Exact Model Reduction for Quantum Dynamics

Tommaso Grigoletto

March 13th, 2025
Quantum Nano Seminar, Dartmouth College

Università degli Studi di Padova

What are the **minimal resources** needed to reproduce a target quantum process?

Finding the minimal resources allows us to reduce the model's description.

Disclaimer: References omitted for aesthetics. All the details are included in the preprints.

Why is quantum model reduction interesting?

- Efficient **quantum simulation** (on classical and quantum computers);
- Efficient implementations of:
    - **controllers**,
    - **error suppression schemes**,
    - **quantum filters**;
- Easier models to study;
- Proving optimality of quantum algorithms;
- Probing "quantumness" of processes;
- Efficient generation of quantum trajectories (Monte Carlo methods).

Continuous-time open quantum dynamics

The model

$\mathcal{B}(\mathcal{H}) = \mathbb{C}^{n\times n}$

ρ are density operators:
$\rho \in \mathbb{C}^{n\times n}$, $\rho = \rho^\dagger \geq 0$, $\mathrm{tr}[\rho] = 1$.

$\mathcal{L}$ is a GKLS generator.

ρ_0 is the initial condition.

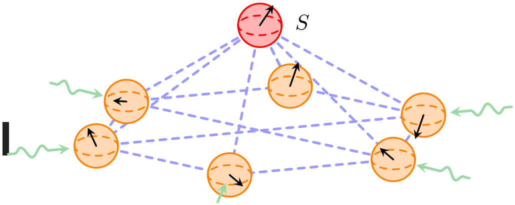
$$\begin{cases} \dot\rho(t) = \boxed{\mathcal{L}}\left[\boxed{\rho(t)}\right] \\ \boxed{Y(t)} = \boxed{\mathcal{C}}[\rho(t)] \end{cases}, \qquad \boxed{\rho_0 \in \mathfrak{D}(\mathcal{H})}$$

$Y(t) \in \mathbb{C}^{m\times m}$ is the output of interest, the one we want to preserve.

$\mathcal{C}$ is a linear output map, e.g. $\mathrm{tr}[O\rho(t)]$ or $\mathrm{tr}_B[\rho(t)]$.

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$$

Illustrative application: Dissipative central spin model



$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B \qquad \mathcal{H}_S \simeq \mathbb{C}^2, \quad \mathcal{H}_B \simeq \mathbb{C}^{2^{N_B}}$

$$H_{SB} = H_S \otimes \mathbb{1}_B + H_B + \frac{1}{2}\left(A_x \sigma_x^{(1)} J_x + A_y \sigma_y^{(1)} J_y + A_z \sigma_z^{(1)} J_z\right)$$

With $H_B = \frac{\lambda}{4} \sum_{2 \leq i < k} \sigma_x^{(i)} \sigma_x^{(k)}$ (for now).

Dissipation: either collective $L_B^c = \Lambda J_+$, or local $L_B^i = \delta \sigma_+^{(i)}$.

We are only interested in reproducing $\rho_S(t) = \mathrm{tr}_B[\rho(t)]$!

Related to NV centers and similar models.

The problem: quantum model reduction

Given a Quantum System $(\mathcal{L}, \mathcal{C})$ defined by a generator $\mathcal{L}$ and an output map $\mathcal{C}$ we want to find **another QS** $(\check{\mathcal{L}}, \check{\mathcal{C}})$ (possibly of smaller dimension) and a linear map $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{r\times r}$ such that for all $t \geq 0$ and all $\rho_0 \in \mathfrak{D}(\mathcal{H})$, $\check{\rho}_0 = \Phi[\rho_0]$

- **exact model reduction**
$$\mathcal{C}[e^{\mathcal{L}t}[\rho_0]] = \check{\mathcal{C}}[e^{\check{\mathcal{L}}t}[\check{\rho}_0]];$$
- approximate model reduction (future work)
$$\mathcal{C}[e^{\mathcal{L}t}[\rho_0]] \approx \check{\mathcal{C}}[e^{\check{\mathcal{L}}t}[\check{\rho}_0]].$$

Symmetries

Let U be a unitary operator, $UU^\dagger = \mathbb{1}$, and define $\mathcal{U}(\rho) = U\rho U^\dagger$.
Let $\{\mathcal{T}_t\}_{t\geq 0} = \{e^{\mathcal{L}t}\}_{t\geq 0}$ be the quantum dynamical semigroup generated by $\mathcal{L}$.

