HW1 - Tommy Azzino (ta1731)

Monday, February 8, 2021 10:12 PM

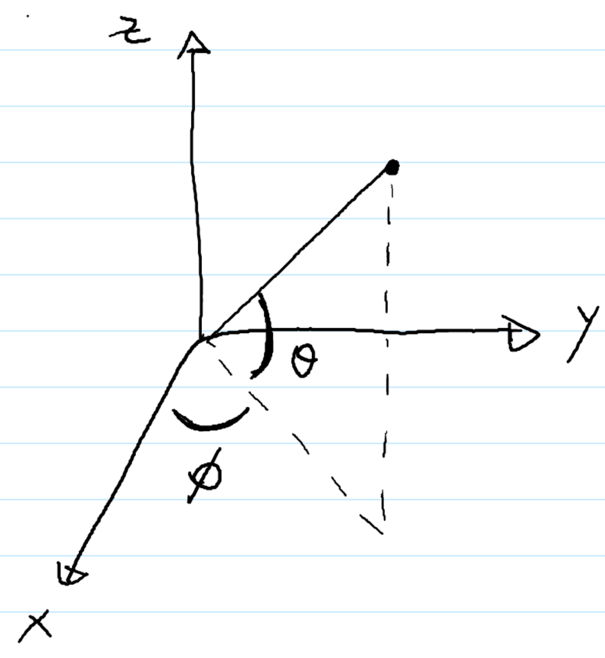
PROBLEM 4

(a) A rotation matrix can be obtained by:

$$R = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

$$= \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{YAW}} \underbrace{\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}}_{\text{PITCH}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix}}_{\text{ROLL}}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

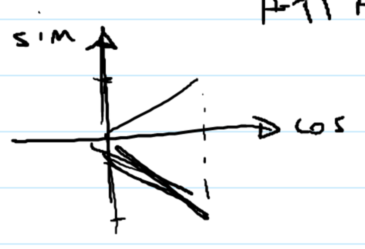


With this definition of rotation matrix we perform rotation about the z, y and x axes in the given order.

To rotate the x-axis to point in (θ,ϕ) the rotation matrix is:

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\theta) & 0 & \sin(-\theta) \\ 0 & 1 & 0 \\ -\sin(-\theta) & 0 & \cos(-\theta) \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I\ [\gamma=0]}$$

Therefore, we have $[\alpha = \phi,\ \beta = -\theta,\ \gamma = 0]$ (CHECK ALSO MATLAB CODE ATTACHED)



(b) Is $R(\alpha,0,0)^{-1} = R(-\alpha,0,0)$? [YES]

$$R(\alpha,0,0) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R(-\alpha,0,0) = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R(\alpha,0,0)^{-1} = R(\alpha,0,0)^T$ since a rotation matrix is orthonormal.

$$R(\alpha,0,0)^T = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = R(-\alpha,0,0) \checkmark$$

(c) $R(\alpha,\beta,0)^{-1} \overset{?}{=} R(-\alpha,-\beta,0)$ [NO]

$$R(\alpha,\beta,0) = \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \qquad R(-\alpha,-\beta,0) = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha & -\cos\alpha\sin\beta \\ -\sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

We have that $R(\alpha,\beta,0)^T \neq R(-\alpha,-\beta,0)$, so $R(\alpha,\beta,0)^{-1} \neq R(-\alpha,-\beta,0)$ ✓

PROBLEM 5

a) $A_1 = \{(\phi,\theta) \mid \underbrace{\phi}_{\text{azimuth}} \in [-30°, 30°],\ \underbrace{\theta}_{\text{elevation}} \in [-90, 90°]\}$

$$A = \iint d\Omega = \iint \sin\theta\, d\theta\, d\phi \qquad \theta_{inc} = \frac{\pi}{2} - \theta_{el}$$

(inclination)

$$= \int_{-\pi/6}^{\pi/6}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = \frac{\pi}{3}\left[-\cos\theta\Big|_0^{\pi}\right] = \frac{\pi}{3}\left[1-(-1)\right] = \frac{2\pi}{3}$$

b) $$A = \int_{-\pi/6}^{\pi/6}\int_{\pi/4}^{\frac{3}{4}\pi} \sin\theta\, d\theta\, d\phi = \frac{\pi}{3}\left[-\cos\theta\Big|_{\pi/4}^{3\pi/4}\right] \qquad \theta_{inc} = \begin{cases} \frac{\pi}{2}+\frac{\pi}{4} = \frac{3\pi}{4} \\ \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4} \end{cases}$$

$$= \frac{\pi}{3}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right] = \frac{\sqrt{2}\pi}{3}$$

PROBLEM 6

$$D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{4\pi\, \cancel{U(\theta,\phi)}}{\int_{-\pi/6}^{\pi/6}\int_{-\pi/4}^{\pi/4} \cancel{U(\theta,\phi)}\cos\theta\, d\theta\, d\phi} = \frac{4\pi}{\int_{-\pi/6}^{\pi/6}\int_{-\pi/4}^{\pi/4}\cos\theta\, d\theta\, d\phi} =$$

uniform: does not depend on θ and ϕ

$$= \frac{4\pi}{\frac{\pi}{3}\int_{-\pi/4}^{\pi/4}\cos\theta\, d\theta} = \frac{4\cancel{\pi}}{\frac{\sqrt{2}\cancel{\pi}}{3}} = \frac{12}{\sqrt{2}} = \frac{\cancel{12}\cdot\sqrt{2}\cdot 6}{\cancel{\sqrt{2}}} = 6\sqrt{2} \Rightarrow 9{,}2867 \text{ dBi}$$

PROBLEM 8

1) $$P_{rad} = \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} U(\theta,\phi)\cos\theta\, d\theta\, d\phi = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} A\cos^3(\theta)\, d\theta\, d\phi =$$

$$= 2\pi A\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3(\theta)\, d\theta = 2\pi A\,\frac{4}{3} = \frac{8}{3}\pi A = \frac{8}{3}\pi\, 10 = \frac{80}{3}\pi \text{ mW}$$

$$P_{rad}[\text{dBm}] = 10\log_{10}\Big(\underbrace{\frac{80}{3}\pi}_{\text{"mW"}}\Big) \simeq 19{,}23 \text{ dBm}$$

2) $$D(\theta,\phi) = \frac{4\pi\, U(\theta,\phi)}{P_{rad}} = \frac{4\pi A\cos^2(\theta)}{\frac{80}{3}\pi}$$

$$\max D(\theta,\phi) = \frac{40\pi}{\frac{80\pi}{3}} = \frac{3}{2} \Rightarrow 1{,}76 \text{ dBi}$$

PROBLEM 9

ATTACHED MATLAB CODE