Exact model reduction for Quantum Systems: an algebraic approach

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Question:

What are the **minimal resources** needed to reproduce a target quantum process?

Resources: Mathematical or physical (or memory).

Why is it interesting?

- Model reduction for open dynamics;
- Efficient quantum simulation;
- Probing "quantumness" of processes;
- Efficient generation of quantum trajectories (Montecarlo methods);
- Reduced state observers/ robustness to noise etc.etc. .

Motivating example: Grover's algorithm

Basics

The unstructured search problem:

Consider a set of N elements $\{x\}$ and a query function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is marked} \\ 0 & \text{if } x \text{ is not} \end{cases}$$

find a marked element x.

The classical algorithm:

Check every x until you find a solution. Solves the problem w/ prob. $1/2$ in $O(N)$.

Dynamical model for Grover's algorithm:

$$\begin{cases} \rho(t+1) = RO\rho(t)O^\dagger R^\dagger \\ \boldsymbol{p}(t) = \mathrm{diag}[\rho(t)] \end{cases}, \qquad \rho_0 = |\psi\rangle\langle\psi|$$

with $O|i\rangle = (-1)^{f(i)}|i\rangle$ and $R = 2|\psi\rangle\langle\psi| - I_N$
and $\langle\psi| = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} / \sqrt{N} \in \mathbb{C}^N$

Grover's algorithm solves the problem w/ probability $1/2$ in $O(\sqrt{N})$.

Evolution: a page from Nielsen and Chuang

$S = \{j | f(x) = 1\}$ the set of solutions to the search problem and $M = |S|$ the number of solutions.

$$|\alpha\rangle := (N-M)^{-1/2} \sum_{j \notin S} |j\rangle \qquad |\beta\rangle := (M)^{-1/2} \sum_{j \in S} |j\rangle$$

$$(RO)^t |\psi\rangle = \alpha(t) |\alpha\rangle + \beta(t) |\beta\rangle$$

with $\alpha(t) = \cos\left(\frac{2t+1}{2}\theta\right)$, $\beta(t) = \sin\left(\frac{2t+1}{2}\theta\right)$ and $\theta = 2 \arccos(\sqrt{(N-M)/M})$

$$\rho(t) = \alpha^2(t) |\alpha\rangle \langle\alpha| + \beta^2(t) |\beta\rangle \langle\beta| + \alpha(t)\beta(t)[|\alpha\rangle \langle\beta| + |\beta\rangle \langle\alpha|]$$

A subspace containing the dynamics

Let us define ($M \neq 0, N/2, N$)

$$\mathscr{R} := \mathrm{span}\{\rho(t), t \geq 0\} = \mathrm{span}\{|\alpha\rangle\langle\alpha|\,, |\beta\rangle\langle\beta|\,, |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|\}$$

The dynamics never exits $\mathscr{R}$, i.e. $\rho(t) \in \mathscr{R}$ for all $t \geq 0$.

We don't need to describe what is outside $\mathscr{R}$ to reproduce the dynamics of Grover's algorithm.

Restricting the model to $\mathscr{R}$

$$\begin{cases} \boldsymbol{x}(t+1) = \begin{bmatrix} (N-2M)^2 & 4M(N-M) & -a \\ 4M(N-M) & (N-2M)^2 & a \\ a & -a & b \end{bmatrix} \boldsymbol{x}(t)/N^2 \\ \boldsymbol{p}(t) = \begin{bmatrix} \frac{1}{N-M} \sum_{i \notin S} |i\rangle & \frac{1}{M} \sum_{i \in S} |i\rangle & \mathbf{0} \end{bmatrix} \boldsymbol{x}(t) \end{cases} \qquad \boldsymbol{x}_0 = \begin{bmatrix} \frac{N-M}{N} \\ \frac{M}{N} \\ \frac{\sqrt{2(N-M)M}}{N} \end{bmatrix}$$

with $a = 2(N-2M)\sqrt{2(N-M)M}$, $b = N^2 + 8M^2 - 8NM$ and.

