, Y_{K-1} = 0)$$

$$\alpha_{K+1}^{o}(i) = P(X_{K+1} = i \mid Y_0 = 0, \ldots, Y_K = 0) = \frac{P(X_{K+1} = i, Y_K = 0 \mid Y_0 = 0, \ldots, Y_{K-1} = 0)}{P(Y_K = 0 \mid Y_0 = 0, \ldots, Y_{K-1} = 0)} =$$

$$= \frac{\sum_{j=0}^{1} P(X_{K+1} = i, Y_K = 0 \mid Y_0 = 0, \ldots, Y_{K-1} = 0, X_K = j)\, P(X_K = j \mid Y_0 = 0, \ldots, Y_{K-1} = 0)}{\sum_{j=0}^{1} P(Y_K = 0 \mid Y_0 = 0, \ldots, Y_{K-1} = 0 \mid X_K = j)\, P(X_K = j \mid Y_0 = 0, \ldots, Y_{K-1} = 0)} \to \alpha_K^{o}(j)$$

$$= \frac{\sum_{j=0}^{1} P(X_{K+1} = i \mid X_K = j)\, P(Y_K = 0 \mid X_K = j)\, \alpha_K^{o}(j)}{\sum_{j=0}^{1} P(Y_K = 0 \mid X_K = j)\, \alpha_K^{o}(j)} \Rightarrow \begin{cases} \alpha_{K+1}^{o}(1) = \dfrac{0{,}8\,0{,}2\, \alpha_K^{o}(1) + 0{,}3\,0{,}6\, \alpha_K^{o}(0)}{0{,}2\, \alpha_K^{o}(1) + 0{,}6\, \alpha_K^{o}(0)} \\ \alpha_{K+1}^{o}(0) = \dfrac{0{,}2\,0{,}2\, \alpha_K^{o}(1) + 0{,}7\,0{,}6\, \alpha_K^{o}(0)}{0{,}2\, \alpha_K^{o}(1) + 0{,}6\, \alpha_K^{o}(0)} \end{cases}$$

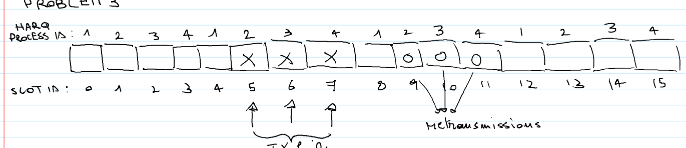
$$\begin{cases} \alpha_{K+1}^{o}(1) = \dfrac{0{,}16\, \alpha_K^{o}(1) + 0{,}18\, \overbrace{\alpha_K^{o}(0)}^{(1 - \alpha_K^{o}(1))}}{0{,}2\, \alpha_K^{o}(1) + 0{,}6\, \alpha_K^{o}(0)} = \dfrac{0{,}18 - 0{,}02\, \alpha_K^{o}(1)}{0{,}6 - 0{,}4\, \alpha_K^{o}(1)} \\ \alpha_{K+1}^{o}(0) = \dfrac{0{,}04\, \alpha_K^{o}(1) + 0{,}42\, \alpha_K^{o}(0)}{0{,}2\, \underbrace{\alpha_K^{o}(1)}_{(1 - \alpha_K^{o}(1))} + 0{,}6\, \alpha_K^{o}(0)} = \dfrac{0{,}04 + 0{,}38\, \alpha_K^{o}(0)}{0{,}2 + 0{,}4\, \alpha_K^{o}(0)} \end{cases}$$

c) $$P(T = K) = P(Y_K = 1, Y_0 = 0, \ldots, Y_{K-1} = 0) = P(Y_K = 1 \mid Y_0 = 0, \ldots, Y_{K-1} = 0)\, P(Y_0 = 0, \ldots, Y_{K-1} = 0) = \underbrace{\sum_{j=0}^{1} P(Y_K = 1 \mid X_K = j)\, \alpha_K^{o}(j)}_{0{,}8\, \overset{1 - \alpha_K^{o}(0)}{\alpha_K^{o}(1)} + 0{,}4\, \alpha_K^{o}(0) = 0{,}8 - 0{,}4\, \alpha_K^{o}(0)} \cdot P(Y_0 = 0, \ldots, Y_{K-1} = 0)$$