Definition 1

U is a symmetry for $\mathcal{L}$ if

$$[\mathcal{T}_t, \mathcal{U}] = 0, \quad \forall t \geq 0.$$

By continuity of the semigroup we have that U is a symmetry if an only if $[\mathcal{L}, \mathcal{U}] = 0$.

Definition 2

- **Strong symmetry** if $[H, U] = [L_k, U] = 0$;
- **Weak symmetry** if $[\mathcal{L}, \mathcal{U}] = 0$.

Note that: (1) Strong implies weak; (2) Symmetries form a group; (3) A symmetry for $\mathcal{L}$ is also a symmetry for $\mathcal{L}^\dagger$.

Symmetries and invariant subspaces

Proposition 1

If U is a symmetry for $\mathcal{L}$, operator eigenspaces of $\mathcal{U}$ are $\mathcal{L}$- (and $\mathcal{L}^\dagger$-) invariant.

Proof.

Take $X \in \mathfrak{B}(\mathcal{H})$ such that $\mathcal{U}(X) = \nu X$ and denote by $Y = \mathcal{L}(X)$. Then

$$\mathcal{U}(Y) = \mathcal{U}\mathcal{L}(X) = \mathcal{L}\mathcal{U}(X) = \nu\mathcal{L}(X) = \nu Y$$

hence any ν-eigenoperator of $\mathcal{U}$ is mapped trough $\mathcal{L}$ to a ν-eigenoperator of $\mathcal{U}$, thus the ν-eigenspace is $\mathcal{L}$-invariant. $\square$

Because $\mathcal{U}$ is normal, we can decompose the space of operators into $\mathcal{L}$-invariant subspaces as

$$\mathfrak{B}(\mathcal{H}) = \bigoplus_j \mathrm{eigsp}_{\nu_j}(\mathcal{U}).$$

Invariant algebras

Consider now a unitary subgroup $\mathscr{G}$ of weak symmetries for $\mathcal{L}$,
i.e. $\forall U \in \mathscr{G}$ we have $\mathcal{L}(U \cdot U^\dagger) = U\mathcal{L}(\cdot)U^\dagger$.

The commutant of $\mathscr{G}$, that is $\mathscr{G}' \equiv \{X \in \mathfrak{B}(\mathcal{H}) | [X, U] = 0, \forall U \in \mathscr{G}\}$:

- is the intersection of the 1-eigenspaces of symmetries in $\mathscr{G}$,

$$\mathscr{G}' = \bigcap_{U \in \mathscr{G}} \mathrm{eigsp}_1(\mathcal{U});$$

- is $\mathcal{L}$- and $\mathcal{L}^\dagger$-invariant;
- is a **unital $*$-algebra**!

*-algebras - Definition

We define a $*$-algebra $\mathscr{A}$ as an operator space closed under matrix multiplication and adjoint action.

$$X, Y \in \mathscr{A} \quad \Rightarrow \quad X + Y \in \mathscr{A} \qquad X^\dagger, Y^\dagger \in \mathscr{A} \qquad \text{and} \qquad XY \in \mathscr{A}.$$

An algebra is said to be unital if it contains the identity, i.e. $\mathbb{1} \in \mathscr{A}$.

It is the fundamental mathematical structure that supports a **quantum probability space** (see e.g. Von Neumann).

They have been extensively used in quantum information theory (e.g. QEC) and mathematical physics, also related to the structure of fixed points of CP maps.