This model:

- Reproduces the same output as Grover's algorithm;
- it's minimal;
- **BUT** not a quantum model.

NOTE: The reduction "is equivalent" to finding the search problem solution.

However ...

Consider the subspace

$$\mathscr{D} = \mathrm{span}\{|\alpha\rangle\langle\alpha|\,, |\beta\rangle\langle\beta|\,, |\alpha\rangle\langle\beta|\,, |\beta\rangle\langle\alpha|\}$$

If we reduce the model to this subspace we obtain:

$$\begin{cases} \check{\rho}(t+1) = \check{U}\check{\rho}(t)\check{U}^\dagger \\ \boldsymbol{p}(t) = \sum_{i\in S} |i\rangle\langle 1|\,\check{\rho}(t)\,|1\rangle\langle i| + \sum_{i\notin S} |i\rangle\langle 0|\,\check{\rho}(t)\,|0\rangle\langle i| \end{cases}$$

with

$$\check{\rho}_0 = \frac{N-M}{N}\,|0\rangle\langle 0| + \frac{M}{N}\,|1\rangle\langle 1| + \frac{\sqrt{(N-M)M}}{N}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$\check{U} = \frac{N-2M}{N} I_2 - i\frac{\sqrt{(N-M)M}}{N}\sigma_y$$

Problem and model definition

Quantum Hidden Markov model

$$\begin{cases} \rho(t+1) = \mathcal{A}\left[\ \rho(t)\ \right] \\ \Upsilon(t) = \mathcal{C}\left[\rho(t)\right] \end{cases}$$

Quantum Hidden Markov model

$\mathcal{B}(\mathcal{H}) = \mathbb{C}^{n \times n}$
ρ are density operators
$\rho(0) \in \mathcal{S}$ a set of initial conditions

$$\begin{cases} \rho(t+1) = \mathcal{A}\left[\boxed{\rho(t)}\right] \\ Y(t) = \mathcal{C}\left[\rho(t)\right] \end{cases}$$

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the one we want to preserve

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This model includes classical HMMs and quantum walks
but **does not include** the effects of conditioning.

Examples of output maps of interest

Expectation values

$$\mathcal{C}(\cdot) = \mathrm{tr}(O\cdot)$$

Probability of outcomes of an observable

$$\mathcal{C}_j(\cdot) = \mathrm{tr}(\Pi_j\cdot)$$

Partial states

$$\mathcal{C}(\cdot) = \mathrm{tr}_E(\cdot)$$

The model reduction problem:

Given a QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$ can we find another QHM model, possibly smaller in dimension, $(\check{\mathcal{A}}, \check{\mathcal{C}}, \check{\mathcal{S}})$ such that for all $\rho_0 \in \mathcal{S}$, $\exists\ \check{\rho}_0 \in \check{\mathcal{S}}$ such that

$$\mathcal{C}\mathcal{A}^t(\rho_0) = \check{\mathcal{C}}\check{\mathcal{A}}^t(\check{\rho}_0)$$

for all $t \geq 0$.

Remark

- **Exact** not approximate like adiabatic elimination (for now);
- **Single-time** no quantum trajectories (for now);
- We want to obtain a **quantum system**;

System theory basics

Reachable subspace (from $\mathcal{S}$)

Assume $\mathcal{S}$ to be a finite set.

$$\mathscr{R} := \mathrm{span}\{\mathcal{A}^t[\rho_0], t \geq 0;\ \rho_0 \in \mathcal{S}\}$$

is the minimal subspace that contains all the trajectories whose initial conditions are in $\mathcal{S}$. We call it the **reachable subspace (from $\mathcal{S}$).**

Fact: $\mathscr{R}$ is the smallest $\mathcal{A}$-invariant subspace containing $\mathrm{span}\{\mathcal{S}\}$.