$$P(Y_0 = 0, \ldots, Y_{K-1} = 0) = P(T > K-1) = 1 - P(T \le K-1) = 1 - \sum_{j=0}^{K-1} P(T = j)$$

$$P(T = K) = \left[1 - \sum_{j=0}^{K-1} P(T = j)\right]\left[0{,}8 - 0{,}4\, \alpha_K^{o}(0)\right]$$

PROBLEM 3



(a)

PDU Index	First Slot PDU decoded index
0	0
1	1
2	2
3	3
4	4
5	9

0	0
1	1
2	2
3	3
4	4
5	9
6	10
7	11
8	8
9	12

(b) Decoded PDUs are delivered in order to the upper layer.

PDU Index	PDU Arrival time at the Higher layer [Slot Index]
0	3
1	4
2	5
3	6
4	7
5	12
6	13
7	14
8	14
9	15

→ PDU 8 was received correctly before PDU 7. However, PDU 8 is delivered to the higher layer only when PDU 7 is received correctly.

PROBLEM 6

(a) • $r_K = w_K \quad w_K \sim CN(0,N) \Rightarrow r_K \sim CN(0,N)$

$$\hat{N} = \frac{1}{K}\sum_{K=1}^{K}|r_K|^2$$

$$r_K = r_K^R + j r_K^I \quad \begin{cases} r_K^R \sim N(0,\frac{N}{2}) \\ r_K^I \sim N(0,\frac{N}{2}) \end{cases} \quad |r_K|^2 = \left(\sqrt{r_K^{R\,2} + r_K^{I\,2}}\right)^2 = r_K^{R\,2} + r_K^{I\,2}$$

$\frac{N}{2} \to \sigma^2$

$$\hat{N} = \frac{1}{K}\sum_{K=1}^{K} r_K^{R\,2} + r_K^{I\,2} = \frac{1}{K}\sum_{K=1}^{K} r_K^{R\,2} + \frac{1}{K}\sum_{K=1}^{K} r_K^{I\,2} = \frac{1}{K}\left[\sum_{K=1}^{K} r_K^{R\,2} + \sum_{K=1}^{K} r_K^{I\,2}\right]$$

Now, we know that the chi-square distribution with K degrees of freedom is the distribution of the sum of the squares of K independent standard normal variables.

$$S_K^R = r_K^R\sqrt{\frac{N}{2}} \; ; \quad S_K^I = r_K^I\sqrt{\frac{N}{2}}$$

$$\hat{N} = \frac{1}{K}\left[\sum_{K=1}^{K}\frac{S_K^{R\,2}}{\frac{2}{N}} + \sum_{K=1}^{K}\frac{S_K^{I\,2}}{\frac{2}{N}}\right] = \frac{N}{2K}\left[\underbrace{\sum_{K=1}^{K} S_K^{R\,2} + \sum_{K=1}^{K} S_K^{I\,2}}\right] \sim \frac{N}{2K}\,\chi^2(2K)$$

Chi-square symbol

$\hat{N}$ is a scaled chi-square distribution with 2K degrees of freedom

sum of 2 independent identical (iid) chi-square distributions is chi-square with K+K = 2K degrees of freedom (assuming independency)

