Exact model reduction for Quantum Systems

An algebraic approach

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What are the minimal resources needed to reproduce a target quantum model?

The model

$\mathcal{B}(\mathcal{H}) = \mathbb{C}^{n \times n}$
ρ are density operators

$\mathcal{A}$ is a CPTP map

$\mathcal{S}$ is a set of initial conditions

$$\begin{cases} \rho(t+1) = \boxed{\mathcal{A}}\,[\,\boxed{\rho(t)}\,] \\ \boxed{y_i(t)} = \mathrm{tr}[\,\boxed{C_i}\,\rho(t)] \end{cases}$$

$\boxed{\rho_0 \in \mathcal{S}}$

$Y(t)$ is the output of interest, the one we want to preserve

$\{C_i\}$ is a set of observables of interest

This **does not include** the effects of conditioning.

The problem: quantum model reduction

Given a QS $(\mathcal{A}, \mathcal{S}, \{C_i\})$ find **another QS** $(\check{\mathcal{A}}, \check{\mathcal{S}}, \{\check{C}_i\})$

such that for all $t \geq 0$ and $\rho_0 \in \mathcal{S}$

- **exact model reduction**

$$\mathrm{tr}[C_i \mathcal{A}^t[\rho_0]] = \mathrm{tr}[\check{C}_i \check{\mathcal{A}}^t[\check{\rho}_0]];$$

- approximate model reduction (not going to talk about this)

$$\mathrm{tr}[C_i \mathcal{A}^t[\rho_0]] \approx \mathrm{tr}[\check{C}_i \check{\mathcal{A}}^t[\check{\rho}_0]].$$

Note: We assume we are interested in a reduced number of initial conditions and observables of interest.

Why model reduction?

- Efficient **simulations** and generation of **quantum dynamics**;
- Efficient implementations of **controllers, error suppression schemes and quantum filters** ;
- Easier models to study;
- Probing “quantumness” of processes.

Why QUANTUM model reduction?

Linear model:

+ Smaller in size (minimal, actually);
+ Efficient simulations on classical computers;
- Loses structures and properties of quantum systems.

Quantum model:

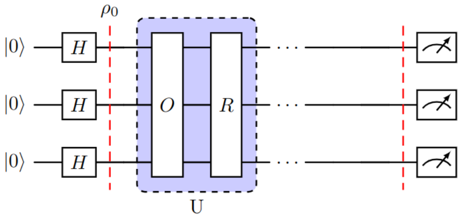
+ Efficient simulations on quantum computers;
+ Retains structures and properties of a quantum system (it is easier to draw conclusions on the original model);
- Larger in size (in general).

State of the art

- Reduced linear models (not quantum);
- Efficient simulations on both classical and quantum computers (not models);
- Quantum approximate MR, e.g. adiabatic elimination, derivation of master equations (approximate, but strong assumptions);
- A few studies **exact quantum model reduction** but have **very limited scopes** (e.g. only Hamiltonian and pure states).

Applications of interest:

Grover's algorithm



REPEAT $U \approx \frac{\pi}{4}\sqrt{N}$ TIMES

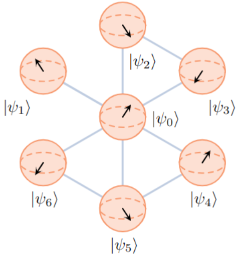
$$O\,|i\rangle = (-1)^{f(i)}\,|i\rangle$$

$$R = 2\,|+\rangle\langle+|^{\otimes N} - I_N$$

$$\begin{cases} \rho(t+1) = U\rho(t)U^{\dagger} \\ \boldsymbol{p}(t) = \mathrm{diag}[\rho(t)] \end{cases}$$

$$\rho_0 = |+\rangle\langle+|^{\otimes N}$$

Dephasing central spin model coupled to a Markovian environment



$$H = \sum_{k=0}^{N-1} J_{k,k}\sigma_z^{(k)} + \sum_{l>k} J_{k,l}\sigma_z^{(k)}\sigma_z^{(l)}$$

$$\{L_k\} = \{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}, \sigma_+^{(j)}, \sigma_-^{(j)}\}_{j=0,\ldots,N-1}$$

$$\begin{cases} \dot{\rho} = -i[H,\rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \\ \rho_S(t) = \mathrm{tr}_E[\rho(t)] \end{cases}$$

$$\rho_0 \in \mathscr{D}(\mathcal{H})$$

The protagonist of this work: $*$-algebras

Definition

We define a $*$-algebra $\mathscr{A}$ as an operator space closed under matrix multiplication and adjoint action.

$$X, Y \in \mathscr{A} \Rightarrow X + Y \in \mathscr{A} \qquad X^\dagger, Y^\dagger \in \mathscr{A} \qquad \text{and} \qquad XY \in \mathscr{A}$$

It is the fundamental mathematical structure that supports a quantum probability space.

