Note on fibrations

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Abstract

This note is meant to be a written version of a talk given for a student seminar on higher category theory organised by Jordan Levin (USPN) & Ran Azouri (USPN), and as such will not be very proof heavy, aiming only to give the audience a first look at the results, without much details. This is the ninth of ten talk. The subject is the notion of cartesian and cocartesian fibrations, some of their basic properties and how one constructs functors using these notions, notably using Lurie's straightening-unstraightening equivalence. The main references are [Har], [HP15], [HHR21],[Lur09] and [Rui20].

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1 Introduction

1.1 Spaces and fibrations

We first try and motivate the main result, by recalling a few classical result that the main result can be seen as a generalisation of.

Definition 1.1. Let X be a topological space. We denote $\Pi_1 X$ the **fundamental groupoid** of X, i.e. the category defined by :

- $\mathrm{Ob}(\Pi_1 X) := X$
- For all $x, y \in X$, we take $\mathrm{Hom}_{\Pi_1 X}(x, y) = \{\gamma \in \mathrm{Hom}_{\mathbf{Top}}(I, X), \gamma(0) = x, \gamma(1) = y\} \Big/ \sim$, where $\sim$ denotes the relation of path homotopy.

Definition 1.2. Recall that a **covering space** of a topological space X is the data of a space Y and a map $p : Y \to X$ in $\mathbf{Top}$ such that for all $x \in X$ there exists an open set $x \in U$ such that the following diagram commutes :

$$\begin{tikzcd}
U \times E_x \arrow[r, "\simeq"] \arrow[dr, "\pi"'] & p^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "p"] \\
& U \arrow[r, hook] & X
\end{tikzcd}$$

with $E_x = p^{-1}(x)$ is discrete.

These assemble into a category $\mathbf{Cov}(X)$ whose objects are covering spaces over X and where a morphism between $p : Y \to X$ and $p' : Y' \to X$ is given by a triangle :

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\
& X &
\end{tikzcd}$$

A classical result from a first course of algebraic topology is a classification of covering spaces, using the language of category theory, it can be packaged as follows.

Theorem 1.3. *For X a sufficiently nice space (for instance a connected CW complex), we have an equivalence of categories*

$$\mathbf{Cov(X)} \simeq \mathbf{Fun}(\Pi_1 X, \mathbf{Set}).$$

Proof (sketch). Define $F : \mathbf{Cov}(X) \to \mathbf{Fun}(\Pi_1 X, \mathbf{Set})$ taking a covering space $p : Y \to X$ to the functor F_p such that $F_p(x) = p^{-1}(x)$, and for a morphism $x \to y$ in $\Pi_1 X$ i.e. a (homotopy class of) path, we can find a map $p^{-1}(x) \to p^{-1}(y)$ using the path lifting property of covering spaces. There are more details to check, which are left to the reader. ∎

A fundamental point of higher categories is that spaces and ∞-groupoids (or Kan complexes) should be the same. Within the real of quasicategories, spaces are indeed the same as Kan complexes. Let us recall some definitions, but given the context of this note, we assume the reader to be familiar with simplicial sets.

Definition 1.4. A **Kan complex** is a simplicial set X such that for any n and $0 \leq i \leq n$, there exists a dotted arrow making the diagram commute :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

A quasicategory is a simplicial set satisfying the condition above for $0 < i < n$.

A map of simplicial set $Y \to X$ is a Kan fibration if for any $n, 0 \leq i \leq n$ there exists a dotted arrow in the diagram below :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

It is well known (see for instance [Lur09]) that Kan complexes are the same as ∞-groupoid, i.e. quasi-categories with every morphism invertible.

We denote $\mathbf{Spc}$, what we call the ∞-category of spaces. There are various models to make its definition precise, one can take the coherent nerve of the simplicial categories of Kan complexes, or the simplicially enriched category of CW-complexes. Morally, in the theory of higher categories, this is the object meant to play the role of $\mathbf{Set}$.

In any case, viewing spaces as Kan complexes, we can now observe that the category $\Pi_1 X$ is now equivalent to hX, the homotopy category of the simplicial set X. It turns out that our theorem can be promoted to a much more comprehensive result.

Theorem 1.5. *Tha data of a Kan fibration $Y \to X$ is equivalent to a functor $X \to \mathbf{Spc}$.*

We do not wish to be more precise here, especially we do not want to spend time giving precise definitions of the objects at play, or stating the full theorem at this stage, as this is meant to be very expository. The main takeaway is that we've already seen two instances of certain fibrations being equivalent to specifying a functor to $\mathbf{Set}$ or $\mathbf{Spc}$. Let's turn to another kind of example.

1.2 Categories

1.2.1 The Grothendieck Construction

Let $F : \mathcal{C} \to \mathbf{Cat}$ be a functor. We can assemble the data of all the $F(c)$ into a single category called the Grothendieck construction for F as follows :

Denote $\int_{\mathcal{C}} F$ the categories whose objects are pairs (C, X) with $C \in \mathcal{C}$ and $X \in F(C)$. A morphism $(C, X) \to (C', X')$ is given by a pair (f, ϕ) where $f : C \to C'$ is a morphism in $\mathcal{C}$ and $\phi : F(f)(X) \to X'$ is a morphism of $F(C')$. For $(f, \phi) : (C, X) \to (C', X')$ and $(g, \psi) : (C', X') \to (C'', X'')$, we define composition with the formula :

$$(g, \psi) \circ (f, \phi) := (g \circ f, \psi \circ F(g)(\phi))$$

The category $\int_{\mathcal{C}} F$ comes equipped with a functor $\int_{\mathcal{C}} F \to \mathcal{C}$ mapping $(C, X) \mapsto C$ and $(f, \phi) \mapsto f$.