• $r_K = h_K x_K + w_K \quad h_K \sim CN(0, E_s) \quad w_K \sim CN(0,N)$

$$\hat{S} = \frac{1}{M}\sum_{K=1}^{M}|r_K|^2 \,, \quad |x_K| = 1$$

$$h_K = h_K^R + j h_K^I \quad h_K^R \sim N(0,\frac{E_s}{2}) \quad h_K^I \sim N(0,\frac{E_s}{2})$$

$$|r_K| = |h_K|\underset{1}{\underbrace{|x_K|}} + |w_K| = |h_K| + |w_K| \Rightarrow r_K \sim CN(0, E_s+N)$$ → assuming independency

$$r_K^R \sim N(0,\frac{E_s+N}{2}) \quad r_K^I \sim N(0,\frac{E_s+N}{2}) \Rightarrow S_K^R = r_K^R\sqrt{\frac{E_s+N}{2}} \quad S_K^I = r_K^I\sqrt{\frac{E_s+N_0}{2}}$$

$$\hat{S} = \frac{E_s+N}{2M}\left[\sum_{K=1}^{M} S_K^{R\,2} + \sum_{K=1}^{M} S_K^{I\,2}\right] \underset{iid}{\sim} \frac{E_s+N}{2M}\,\chi^2(2M)$$

Therefore $\hat{S}$ is a scaled chi-square distribution with 2M degrees of freedom

b) From wikipedia: A F-distributed random variable with parameters d_1 and d_2 arises from the ratio of two appropriately scaled chi-square random variables $\left[X = \frac{U_1/d_1}{U_2/d_2}\right]$

$$\frac{\hat{S}}{\hat{N}} = \frac{\frac{E_s+N}{2M}\left[\sum_{K=1}^{M} S_K^{R\,2} + \sum_{K=1}^{M} S_K^{I\,2}\right]}{\frac{N}{2K}\left[\sum_{K=1}^{K} S_K^{R\,2} + \sum_{K=1}^{K} S_K^{I\,2}\right]} = \frac{(E_s+N)K}{N\,M}\,\frac{\chi^2(2M)}{\chi^2(2K)} = \frac{(E_s+N)\cancel{K}}{N\cancel{M}}\,\frac{\cancel{2M}}{\cancel{2K}}\left[\frac{\chi^2(2M)/2M}{\chi^2(2K)/2K}\right]$$

$\sim F(2M,2K)$ assuming independency

$$\frac{\hat{S}}{\hat{N}} \sim \frac{E_s+N}{N}F(2M,2K)$$

$$\frac{\hat{S}}{\hat{N}} \sim \left(\frac{E_S+N}{N}\right) F(2M, 2K)$$

F-distribution

$\sim F(2M,2K)$ assuming independency of the two chi-square distributions

c)

$$\hat{\gamma} = \max\left\{0, \frac{\hat{S}}{\hat{N}} - 1\right\} \qquad \gamma_{true} = \frac{E_S}{N} = 3\,dB \quad \gamma_{tol} = 0{,}5\,dB \quad , \quad K = M$$

$$\hat{\gamma} = \max\left\{0, \frac{\hat{S}}{\hat{N}} - 1\right\} = \begin{cases} 0 & \text{if } \frac{\hat{S}}{\hat{N}} < 1 \Rightarrow \gamma_{err} = |\gamma_{true} - 0| = \gamma_{true} > \gamma_{tol} \text{ with prob } 1 \\ \frac{\hat{S}}{\hat{N}} & \text{if } \frac{\hat{S}}{\hat{N}} > 1 \Rightarrow \gamma_{err} = |\hat{\gamma} - \gamma_{true}| \end{cases}$$

$$P(\gamma_{err} < \gamma_{tol}) = P(|\hat{\gamma} - \gamma_{true}| < \gamma_{tol}) = P(-\gamma_{tol} < \hat{\gamma} - \gamma_{true} < \gamma_{tol})$$

$$= P\left(-\gamma_{tol} < \frac{\hat{S}}{\hat{N}} - 1 - \gamma_{true} < \gamma_{tol}\right) = P\left(1 + \gamma_{true} - \gamma_{tol} < \frac{\hat{S}}{\hat{N}} < \gamma_{tol} + \gamma_{true} + 1\right)$$