Wedderburn decomposition

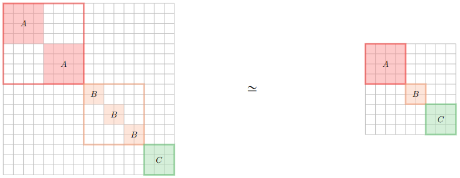
For any unital (finite dimensional) algebra $\mathscr{A}$ ($\mathbb{1} \in \mathscr{A}$), there exist an Hilbert space decomposition

$$\mathcal{H} = \bigoplus_k \mathcal{H}_{S,k} \otimes \mathcal{H}_{F,k}$$

and a unitary operator U such that

$$\mathscr{A} = U \left(\bigoplus_k \mathcal{B}(\mathcal{H}_{S,k}) \otimes \mathbb{1}_{F,k} \right) U^\dagger .$$

Note: some values are repeated multiple times hence we can find a smaller isomorphic representation $\mathscr{A} \simeq \bigoplus_k \mathcal{B}(\mathcal{H}_{S,k}) \equiv \check{\mathscr{A}}$.



Conditional expectations

A *conditional expectation* $\mathbb{E}_{\mathscr{A},\rho}[\cdot]$ is a **CP unital projection onto a $*$-algebra $\mathscr{A}$**
(and such that $\mathrm{tr}[\rho\,\mathbb{E}_{\mathscr{A},\rho}[X]] = \mathrm{tr}[\rho X]$ for all $X \in \mathcal{B}(\mathcal{H})$).

When $\mathbb{1} \in \mathscr{A}$, $\mathbb{E}_{\mathscr{A},\mathbb{1}/n}[\cdot]$ exists and is an orthogonal projection hence CPTP and has form:

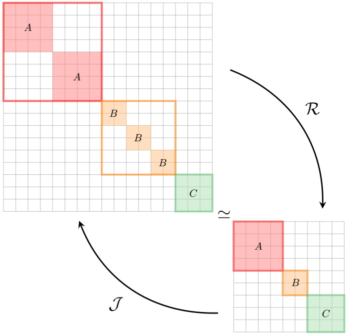
$$\mathbb{E}_{\mathscr{A},\mathbb{1}/n}[X] = U\left(\bigoplus_k \mathrm{tr}_{\mathcal{H}_{F,k}}[V_k X V_k^\dagger] \otimes \mathbb{1}_{F,k}\right)U^\dagger.$$

$\mathbb{E}_{\mathscr{A},\mathbb{1}/n}[\cdot]$ can be factorized in two non-square CPTP maps

$$\mathbb{E}_{\mathscr{A},\mathbb{1}/n}[\cdot] = \mathcal{J}\mathcal{R}.$$

$$\check{X} = \mathcal{R}[X] = \bigoplus_k \mathrm{tr}_{\mathcal{H}_{F,k}}[V_k X V_k^\dagger] = \bigoplus_k \check{X}_k$$

$$\mathcal{J}[\check{X}] = U\left(\bigoplus_k \check{X}_k \otimes \mathbb{1}_{F,k}\right)U^\dagger$$



Main result (new):

Given a GKSL generator $\mathcal{L}$, its action onto the algebra $\mathscr{A}$,

$$\mathcal{L}|_{\mathscr{A}} = \mathcal{R}\mathcal{L}\mathcal{J}$$

is still a GKSL generator.

What does this mean?

Consider $\rho_0 \in \mathscr{G}'$.

- $\mathscr{G}'$ is $\mathcal{L}$-invariant $\quad\Rightarrow\quad \rho(t) = e^{\mathcal{L}t}(\rho_0) \in \mathscr{G}', \quad \forall t \geq 0$.
- $\rho(t) = \mathbb{E}_{|\mathscr{G}',\mathbb{1}/n}[\rho(t)], \quad \forall t \geq 0$.
- $\mathbb{E}_{|\mathscr{G}',\mathbb{1}/n} e^{\mathcal{L}t} = \mathcal{J} e^{\mathcal{R}\mathcal{L}\mathcal{J}t}\mathcal{R}$

