Exact model reduction for Quantum Systems

- An algebraic approach -

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What are the **minimal resources** needed to reproduce a target quantum process?

Resources? Mathematical or physical degrees of freedom (“memory”).

Why is it interesting?

- Model reduction for **quantum dynamics**;
- Efficient quantum simulation (quantum systems are notoriously difficult to simulate);
- Efficient implementations of **controllers, error suppression schemes and quantum filters** ;
- Easier models to study;
- Proving optimality of quantum algorithms;
- Probing “quantumness” of processes;
- Efficient generation of quantum trajectories (Montecarlo methods).

The model (discrete-time quantum walk)

$$\begin{cases} \rho(t+1) = \ \mathcal{A}\ [\ \rho(t)\ ] \\ \ Y(t) \ = \ \mathcal{C}\ [\rho(t)] \end{cases} , \qquad \rho_0 \in \mathcal{S}$$

The model (discrete-time quantum walk)

$\mathcal{B}(\mathcal{H}) = \mathbb{C}^{n\times n}$

ρ are density operators:

$\rho \in \mathbb{C}^{n\times n}$, $\rho = \rho^\dagger \geq 0$, $\mathrm{tr}[\rho] = 1$

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$\mathcal{C}$ is a linear output map

The classical counterpart (Markov Chains)

$\mathbb{R}^n$

p are probability distributions:
$p \in \mathbb{R}^n$, $p_i \geq 0$, $\sum_i p_i = 1$

A is a **Stochastic matrix**

$\mathcal{S}$ is a set of initial conditions

$$\begin{cases} p(t+1) = \boxed{A}\,[\,\boxed{p(t)}\,] \\ \boxed{y(t)} = \boxed{C}\,[p(t)] \end{cases}, \qquad \boxed{p_0 \in \mathcal{S}}$$

$y(t) \in \mathbb{R}^m$ is the output of interest, the one we want to preserve

C is a linear output map

The problem: quantum model reduction

Given a QS $(\mathcal{A}, \mathcal{S}, \mathcal{C})$ find **another QS** $(\check{\mathcal{A}}, \check{\mathcal{S}}, \check{\mathcal{C}})$ and a linear map $\Phi : \mathbb{C}^{n\times n} \to \mathbb{C}^{r\times r}$

such that for all $t \geq 0$ and $\rho_0 \in \mathcal{S}$, $\check{\rho}_0 = \Phi[\rho_0]$

- **exact model reduction**

$$\mathcal{C}[\mathcal{A}^t[\rho_0]] = \check{\mathcal{C}}[\check{\mathcal{A}}^t[\check{\rho}_0]];$$

- approximate model reduction (future work)

$$\mathcal{C}[\mathcal{A}^t[\rho_0]] \approx \check{\mathcal{C}}[\check{\mathcal{A}}^t[\check{\rho}_0]].$$

First step. Assume:

$$\ker[\mathcal{C}] \neq \{0\},$$

$$\rho_0 \in \mathcal{S} = \mathfrak{D}(\mathcal{H}).$$

We are only interested in reproducing trajectories of certain values of interest starting from any initial condition.

Linear model reduction

$$\begin{cases} \rho(t+1) = \mathcal{A}[\rho(t)] \\ Y(t) = \mathcal{C}[\rho(t)] \end{cases}$$

$$\mathscr{N} = \{X \in \mathbb{C}^{n\times n} | \quad \mathcal{C}[\mathcal{A}^t[X]] = 0, \forall t \geq 0\}$$

$\mathscr{N}$ is the largest $\mathcal{A}$-invariant operators subspace contained in $\ker[\mathcal{C}]$.

$$\begin{cases} \left[\begin{array}{c} \rho_{\mathscr{N}} \\ \hline \rho_{\mathscr{N}^\perp} \end{array}\right](t+1) = \left[\begin{array}{c|c} \mathcal{A}_{\mathscr{N},\mathscr{N}} & \mathcal{A}_{\mathscr{N},\mathscr{N}^\perp} \\ \hline 0 & \mathcal{A}_{\mathscr{N}^\perp,\mathscr{N}^\perp} \end{array}\right] \left[\begin{array}{c} \rho_{\mathscr{N}} \\ \hline \rho_{\mathscr{N}^\perp} \end{array}\right](t) \\ Y(t) = \left[\begin{array}{c|c} 0 & C_{\mathscr{N}^\perp} \end{array}\right] \left[\begin{array}{c} \rho_{\mathscr{N}} \\ \hline \rho_{\mathscr{N}^\perp} \end{array}\right](t) \end{cases}$$