Examples

$$\mathscr{A}_1 = \mathrm{span}\{\sigma_j, \quad j = 0, x, y, z\} = \mathbb{C}^{2\times 2}, \qquad \dim(\mathscr{A}_1) = 4$$

$$\mathscr{A}_2 = \mathrm{span}\{\sigma_j \otimes \sigma_k, \quad j, k = 0, x, y, z\} = \mathbb{C}^{4\times 4}, \qquad \dim(\mathscr{A}_2) = 16$$

$$\mathscr{A}_3 = \mathrm{span}\{\sigma_j \otimes \sigma_k, \quad j = 0, x, y, z, k = 0, z\} \subsetneq \mathbb{C}^{4\times 4}, \qquad \dim(\mathscr{A}_3) = 8$$

$$\mathscr{A}_4 = \mathrm{span}\{\sigma_j \otimes \sigma_j, \quad j = 0, x, y, z\} \subsetneq \mathbb{C}^{4\times 4}, \qquad \dim(\mathscr{A}_4) = 4$$

Wedderburn decomposition

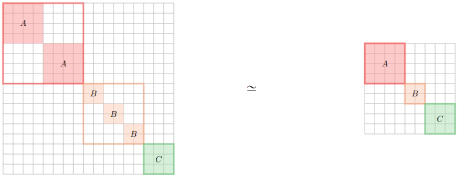
For any algebra $\mathscr{A}$, there exist an Hibert space decomposition

$$\mathcal{H} = \bigoplus_k \mathcal{H}_{S,k} \otimes \mathcal{H}_{F,k} \oplus \mathcal{H}_R$$

and a unitary operator U such that

$$\mathscr{A} = U \left(\bigoplus_k \mathcal{B}(\mathcal{H}_{S,k}) \otimes I_{F,k} \oplus 0_R \right) U^\dagger$$

$\mathscr{A} \simeq \bigoplus_k \mathcal{B}(\mathcal{H}_{S,k})$



Examples

$$\mathscr{A}_1 = \mathrm{span}\{\sigma_j, \quad j = 0, x, y, z\} = \mathbb{C}^{2\times 2}$$

$$\mathscr{A}_2 = \mathrm{span}\{\sigma_j \otimes \sigma_k, \quad j, k = 0, x, y, z\} = \mathbb{C}^{4\times 4}$$

$$\mathscr{A}_3 = \mathrm{span}\{\sigma_j \otimes \sigma_k, \quad j = 0, x, y, z, k = 0, z\} \simeq \mathbb{C}^{2\times 2} \bigoplus \mathbb{C}^{2\times 2}$$

$$\mathscr{A}_4 = \mathrm{span}\{\sigma_j \otimes \sigma_j, \quad j = 0, x, y, z\} \simeq \mathbb{C}^{2\times 2}$$

Conditional expectations & state extensions

A *conditional expectation* $\mathbb{E}_{\mathscr{A},\rho}[\cdot]$ is a **CP unital projector onto a $*$-algebra** $\mathscr{A}$ and such that $\mathrm{tr}[\rho\mathbb{E}_{\mathscr{A},\rho}[X]] = \mathrm{tr}[\rho X]$ for all $X \in \mathcal{B}(\mathcal{H})$.
(Superoperator in Heisenberg picture)

Its dual (Schroedinger's picture) is called a *state extension* $\mathbb{J}_{\mathscr{A},\rho}[\cdot] = \mathbb{E}^{\dagger}_{\mathscr{A},\rho}[\cdot]$ is a **CPTP projection onto an operator subspace** $\mathscr{X}$ and is such that $\mathbb{J}_{\mathscr{A},\rho}[\rho] = \rho$.

$\mathscr{X}$ is an operator subspace closed under adjoint action and a modified matrix product $X \cdot_\rho Y = X\rho^{-1}Y$ and can be computed as $\mathscr{X} = \rho^{1/2}\mathscr{A}\rho^{1/2}$.
We call this a **distorted algebra**.

Properties of state extensions (new!)

$\mathbb{J}_{\mathscr{A},\rho}[\cdot]$ can be factorized in two non-square CPTP isometries $\mathbb{J}_{\mathscr{A},\rho}[\cdot] = \mathcal{J}\mathcal{R}$.

Given a CPTP map $\mathcal{A}$, its action onto the distorted algebra $\mathscr{X}$, $\mathcal{A}|_{\mathscr{X}} = \mathcal{R}\mathcal{A}\mathcal{J}$ is CPTP.

Given a GKSL generator $\mathcal{L}$, its action onto the distorted algebra $\mathscr{X}$, $\mathscr{L}|_{\mathscr{X}} = \mathcal{R}\mathcal{L}\mathcal{J}$ is still a GKSL generator.