Then we have

$$\begin{cases} \dot{\rho}(t) = \mathcal{L}[\rho(t)] \\ Y(t) = \mathcal{C}[\rho(t)] \end{cases} \qquad \rho(0) = \rho_0 \qquad \equiv \qquad \begin{cases} \dot{\check{\rho}}(t) = \underbrace{\mathcal{R}\mathcal{L}\mathcal{J}}_{\check{\mathcal{L}}}[\check{\rho}(t)] \\ Y(t) = \underbrace{\mathcal{C}\mathcal{J}}_{\check{\mathcal{C}}}[\check{\rho}(t)] \end{cases} \qquad \check{\rho}(0) = \mathcal{R}(\rho_0)$$

in the sense that

$$\mathcal{C}e^{\mathcal{L}t}(\rho_0) = \check{\mathcal{C}}e^{\check{\mathcal{L}}t}\mathcal{R}(\rho_0) \qquad \forall t \geq 0.$$

Furthermore, $\check{\mathcal{L}} = \mathcal{R}\mathcal{L}\mathcal{J}$ is a Lindblad generator!

Observable, symmetry-based reduction

Assume now that $\mathcal{C}(\rho) = \mathrm{tr}[O\rho]$ and $O \in \mathscr{G}'$.

- $\mathscr{G}'$ is ALSO $\mathcal{L}^\dagger$-invariant $\quad \Rightarrow \quad O(t) = e^{\mathcal{L}^\dagger t}(O) \in \mathscr{G}'$, $\forall t \geq 0$.

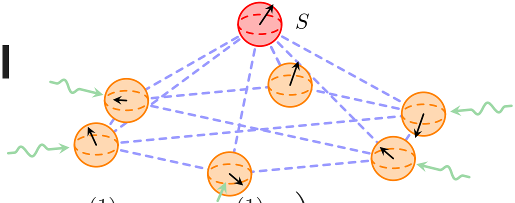
Then, again, we have

$$\begin{cases} \dot{\rho}(t) = \mathcal{L}[\rho(t)] \\ Y(t) = \mathcal{C}[\rho(t)] \end{cases} \qquad \rho(0) = \rho_0 \qquad \equiv \qquad \begin{cases} \dot{\check{\rho}}(t) = \mathcal{R}\mathcal{L}\mathcal{J}[\check{\rho}(t)] \\ Y(t) = \mathcal{C}\mathcal{J}[\check{\rho}(t)] \end{cases} \qquad \check{\rho}(0) = \mathcal{R}(\rho_0)$$

in the sense that

$$\mathcal{C}e^{\mathcal{L}t}(\rho_0) = \check{\mathcal{C}}e^{\check{\mathcal{L}}t}\mathcal{R}(\rho_0) \qquad \forall t \geq 0.$$

Dissipative central spin model



$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B \qquad \mathcal{H}_S \simeq \mathbb{C}^2, \quad \mathcal{H}_B \simeq \mathbb{C}^{2^{N_B}}$

$$H_{SB} = H_S \otimes \mathbb{1}_B + H_B + \frac{1}{2}\Big(A_x \sigma_x^{(1)} J_x + A_y \sigma_y^{(1)} J_y + A_z \sigma_z^{(1)} J_z\Big)$$

With $H_B = \frac{\lambda}{4} \sum_{2 \le i < k} \sigma_x^{(i)} \sigma_x^{(k)}$ (for now).

Dissipation: either collective $L_B^c = \Lambda J_+$, or local $L_B^i = \delta \sigma_+^{(i)}$.