Linear model reduction pt. 2

This is equivalent to finding factors $\mathcal{R}$ and $\mathcal{J}$ such that $\mathcal{J}\mathcal{R} = \Pi_{\mathcal{N}^\perp}$ and $\mathcal{R}\mathcal{J} = \mathcal{I}$ so that

$$\check{\mathcal{A}} = \mathcal{R}\mathcal{A}\mathcal{J}, \qquad \check{\mathcal{C}} = \mathcal{C}\mathcal{J}, \qquad \Phi = \mathcal{R}.$$

Note for later:
Restricting onto $\mathcal{N}^\perp$ provides the minimal linear equivalent model.
Nonetheless, we can include some variables from $\mathcal{N}$ and the reduced model is still equivalent.

Problem:

$\check{\mathcal{A}}$ is not necessarily CPTP

How do we ensure that physical (probability) constraints are satisfied?

The protagonist of this work: $*$-algebras

Definition

We define a $*$-algebra $\mathscr{A}$ as an operator space closed under matrix multiplication and adjoint action.

$$X, Y \in \mathscr{A} \quad \Rightarrow \quad X + Y \in \mathscr{A} \qquad X^\dagger, Y^\dagger \in \mathscr{A} \qquad \text{and} \qquad XY \in \mathscr{A}$$

It is the fundamental mathematical structure that supports a **quantum probability space.**

Classically you can see an algebra as the set of random variables that are measurable w.r.t. a given σ-algebra.

Conditional expectations & state extensions

A *conditional expectation* $\mathbb{E}_{\mathscr{A},\rho}[\cdot]$ is a **CP unital proj. onto a $*$-algebra $\mathscr{A}$**
(and such that $\mathrm{tr}[\rho\,\mathbb{E}_{\mathscr{A},\rho}[X]] = \mathrm{tr}[\rho X]$ for all $X \in \mathcal{B}(\mathcal{H})$).

Its dual is called a *state extension* $\mathbb{J}_{\mathscr{A},\rho}[\cdot] = \mathbb{E}^{\dagger}_{\mathscr{A},\rho}[\cdot]$ is a **CPTP projection onto an operator subspace $\mathscr{X}$**
(and is such that $\mathbb{J}_{\mathscr{A},\rho}[\rho] = \rho$).

$\mathscr{X}$ is an operator subspace closed under adjoint action and a modified matrix product $X \cdot_\rho Y = X\rho^{-1}Y$ and can be computed as $\mathscr{X} = \rho^{1/2}\mathscr{A}\rho^{1/2}$.
We call this a **distorted algebra**.

Properties of state extensions (new!)

$\mathbb{J}_{\mathscr{A},\rho}[\cdot]$ can be factorized in two non-square CPTP isometries $\mathbb{J}_{\mathscr{A},\rho}[\cdot] = \mathcal{J}\mathcal{R}$.

Given a CPTP map $\mathcal{A}$, its reduction onto the distorted algebra $\mathscr{X}$, $\mathcal{A}|_{\mathscr{X}} = \mathcal{R}\mathcal{A}\mathcal{J}$ is CPTP.

Take home ideas

1. We need algebras to define a quantum probability space;
2. Projections onto distorted algebras provide CPTP reduction.

Observable CPTP reduction

We leverage the knowledge of the quantities of interest to reduce the model.

1. Find the **non-observable subspace**

$$\mathscr{N} = \{X \in \mathbb{C}^{n\times n} | \quad \mathcal{C}[\mathcal{A}^t[X]] = 0, \forall t \geq 0\}$$

2. Close its orthogonal complement to a ***-algebra** $\mathscr{A} = \mathrm{alg}(\mathscr{N}^\perp)$;

3. Compute the **state extension** $\mathbb{J}_{\mathscr{A},I/n}[\cdot]$ and its CPTP factors $\mathcal{R}$ and $\mathcal{J}$;

4. Use the two factors to **reduce the dynamics**:

$$\check{\mathcal{A}} = \mathcal{R}\mathcal{A}\mathcal{J}, \qquad \check{\mathcal{C}} = \mathcal{C}\mathcal{J}, \qquad \Phi = \mathcal{R}.$$

Main result: The closure of $\mathscr{N}^\perp$ (w.r.t. $\langle\cdot,\cdot\rangle_{HS}$) to an algebra is “optimal”.

Second step. Assume:

$$\mathcal{C} = \mathcal{I},$$

$$\rho_0 \in \mathcal{S} \subsetneq \mathfrak{D}(\mathcal{H}).$$

We are interested in reproducing the evolution of the entire state starting only from certain initial conditions.