$\sim \frac{E_S+N}{N} F(2K, 2K)$

$$= P\left(\frac{1+\gamma_{true} - \gamma_{tol}}{\frac{E_S+N}{N}} < \left(\frac{\hat{S}}{\hat{N}}\right)^* < \frac{\gamma_{tol} + \gamma_{true} + 1}{\frac{E_S+N}{N}}\right)$$

$\left(\frac{\hat{S}}{\hat{N}}\right)^* \sim F(2K, 2K)$

$$= P\left(\frac{1+\gamma_{true} - \gamma_{tol}}{1+\gamma_{true}} < \left(\frac{\hat{S}}{\hat{N}}\right)^* < \frac{\gamma_{tol} + \gamma_{true} + 1}{\gamma_{true} + 1}\right)$$

$$= FCDF\left(\frac{\gamma_{tol}}{\gamma_{true}+1} + 1\right) - FCDF\left(1 - \frac{\gamma_{tol}}{\gamma_{true}+1}\right)$$

MATLAB

[SEE MATLAB CODE]

PROBLEM 7

$$r_K = h x_K + w_K \,, \quad w_K \sim CN(0, N) \qquad |h x_K|^2 = E_S$$

$$\hat{h} = \frac{\sum_{K=1}^{K} x_K^* r_K}{\sum_{K=1}^{K} |x_K|^2} \,, \qquad \hat{N} = \frac{\alpha}{K} \sum_{K=1}^{K} |r_K - \hat{h} x_K|^2$$

a) Find α in order to have an unbiased estimator for $\hat{N}$ (i.e. $E[\hat{N}] = N$)

$$\hat{h} = \frac{\sum_{K=1}^{K} \overbrace{x_K^* (h x_K}^{|x_K|^2 h} + w_K)}{\sum_{K=1}^{K} |x_K|^2} = h \frac{\sum_{K=1}^{K} |x_K|^2}{\sum_{K=1}^{K} |x_K|^2} + \frac{\sum_{K=1}^{K} x_K^* w_K}{\sum_{K=1}^{K} |x_K|^2} = h + \frac{\sum_{K=1}^{K} x_K^* w_K}{\sum_{K=1}^{K} |x_K|^2}$$

$$E[\hat{N}] = E\left[\frac{\alpha}{K} \sum_{K=1}^{K} \left| h x_K + w_K - \left(h + \frac{\sum_{K=1}^{K} x_K^* w_K}{\sum_{K=1}^{K} |x_K|^2}\right) x_K \right|^2\right] \qquad |x_K|^2 = \frac{E_S}{h^2}$$

$$= \frac{\alpha}{K} E\left[\sum_{K=1}^{K} \left| w_K - x_K \frac{\sum_{K=1}^{K} x_K^* w_K}{\sum |x_K|^2} \right|^2\right]$$

$$= \frac{\alpha}{K} E\left[\sum_{K=1} \left| w_K - x_K \frac{\sum_{K=1} x_K^* w_K}{\sum_{K=1} |x_K|^2} \right|^2\right]$$

$$= \frac{\alpha}{K} \sum_{K=1}^{K} E\left[|w_K|^2 \left(1 - \frac{1}{K}\right)^2\right] + \frac{E_s}{h^2} \frac{E\left[\sum |x_l^* w_K|^2\right]}{\frac{K^2 E_s^2}{h^4}} \Rightarrow |x_K|^2 = \frac{E_s}{h^2}$$

$$= \frac{\alpha}{K} N \left(\frac{(K-1)^2}{K^2} + \frac{K-1}{K^2}\right) = \frac{\alpha}{K} N \left[\frac{K^2 - 2K \cancel{+1} + K - \cancel{1}}{K^2}\right] = \frac{\alpha}{\cancel{K}} N \frac{\cancel{K}(K-1)}{K} = \frac{\alpha}{N} \frac{(K-1)}{K}$$