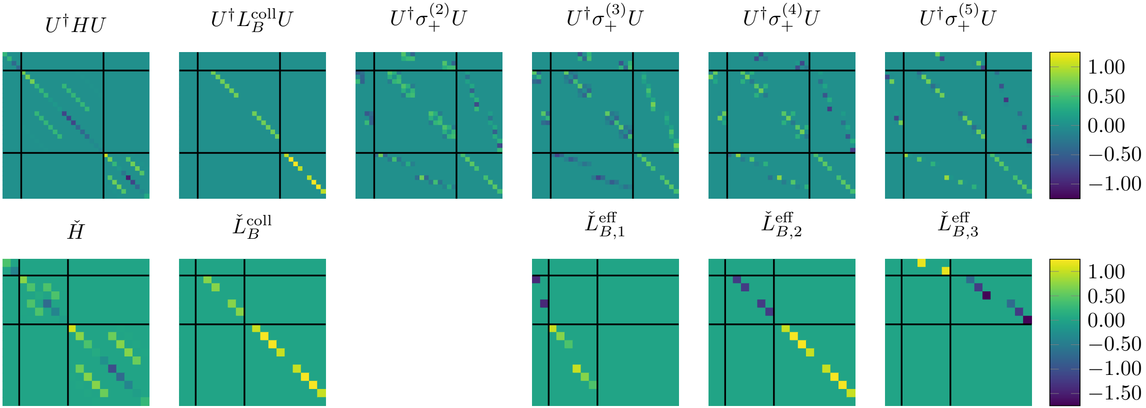
We are only interested in reproducing $\rho_S(t) = \mathrm{tr}_B[\rho(t)]$!

Bath permutations form strong (or weak for local dissipation) **symmetries** $U \in \mathscr{G}$ AND all observables of interest $O_S \otimes \mathbb{1}_B$

$$[U, O_S \otimes \mathbb{1}_B] = 0, \quad \Rightarrow O_S \otimes \mathbb{1}_B \in \mathscr{G}'.$$

We are able to find a reduced quantum model to reproduce the same dynamics.

Reduced dynamics ($N_B = 4$)

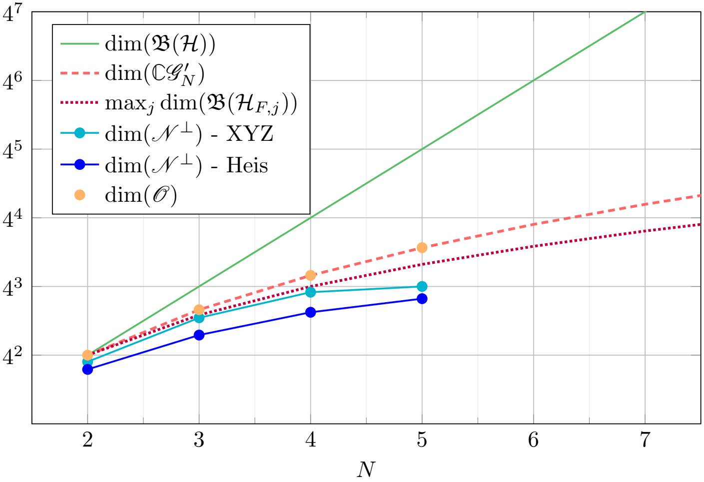


Strong symmetry: each block is invariant.
Weak symmetry: there is comunication between blocks.

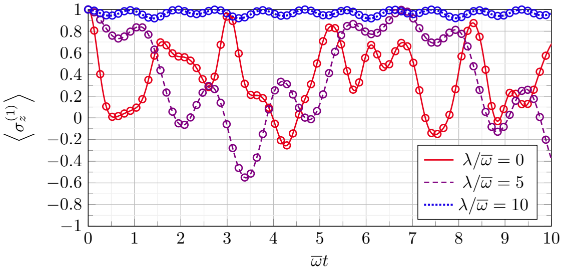
How much are we reducing?

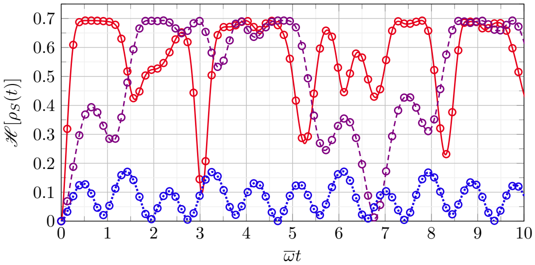
The dimension of $\mathscr{O} = \mathscr{G}'$ scales with N^3 while the dimension of $\mathfrak{B}(\mathcal{H})$ is 4^N.

The dimension of the largest block grows with N^2.
We can efficiently parallelize the simulation if the symmetry is strong (case with collective dissipation).