1.2.2 (co)Cartesian fibrations

Let $\pi : A \to B$ be a functor.

Definition 1.6. A morphism of A, $\phi : X \to Y$ is called **π-cocartesian** (or simply cocartesian), if given any morphism $\psi : X \to Z$ and any dotted arrow as in the diagram :

$$\begin{tikzcd} \pi X \arrow[r, "\pi(\psi)"] \arrow[d, "\pi(\phi)"'] & \pi Z \\ \pi Y \arrow[ur, dashed, "\rho"'] & \end{tikzcd}$$

There exists a lift, i.e. a map $\varepsilon : Y \to Z$ such that $\pi(\varepsilon) = \rho$ and such that the following diagram commutes :

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & Z \\ Y \arrow[ur, dashed, "\varepsilon"'] & \end{tikzcd}$$

Remark 1.7. We can make two remarks with the previous definition :

1. If $A = *$, a morphism is cocartesian if and only if it is an isomorphism.
2. If π is the projection from the Grothendieck construction of a functor F, i.e. $\pi : \int_{\mathcal{C}} F \to \mathcal{C}$ for a functor $F : \mathcal{C} \to \mathbf{Cat}$, then $(f, \phi) : (C, X) \to (C', X')$ is cocartesian if and only if ϕ is an isomorphism.

Definition 1.8. $\pi : \mathcal{D} \to \mathcal{C}$ is called a **cocartesian fibration** if for every morphism $f : A \to B$ in $\mathcal{C}$, and every object $X \in \mathcal{D}$ s.t. $\pi(X) = A$, there exists a cocartesian morphism $\phi : X \to Y$ such that $\pi(\phi) = f$.

Given two cocartesian fibrations over $\mathcal{C}$, $\pi : \mathcal{D} \to \mathcal{C}$ and $\pi' : \mathcal{D}' \to \mathcal{C}$, a morphism between the two cocartesian fibrations is a functor $F : \mathcal{D} \to \mathcal{D}'$ mapping π-cocartesian morphisms to π'-cocartesian morphisms.

Thus, we get a $(2,1)$-category $\mathbf{coCart}(\mathcal{C})$ whose objects are cocartesian fibrations over $\mathcal{C}$, 1-morphisms are morphisms of cocartesian fibrations, and 2-morphisms are natural isomorphisms of functors.

It turns out that the projection $\int_{\mathcal{C}} F \to \mathcal{C}$ is a cocartesian fibration.

Proposition 1.9. *For a functor $F : \mathcal{C} \to \mathbf{Cat}$, the projection $\pi : \int_{\mathcal{C}} F \to \mathcal{C}$ is a cocartesian fibration.*

Proof. Given $f : C \to C'$ in $\mathcal{C}$ and $(A, U) \in \int_{\mathcal{C}} F$ with $\pi((A, U)) = C$ (i.e. $A = C$), we ought to provide a cocartesian morphism $(g, \varphi) : (A, U) \to (B, V)$ in $\int_{\mathcal{C}} F$ with $\pi((g, \varphi)) = f$. Taking $(B, U) := (C', F(f)(U))$, $g = f$ and $\varphi = \mathrm{id}_{F(f)(U)}$. We do get a morphism $(f, \mathrm{id}_{F(f)(U)}) : (C, U) \to (C', F(f)(U))$. By the second part of remark 1.7, we see that it is indeed a cocartesian morphism. By definition $\pi((f, \varphi)) = f$, we are done. ■

We now get a new incarnation of the theorems we got in the previous section.

Theorem 1.10. *The Grothendieck construction gives an equivalence of $(2,1)$-categories*

$$\mathbf{Fun}(\mathcal{C}, \mathbf{Cat}) \simeq \mathbf{coCart}(\mathcal{C})$$

The proof is somewhat technical and lenthgy, we defer to SGA1 [GR04] for the original proof, and [JY20] or [Vis07] for a somewhat more recent account.

We've now seen a few instances of a phenonemon where some notion of fibration correspond to a notion of functor or map to some "global" category $(\mathbf{Set}, \mathbf{Cat}, \mathbf{Spc})$. One of the point of higher categories is to have a single setting being able to generalise both categories & spaces, as such, one might expect there to be a higher-categorical analog of the theorems we've exposed so far. As it turns out, there is, this is the main focus of the next section.

1.3 Brief mention of duality

Throughout this note, we mentioned cocartesian morphisms and fibrations, there is of course a dual notion of cartesian morphism and fibrations, and the equivalence of theorem 1.10 can be dualised to get the following result :

Theorem 1.11. *The dual of the Grothendieck construction gives an equivalence of $(2,1)$-categories*

$$\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cart}(\mathcal{C})$$

The next section may also be dualised, we do not give all of the dual statements and definitions, hoping the reader can figure out the missing details.

2 (Co)cartesian fibrations and the construction of functors

The main goal of this section is to arrive at a generalisation of the results discussed in the previous section. We first give an short account of the theory of left fibrations and eventually cocartesian fibrations.

2.1 Left fibrations of simplicial sets

For this section, we mostly refer to Higher Topos Theory [Lur09]. The point of left fibrations is to generalise the equivalence between functors $\mathcal{D} \to \mathbf{Grpd}$ and categories cofibered in groupoid over $\mathcal{D}$.