$\alpha = \frac{K}{K-1}$ gives the unbiased estimator

(b) Unbiased estimate of E_s

$$\hat{E}_s = \frac{1}{K} \sum_{K=1}^{K} |r_K|^2 - \hat{N} \Rightarrow E\left[\hat{E}_s\right] = \frac{1}{K} \sum_{K=1}^{K} E\left[|r_K|^2\right] - N = \frac{\cancel{K}}{\cancel{K}}(E_s + N) - N = E_s$$

unbiased

$E\left[\hat{E}_s\right] = E_s$

PROBLEM 9

$$R(\gamma) = \min\{\rho_{max}, \alpha \log_2(1+\gamma)\}$$

All symbols in transmission K experience some SNR $\gamma_K = K = 1, \dots, K$

(a) For a given SNR target (γ_{target}) for the codeword

The condition on each γ_K is $\sum_{j=1}^{K} \gamma_K \geq \gamma_{target}$ for $K = 1, \dots, K$

if the packet passes after K transmission, the rate with chase combining will be

$$R_{chase}(\gamma) = \min\left\{\frac{\rho_{max}}{K}, \frac{\alpha}{K} \log_2(1 + \gamma_{target})\right\}$$

(b) The condition on each γ_K is $\frac{1}{K} \sum_{K=1}^{K} M_K \geq \log_2(1 + \gamma_{target})$

$$\frac{1}{K} \sum_{K=1}^{K} \log_2(1+\gamma_K) \geq \log_2(1 + \gamma_{target})$$

$$\frac{1}{K}\sum_{k=1}^{K}\log_2(1+\gamma_k) \geq \log_2(1+\gamma_{target})$$

The rate with IR will be:

$$R_{IR}(\gamma) = \min\left\{\rho_{max},\ \alpha\log_2(1+\gamma_{target})\right\}$$

(c) $\boxed{\text{SEE MATLAB}}$

Wireless Communications EL-GY 6023

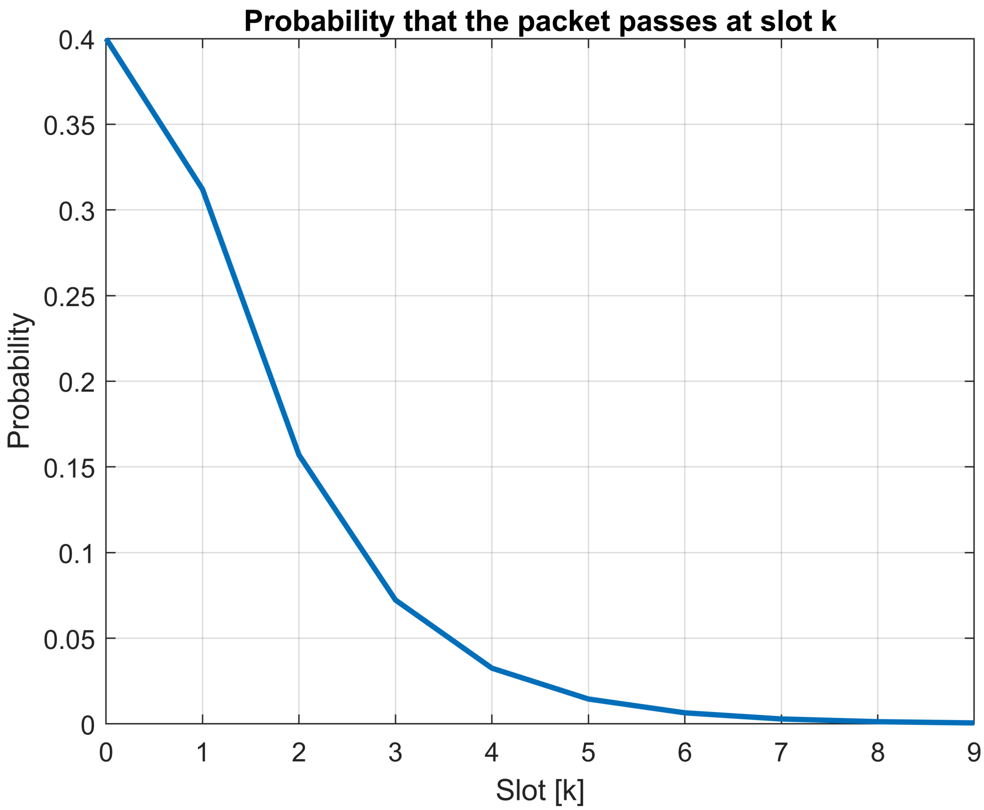
Homework 5 - Tommy Azzino (ta1731)