Non-observable space

$$\mathscr{N} := \{X \in \mathfrak{B}(\mathcal{H}) | \mathcal{C}\mathcal{A}^t[X] = 0,\ \forall t \geq 0\}$$

Two initial conditions are indistinguishable, i.e. ρ_1, ρ_2 such that $\mathcal{C}\mathcal{A}^t\rho_1 = \mathcal{C}\mathcal{A}^t\rho_2$ for all t if $\rho_1 - \rho_2 \in \mathscr{N}$

Fact: $\mathscr{N}$ is the largest $\mathcal{A}$-invariant subspace contained in $\ker(\mathcal{C})$.

In Heisenberg picture: $\mathcal{C}(\cdot) = \sum_i E_i \langle C_i, \cdot \rangle_{HS}$ and

$$\mathscr{N}^\perp = \mathrm{span}\{\mathcal{A}^{\dagger t}[C_i], t \geq 0\}$$

System-environment with qubit interface

Consider $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_I \otimes \mathcal{H}_E$

$$\begin{cases} \rho(t+1) = \mathcal{A}(\rho(t)) = \mathcal{E}(e^{-iH\Delta t}\rho(t)e^{iH\Delta t}) \\ \tau(t) = \mathcal{C}(\rho(t)) = \mathrm{tr}_{I,E}(\rho(t)) \end{cases}$$

$$H = H_S \otimes \sigma_z \otimes I + I \otimes \sigma_z \otimes H_E$$

$$\mathcal{E}(\cdot) = p_I \sigma_x^{(I)} \cdot \sigma_x^{(I)} + p_S U_S \cdot U_S^\dagger + p_E U_E \cdot U_E^\dagger + (1 - p_I - p_S - p_E) I \cdot I$$

H_S, H_E, U_S, U_E generic

$$C_i = S_i \otimes I_{I,E}$$

System-environment with qubit interface

$$\mathcal{E}^\dagger[S_i \otimes I_{I,E}] = p_S U_S S_i \otimes I_{I,E} U_S^\dagger + (1 - p_S) S_i \otimes I_{I,E}$$

$$e^{-i(I_S \otimes \sigma_z \otimes H_E)\Delta t}(S_i \otimes I_{I,E}) e^{i(I_S \otimes \sigma_z \otimes H_E)\Delta t} = S_i \otimes [e^{-i(\sigma_z \otimes H_E)\Delta t} e^{i(\sigma_z \otimes H_E)\Delta t}]$$

$$e^{-i(H_S \otimes \sigma_z \otimes I_E)\Delta t}(E_i \otimes I_{I,E}) e^{i(H_S \otimes \sigma_z \otimes I_E)\Delta t} = [e^{-iH_S \Delta t} S_i e^{iH_S \Delta t} \otimes |0\rangle\langle 0| + e^{iH_S \Delta t} S_i e^{-iH_S \Delta t} \otimes |1\rangle\langle 1|] \otimes I_E$$

$$\mathcal{N}^\perp = \mathrm{span}\{S_i \otimes |j\rangle\langle j| \otimes I_E,\ i = 0, 1\}$$

Optimal linear reduction (Rosembrok 1970)

Take the QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$

1. Compute $\mathscr{R}$ and $\Pi_{\mathscr{R}}$ and its two non-square isometries $\mathcal{R}_{\mathscr{R}}, \mathcal{J}_{\mathscr{R}}$ such that $\Pi_{\mathscr{R}} = \mathcal{J}_{\mathscr{R}}\mathcal{R}_{\mathscr{R}}$ and $\mathcal{R}_{\mathscr{R}}\mathcal{J}_{\mathscr{R}} = \mathcal{I}_{\mathscr{R}}$;
2. Restrict the original model to $\mathscr{R}$ and obtain $(\mathcal{R}_{\mathscr{R}}\mathcal{A}\mathcal{J}_{\mathscr{R}}, \mathcal{C}\mathcal{J}_{\mathscr{R}}, \mathcal{R}_{\mathscr{R}}\mathcal{S})$;
3. Compute $\mathscr{N}$ and $\Pi_{\mathscr{N}}^{\perp} = \mathcal{I} - \Pi_{\mathscr{N}} = \mathcal{J}_{\mathscr{N}}^{\perp}\mathcal{R}_{\mathscr{N}}^{\perp}$ for this new model;
4. Restrict the new model to $\mathscr{N}^{\perp}$:

