HW 8

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$\boxed{\text{PROBLEM 1}}$ ULA

$$\underline{u}(\phi) = [1, e^{i\beta \sin(\phi)}, \dots, e^{i\beta(N-1)\sin(\phi)}]^T \quad , \beta = \frac{2\pi d}{\lambda}$$

ϕ is the azimuth angle relative to boresight

(a) optimal beamforming vector for $\phi_o = 30°$, $N=8$, $d = \frac{\lambda}{2}$

$$\beta = \frac{2\pi d}{\lambda} = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi \quad ; \quad \sin(\phi_o) = \sin(30°) = \frac{1}{2}$$

The optimal BF vector for AoA $\phi_o = 30°$ is

$$\hat{w}(\phi_o) = \frac{1}{\sqrt{N}}\bar{u}(\phi_o) = \frac{1}{2\sqrt{2}}\left[1, e^{-i\frac{\pi}{2}}, e^{-i\pi}, e^{-i\frac{3\pi}{2}}, e^{-i2\pi}, e^{-i\frac{5\pi}{2}}, e^{-i3\pi}, e^{-i\frac{7\pi}{2}}\right]^T = \frac{1}{2\sqrt{2}}\left[1, -i, -1, i, 1, -i, -1, i\right]^T$$

element-wise conjugate

(b) Path arrives at $\phi = 60°$ Beamforming gain? (BF vector of part (a))

$$\underline{u}(\phi) = [1, e^{i\beta\sin(\phi)}, \dots, e^{i\beta(N-1)\sin(\phi)}]^T =$$

$$\underline{u}^*(\phi_o)\,\underline{u}(\phi) = [1, e^{-i\beta\sin(\phi_o)}, \dots, e^{-i\beta(N-1)\sin(\phi_o)}]\begin{bmatrix} 1 \\ e^{i\beta\sin(\phi)} \\ \vdots \\ e^{i\beta(N-1)\sin(\phi)}\end{bmatrix}$$

$$= 1 + e^{i\beta(\sin\phi - \sin\phi_o)} + \dots + e^{i\beta(N-1)(\sin\phi - \sin\phi_o)}$$

$$= 1 + e^{i\gamma} + \dots e^{i(N-1)\gamma} \quad , \gamma = \beta(\sin\phi - \sin\phi_o)$$

$$AF' = 1 + e^{i\gamma} + \dots e^{i(N-1)\gamma} \quad , \quad AF' e^{i\gamma} = e^{i\gamma} + e^{2i\gamma} + e^{iN\gamma}$$

$$AF'(1 - e^{i\gamma}) = 1 - e^{iN\gamma} \Rightarrow AF' = \frac{1 - e^{iN\gamma}}{1 - e^{i\gamma}} = \frac{e^{iN/2\gamma}\left(e^{-iN/2\gamma} - e^{iN/2\gamma}\right)}{e^{i\gamma/2}\left(e^{-i\gamma/2} - e^{i\gamma/2}\right)} = e^{i(N-1)\frac{\gamma}{2}}\frac{\sin\left(N\frac{\gamma}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}$$

$$AF(\phi, \phi_o) = \frac{1}{\sqrt{N}}\underline{u}^*(\phi_o)\underline{u}(\phi) = \frac{e^{i(N-1)\frac{\gamma}{2}}}{\sqrt{N}}\frac{\sin(N\gamma/2)}{\sin(\gamma/2)} \quad , \quad \gamma = \beta(\sin\phi - \sin\phi_o)$$

$$\gamma = \pi\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{\sqrt{3}-1}{2}\pi$$

$N=8$

$$\text{Beamforming gain} = |AF(\phi,\phi_o)|^2 = \frac{\sin^2(N\gamma/2)}{N\sin^2(\gamma/2)} = \frac{\sin^2((\sqrt{3}-1)2\pi)}{8\sin^2\left(\frac{\sqrt{3}-1}{4}\pi\right)} = 0{,}4174$$

$$\gamma = 0 \left\{ \frac{\sin(N\gamma/2)}{\sin(\gamma/2)} \overset{H}{=} \frac{\cos(N\gamma/2)N/2}{\cos(\gamma/2)1/2} \overset{\gamma=0}{=} \frac{N}{\cancel{2}}\cancel{2} = N \right.$$

$\boxed{\text{PROBLEM 2}}$ SIMO Channel

$$\underline{r} = \underline{h}x + \underline{v} \quad , \quad \underline{v} \sim CN(0, N_o I) \quad , \quad |x|^2 = E_x$$

(a) SNR after beamforming with a vector $\underline{w}$?

$$z = \underline{w}^T\underline{r} = \underline{w}^T(\underline{h}x + \underline{v}) = \underbrace{\underline{w}^T\underline{h}}_{\alpha}x + \underbrace{\underline{w}^T\underline{v}}_{\beta}$$

$$\gamma = \frac{|\alpha|^2|x|^2}{E|\beta|^2} = \frac{|\underline{w}^T\underline{h}|^2 E_x}{\|\underline{w}\|^2 N_o} \qquad E|\beta|^2 = E|\underline{w}^T\underline{v}|^2 = \|\underline{w}\|^2 N_o$$

(b) MAX SNR if we are allowed any vector $\underline{w}$?