Problem 2

```
ks = 0:9;
Pt = zeros(length(ks),1);
alpha_0 = 1; % we know that X_0=0
Pt(1) = 0.4; % P(Y0=1|X0=0);
for k=ks(2:end)
    alpha_0 = (0.04+0.38*alpha_0)/(0.2+0.4*alpha_0);
    Pt(k+1) = (1-sum(Pt(1:(k))))*(0.8-0.4*alpha_0);
end
disp(Pt);
```

```
    0.4000
    0.3120
    0.1570
    0.0723
    0.0325
    0.0145
    0.0065
    0.0029
    0.0013
    0.0006
```

```
figure;
plot(ks, Pt, "LineWidth",2); grid on;
xlabel("Slot [k]"); ylabel("Probability");
title("Probability that the packet passes at slot k");
```




```
disp(sum(Pt));
```

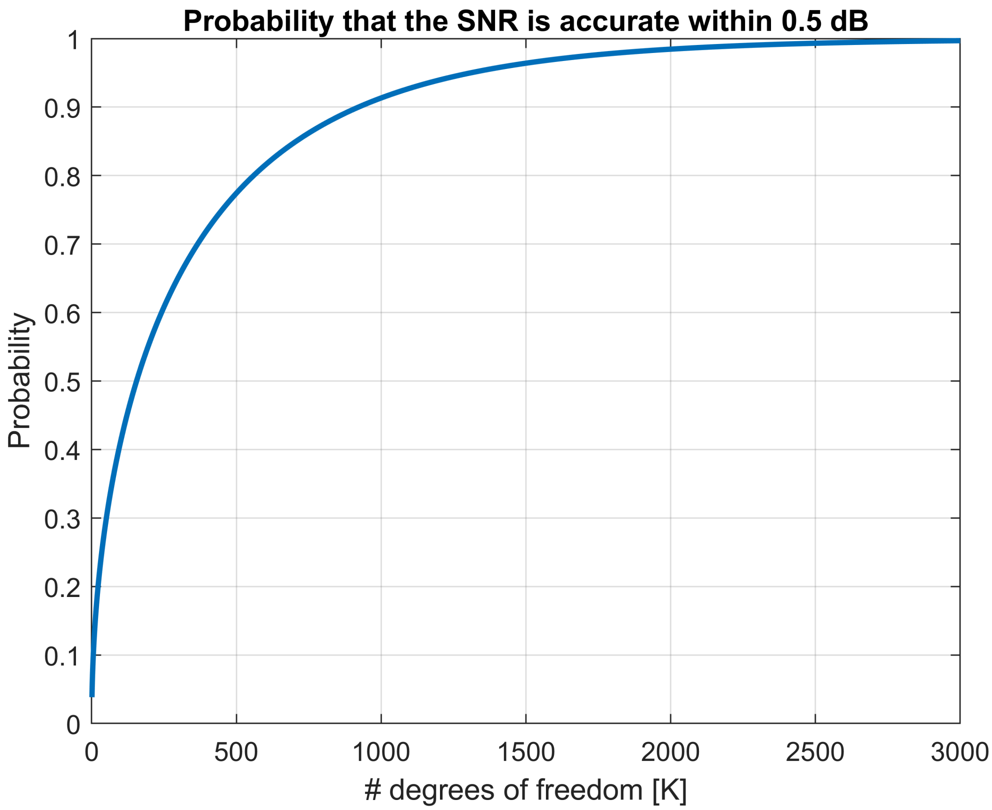
```
0.9995
```

If I consider more slots the sum of the probability values is 1, as expected. Also, it never exceeds 1, even with a huge number of slots considered.