Reachable reduction: A naive approach

We leverage the knowledge of $\rho_0 \in \mathcal{S}$ to reduce the model.

1. Find the **minimal subspace that contains the dynamics**
$$\mathscr{R} = \mathrm{span}\{\mathcal{A}^t(\rho_0), t \geq 0\};$$
2. Close it to a ***-algebra** $\mathscr{A} = \mathrm{alg}(\mathscr{R})$;
3. Compute the **state extension** $\mathbb{J}_{\mathscr{A},\rho}[\cdot]$ and its CPTP factors $\mathcal{R}$ and $\mathcal{J}$;
4. Use the two factors to **reduce the dynamics**:
$$\check{\mathcal{A}} = \mathcal{R}\mathcal{A}\mathcal{J}, \qquad \check{\mathcal{C}} = \mathcal{C}\mathcal{J}, \qquad \Phi = \mathcal{R}.$$

This approach works. Is it the best we can do?

A simple example: Invariant State

Consider CPTP dynamics $\mathcal{A}$ and $\mathcal{S} = \{\rho_0 = \sum_j p_j \Pi_j\}$ such that

$$\rho_0 = \mathcal{A}[\rho_0].$$

Then

$$\mathrm{alg}\{\mathcal{A}^t[\rho_0], t = 0, \ldots, n^2 - 1\} = \mathrm{alg}(\rho_0) = \mathrm{span}\{\Pi_j, \forall j\}.$$

Commutative (classical) algebra!
However, a trivial (1-dim) dynamics $\check{\rho}(t) = 1$ is sufficient to reproduce the states at each t: choose the CPTP reduction $\check{\rho} = \mathcal{R}(\rho) = \mathrm{Tr}(\rho)$, and injection $\mathcal{J}(\check{\rho}) = \rho_0 \check{\rho}$, so that $\rho_0 = \mathcal{J}(1)$ and

$$\check{\rho}(t+1) = \check{\rho}(t).$$

Improving on algebras

Notice that $\mathscr{R} = \mathrm{span}\{\rho_0\}$ is a distorted algebra, closed under the product $X \cdot_{\rho_0} Y = X\rho_0^{-1}Y$.

We need to close $\mathscr{R}$ to a distorted algebra.

In general, if $\mathscr{R}$ is the reachable space we need to find a state σ such that $\mathrm{alg}_\sigma(\mathscr{R})$:

- is the smallest distorted algebra that contains $\mathscr{R}$;
- there exists a state extension $\mathbb{J}$ that projects onto $\mathrm{alg}_\sigma(\mathscr{R})$ (Takesaki's theorem).

How to "properly" close $\mathscr{R}$ to a distorted algebra

Given an operator space $\mathscr{R}$:

1. Compute $\mathscr{A} = \mathrm{alg}(\mathscr{R})$;
2. Compute its center $\mathscr{Z} = \{X \in \mathscr{A} | [X, A] = 0, \quad \forall A \in \mathscr{A}\}$;
3. Pick a full-rank element $\sigma \in \mathscr{Z}$;

Main result: $\mathrm{alg}_\sigma(\mathscr{R})$ is the **smallest distorted algebra that allows for a state extension** and contains $\mathscr{R}$.

Reachable reduction

1. Find the **reachable subspace**

$$\mathscr{R} = \mathrm{span}\{\mathcal{A}^t(\rho_0), t \geq 0\};$$

2. Close it to a **distorted algebra** $\mathscr{A}_\sigma = \mathrm{alg}_\sigma(\mathscr{R})$;

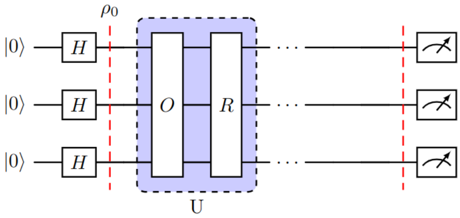
3. Compute the **state extension** $\mathbb{J}_{\mathscr{A},\sigma}$ and its factors $\mathcal{R}$ and $\mathcal{J}$;

4. Use the two factors to **reduce the dynamics**:

$$\check{\mathcal{A}} = \mathcal{R}\mathcal{A}\mathcal{J}, \qquad \check{\mathcal{C}} = \mathcal{C}\mathcal{J}, \qquad \Phi = \mathcal{R}.$$

Applications of interest:

Grover's algorithm



REPEAT $U \approx \frac{\pi}{4}\sqrt{N}$ TIMES

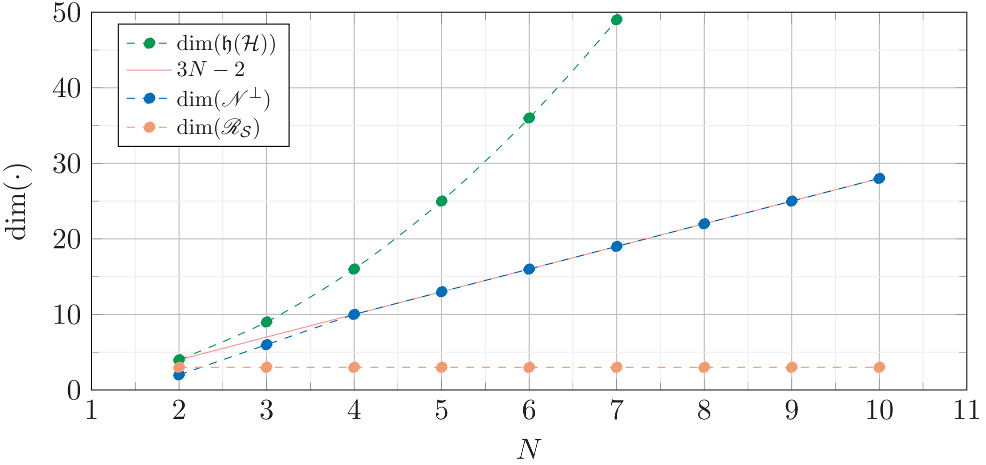
$$O\,|i\rangle = (-1)^{f(i)}\,|i\rangle$$

$$R = 2\,|+\rangle\langle+|^{\otimes N} - I_N$$

$$\begin{cases} \rho(t+1) = U\rho(t)U^\dagger \\ \boldsymbol{p}(t) = \mathrm{diag}[\rho(t)] \end{cases}$$

$$\rho_0 = |+\rangle\langle+|^{\otimes N}$$

Numerical Analysis of Grover's algorithm



Reachable reduction applied to Grover

Fix $S = \{j | f(x) = 1\}$ the set of solutions to the search problem and $M = |S|$ the number of solutions.

$$|\alpha\rangle := (N - M)^{-1/2} \sum_{j \notin S} |j\rangle \qquad |\beta\rangle := (M)^{-1/2} \sum_{j \in S} |j\rangle$$

$$\mathscr{R} := \mathrm{span}\{\rho(t), t \geq 0\} = \mathrm{span}\{|\alpha\rangle\langle\alpha| , |\beta\rangle\langle\beta| , |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|\}$$

$$\mathscr{A} = \mathrm{alg}(\mathscr{R}) = \mathrm{span}\{|\alpha\rangle\langle\alpha| , |\beta\rangle\langle\beta| , |\alpha\rangle\langle\beta| , |\beta\rangle\langle\alpha|\} \simeq \boxed{\mathbb{C}^{2\times 2}}$$

It's a qubit!
Regardless of the system's size N

Grover's reduced quantum model

$$\begin{cases} \check{\rho}(t+1) = \check{U}\,\check{\rho}(t)\check{U}^\dagger \\ \boldsymbol{p}(t) = \sum_{i\in S} |i\rangle\langle 1|\,\check{\rho}(t)\,|1\rangle\langle i| + \sum_{i\notin S} |i\rangle\langle 0|\,\check{\rho}(t)\,|0\rangle\langle i| \end{cases}$$

with

$$\check{\rho}_0 = \frac{N-M}{N}\,|0\rangle\langle 0| + \frac{M}{N}\,|1\rangle\langle 1| + \frac{\sqrt{(N-M)M}}{N}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$\check{U} = \frac{N-2M}{N} I_2 - i\frac{\sqrt{(N-M)M}}{N}\sigma_y$$

NOTE: Knowing $\mathscr{R}$ “is equivalent” to finding the search problem solution.

Continuous-time

Consider

$$\dot{\rho}(t) = \mathcal{L}[\rho(t)].$$

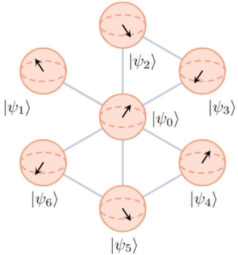
For $e^{\mathcal{L}t}[\cdot]$ to be CPTP for all $t \geq 0$, $\mathcal{L}[\cdot]$ needs to be a GKSL generator:

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}.$$

Main result:

Given a GKSL generator $\mathcal{L}$, its action onto the distorted algebra $\mathscr{X}$, $\mathscr{L}|_{\mathscr{X}} = \mathcal{R}\mathcal{L}\mathcal{J}$ is still a GKSL generator.