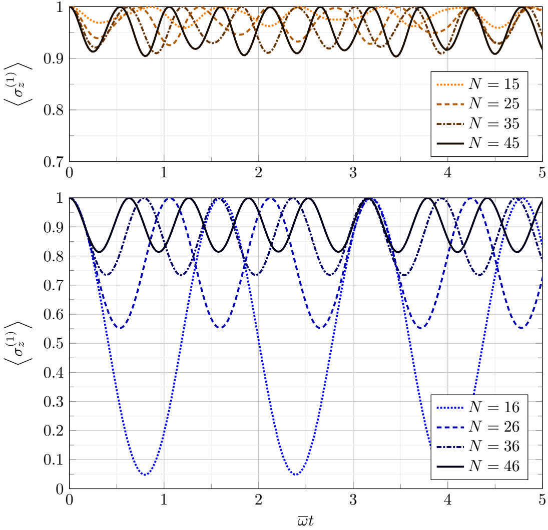


Simulations ($N_B = 6$, No dissipation)

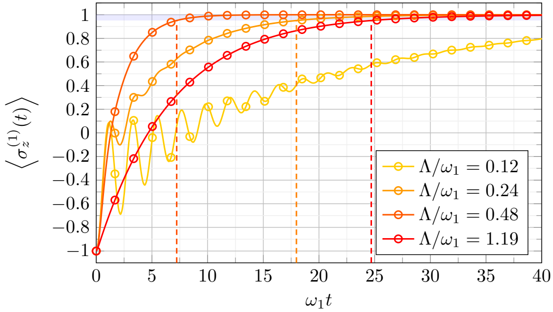


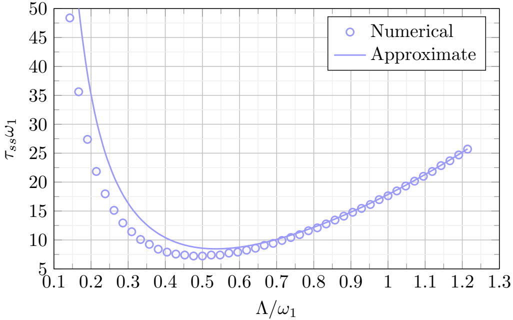


Simulations (No dissipation, $\lambda/\bar{\omega} = 20$)



Simulations ($N_B = 5$, collective dissipation)





Considering a single initial condition (reachable reduction), e.g. $\rho_0 = |1\rangle\langle 1| \otimes |0\ldots 0\rangle\langle 0\ldots 0|$ we can further reduce the model and obtain an even easier model to study.

Central spin variation

What happens if we consider no dissipation and

$$H_B = \sum_{2\leq i<k} B_{ik}\, \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(k)}?$$

Bath permutations are no longer symmetries for the model. Can we salvage this?
Note that

$$[H_{SB}, H_B] = [O_u, H_B] = 0, \quad \forall O_u$$

thus

$$\begin{aligned}
\langle \sigma_u^{(1)}(t) \rangle &= \mathrm{tr}(\sigma_u^{(1)} e^{-i(H_{SB}+H_B)t} \rho_0 e^{i(H_{SB}+H_B t)} \\
&= \mathrm{tr}(\sigma_u^{(1)} e^{-iH_B t} e^{-iH_{SB} t} \rho_0 e^{iH_{SB} t} e^{iH_B t}) \\
&= \mathrm{tr}(\sigma_u^{(1)} e^{-iH_{SB} t} \rho_0 e^{iH_{SB} t}), \quad \forall t.
\end{aligned}$$

A **NEW** notion of symmetry

What if we are interested in operators that are not in $\mathscr{G}'$?

> **Definition 3**
> A unitary U is an observable-dependent symmetry (ODS) for an observable O if $\mathcal{U}(\rho) = U\rho U^\dagger$ satisfies:
> $$[\mathcal{T}_t^\dagger, \mathcal{U}](O) = 0, \quad \forall t \geq 0$$
> $$\mathcal{U}(O) = O$$

Equivalent condition $\mathcal{U}\mathcal{L}^{\dagger k}(O) = \mathcal{L}^{\dagger k}\mathcal{U}(O)$ for all $k \in \mathbb{N}$.