Max SNR when $\underline{w} = c\bar{\underline{h}}$ (Any constan $c \neq 0$ can be used). Hence $\underline{w} = \bar{\underline{h}}$ ($c=1$)

In this case: $$\gamma_{MAX} = \frac{|\underline{h}^*\underline{h}|^2 E_x}{\|\underline{h}\|^2 N_o} = \frac{\|\underline{h}\|^4}{\|\underline{h}\|^2}\frac{E_x}{N_o} = \|\underline{h}\|^2\frac{E_x}{N_o}$$

(c) $\underline{w}$ must have constant magnitude ($|w_m| = 1$)

$$|w_m| = 1 \Rightarrow w_m = e^{i\theta_m} \qquad \|\underline{w}\|^2 = \sum_{i=1}^N |w_m|^2 = N$$

$$\gamma_{MAX} = \frac{|\underline{w}^T\underline{h}|^2}{N}\frac{E_x}{N_o} = \frac{\left(\sum_{i=1}^N |h_i|\right)^2}{N}\frac{E_x}{N_o}$$

$$|\underline{w}^T\underline{h}|^2 = \left|\sum_{i=1}^N w_i h_i\right|^2 \le \left(\sum_{i=1}^N \overset{1}{\overset{\shortparallel}{|w_i|}}\,|h_i|\right)^2 = \left(\sum_{i=1}^N |h_i|\right)^2$$

(d) Max SNR after beamforming when $w_m \neq 0$ only for one antenna (m)

$\underline{w} = [0, \cdots, 0, ch_m, 0, \cdots, 0]$

pick antenna with largest channel gain

$$\gamma_{max} = \max_m \frac{|\underline{w}^T \underline{h}|^2 E_x}{\|\underline{w}\|^2 N_0} = \max_m \frac{\cancel{c^2}|h_m|^{\cancel{4}\,2} E_x}{\cancel{c^2|h_m|^2} N_0} = \max_m |h_m|^2 \frac{E_x}{N_0}$$

(c)

$\underline{h} = [4, 2+i, -1, i]^T \qquad E_x/N_0 = 5\text{dB}$

b) $\gamma_{MAX} = \|\underline{h}\|^2 \frac{E_x}{N_0} \sim 18{,}62\ \text{dB}$

c) $\gamma_{MAX} = \frac{\left(\sum_{i=1}^{N} |h_i|\right)^2}{N} \frac{E_x}{N_0} \sim 17{,}29\ \text{dB}$, $N = 4$

d) $\gamma_{max} = \max_{w_m} |h_m|^2 \frac{E_x}{N_0} \sim 17{,}04\ \text{dB}$ (which corresponds to $m=1$, the first antenna)

linear $\rightarrow \left(16 \cdot \frac{E_x}{N_0}\right) \rightarrow$ dB

$\boxed{\text{PROBLEM 3}}$

$\underline{r} = \underline{h}(f)x + \underline{v} \qquad \underline{h}(f) = \sum_{\ell=1}^{L} a_\ell e^{2\pi i f \tau_\ell} \underline{u}(\theta_\ell)$ (L: paths)

$\underline{v} \sim CN(0, N_0 I) \qquad |x|^2 = E_x$

$N = 8 \quad d = \frac{\lambda}{2} \quad L = 3 \qquad SNR = \frac{E_x}{N_0}|a_\ell|^2$ "path phase" is the angle of a_ℓ

[SEE MATLAB] $\qquad \underline{h}(f) = \sum_{\ell=1}^{L} |a_\ell| e^{i \angle a_\ell} e^{2\pi i f \tau_\ell} \underline{u}(\theta_\ell)$

(a) $SNR = |h_1|^2 \frac{E_x}{N_0} = \left|\sum_{\ell=1}^{L} |a_\ell| e^{i \angle a_\ell} e^{2\pi i f \tau_\ell}\right|^2 \frac{E_x}{N_0} =$

(b) $SNR = \|\underline{h}\|^2 \frac{E_x}{N_0}$

(c) $SNR = \frac{|\underline{w}^T \underline{h}|^2}{\|\underline{w}\|^2} \frac{E_x}{N_0}$ with $\underline{w} = \underline{h}(f=0)$

$\boxed{\text{PROBLEM 4}}$ $\quad \underline{u}_{tx}(\Omega_\ell^{tx}) \quad \underline{u}_{rx}(\Omega_\ell^{RX}) \qquad$ complex gains g_ℓ.

