Marcus Exercises

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Chapter 2

8 (a) By the definition of disc, it is the square of an element in $\mathbb{Q}[\omega]$, and so we have $(\pm p)^{\frac{p-2}{2}} \in \mathbb{Q}[\omega]$ with $+$ when $p \equiv 1 \pmod 4$ and $-$ when $p \equiv -1 \pmod 4$. Since p is odd, we also know $p^{\frac{p-3}{2}} \in \mathbb{Q}[\omega]$. Dividing the two expressions gives $(-p)^{\frac{1}{2}} \in \mathbb{Q}[\omega]$ as desired. By Theorem 8, we have $\operatorname{disc}(\omega_3) = (\omega - \omega^2)^2$ and so its square root divided by 3^0 or $\omega - \omega^2$ is $\sqrt{-3}$. Similarly, we can find $\operatorname{disc}(\omega_5) = \frac{\prod_{1 \le r < s \le n}(\omega_5^r - \omega_5^s)}{5}$.

(b) Let $z = e^{\frac{2\pi i}{8}}$. Then, $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, z^3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \in \mathbb{Q}[z]$. Adding gives $\sqrt{2} \in \mathbb{Q}[z]$.

(c)

19 Let $f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$. By subtracting the last column by b_i times the ith column for $i = 0$ to $i = n-1$, as an elementary matrix operation which keeps the determinant invariant, we find

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^n \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{vmatrix} = \begin{vmatrix} 1 & a_1 & \cdots & f(a_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & f(a_n) \\ 1 & a_{n+1} & \cdots & f(a_{n+1}) \end{vmatrix}$$

Now let $f(x)$ be the monic polynomial of degree n defined by $f(x) = \prod_{i \in [n]}(x - a_i)$. Plugging and using the recursive definition of the determinant via cofactors gives

$$\begin{vmatrix} 1 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & 0 \\ 1 & a_{n+1} & \cdots & \prod_{i \in [n]}(a_{n+1} - a_i) \end{vmatrix} = \prod_{1 \le r < s \le n+1} (a_s - a_r)$$

20 Let $f(x) = (x-\alpha)g(x)$. Then,

$$\begin{aligned} f'(x) &= (x-\alpha)g'(x) + g(x) \\ \implies f'(\alpha) &= 0 + g(\alpha) = \prod_{\text{roots } \beta\neq\alpha} (\alpha-\beta) \end{aligned}$$

by irreducibility.

23 (a) Theorem 6'.

$$\operatorname{disc}_K^L(\alpha_1, \cdots, \alpha_n) = \left|\mathrm{T}_K^L(\alpha_i\alpha_j)\right|$$

Corollary'.
stuff
Theorem 7'.
$\operatorname{disc}_K^L(\alpha_1, \cdots, \alpha_n) = 0$ iff $\alpha_1, \cdots, \alpha_n$ are linearly independent over $\mathbb{Q}$.
Theorem 8'.
Suppose $J = K[\alpha]$, and let $\alpha_1, \cdots, \alpha_n$ be the conjugates of α over K. Then,

$$\operatorname{disc}_K^L(\alpha_1, \cdots, \alpha_n) = \prod_{1\leq r<s\leq n} (\alpha_r - \alpha_s)^2 = \pm \mathrm{N}_K^L(f'(\alpha))$$

24 First, we prove that any subgroup H of $\mathbb{Z}$ must be $n\mathbb{Z} \cong \mathbb{Z}$ or $\{0\}$. Let $n = \min\{|h|\} \in H$. If $n = 0$, then $H = \{0\}$. If $n \neq 0$, then suppose for contradiction that $H \neq n\mathbb{Z}$. That would imply $\exists m \in H$ such that $m = nq + r$ for some $0 < r \neq 0 < n$ by Euclidean division. This would imply $m - nq = r \in H$, which is a value in H with absolute value less that n, contradicting our defintion of n. Therefore, we must have $H = n\mathbb{Z}$ in this case.

(a) We have $K = \ker \pi = \mathbb{Z}^{n-1} \leq G$, and so $H \cap K \leq K$, which is a free abelian group of rank $n-1$. By the inductive hypothesis, we find the $H \cap K$ is a free abelian group of rank $\leq n-1$.

(b) We have $\pi(H) \leq \mathbb{Z}$, and so it is either $h\mathbb{Z}$ or $\{0\}$. If it is $\{0\}$, then $H = H \cap K$ by the definition of the kernel, which means we are done with (a). In the other case, we have $\pi(H) = h\mathbb{Z}$ for some $h \in H$ such that $\pi(h)$ generates $\pi(H)$.
First, we prove $h\mathbb{Z}$ and $H \cap K$ trivially intersect. Take $hk \in h\mathbb{Z} \cap (H \cap K)$ for some $k \in \mathbb{Z}$. We know $\pi(hk) = 0 \Rightarrow k\pi(h) = 0$ since $hk \in K$. This gives $k = 0$ or $\pi(h) = 0$; however, $\phi(h) \neq 0$ since or else it would generated $\{0\}$ instead of $h\mathbb{Z}$, and so $k = 0 \Rightarrow hk = 0$.
Next, we show $H = h\mathbb{Z} + H \cap K$. Take anhy $h' \in H$. Then, $\pi(h') \in \pi(H)$, so let $\pi(h') = k'\pi(h) = \pi(k'h)$. Subtracting gives $\pi(h' - k'h) = 0$, and so $h' - k'h \in H \cap K$, which means $h' = k'h + a$ for some $a \in H \cap K$. Therefore, $H = h\mathbb{Z} \oplus K$.