Note that all symmetries are ODS for the observables in the commutant of their group. This generalizes the use of symmetries that we made above.

Observable-dependent symmetries

If we now define $\mathscr{G}$ as a group of ALL ODS for the observable O and generator $\mathcal{L}$, we can prove that $\mathscr{G}'$ **is the minimal** $*$**-algebra containing** $e^{\mathcal{L}^\dagger t}[O]$ **for all** $t \geq 0$:

$$\mathscr{G}' = \mathrm{alg}\{\mathcal{L}^{\dagger k}[O],\ \forall k \in \mathbb{N}\}.$$

This gives us a numerical method to compute $\mathscr{G}'$.
Using the same procedure as before we can perform model reduction.

Note: $\mathscr{G}'$ is not necessarily $\mathcal{L}^\dagger$-invariant but contains the smallest $\mathcal{L}^\dagger$-invariant subspace that contains the observables evolved in Heisenberg picture, and this is the important fact.

Continuous-time quantum trajectories

The model

Stochastic master equations describe continuously monitored quantum systems (e.g. photon counting, homodyne detection, photon box, etc. etc.).
We can consider either:

- Diffusive-type: $\mathcal{G}_D(\rho_t) = D\rho_t + \rho_t D^\dagger$

$$d\rho_t = \mathcal{L}(\rho_t)dt + \{\mathcal{G}_D(\rho_t) - \mathrm{tr}[\mathcal{G}_D(\rho_t)]\rho_t\}\, dW_t$$

- Counting-type: $\mathcal{K}_C(\rho_t) = C\rho_t C^\dagger$

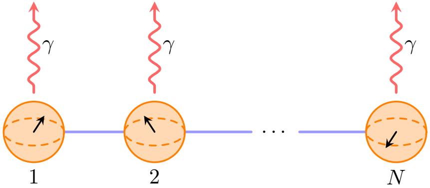
$$d\rho_t = \mathcal{L}(\rho_t)dt + \left\{\frac{\mathcal{K}_C(\rho_t)}{\mathrm{tr}[\mathcal{K}_C(\rho_t)]} - \rho_t\right\}(dN_t - \mathrm{tr}[\mathcal{K}_C(\rho_t)])$$

The output of interest then is:

$$Y_j(t) = \mathrm{tr}[O_j \rho_t]$$

with two assumptions 1) $C^\dagger C \in \mathrm{span}\{O_j\}$, $D + D^\dagger \in \mathrm{span}\{O_j\}$ and 2) $\mathbb{1} \in \mathrm{span}\{O_j\}$.

Illustrative application #2: Measured spin chain



$\mathcal{H} \simeq \mathbb{C}^{2^N}$

$$H = \sum_{j=1}^{N-1} \delta_j \sigma_x^{(j)} \sigma_x^{(j+1)} + \sum_{j=1}^{N} \mu_j \sigma_z^{(j)},$$

Continuous-time local measurement: $D_j = \gamma_j \sigma_z^{(j)}$.
Counting-type local measurement: $C_j = \alpha_j \sigma_-^{(j)}$.

We are interested in reproducing $p_j(t) = \mathrm{tr}[|j\rangle\langle j|\,\rho(t)]$!

Related to measurement-induced phase transitions.

Similar idea to before

Let consider e.g the **diffusive case** (identical for jumps).

Let then define a super-operator algebra

$$\mathscr{T} \equiv \mathrm{alg}\{\mathcal{I}, \mathcal{L}, \mathcal{G}_D\} = \mathrm{span}\{\mathcal{I}, \mathcal{L}, \mathcal{G}_D, \mathcal{L}^2, \mathcal{G}_D^2, \mathcal{L}, \mathcal{G}_D, \mathcal{G}_D\mathcal{L}, \ldots\}$$

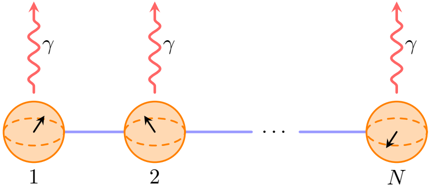
and the operator algebra

$$\mathscr{A} \equiv \mathrm{alg}\{\mathcal{A}(O_j), \quad \forall j, \quad \forall \mathcal{A} \in \mathscr{T}\}.$$

Using $\mathbb{E}_{\mathscr{A},\mathbb{1}/n}$ we can reduce the model similarly to what we did before. The reduced dynamics turns out to be a valid SME.