Dephasing central spin model coupled to a Markovian environment



$$H = \sum_{k=0}^{N-1} J_{k,k}\sigma_z^{(k)} + \sum_{l>k} J_{k,l}\sigma_z^{(k)}\sigma_z^{(l)}$$

$$\{L_k\} = \{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}, \sigma_+^{(j)}, \sigma_-^{(j)}\}_{j=0,\dots,N-1}$$

$$\begin{cases} \dot\rho = -i[H,\rho] + \sum_k L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \\ \rho_S(t) = \mathrm{tr}_E[\rho(t)] \end{cases}$$

$$\rho_0 \in \mathscr{D}(\mathcal{H})$$

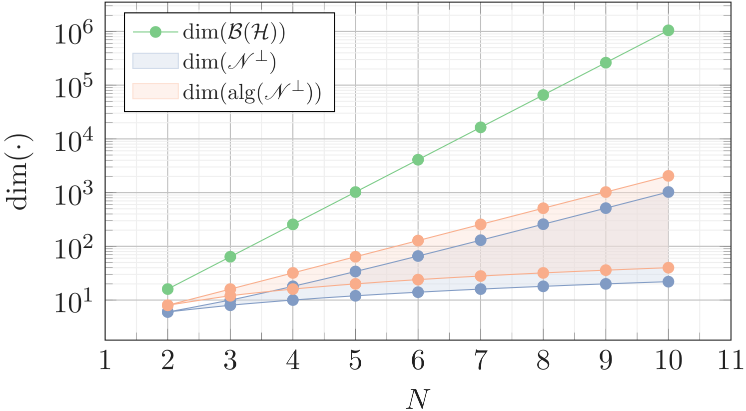
Observable space and algebra

$$\mathscr{N}^\perp \subseteq \mathrm{span}\{\sigma_0 \otimes I_{2^{N-1}}, \sigma_z \otimes I_{2^{N-1}}, \sigma_x \otimes |j\rangle\langle j|\,, \sigma_y \otimes |j\rangle\langle j|\,, \quad j = 0, \ldots, 2^{N-1}\}$$

$$\mathrm{alg}(\mathscr{N})^\perp \subseteq \mathrm{span}\{\sigma_k \otimes |j\rangle\langle j|\,, \quad j = 0, \ldots, 2^{N-1}, \quad k = 0, x, y, z\} \simeq \bigoplus_{j=0}^{2^{N-1}-1} \mathbb{C}^{2\times 2}$$

The bath can be modeled classically!

How much are we reducing?



$4N \leq \dim(\mathscr{A}) \leq 2^{N+1}$ (depending on the parameters) instead of 2^{2N}.

Conclusion

- **Framework for model reduction of statistical dynamics (CPTP)**
- Motivated by quantum walks/HMM;
- The Framework is very general and has been extended to:
  - Discrete-time case;
  - Continuous-time case;
  - Open systems, homogeneus;
  - Quantum trajectories.
- Solves an open problem in HMM (since ’92);
- Explores quantumness of processes;
- Explores optimality of Grover’s algorithm and other quantum walks;

Outlook

- Approximate method;
- Asymptotic model reduction (ongoing w/ Viola);
- Physically relevant examples (ongoing w/ Viola);
- Positive linear systems (ongoing w/ Cortese and Ferrante);
- Infinite-dimensional case (e.g. bosons);
- Locality constraints, networks (ongoing w/ Peruzzo);
- Continuous-time with measurement, SME (ongoing w/ Pellegrini);
- Adiabatic elimination techniques (ongoing w/ Sarlette);

Related pubblications

- TG and FT, "Minimal resources for exact simulation of quantum walks," IEEE CDC 2022;
- TG and FT, "Algebraic Reduction of Hidden Markov Models," IEEE TAC 2023;
- TG and FT, "Model Reduction for Quantum Systems: Discrete-time Quantum Walks and Open Markov Dynamics," (submitted to) IEEE TIT 2023;
- TG and FT, "Exact model reduction for conditional quantum dynamics, " (submitted to) IEEE CDC and L-CSS 2024;
- TG, Yukuan Tao, FT, and Lorenza Viola, "Exact Model Reduction for Continuous-Time Open Quantum Dynamics," (to be submitted to) PRXQ 2024;

Thanks for your attention!

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