$$(\underbrace{\mathcal{R}_{\mathscr{N}}^{\perp}\mathcal{R}_{\mathscr{R}}\mathcal{A}\mathcal{J}_{\mathscr{R}}\mathcal{J}_{\mathscr{N}}^{\perp}}_{\mathcal{A}'}, \underbrace{\mathcal{C}\mathcal{J}_{\mathscr{R}}\mathcal{J}_{\mathscr{N}}^{\perp}}_{\mathcal{C}'}, \underbrace{\mathcal{R}_{\mathscr{N}}^{\perp}\mathcal{R}_{\mathscr{R}}\mathcal{S}}_{\mathcal{S}'})$$

Remark

This is the minimal linear model.

This is not a QHM model! $\mathcal{A}'$ is not CPTP and $\mathcal{S}'$ are not states!

What are we missing?

*-algebras

A **-algebra* is an operator space, closed under adjoint operations, and matrix multiplication.

It is the fundamental mathematical structure that supports a quantum probability space.

They allow for a block-diagonal decomposition called Wedderburn decomposition. If $\mathscr{A}$ is unital ($I \in \mathscr{A}$) we have:

$$\mathcal{H} = \bigoplus_j \mathcal{H}_{S,j} \otimes \mathcal{H}_{F,j}$$

$$U\mathscr{A}U^\dagger = \bigoplus_j \mathcal{B}(\mathcal{H}_{S,j}) \otimes I_{F,j} \qquad \simeq \bigoplus_j \mathcal{B}(\mathcal{H}_{S,j}) = \check{\mathscr{A}}$$

Conditional expectations

A *conditional expectation* is a CP unital map $\mathbb{E}_{\mathscr{A}}[\cdot]$ that is a projection onto $\mathscr{A}$.

Its dual is called a *state extension* $\mathbb{J}_{\mathscr{A}}[\cdot] = \mathbb{E}^{\dagger}_{\mathscr{A}}[\cdot]$ and is a CPTP projection onto a distorted algebra (closed for a distorted product defined as $X \cdot_{\bar{\rho}} Y = X\bar{\rho}^{-1}Y$).

$$\bar{\rho} = \mathbb{J}_{\mathscr{A}}[\bar{\rho}] = \bigoplus_j \rho_{S,j} \otimes \tau_{F,j}$$

$$U\mathscr{A}_{\bar{\rho}}U^{\dagger} = \bigoplus_j \mathcal{B}(\mathcal{H}_{S,j}) \otimes \tau_{F,j}$$

It can be factorized in two CPTP isometries $\mathbb{J}_{\mathscr{A}}[\cdot] = \mathcal{J}\mathcal{R}$, a reduction $\mathcal{R} : \mathcal{B}(\mathcal{H}) \to \check{\mathscr{A}}$ and an injection $\mathcal{J} : \check{\mathscr{A}} \to \mathscr{A}_{\rho}$ such that $\mathcal{R}\mathcal{J} = \mathcal{I}_{\mathscr{A}}$.

$$\mathcal{R} = \bigoplus_j Tr_{F,j}[\Pi_{SF,j} \cdot \Pi_{SF,j}] \qquad \mathcal{J} = \bigoplus_j Id_{S,j} \otimes \tau_{F,j}$$