For each channel matrix: find the rank r, the maximum singular value, and the optimal TX and RX beamforming vectors.

Assume all matrices have dimension $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$

and $\|\underline{u}_{tx}(\Omega^{tx})\|^2 = N_t$, $\|\underline{u}_{rx}(\Omega^{rx})\|^2 = N_r$

(a) Single path channel

$$H = g_1 \underline{u}_{rx}(\Omega_1^{rx}) \underline{u}_{tx}^T(\Omega_1^{tx}) = |g_1| e^{i\theta_1} \underline{u}_{rx}(\Omega_1^{rx}) \underline{u}_{tx}^T(\Omega_1^{tx})$$

$$= |g_1| \sqrt{N_t N_r}\, e^{i\theta_1} \frac{\underline{u}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}} \frac{\underline{u}_{tx}^T(\Omega_1^{tx})}{\sqrt{N_t}}$$

The rank is 1 // Max singular value: $\sigma_1 = |g_1|\sqrt{N_t N_r}$ //

$\underline{w}_{rx} = \frac{\bar{\underline{u}}_{rx}(\Omega_1^{rx})\, e^{-i\theta_1}}{\sqrt{N_r}}$ // $\qquad \|\underline{w}_{rx}\|^2 = 1$

$$\underline{w}_{tx} = \frac{\bar{\underline{u}}_{tx}(\Omega_1^{tx})}{\sqrt{N_t}} \quad // \qquad \|\underline{w}_{tx}\|^2 = 1$$

(b) Two paths. same RX angle Ω_1^{rx} but two TX angles $\Omega_1^{tx}, \Omega_2^{tx}$

$$H = g_1 \underline{u}_{rx}(\Omega_1^{rx})\underline{u}_{tx}^T(\Omega_1^{tx}) + g_2 \underline{u}_{rx}(\Omega_1^{rx})\underline{u}_{tx}^T(\Omega_2^{tx})$$

$$= \underline{u}_{rx}(\Omega_1^{rx})\Big[\underbrace{g_1 \underline{u}_{tx}^T(\Omega_1^{tx}) + g_2 \underline{u}_{tx}^T(\Omega_2^{tx})}_{\underline{s}_{tx}^T}\Big]$$

$$\|\underline{s}_{tx}^T\|^2 = [|g_1|^2+|g_2|^2]N_t \qquad \hat{\underline{s}}_{tx} = \frac{\bar{\underline{s}}_{tx}}{\sqrt{|g_1|^2+|g_2|^2}\sqrt{N_t}}, \quad \hat{\underline{u}}_{rx} = \frac{\underline{u}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}}$$

$$H = \sqrt{|g_1|^2+|g_2|^2}\sqrt{N_r N_t}\, \hat{\underline{u}}_{rx}\hat{\underline{s}}_{tx}^*$$

The rank is 1 // Max singular value: $\sigma_1 = \sqrt{|g_1|^2+|g_2|^2}\sqrt{N_t N_r}$ //

$$\underline{w}_{rx} = \frac{\bar{\underline{u}}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}} \quad // \qquad \|\underline{w}_{rx}\|^2 = 1$$

$$\underline{w}_{tx} = \frac{\left(g_1 \bar{\underline{u}}_{tx}(\Omega_1^{tx}) + g_2 \bar{\underline{u}}_{tx}(\Omega_2^{tx})\right)}{\sqrt{(|g_1|^2+|g_2|^2)N_t}} \quad // \qquad \|\underline{w}_{tx}\|^2 = 1$$