Problem 6

```
num_ks = 3000;
k = 1:num_ks;
true_avgSnr_db = 3;
true_avgSnr = db2pow(true_avgSnr_db);
target_db = 0.5;
gamma_max = db2pow(true_avgSnr_db+target_db);
gamma_min = db2pow(true_avgSnr_db-target_db);
a = ((gamma_max+1)/(true_avgSnr+1));
b = ((gamma_min+1)/(true_avgSnr+1));
prob = fcdf(a,2*k,2*k) - fcdf(b,2*k,2*k);

figure;
plot(k, prob, "LineWidth",2); grid on;
xlabel("# degrees of freedom [K]");
ylabel("Probability");
title("Probability that the SNR is accurate within 0.5 dB");
```



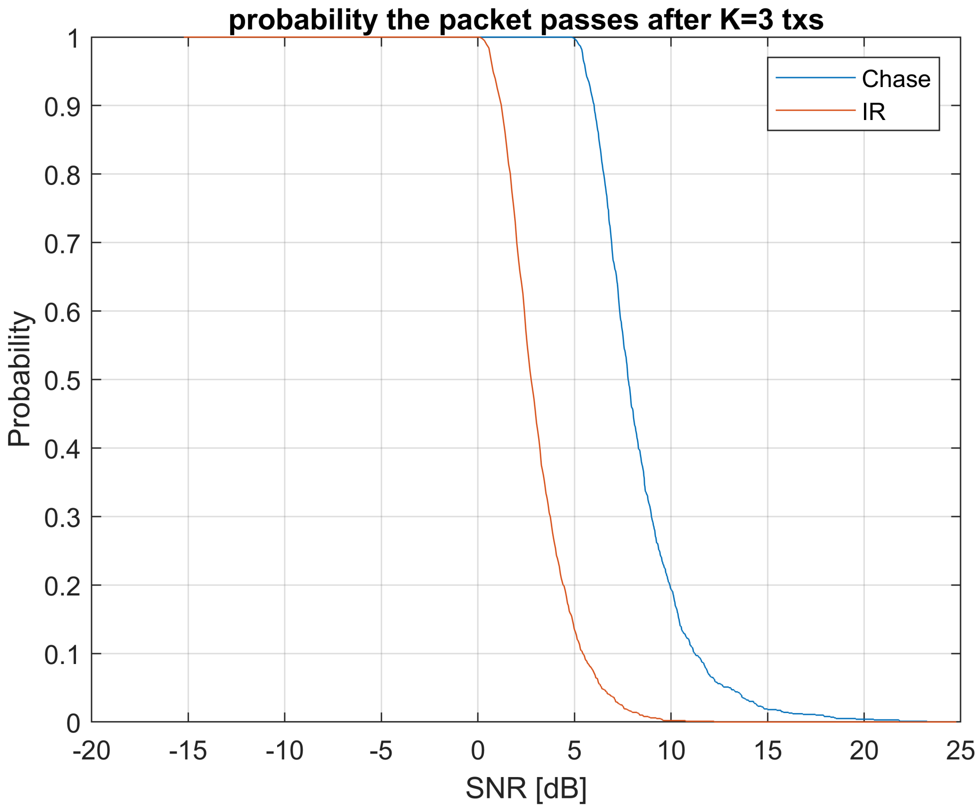
As we can see from the figure above, as we increase the number of degrees of freedom (i.e. the number of symbols we use to estimate the SNR), the probability of having an estimation accuracy within 0.5 dB increases