Fact: $\mathscr{A}$ contains all the stochastic tracjectories of the observables evolved in Heisenberg picture.

Measured spin chain
Reduced model



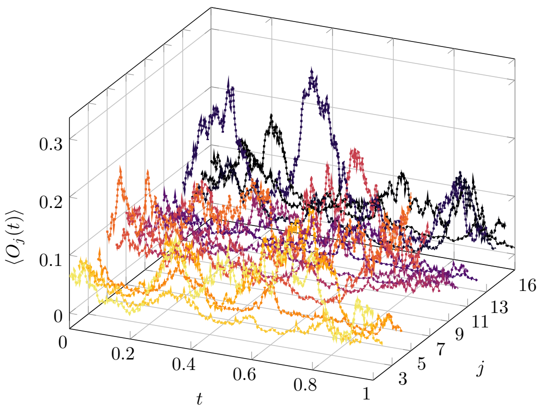
$\mathcal{H} \simeq \mathbb{C}^{2^N}$ with sub-blocks of size 2^{N-1} (not a big reduction):

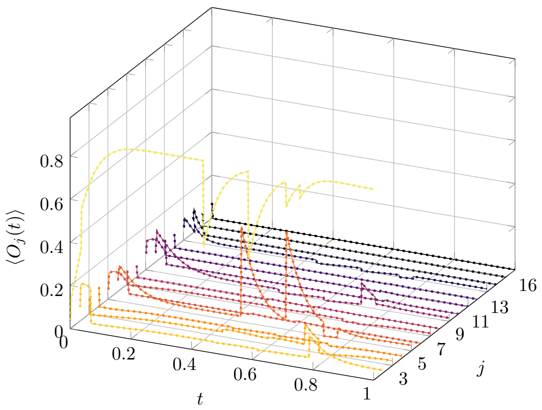
$$\check{\rho}(t) = \left[\begin{array}{c|c} \check{\rho}_1(t) & 0 \\ \hline 0 & \check{\rho}_2(t) \end{array}\right]$$

$$\check{H} = \left[\begin{array}{c|c} \check{H}_1 & 0 \\ \hline 0 & \check{H}_2 \end{array}\right], \qquad \check{D}_j = \left[\begin{array}{c|c} \check{D}_{j,1} & 0 \\ \hline 0 & \check{D}_{j,2} \end{array}\right], \qquad \check{C}_j = \left[\begin{array}{c|c} 0 & \check{C}_{j,1} \\ \hline \check{C}_{j,2} & 0 \end{array}\right].$$

Only counting-type noise operators can have off-diagonal blocks.

Simulations





Take home ideas

1. **Algebras** and **conditional expectations** provide CPTP-preserving model reduction.

2. Observable-dependent symmetries allow us to go beyond simple symmetry based model reduction.

Conclusion

- We presented a **general framework for model reduction of quantum dynamics, ensuring CPTP**. It has been applied to:
  - (classical) Hidden Markov models **[arXiv:2208.05968]**;
  - (deterministic) Discrete-time case **[arXiv:2307.06319]**;
  - (deterministic) Continuous-time case (joint with LV) **[arXiv:2412.05102]**;
  - (stochastic) Discrete-time quantum trajectories **[arXiv:2403.12575]**;
  - (stochastic) Continuous-time quantum trajectories **[arXiv:2501.13885]**.
- **Outlook**
  - Approximate model reduction (in progress);
  - Connection with adiabatic elimination techniques (in progress).

Thanks for your attention!

Our group's website →