Observable algebra

Take the QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$

1. Compute $\mathscr{N}$;
2. Compute $\mathscr{A} = \mathrm{alg}(\mathscr{N}^{\perp})$ and $\mathcal{R}$ and $\mathcal{J}$;
3. Reduce the model to $\check{\mathscr{A}}$:

$$(\underbrace{\mathcal{R}\mathcal{A}\mathcal{J}}_{\mathcal{A}'}, \underbrace{\mathcal{C}\mathcal{J}}_{\mathcal{C}'}, \underbrace{\mathcal{R}\mathcal{S}}_{\mathcal{S}'})$$

Remark
This IS a QHM model! $\mathcal{A}'$ is CPTP and $\mathcal{S}'$ are states!
Note: The choice of $\mathscr{N}^{\perp}$ is non-trivial.

System-environment with qubit interface

$\mathcal{N}^{\perp} = \mathrm{span}\{S_i \otimes |j\rangle\langle j| \otimes I_E,\ i = 0, 1\}$ is an algebra.

$$\mathcal{J}[\check{X}] = \check{X} \otimes \frac{I_E}{\dim(\mathcal{H}_E)} \qquad \mathcal{R}(X) = \sum_i \sum_{j=0,1} S_i \otimes |j\rangle\langle j| \, \mathrm{tr}_E(X) S_i \otimes |j\rangle\langle j| \, .$$

System-environment with qubit interface

$$\check{\mathscr{A}} = W(\mathcal{B}(\mathcal{H}_S) \bigoplus \mathcal{B}(\mathcal{H}_S))W^\dagger \subseteq \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_I)$$

$$\begin{cases} \check{\rho}(t+1) = \check{\mathcal{E}}(\check{U}\check{\rho}(t)\check{U}^\dagger) \\ \tau(t) = \mathrm{tr}_I(\check{\rho}(t)) \end{cases} \qquad \check{\rho}_0 = \rho_S \otimes \begin{bmatrix} \alpha & \\ & 1-\alpha \end{bmatrix}$$

$$\check{U} = e^{-iH_S \otimes \sigma_z \Delta t}$$
$$\check{\mathcal{E}}(\cdot) = p_I \sigma_x^{(I)} \cdot \sigma_x^{(I)} + p_S U_S \cdot U_S^\dagger + (1 - p_I - p_S) I \cdot I$$

Reachable algebra: a naive approach

Take the QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$

1. Compute $\mathscr{R}$;
2. Compute $\mathscr{A} = \mathrm{alg}(\mathscr{R})$ and $\mathcal{R}$ and $\mathcal{J}$;
3. Reduce the model to $\check{\mathscr{A}}$:

$$(\underbrace{\mathcal{R}\mathcal{A}\mathcal{J}}_{\mathcal{A}'}, \underbrace{\mathcal{C}\mathcal{J}}_{\mathcal{C}'}, \underbrace{\mathcal{R}\mathcal{S}}_{\mathcal{S}'})$$

Improving algebras: Invariant State

Consider CPTP dynamics $\mathcal{A}$ and $\mathcal{S} = \{\rho_0 = \sum_j p_j \Pi_j\}$ such that

$$\rho_0 = \mathcal{A}[\rho_0].$$

Then

$$\mathrm{alg}\{\mathcal{A}^t[\rho_0], t = 0, \ldots, n^2 - 1\} = \mathrm{alg}(\rho_0) = \mathrm{span}\{\Pi_j, \forall j\}.$$

Commutative (classical) algebra!

However, a trivial (1-dim) dynamics $\check{\rho}(t) = 1$ is sufficient to reproduce the states at each t: choose the CPTP reduction $\check{\rho} = \mathcal{R}(\rho) = \mathrm{Tr}(\rho)$, and injection $\mathcal{J}(\check{\rho}) = \rho_0 \check{\rho}$, so that

$$\rho_0 = \mathcal{J}(1)$$

Improving algebras

Notice that $\mathscr{R} = \mathrm{span}\{\rho_0\}$ is a distorted algebra, closed under the product $X \cdot_{\rho_0} Y = X\rho_0^{-1}Y$.