Take home ideas

1. We need algebras to define a quantum probability space;
2. Projections onto distorted algebras provide CPTP reduction.

Proposed method

Reachable reduction: LINEAR case

We leverage the knowledge of the initial conditions $\rho_0 \in \mathcal{S}$ to reduce the model.

1. Find the **minimal subspace that contains the dynamics**

$$\mathscr{R} = \mathrm{span}\{e^{\mathcal{L}t}(\rho_0), t \geq 0\} = \mathrm{span}\{\mathcal{L}^k(\rho_0), k \geq 0\};$$

2. **Reduce the dynamics** onto $\mathscr{R}$. $\rightarrow$ This is the minimal model.

Reducing the model onto any space that contains $\mathscr{R}$ provides a non-minimal valid reduction.

Reachable reduction: main idea

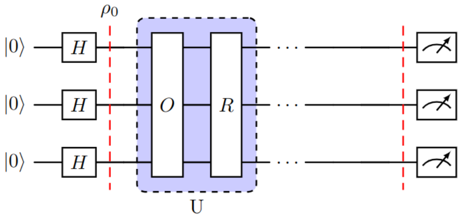
We leverage the knowledge of the initial conditions $\rho_0 \in \mathcal{S}$ to reduce the model.

1. Find the **minimal subspace that contains the dynamics**

$$\mathscr{R} = \mathrm{span}\{e^{\mathcal{L}t}(\rho_0), t \geq 0\} = \mathrm{span}\{\mathcal{L}^k(\rho_0), k \geq 0\};$$

2. Close it to a **distorted algebra** (how? - new results and a lot of work);
3. Compute the **state extension** and its factors;
4. Use the two factors to **reduce the dynamics**.

Grover's algorithm



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$$\rho_0 = |+\rangle\,\langle+|^{\otimes N}$$

Reachable reduction applied to Grover

Fix $S = \{j | f(x) = 1\}$ the set of solutions to the search problem and $M = |S|$ the number of solutions.

$$|\alpha\rangle := (N - M)^{-1/2} \sum_{j \notin S} |j\rangle \qquad |\beta\rangle := (M)^{-1/2} \sum_{j \in S} |j\rangle$$

$$\mathscr{R} := \mathrm{span}\{\rho(t), t \geq 0\} = \mathrm{span}\{|\alpha\rangle\langle\alpha| \,, |\beta\rangle\langle\beta| \,, |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|\}$$

$$\mathscr{A} = \mathrm{alg}(\mathscr{R}) = \mathrm{span}\{|\alpha\rangle\langle\alpha| \,, |\beta\rangle\langle\beta| \,, |\alpha\rangle\langle\beta| \,, |\beta\rangle\langle\alpha|\} \simeq \boxed{\mathbb{C}^{2\times 2}}$$

It’s a qubit!
Regardless of the system’s size N

Grover's reduced quantum model

$$\begin{cases} \check{\rho}(t+1) = \check{U}\check{\rho}(t)\check{U}^\dagger \\ \boldsymbol{p}(t) = \sum_{i\in S} |i\rangle\langle 1|\,\check{\rho}(t)\,|1\rangle\langle i| + \sum_{i\notin S} |i\rangle\langle 0|\,\check{\rho}(t)\,|0\rangle\langle i| \end{cases}$$

with

$$\check{\rho}_0 = \frac{N-M}{N}\,|0\rangle\langle 0| + \frac{M}{N}\,|1\rangle\langle 1| + \frac{\sqrt{(N-M)M}}{N}(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$\check{U} = \frac{N-2M}{N} I_2 - i\frac{\sqrt{(N-M)M}}{N}\sigma_y$$

NOTE: Knowing $\mathscr{R}$ "is equivalent" to finding the search problem solution.

Observable reduction

We leverage the knowledge of the observables of interest $\{C_i\}$ to reduce the model.

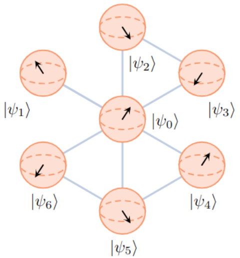
Switch to the **Heisenberg picture**.

The minimal subspace that contains the “observables” dynamics is

$$\mathscr{O} = \mathrm{span}\{e^{\mathcal{L}^\dagger t}(C_i), \quad t \geq 0\} = \mathrm{span}\{\mathcal{L}^{\dagger k}(C_i), \quad k \geq 0\}$$

What is orthogonal to $\mathscr{O}$ <u>can not be observed</u>.