Definition 2.1. A map of simplicial sets $f : X \to S$ is a **left fibration** if it has the right lifting property w.r.t. all horn inclusions $\Lambda_i^n \subset \Delta^n$, $0 \leq i < n$.

We begin by first observing that fibers of a left fibrations are Kan complexes. To be explicit :

Proposition 2.2. *Let $p : X \to S$ be a left fibration, s a vertex of S, then $X_s := X \times_S \{s\}$ is a Kan complex. Furthermore, given two vertices s, s' of S, joined by an edge $f : s \to s'$, we have an induced map, unique up to homotopy $f_! : X_s \to X_{s'}$.*

Proof. The first part follows by stability of left fibrations under pullback and the remark that if S is a point, p is a left fibrations if and only if X is a Kan complex which itself amounts to unravelling the definition. The second part of the statement requires introducing anodyne maps, which are maps having a certain lifting property and proving that a certain map is left anodyne. ■

Example 2.3. Our main source of exemple is the following one. If S is a ∞-category, and we have a diagram $p : K \to S$, then remember the construction of the under ∞-category, i.e. we define $S_{p/}$ as the unique (up to isomorphism) simplicial set satisfying $\mathrm{Hom}_{\mathbf{sSet}}(Y, S_{p/}) = \mathrm{Hom}_p(K \star Y, S)$ or more explicitely $(S_{p/})_n := \mathrm{Hom}_p(K \star \Delta^n, S)$ (the subscript $_p$ denotes that we only take maps $f : K \star Y \to S$ such that $f|_K = p$. The undercategory comes equipped with a projection map $\pi : S_{p/} \to S$. This map is a left fibration. The proof is not particularly technical, merely combinatorial and can be found as a consequence of proposition 2.1.2.3 in [Lur09].

Fix a simplicial set S, we can form the category $LFib(S)$ as the full subcategory of $\mathbf{sSet}_{/S}$ of left fibrations. $LFib(S)$ is actually enriched in Kan complexes and as such can readily be seen as an $(\infty, 1)$-category itself. Being a simplicially enriched category, we can consider its simplicial nerve $\mathbf{LFib}(S) := N^{hc}_\bullet(LFib(S)) \in \mathbf{sSet}$, it is a quasicategory.

Similarly, we define $\mathbf{Grpd}_\infty$ to be $N^{hc}_\bullet(\mathbf{Grpd}^\Delta_\infty)$, where $\mathbf{Grpd}^\Delta_\infty$ is the simplicial category whose object are Kan complexes, and for K, K' two Kan complexes, $\mathrm{Hom}_{\mathbf{Grpd}^\Delta_\infty}(K, K')$ is $\mathbf{Fun}(K, K')$ (this is a Kan complex!).

The theory of left fibrations eventually leads to the following result :

Theorem 2.4. *Let S be an ∞-category, we then have an equivalence of ∞-categories :*

$$\mathbf{Fun}(S, \mathbf{Grpd}_\infty) \simeq \mathbf{LFib}(S)$$

The proof of this result is already quite involved, but we give an outline of how one would prove this results. Furthermore, the proof naturally generalises to the setting of cocartesian fibrations.

Remark 2.5. Naturally, we have the dual theorem classifying presheaves in ∞-groupoids as right fibrations, i.e.

$$\mathbf{Fun}(S^{\mathrm{op}}, \mathbf{Grpd}_\infty) \simeq \mathbf{RFib}(S)$$

The general outline of the proof goes as follows :

1. We define a model structure on $\mathbf{sSet}_{/S}$, we will call it the "covariant model structure" (resp. contravariant) which models the ∞-category of left fibrations (resp. right fibrations) i.e. its localisation is equivalent to $\mathbf{LFib}(S)$.
2. Take $C = \mathfrak{C}[S]$, the simplicial category generated by S. We define a model structure on $\mathbf{sSet}$, from which we obtain one on the category of simplicial functors $\mathbf{sSet}^C$ (resp. $\mathbf{sSet}^{C^{\mathrm{op}}}$), which models $\mathbf{Fun}(S, \mathbf{Grpd}_\infty)$ (resp. $\mathbf{Fun}(S^{\mathrm{op}}, \mathbf{Grpd}_\infty)$).
3. We define a Quillen adjuction depending on ϕ, a certain morphism of simplicial categories :
$$St_\phi : \mathbf{sSet}_{/S} \leftrightarrows [C, \mathbf{sSet}] : Un_\phi$$
called the straightening and unstraightening functors.
4. One proves that when ϕ is a weak equivalence for a specific model structure on simplicial categories, then the Quillen adjunction above is a Quillen equivalence, which finally proves the result.

We now give some details for each step, but we will not give a full prove, as this would be beyond the scope of an expository talk.

2.1.1 The model structures

We first recall the Joyal model structure on $\mathbf{sSet}$.

Theorem 2.6. *There is a model structure on $\mathbf{sSet}$ called the Joyal model structure, in which a morphism $f : X \to Y$ is :*

Cof: *a cofibration if it is a monomorphism.*

W: *a weak equivalence if the induced map $\mathfrak{C}[X] \to \mathfrak{C}[Y]$ is a (simplicial) equivalence.*

F: *a fibration if it has the right lifting property with respect to Cof $\cap$ W.*

In this model structure, fibrant objects are precisely the quasi-categories.

We recall the notation introduced in HTT [Lur09] : For a simplicial set X, we denote $X^{\triangleleft}$ its cone, i.e. $\Delta^0 \star X$. Given $f : X \to S$ in $\mathbf{sSet}$ we can form the pushout

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & X^{\triangleleft} \arrow[d] \\ S \arrow[r] & S \coprod_X X^{\triangleleft} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We can now describe the covariant model structure.