(c) Two paths, Two different RX angles and two different TX angles

$$H = g_1 \underline{u}_{rx}(\Omega_1^{rx})\underline{u}_{tx}^T(\Omega_1^{tx}) + g_2 \underline{u}_{rx}(\Omega_2^{rx})\underline{u}_{tx}^T(\Omega_2^{tx}), \qquad g_1 = |g_1|e^{i\theta_1}, \quad g_2 = |g_2|e^{i\theta_2}$$

$$\underline{u}_l = \frac{\underline{u}_{rx}(\Omega_l^{rx})e^{i\theta_l}}{\sqrt{N_r}}, \qquad \underline{v}_l = \frac{\bar{\underline{u}}_{tx}(\Omega_l^{tx})}{\sqrt{N_t}} \qquad l = 1,2$$

$$\underline{u}_{rx}(\Omega_1^{rx}) \perp \underline{u}_{rx}(\Omega_2^{rx}) \qquad \underline{u}_{tx}(\Omega_1^{tx}) \perp \underline{u}_{tx}(\Omega_2^{tx})$$

$$H = |g_1|\sqrt{N_r N_t}\,\underline{u}_1\underline{v}_1^* + |g_2|\sqrt{N_r N_t}\,\underline{u}_2\underline{v}_2^*$$

The rank is 2 // Max singular value: $\sigma_{max} = \max_l |g_l|\sqrt{N_t N_r} = |g_1|\sqrt{N_t N_t}$ // given that $|g_1| > |g_2|$

left singular vector for maximum singular value

$$\underline{w}_{rx} = \bar{\underline{u}}_1 = \frac{\bar{\underline{u}}_{rx}(\Omega_1^{rx})e^{-i\theta_1}}{\sqrt{N_r}} \quad // \qquad \|\underline{w}_{rx}\|^2 = 1$$

$$\underline{w}_{tx} = \underline{v}_1 = \frac{\bar{\underline{u}}_{tx}(\Omega_1^{tx})}{\sqrt{N_t}} \quad // \qquad \|\underline{w}_{tx}\|^2 = 1$$

right singular vector for maximum singular value

(d) Same as (b) but: $\underline{u}_{tx}(\Omega_1^{tx})^*\underline{u}_{tx}(\Omega_2^{tx}) = \rho N_t$, $|\rho| \leq 1$

$$H = \underline{u}_{rx}(\Omega_1^{rx})\Big[\underbrace{g_1 \underline{u}_{tx}(\Omega_1^{tx}) + g_2 \underline{u}_{tx}(\Omega_2^{tx})}_{\underline{v}_{tx}}\Big]$$

$$\|\underline{v}_{tx}\|^2 = \left(g_1 \underline{u}_{tx}(\Omega_1^{tx}) + g_2 \underline{u}_{tx}(\Omega_2^{tx})\right)^*\left(g_1 \underline{u}_{tx}(\Omega_1^{tx}) + g_2 \underline{u}_{tx}(\Omega_2^{tx})\right)$$

$$= |g_1|^2 N_t + g_1^* g_2 \rho N_t + g_2^* g_1 \rho N_t + |g_2|^2 N_t = N_t\left(|g_1|^2 + |g_2|^2 + 2\mathrm{Re}\{g_1^* g_2 \rho\}\right)$$

$$\underline{u}_1 = \frac{\underline{u}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}}, \qquad \underline{v}_1 = \frac{\bar{\underline{v}}_{tx}}{\|\bar{\underline{v}}_{tx}\|} = \frac{\bar{\underline{v}}_{tx}}{\left(N_t(|g_1|^2+|g_2|^2+2\mathrm{Re}\{g_1^* g_2 \rho\})\right)^{1/2}}$$

$$\underline{u}_1 = \frac{\underline{u}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}} \quad , \quad \underline{v}_1 = \frac{\bar{v}_{tx}}{\|\bar{v}_{tx}\|} = \frac{\bar{v}_{tx}}{(N_t(|g_1|^2+|g_2|^2+2\mathrm{Re}\{g_1^* g_2 \rho\}))^{1/2}}$$

$$H = \sqrt{N_r N_t}\,(|g_1|^2+|g_2|^2+2\mathrm{Re}\{g_1^* g_2 \rho\})^{1/2}\, \underline{u}_1 \underline{v}_1^*$$

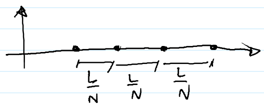
The rank is 1 // Max singular value: $\sigma_1 = \sqrt{N_r N_t}\,(|g_1|^2+|g_2|^2+2\mathrm{Re}\{g_1^* g_2 \rho\})^{1/2}$ //

$$\underline{w}_{rx} = \underline{u}_1 = \frac{\bar{u}_{rx}(\Omega_1^{rx})}{\sqrt{N_r}} \;//\; , \quad \|\underline{w}_{rx}\|^2 = 1 \quad ; \quad \underline{w}_{tx} = \underline{v}_1 = \frac{g_1 \bar{u}_{tx}(\Omega_1^{tx}) + g_2 \bar{u}_{tx}(\Omega_2^{tx})}{[N_t(|g_1|^2+|g_2|^2+2\mathrm{Re}\{g_1^* g_2 \rho\})]^{1/2}} \;//$$

$$\|\underline{w}_{tx}\|^2 = 1$$

$\boxed{\text{PROBLEM 5}}$

ULA, N elements, L total length, $\frac{L}{N}$ $\hookrightarrow$ space between the antenna elements