We need to close $\mathscr{R}$ to a distorted algebra.

In general, if $\mathscr{R}$ is the reachable space we need to find a state σ such that $\mathrm{alg}_\sigma(\mathscr{R})$:

- is the smallest distorted algebra that contains $\mathscr{R}$;
- there exists a state extension $\mathbb{J}$ that projects onto $\mathrm{alg}_\sigma(\mathscr{R})$ (Takesaki's theorem).

Reachable algebra: the correct approach

Take the QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$

1. Compute $\mathscr{R}$;
2. Compute $\mathscr{A} = \mathrm{alg}(\mathscr{R})$ and its center $\mathcal{Z}$;
3. Pick any full support state $\bar{\rho}$ from $\mathscr{R}$;
4. Project $\bar{\rho}$ onto $\mathcal{Z}$, $\sigma = \Pi_{\mathcal{Z}}[\bar{\rho}]$;
5. Compute the **minimal distorted algebra** $\mathscr{D} = \mathrm{alg}_\sigma(\mathscr{R})$
6. Reduce the model to $\check{\mathscr{D}}$:

$$(\underbrace{\mathcal{RAJ}}_{\mathcal{A}'}, \underbrace{\mathcal{CJ}}_{\mathcal{C}'}, \underbrace{\mathcal{RS}}_{\mathcal{S}'})$$

This is “Optimal”

Combined reduction

Take the QHM model $(\mathcal{A}, \mathcal{C}, \mathcal{S})$

1. Perform the reachable reduction;
2. Perform the observable reduction on the new model;
3. Iterate until the size does not decrease any more.

Multi-time probabilities

We are actually guaranteeing that for someobservable X we have:

$$\mathrm{Tr}(X\rho(t)) = \mathrm{Tr}(\check{X}\check{\rho}(t))$$

What about measurement joint probabilities, not just averages?
If $X = \sum_j \Pi_j$:

$$\mathbb{P}[X(t+1) = i, X(t) = j] = \mathrm{Tr}(\Pi_i \mathcal{A}(\Pi_j \rho(t)\Pi_j)) = \mathrm{Tr}(\tilde{\Pi}_i \tilde{\mathcal{A}}(\tilde{\Pi}_t \tilde{\rho}(t)\tilde{\Pi}_t)),$$

where $\tilde{\Pi}_j$ is the reduction of Π_j.

Multi-time probabilities

The method can be adapted, we need to redefine the observable and reachable subspaces to include the effect of conditioning.

$$\mathscr{R}_C = \mathrm{span}\{\bigcirc_{t=0}^{+\infty}(\mathcal{P}_{j_t} \circ \mathcal{A})[\rho_0], t \geq 0, \forall j_t, \forall \rho_0 \in \mathcal{S}\}$$

with $\mathcal{P}_{j_t}(\cdot) = \Pi_{j_t} \cdot \Pi_{j_t}$

Under certain conditions (allowing to not do a measurement or having only projective measurements) we can prove the adapted methods work in the same way.

Pros and Cons of the method

- There is no assumption on the set of initial conditions (no factorized initial states);
- No assumption of timescale separation;

- Exact $\rightarrow$ model size/accuracy trade-off totally on the side of accuracy;
- Infinite dimensional?

- No proof of optimality (is there a smaller quantum model that reproduces the same output trajectory?)
- Strong dependence on existence of symmetries.

Todo list:

In preparation (part of my PhD thesis):

- Quantum trajectories in discrete-time;
- Continuous-time case with control (with Lorenza Viola's group);

Future directions (partially under constructions):

- *Approximate method*;
- Physically relevant examples (working with Lorenza Viola's group suggestions are welcome);
    - Locality constraints, networks;
- General positive dynamics with input (classical) and optimality;
- Infinite-dimensional case, e.g. bosons;
- Continuous-time with measurement, SME (non-linear);

Thank you for your attention!

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