Theorem 2.7. *Fix a simplicial set S. There is a model structure on $\mathbf{sSet}_{/S}$ in which a morphism $f : X \to Y$ is a:*

Cof: (covariant) cofibration if the underlying morphism of simplicial sets is a monomorphism.

W: (covariant) weak equivalence if the induced map $X^{\triangleleft} \coprod_X S \to Y^{\triangleleft} \coprod_Y S$ is a weak equivalence in Joyal's model structure on simplicial sets.

F: (covariant) fibration if it has the right lifting property with respect to maps in Cof $\cap$ W.

This model structure has a few nice property, for one it is left proper, and it's a simplicial model structure. One can also realise it as a left Bousfield localisation of the model structure induced on $\mathbf{sSet}_{/S}$ by Joyal's model structure on $\mathbf{sSet}$.

Proposition 2.8. *Fibrant objects of $\mathbf{sSet}_{/S}$ endowed with the covariant model structure are precisely the left fibrations.*

Remark 2.9. We can get the "contravariant model structure" from the covariant one, the contravariant fibrations are the same, and for fibrations and equivalences, a map $f \in \mathbf{sSet}_{/S}$ is a contravariant fibration (resp. weak equivalence) if f^{op} is a covariant fibration (resp. weak equivalence) in $\mathbf{sSet}_{/S^{\mathrm{op}}}$.

The model structure we endow $\mathbf{sSet}^C$ is the projective model structure (w.r.t to the Joyal model structure). Hence we have the following theorem.

Theorem 2.10. *There is a model structure on $\mathbf{sSet}^C$ in which a map $\eta : F \to G$ is :*

W: a weak equivalence if $\eta_c : F(c) \to G(c)$ is a weak equivalence for each $c \in C$.

F: a (projective) fibration if $\eta_c : F(c) \to G(c)$ is a fibration for each $c \in C$.

Cof: a cofibration if it has the left lifting property with respect to F $\cap$ W.

Now we've described our model categories (but not proved their existence!), one would want to show that their respective localisation are what we want them to be. Again this is certainly out of the scope of this note.

2.1.2 Lurie's (un)straightening equivalence (unmarked case)

We will now describe Lurie's construction of the unstraightening functor, which for technical reason is done in the case of right fibrations. For the remainder of this section, fix a simplicial set S, a simplicial category C and a simplicial functor $\phi : \mathfrak{C}[S] \to C^{\mathrm{op}}$.

Recall that for a simplicial set X, the cone $X^{\triangleright}$ has a distinguished vertex v, which we call cone point, i.e. the image of the inclusion $\Delta^0 \hookrightarrow X \star \Delta^0$. Given $X \in \mathbf{sSet}_{/S}$, i.e. $p : X \to S$, we may form the following pushout in the category of simplicial categories :

$$\begin{tikzcd} \mathfrak{C}[X] \arrow[r] \arrow[d] & C^{\mathrm{op}} \arrow[d] \\ \mathfrak{C}[X^{\triangleright}] \arrow[r] & \mathcal{M}(p, \phi) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $\mathfrak{C}[X] \to C^{\mathrm{op}}$ is obtained as the composite $\phi \circ \mathfrak{C}[p]$. We obtain the desired functor by defining

$$\begin{aligned} St_\phi \colon \mathbf{sSet}_{/S} &\longrightarrow [C^{\mathrm{op}}, \mathbf{sSet}] \\ (p : X \to S) &\longmapsto St_\phi X := \mathrm{Map}_{\mathcal{M}(p,\phi)}(-, v) \end{aligned}$$

We've now got one part of our adjunction, and the second part it turns out exists by the adjoint functor theorem (because St_ϕ commutes with colimits). Given $F : C^{\mathrm{op}} \to \mathbf{sSet}$, $Un_\phi F \to S$ will be a right fibration We now finally arrive at the main theorem.

Theorem 2.11 (Lurie's (un)straightening equivalence). *Let S be a simplicial set, C a simplicial category, $\phi : \mathfrak{C}[S] \to C^{\mathrm{op}}$ a simplicial functor. Then*

$$St_\phi : \mathbf{sSet}_{/S} \leftrightarrows [C^{\mathrm{op}}, \mathbf{sSet}] : Un_\phi$$

is a Quillen adjunction, when $\mathbf{sSet}_{/S}$ is endowed with the contravariant model structure and $[C^{\mathrm{op}}, \mathbf{sSet}]$ the projective model strucutre. Furthermore, if ϕ is a simplicial equivalence, (St_ϕ, Un_ϕ) is a Quillen equivalence.

One case of interest is when $C = \mathfrak{C}[S]^{\mathrm{op}}$ and ϕ is the identity, because this is the specific case giving us the equivalence between the localisations $\mathbf{RFib}(S) \simeq \mathbf{Fun}(S^{\mathrm{op}}, \mathbf{Grpd}_\infty)$. To get to our final generalisation, we want to replace $\mathbf{Grpd}_\infty$ by $\mathbf{Cat}_\infty$, so we need a new notion of fibration, and we'll have to find new model categories to present the relevant ∞-categories.

2.2 (co)Cartesian fibrations

2.2.1 Preliminary definitions

We first introduce the relevant notions of fibrations and some of their properties.