Given a BF vector $\underline{w} \in \mathbb{C}^N$

$$U(\phi) = c\,|\underline{w}^T \underline{u}(\phi)|^2$$

$\uparrow$ power intensity

$\underline{u}(\phi)$ is the spatial signature $(c>0)$

$w_n = \frac{1}{N}$ $\quad \underline{w} = \left[\frac{1}{N}, \ldots, \frac{1}{N}\right]^T$

energy max at $\phi = 0$

(a) $\underline{u}(\phi)$ as a function of L (length), λ and N (# elements)

$$\underline{u}(\phi) = \left[1, e^{2\pi j \frac{d \sin\phi}{\lambda}}, \ldots, e^{2\pi j (N-1) \frac{d\sin\phi}{\lambda}}\right]^T \qquad d = \frac{L}{N}$$

$$= \left[1, e^{2\pi j \frac{L\sin\phi}{N\lambda}}, \ldots, e^{2\pi j (N-1)\frac{L\sin\phi}{N\lambda}}\right]^T \;//$$

(b) $\lim_{N\to\infty} U(\phi)$

$$U(\phi) = c\,|\underline{w}^T\underline{u}(\phi)|^2 = c\,[\underline{w}^T\underline{u}(\phi)][\underline{w}^T\underline{u}(\phi)]^*$$

$$\underline{w}^T\underline{u}(\phi) = \frac{1}{N}\Big(\overbrace{1 + e^{2\pi j\frac{L\sin\phi}{N\lambda}} + \ldots + e^{2\pi j(N-1)\frac{L\sin\phi}{N\lambda}}}^{M(\phi)}\Big)$$

$$M(\phi) = 1 + e^{j\beta} + e^{j2\beta} + e^{j(N-1)\beta} \qquad \beta = \frac{2\pi L\sin\phi}{N\lambda} \qquad \frac{N}{2}\beta = \frac{\pi L\sin\phi}{\lambda}$$

$$e^{j\beta}M(\phi) = e^{j\beta} + e^{j2\beta} + e^{jN\beta}$$

$$M(\phi)(1-e^{j\beta}) = 1 - e^{jN\beta} \Rightarrow M(\phi) = \frac{1-e^{jN\beta}}{1-e^{j\beta}} = \frac{e^{j\frac{N}{2}\beta}(e^{-j\frac{N}{2}\beta} - e^{j\frac{N}{2}\beta})}{e^{j\beta/2}(e^{-j\beta/2} - e^{j\beta/2})} = e^{j(N-1)\frac{\beta}{2}} \frac{\sin(\frac{N}{2}\beta)}{\sin(\frac{\beta}{2})}$$

$$= e^{j(N-1)\frac{\beta}{2}} \frac{\sin\left(\frac{\pi L\sin\phi}{\lambda}\right)}{\sin\left(\frac{\pi L\sin\phi}{N\lambda}\right)}$$

$$U(\phi) = \frac{c}{N^2}\frac{\sin^2\left(\frac{\pi L\sin\phi}{\lambda}\right)}{\sin^2\left(\frac{\pi L\sin\phi}{N\lambda}\right)}$$

$$\alpha = \frac{\pi L\sin\phi}{\lambda}$$

$$\lim_{N\to\infty} U(\phi) = \lim_{N\to\infty}\frac{c}{N^2}\frac{\sin^2(\alpha)}{\sin^2(\frac{\alpha}{N})} = c\,\frac{\sin^2(\alpha)}{\alpha^2} = c\,\frac{\sin^2\left(\frac{\pi L\sin\phi}{\lambda}\right)}{\left(\frac{\pi L\sin\phi}{\lambda}\right)^2} = c\,\mathrm{sinc}^2\left(\frac{\pi L\sin\phi}{\lambda}\right) \;//$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

(c) Let's have $c=1$ $\quad U(\phi) = \frac{\sin^2\left(\frac{\pi L\sin\phi}{\lambda}\right)}{\left(\frac{\pi L\sin\phi}{\lambda}\right)^2} = \mathrm{sinc}^2\left(\frac{\pi L\sin\phi}{\lambda}\right)$ $\leftarrow$ for a large number of antennas

[SEE MATLAB]