25 Let α be the root of $\frac{a_n}{b_n}x^n + \frac{a_{n-1}}{b_{n-1}}x^{n-1} + \cdots + \frac{a_0}{b_0}$ for $a_i, b_i \in \mathbb{Z}\forall i$. Multiplying the polynomial by $b_n \cdots b_0$ gives a polynomial $a'_n x^n + a'_{n-1}x^{n-1} + \cdots + a'_0$ with root α for $a'_i \in \mathbb{Z}\forall i$. Now, note $a'_n\alpha$ is the root of $\frac{1}{(a'_n)^{n-1}}x^n + \frac{a'_{n-1}}{(a'_n)^{n-1}}x^{n-1} + \cdots + \frac{a'_1}{a'_n}x + a'_0$ or $x^n + a'_{n-1}x^{n-1} + \cdots + a'_1 a'^{n-2}_n x + a'_0 a'^{n-1}_n$, which makes $a'_n\alpha$ an algebraic integer. For a finite set of algerbaic numbers α_i, let m_i be s set of integers such that $m_i\alpha_i$ is an algebraic integer $\forall i$. Then, we find $(\prod m_i)\alpha_i$ is an algerbaic integer since every integer m_i is an algebraic integer, and the algebraic integers are closed under multiplication.

26 As in the proof of Theorem 11, we can produce the same equation

$$\begin{pmatrix}\beta_1\\ \vdots \\ \beta_2\end{pmatrix} = M\begin{pmatrix}\gamma_1\\ \vdots \\ \gamma_2\end{pmatrix}$$

as $\{\beta_i\forall i\}, \{\gamma_i\forall i\}$ generate the same subgroup for an $n \times n$ matrix M over $\mathbb{Z}$. With this, we can finish the proof exactly the same as in the one of Theorem 11.

28 (a) We find

$$f'(x) = 3x^2 + a$$
$$\implies f'(\alpha) = 3\alpha^2 + a = -\frac{2a\alpha + 3b}{\alpha}$$

since

$$\alpha^3 + a\alpha + b = 0 \implies 3\alpha^2 + a = -\frac{2a\alpha + 3b}{\alpha}$$

(b) Plugging in gives $\frac{(2a\alpha+3b)-3b}{2a} = \alpha$, and so we find that $2a\alpha + 3b$ is a root with $f(\alpha) = 0$. The embeddings of $\mathbb{Q}[\alpha] \hookrightarrow \mathbb{C}$ must send a root to a root, and so $\mathrm{N}^{\mathbb{Q}[\alpha]}_{\mathbb{Q}}$ can be calculated via Vieta's from the minimal polynomial of $2a\alpha + 3b$ to give $-27b^3 - 4a^3b$.

(c) By Theorem 8, we find $\mathrm{disc}(\alpha) = -\mathrm{N}^{\mathbb{Q}[\alpha]}_{\mathbb{Q}}(-\frac{2a\alpha+3b}{\alpha}) = -\mathrm{N}(2a\alpha+3b)\,\mathrm{N}(\frac{1}{\alpha})\,\mathrm{N}(-1)$ with the multiplicativity of N. We find $(\frac{1}{x})^3 + \frac{a}{x} + b$ or $1 + ax^2 + bx^3$ to be a minimal polynomial for $\frac{1}{\alpha}$, and so $\mathrm{N}(\frac{1}{\alpha}) = \frac{1}{b}$. We also have $\mathrm{N}(-1) = (-1)^3 = -1$ since $-1 \in \mathbb{Q}$. We finally get

$$\mathrm{disc}(\alpha) = -\left((-27b^3 - 4a^3b)\left(\frac{1}{b}\right)(-1)\right) = -(4a^3 + 27b^2)$$

(d) In the first case, α satisfies $\alpha^3 - \alpha - 1 = 0$, and so $\mathrm{disc}(\alpha) = -(-4 + 27) = -23$, which is indeed squarefree. In the second case, α satisfies $\alpha^3 + \alpha - 1 = 0$, and so $\mathrm{disc}(\alpha) = -(4 + 27) = -1$, which is squarefree again.

Lang Algebra V

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Chapter 3

(1) $\not\Rightarrow$ (1′) Take any increasing sequence of ideals $I_1, \subseteq I_2 \subseteq \cdots$, and let $I = (I_1, I_2, \cdots)$ be the ideal generated by them all.