Definition 2.12. Let $f : X \to Y$ be a map of simplicial sets, it is an **inner fibration** if it has the right lifting property with respect to every inner horn inclusions $\Lambda^n_i \subset \Delta^n$, i.e. if for any $n \in \mathbb{N}$ and $0 < i < n$, any diagram as below admits a diagonal lift making the two triangles commute :

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Definition 2.13. Let $p : X \to S$ be an inner fibration and $f : \Delta^1 \to X$ an edge. We say f is p-**cocartesian** if for any $n \geq 2$, and any diagram as follows, there exists a dotted lift.

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "f"] & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Replacing Λ^n_0 with Λ^n_n and $\Delta^{\{0,1\}}$ with $\Delta^{\{n-1,n\}}$ we get the notion of p-**cartesian edge**.

We can formulate a few properties due to Lurie, stated in the language of cartesian edges, but which have dual forms. Essentially, cartesian edges are well behaved with regards to pullbacks and composition in the following sense :

Proposition 2.14.

1. *If $p : X \to S$ is an isomorphism of simplicial sets, then any $\Delta^1 \to X$ is cartesian.*
2. *If $\mathcal{C}$ is an ∞-category, $p : \mathcal{C} \to \Delta^0$, then an edge f of $\mathcal{C}$ is p-cartesian if and only if it is an equivalence.*
3. *(stronger version of 2) Given $p : \mathcal{C} \to \mathcal{D}$ between ∞-categories, and f an edge of $\mathcal{C}$. Then the following are equivalent :*
   (a) *f is an equivalence in $\mathcal{C}$.*
   (b) *f is p-cartesian and $p(f)$ is an equivalence of $\mathcal{D}$.*
4. *Given a pullback diagram :*

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ S' \arrow[r] & S \end{tikzcd}$$

with p an inner fibration, if f is an edge of X' with $q(f)$ being p-cartesian, then f is p'-cartesian.

5. *Let $p : X \to Y$, $q : Y \to Z$ be two inner fibrations, and f an edge of X such that $p(f)$ is q-cartesian. Then f is p-cartesian if and only it is qp-cartesian.*
6. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration and $\sigma : \Delta^2 \to \mathcal{C}$ a two simplex of $\mathcal{C}$ given by :*

$$\begin{tikzcd} & x' \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \sigma & x'' \end{tikzcd}$$

Suppose g is p-cartesian, then f is also if and only if h is p-cartesian.

With the definition of (co)cartesian edges, we can introduce our new kind of fibration, namely (co)cartesian fibrations. These will be the suitable replacement to left (resp. right) fibrations. Recall earlier that when we looked at left fibrations, the fibers were Kan complexes, which were related covariantly (i.e. an edge $s \to s'$ gave a map $X_s \to X'_s$). The situation here will be a slight generalisation : the fibers will no longer be Kan complexes, merely ∞-categories, but they'll still be related covariantly (or contravariantly depending on our choice of fibration, cocartesian or cartesian).

Definition 2.15. A map $p : X \to S$ is called a p-**cartesian fibration** (resp. p-**cocartesian fibration**) if :

- p is an inner fibration
- For every edge $f : x \to y$ of S and every vertex $\tilde{y}$ of X with $p(\tilde{y}) = y$ (resp. every vertex $\tilde{x}$ of X, with $p(\tilde{x}) = x$) there exists a p-cartesian edge (resp. p-cocartesian) $\tilde{f} : \tilde{x} \to \tilde{y}$ with $p(\tilde{f}) = f$.

Remark 2.16 (Heuristics)**.** In general, if an inner fibration $p : X \to S$ associates to each vertex $s \in S$ an ∞-category X_s, and to every edge $f : s \to s'$ a correspondence between X_s and $X_{s'}$, p will be (co)cartesian if the correspondence comes from a functor $X_{s'} \to X_s$ (resp. $X_s \to X_{s'}$). Roughly, (co)cartesian fibrations over S ought to be equivalent to functors $S^{\mathrm{op}} \to \mathbf{Cat}_\infty$ (resp. $S \to \mathbf{Cat}_\infty$).

Remark 2.17. The notion of (co)cartesian fibration is a genuine generalisation of (co)cartesian fibrations of ordinary categories, this can be justified by the following observation : Let F a functor between ordinary categories $C \to C'$. The induced map $NF : NC \to NC'$ is a (co)cartesian fibration in the sense of definition 2.15 if and only if F is a (co)cartesian fibration in the sense of definition 1.8.

From the definition formally follows a series of formal stability properties.

Proposition 2.18. *1. Isomorphism are cartesian fibration.*

2. *Cartesian fibrations are stable under base change.*
3. *Cartesian fibrations are stable under composition.*

There a connection between (co)cartesian fibrations and left/right fibrations.

Proposition 2.19. *Let $p : X \to S$ be a (co)Cartesian fibration. The following are equivalent :*

1. *Every edge of X is p-(co)Cartesian.*
2. *For every $s \in S$, $p^{-1}(s)$ is a ∞-groupoid.*

If either of those condition is satisfied, p is a left/right fibration.

We've almost have every ingredient to state the relevant and final generalisation we've been seeking. We need a proper definition of $\mathbf{Cat}_\infty$. There's a few equivalent definitions, we choose the one found in HTT [Lur09].

Definition 2.20. We denote $\mathbf{Cat}_\infty$ the ∞-category defined by $N_\bullet^{hc}(\mathbf{Cat}_\infty^\Delta)$, where $\mathbf{Cat}_\infty^\Delta$ is the simplicial category whose objects are (small) ∞-categories and mapping simplicial sets between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ given by the largest Kan complex contained in the ∞-category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

As for left and right fibrations, (co)cartesian fibrations over a simplicial set S may be assembled into a ∞-category as well. Indeed, we can take $\mathbf{Cart}(S)$ to be the subcategory of $\mathbf{Cat}_{\infty/S}$ such that the objects are cartesian fibrations, and the morphisms are those preserving cartesian edges. .We can readily state the theorem.