Spin model



$$H = \sum_{k=0}^{N-1} J_{k,k}\sigma_z^{(k)} + \sum_{l>k} J_{k,l}\sigma_z^{(k)}\sigma_z^{(l)}$$

$$\{L_k\} = \{\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}, \sigma_+^{(j)}, \sigma_-^{(j)}\}_{j=0,\ldots,N-1}$$

$$\begin{cases} \dot{\rho} = -i[H,\rho] + \sum_k L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \\ \rho_S(t) = \mathrm{tr}_E[\rho(t)] \end{cases} \qquad \rho_0 \in \mathscr{D}(\mathcal{H})$$

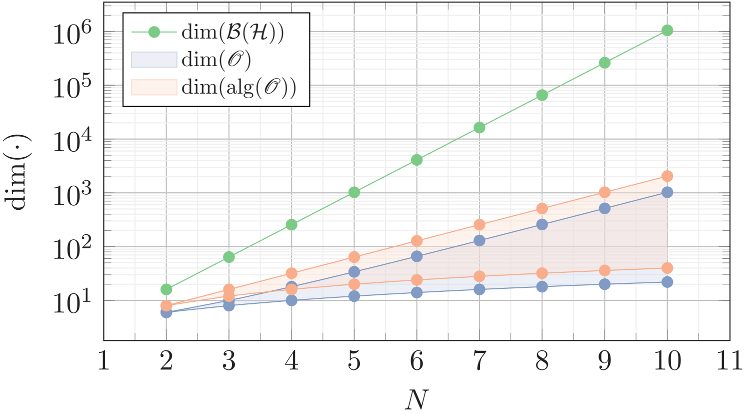
$$\{C_i\} = \{\sigma_j^{(0)}, \quad j = 0, x, y, z\}$$

Observable space and algebra

$$\mathscr{O} \subseteq \mathrm{span}\{\sigma_0 \otimes I_{2^{N-1}}, \sigma_z \otimes I_{2^{N-1}}, \sigma_x \otimes |j\rangle\langle j|, \sigma_y \otimes |j\rangle\langle j|, \quad j = 0, \ldots, 2^{N-1}\}$$

$$\mathrm{alg}(\mathscr{O}) \subseteq \mathrm{span}\{\sigma_k \otimes |j\rangle\langle j|, \quad j = 0, \ldots, 2^{N-1}, \quad k = 0, x, y, z\} \simeq \bigoplus_{j=0}^{2^{N-1}-1} \mathbb{C}^{2\times 2}$$

How much are we reducing?



$4N \leq \dim(\mathscr{A}) \leq 2^{N+1}$ (depending on the parameters) instead of 2^{2N}.

Open problems and future directions

- Approximate method;
- Physically relevant examples (ongoing w/ Lorenza Viola's group);
- Positive linear systems;
- Infinite-dimensional case (e.g. bosons);
- Continuous-time with measurement, SME (non-linear);

Thanks for your attention!

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Improving algebras: Invariant State

Consider CPTP dynamics $\mathcal{A}$ and $\mathcal{S} = \{\rho_0 = \sum_j p_j \Pi_j\}$ such that

$$\rho_0 = \mathcal{A}[\rho_0].$$

Then

$$\mathrm{alg}\{\mathcal{A}^t[\rho_0], t = 0, \ldots, n^2 - 1\} = \mathrm{alg}(\rho_0) = \mathrm{span}\{\Pi_j, \forall j\}.$$

Commutative (classical) algebra!

However, a trivial (1-dim) dynamics $\check{\rho}(t) = 1$ is sufficient to reproduce the states at each t: choose the CPTP reduction $\check{\rho} = \mathcal{R}(\rho) = \mathrm{Tr}(\rho)$, and injection $\mathcal{J}(\check{\rho}) = \rho_0 \check{\rho}$, so that

$$\rho_0 = \mathcal{J}(1)$$

Improving algebras

Notice that $\mathscr{R} = \mathrm{span}\{\rho_0\}$ is a distorted algebra, closed under the product $X \cdot_{\rho_0} Y = X\rho_0^{-1}Y$.

We need to close $\mathscr{R}$ to a distorted algebra.

In general, if $\mathscr{R}$ is the reachable space we need to find a state σ such that $\mathrm{alg}_\sigma(\mathscr{R})$:

- is the smallest distorted algebra that contains $\mathscr{R}$;
- there exists a state extension $\mathbb{J}$ that projects onto $\mathrm{alg}_\sigma(\mathscr{R})$ (Takesaki's theorem).

How to “properly” close $\mathscr{R}$ to a distorted algebra

Given an operator space $\mathscr{R}$:

1. Compute $\mathscr{A} = \mathrm{alg}(\mathscr{R})$;
2. Compute its center $\mathscr{Z} = \{X \in \mathscr{A} | [X, A] = 0, \quad \forall A \in \mathscr{A}\}$;
3. Pick a full-rank element $\sigma \in \mathscr{Z}$;
4. $\mathrm{alg}_\sigma(\mathscr{R})$ is the **smallest distorted algebra that allows for a state extension.**