Problem 9

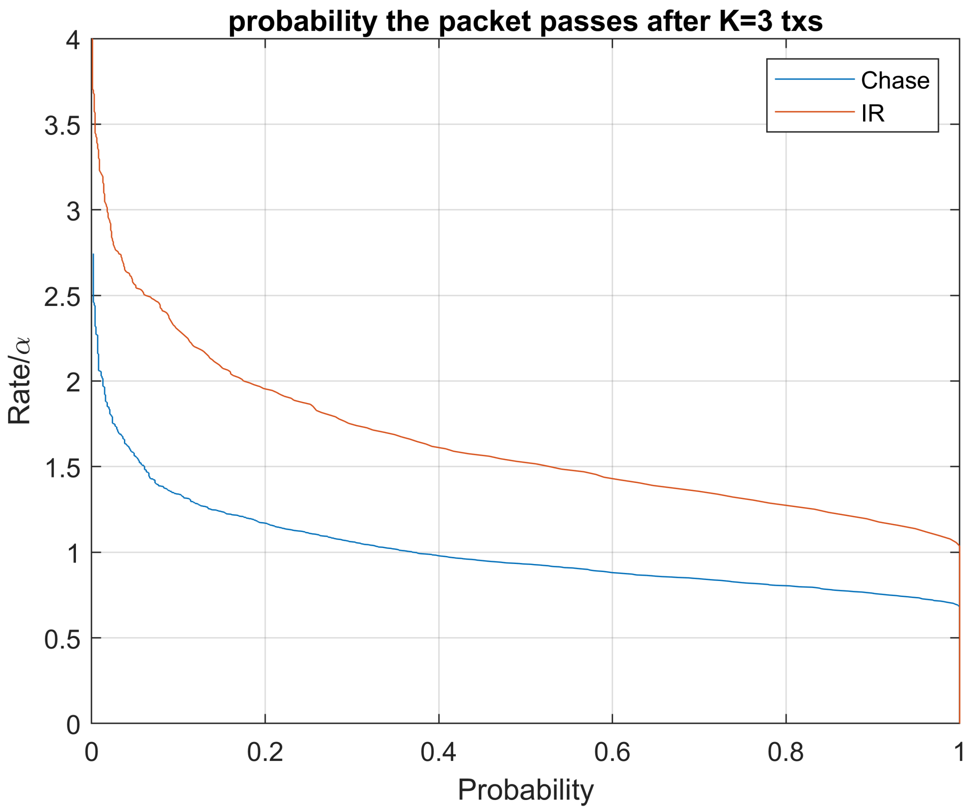
```
K = 3;
npoints = 1000;
avg_snr = 3;
gammas = exprnd(repmat(avg_snr,[npoints,K]));
gammas_lin = db2pow(gammas);

n_targets = 10000;
gamma_target = linspace(0,300,n_targets);
prob_chase = zeros(n_targets,1);
prob_ir = zeros(n_targets,1);
Rchase = zeros(n_targets,1);
Rir = zeros(n_targets,1);
for i=1:n_targets
    gamma_t = gamma_target(i);
    passed = sum(gammas_lin,2) >= gamma_t;
    prob_chase(i) = sum(passed)/length(passed);
    passed_ir = sum(log2(1+gammas_lin),2)*(1/K) >= log2(1+gamma_t);
    prob_ir(i) = sum(passed_ir)/length(passed_ir);
    Rchase(i) = log2(1+gamma_t)/K;
    Rir(i) = log2(1+gamma_t);
end
```

```
figure;
plot(pow2db(gamma_target), prob_chase); grid on; hold on;
plot(pow2db(gamma_target), prob_ir);
legend("Chase", "IR"); xlabel("SNR [dB]"); ylabel("Probability");
title("probability the packet passes after K=3 txs");
```




```
figure;
plot(prob_chase,Rchase); grid on; hold on;
plot(prob_ir, Rir);
legend("Chase", "IR"); ylabel("Rate/\alpha");
xlabel("Probability");
title("probability the packet passes after K=3 txs");
ylim([0 4]); xlim([0 1]);
```


probability the packet passes after K=3 txs
Chase
IR
Rate/α
Probability
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.2
0.4
0.6
0.8
1