Theorem 2.21. *For $S \in \mathbf{sSet}$ there is an equivalence of ∞-categories :*

$$Un_S^{\infty} : \mathbf{Fun}(S^{\mathrm{op}}, \mathbf{Cat}_\infty) \simeq \mathbf{Cart}(S)$$

which is compatible with base change and reduces to the identity when $S = *$.

Dually, the same holds true, i.e. we have an equivalence of ∞-categories :

$$\mathbf{Fun}(S, \mathbf{Cat}_\infty) \simeq \mathbf{coCart}(S)$$

Remark 2.22. This is a genuine generalisation of the Grothendieck construction : indeed, if $S = N(\mathcal{C})$ for an ordinary category, and $F : S \to \mathbf{Cat}_\infty$ factors through $\mathbf{Cat} \subset \mathbf{Cat}_\infty$, then $Un_S^{\infty}(F)$ is equivalent to $\int_{\mathcal{C}} F$, hence we get again theorem 1.10.

As we did for theorem 2.4, we give an outline of the proof, and we introduce the key ingredients in the next few sections, but the full proof is certainly out of the scope of this expository talk. The outline of the proof is very similar to what was done previously, indeed it goes as follows :

Fix a simplicial set S.

1. We define a new category, the marked simplicial sets denoted $\mathbf{sSet}^+$.
2. We define a model structure on $\mathbf{sSet}^+$, the marked Joyal model structure, whose localisation is $\mathbf{Cat}_\infty$, which induces via the projective model structure a model structure on the category of simplicial functors $[\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}^+]$[1] whose localisation is $\mathbf{Fun}(S^{\mathrm{op}}, \mathbf{Cat}_\infty)$.
3. From S we obtain $S^\sharp \in \mathbf{sSet}^+$, and we define a model structure on $\mathbf{sSet}^+_{/S^\sharp}$ whose localisation is $\mathbf{Cart}(S)$.
4. We define a Quillen equivalence

$$St^+ : \mathbf{sSet}^+_{/S^\sharp} \leftrightarrows [\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}^+] : Un^+$$

which finishes the proof.

2.2.2 Marked simplicial sets

We begin with a short introduction to marked simplicial sets, these are simplicial sets with "marked" edges, keeping track of marked edges allows us to keep track of cartesian edges.

Definition 2.23. A **marked simplicial set** is a pair (X, Σ), where X is a simplicial set, and Σ is a subset of X_1 containing the degenerate edges, elements of Σ are called **marked edges**. Marked simplicial sets assembles into a category $\mathbf{sSet}^+$, where a morphism $(X, \Sigma_X) \to (Y, \Sigma_Y)$ is a map of simplicial sets $X \to Y$ carrying marked edges to marked edges.

This category has some nice properties, for instance it is both complete and cocomplete, which can be seen as a consequence of it being a reflective subcategory of a category of presheaves. Indeed one can augment the category Δ by adjoining a element e and two unique maps $[1] \to e$ and $e \to [0]$ factoring the unique map $[1] \to [0]$, thus getting a category Δ^+. Then the inclusion $i : \mathbf{sSet}^+ \to [(\Delta^+)^{\mathrm{op}}, \mathbf{Set}]$ admits a right adjoint, which proves the claim.

There are various ways to obtain markings on a simplicial sets and vice versa, i.e. various functors $\mathbf{sSet} \leftrightarrows \mathbf{sSet}^+$, with some of them being adjoint to one another.

Definition 2.24. If $X \in \mathbf{sSet}$, we denote $X^\sharp$ the marked simplicial set with $\Sigma_{X^\sharp} = X_1$, the construction is clearly functorial. Similarly we denote $X^\flat$ the simplicial set whose marked edges are precisely the degenerate edges. Both functors admit a respective right adjoint. Indeed, for a marked simplicial set X, we can forget the markings and obtain a simplicial set denoted $X_\sharp$, or we can consider the subsimplicial set of X whose simplices have their edges marked, which we denote $X_\flat$. We have the following adjunctions :

$$\begin{tikzcd}
\mathbf{sSet} \arrow[r, bend left=50, "(-)^\flat"] \arrow[r, bend right=50, "(-)^\sharp"'] & \mathbf{sSet}^+ \arrow[l, bend right=15, "(-)_\flat"'] \arrow[l, bend left=15, "(-)_\sharp"]
\end{tikzcd}$$

[1]We sometimes denote $\mathbf{Fun}^{\mathrm{s}}(C, D)$ the category of simplicial functors when we want to make obvious we only consider simplicial functors. We do not always do this. We may also use $^{\mathrm{s}}$ for other purpose, for instance denoting simplicial mapping spaces, we hope the context makes the purpose evident. Again, we do not always use this notation, as it sometimes gets clutered.

Given $S \in \mathbf{sSet}$ we can endow $\mathbf{sSet}_{/S^\sharp}$ with a simplicial structure defining the mapping simplicial sets to be

$$\mathrm{Map}^\sharp_S(X,Y) := (Y^X \times_{(S^\sharp)^X} \Delta^0)_\sharp$$

These allows us to track cartesian edges as follows : given a cartesian fibration $p : X \to S$, one can define $X^\natural$ the marked simplicial sets whose marked edges are precisely the cartesian edges. A lot more could be said about marked simplicial sets and the interplay with cartesian edges and fibrations but we keep it short for the purpose of our talk.

2.2.3 The model structures

We need to define two model structures, we begin with what we call the cartesian model structure.

Theorem 2.25 (Cartesian model structure)**.** *Let S be a simplicial set, then there exists a model structure on $\mathbf{sSet}^+_{/S^\sharp}$ whose cofibrations are precisely the morphisms whose underlying map of simplicial set is a monomorphism, and whose fibrant objects are the marked simplicial sets $X^\natural$ corresponding to cartesian fibrations. The simplicial enrichment we gave on $\mathbf{sSet}^+_{/S}$ make it into a simplicial model category.*

It is well known that specifying the cofibrations and fibrant objects uniquely determines the model structure. Although we could have given a description of the weak equivalences as being what's called "cartesian equivalences". We still state it because we can.

Definition 2.26. We call a map $X \to Y$ in $\mathbf{sSet}^+_{/S}$ a **cartesian equivalence** if for every cartesian fibration $Z \to S$, the induced map $\mathrm{Map}^\sharp_S(Y, Z^\natural) \to \mathrm{Map}^\sharp_S(X, Z^\natural)$ is an equivalence of quasicategories.

We can dualise the definition of either fibrant objects or weak equivalence to get the cocartesian model structure, i.e. stating a map f in $\mathbf{sSet}^+_{/S}$ to be a cocartesian equivalence if f^{op} is a cartesian equivalence of $\mathbf{sSet}^+_{/S^{\mathrm{op}}}$.

The fibrant objects being what they are, can be seen either as a definition as stated, or as a property of the model structure if we specify it with its cofibrations and weak equivalences. Regardless, seeing that fibrant objects are precisely the cartesian fibrations over S, it justifies that this model category presents the $(\infty,1)$-category of cartesian fibrations over S.

To get the desired model structure on $\mathbf{sSet}^+$, one only need to look at the case where $S = *$, in which case the cartesian and cocartesian model structure coincide. Let us capture this with the following theorem.

Theorem 2.27. *When $S = *$, the cartesian and cocartesian model structure on $\mathbf{sSet}^+_{/S} = \mathbf{sSet}^+$ coincide, we call it the marked Joyal structure. A marked simplicial set X is fibrant if and only $X_\flat$ is a quasicategory and the marked edges are the equivalences in $X_\flat$. The adjoint pair*

$$(-)^\flat : \mathbf{sSet} \leftrightarrows \mathbf{sSet}^+ : (-)_\flat$$

is a Quillen equivalence when each category is endowed with the Joyal model structure and the marked Joyal model structure respectively.

2.2.4 Marked straightening

We now need to produce a marked version of the (un)straightening adjunction of the previous section. To do so, one incorporates markings to the unmarked construction. First recall the notation and construction of the unmarked case : given $p : X \to S$ we form the pushout of simplicial categories :

$$\begin{tikzcd} \mathfrak{C}[X] \arrow[r] \arrow[d] & \mathfrak{C}[S] \arrow[d] \\ \mathfrak{C}[X^\triangleright] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{M}(p) \end{tikzcd}$$

and define $St(X)$ to be $\mathrm{Hom}_{\mathcal{M}(p)}(-,*) : \mathfrak{C}[S]^{\mathrm{op}} \to \mathbf{sSet}$, hence we get a functor $St : \mathbf{sSet}_{/S} \to [\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}]$.

We follow [HHR21] to incorporate markings to the construction. Given a map $p : X \to S^\sharp$ of marked simplicial sets and a marked edge $f : \Delta^1 \to X$, call it $f : x \to x'$. We can consider the composition

$$\mathfrak{C}[\Delta^2] = \mathfrak{C}[\Delta^1 \star \Delta^0] \xrightarrow{\mathfrak{C}[f \star \mathrm{id}]} \mathfrak{C}[X^\triangleright] \to \mathcal{M}(p)$$

Elementary combinatorics and unravelling the definitions, one can check that $\Delta^1 \cong \mathrm{Hom}^{\mathrm{s}}_{\mathfrak{C}[\Delta^2]}(0,2)$. This edge is taken to an edge of $\mathrm{Hom}^{\mathrm{s}}_{\mathcal{M}(p)}(px,*) = (St(p))(px)$, which we declare marked, as well as all of its images under the mapping :

$$(St(p))(px) \xrightarrow{\sigma^*} (St(p))(s), \quad \forall \sigma \in \mathrm{Hom}^{\mathrm{s}}_{\mathfrak{C}[S]}(s,px)_1, \forall s \in S_1.$$

We obtain a functor

$$St^+ : \mathbf{sSet}^+_{/S^\sharp} \to \mathbf{Fun}^{\mathrm{s}}(\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}^+)$$

There would be a lot of verification to carry out, but given the context of these notes and the talk they're made for, we feel it's best to skim over the details. To get the right adjoint, either one uses the right adjoint we had before, and explicitely adds the marking as we did above, or one can show St^+ preserves colimits, and using the adjoint functor theorem get the desired right adjoint, which we denote $Un^+ : \mathbf{Fun}^{\mathrm{s}}(\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}^+) \to St^+ : \mathbf{sSet}^+_{/S^\sharp}$.

We have now an upgraded form Lurie's straightening and unstraightening equivalence.

Theorem 2.28. *Given S any ∞-category, the marked straightening/unstraightening adjunction*

$$St^+ : \mathbf{sSet}^+_{/S^\sharp} \to \mathbf{Fun}^{\mathrm{s}}(\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{sSet}^+) : Un^+$$

is a Quillen equivalence, when the category on the left is endowed with the cartesian model structure, and the one on the right the projective model structure based on the marked Joyal model structure.

The proof is very involved, the most arduous part certainly is proving it is a Quillen equivalence, it being a Quillen adjunction is actually fairly easy, once we know some things about each model structure (for instance, the generating sets of cofibrations). The proof may be found in [Lur09], for a different, more recent proof, one can read [HHR21], whose purpose was to provide a proof of a different flavour than Lurie's.

2.2.5 Handling functors via fibrations

After having seen all of one natural question that comes up is why does one care about this result, apart from it being a generalisation of some previous theorem. The main point is the construction of functors : defining a functor $S \to \mathbf{Grpd}_\infty$ or $S \to \mathbf{Cat}_\infty$ (or dually presheaves in $\mathbf{Cat}_\infty$ and $\mathbf{Grpd}_\infty$) is a dauting task as it requires providing an incredible amount of coherence data. Using the equivalence we've described, it suffices to provide a single morphism of quasicategories satisfying some property. It moves the difficulty from defining an object to proving an object has some property (being a left/right/(co)cartesian fibration). In practice, writing down such fibrations is easier than explicitely defining the corresponding functor. Beware, it might still be hard to prove that the morphism given is a fibration but nonetheless, it is often more palatable.

Let's see some examples. The first example that should come to mind are (co)representable functors, i.e. given $\mathcal{C}$ and $x \in \mathcal{C}$ one has the functor $y \mapsto \mathrm{Map}_{\mathcal{C}}(x,y) \in \mathbf{Grpd}_\infty \subset \mathbf{Cat}_\infty$. Given this functor takes values in ∞-groupoids, we can use the unmarked straightening to obtain a left fibration, or equivalently, use the marked version to get a cocartesian fibration which will turn out to be a left fibration. The left fibration associated to this functor is the map $\mathcal{C}_{x/} \to \mathcal{C}$ where $\mathcal{C}_{x/}$ is the coslice defined as in the example 1.8 for $\Delta^0 \xrightarrow{x} \mathcal{C}$. The n-simplices of $\mathcal{C}_{x/}$ are the $n+1$-simplices σ of $\mathcal{C}$ such that $\sigma(\Delta^{\{0\}}) = x$. Dually, the slice projection $\mathcal{C}_{/x} \to \mathcal{C}$ is the right fibration classifying the functor $y \mapsto \mathrm{Map}_{\mathcal{C}}(y,x)$.

These two functors can be packaged into a single functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Grpd}_\infty$ mapping (x,y) to $\mathrm{Map}_{\mathcal{C}}(x,y)$. We can construct the corresponding right fibration as follows.

We can define a new ∞-category, called the twisted arrow category,

$$\mathbf{Tw}(\mathcal{C})_\bullet := \mathrm{Hom}_{\mathbf{sSet}}(\Delta^\bullet \star (\Delta^\bullet)^{\mathrm{op}}, \mathcal{C}),$$

unravelling the construction of $\Delta^n \star (\Delta^n)^{\mathrm{op}}$, we can give the following explicit of simplices of $\mathbf{Tw}(\mathcal{C})$. The vertices are edges of $\mathcal{C}$. The n-simplices are $(2n+1)$-simplices of $\mathcal{C}$ depicting diagrams as below :

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & x_n \arrow[d] \\ y_0 & y_1 \arrow[l] & \cdots \arrow[l] & y_n \arrow[l] \end{tikzcd}$$

The inclusions $\Delta^n \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}} \hookleftarrow (\Delta^n)^{\mathrm{op}}$ induce a map $\mathbf{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. This map is a right fibration. We could have gotten a left fibration following the same construction switching the order in the join, i.e. defining the left twisted arrow category as

$$\mathbf{Tw}^l(\mathcal{C})_\bullet := \mathrm{Hom}_{\mathbf{sSet}}((\Delta^\bullet)^{\mathrm{op}} \star \Delta^\bullet, \mathcal{C})$$

The projection $\mathbf{Tw}^l(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is a left fibration.

But there's more. Looking at the specific case where $S = \Delta^1$, we get that cartesian fibrations $p : K \to \Delta^1$ correspond to functors $(\Delta^1)^{\mathrm{op}} \to \mathbf{Cat}_\infty$, and this is simply a functor of quasi-categories $f : \mathcal{D} \to \mathcal{C}$. In this way we can construct functors of ∞-categories by just building (co)cartesian fibrations over the interval, this implies providing a lot less data, but checking this data satisfies some assumptions. To be more particular, given $p : K \to \Delta^1$ we get two quasicategories, namely the fibers over $0, 1$, so set $\mathcal{C} := K_0$, $\mathcal{D} := K_1$, an associated functor to p is a ∞-functor $f : \mathcal{D} \to \mathcal{C}$ such that there exists a diagram

$$\begin{tikzcd}
\mathcal{D} \times \Delta^1 \arrow[rr, "F"] \arrow[dr] & & K \arrow[dl] \\
& \Delta^1 &
\end{tikzcd}$$

with $F|_1 = \mathrm{id}_D$ and $F|_0 = f$ and for all $d \in D$ $F(\{d\} \times \{0 \to 1\})$ is p-cartesian in K. There is a procedure to obtain such a functor from the data of p. We may form the following commutative square in $\mathbf{sSet}^+$:

$$\begin{tikzcd}
\mathcal{D}^\flat \times \{1\} \arrow[r, hook] \arrow[d] & K^\sharp \arrow[d] \\
\mathcal{D}^\flat \times (\Delta^1)^\sharp \arrow[r] \arrow[ur, dashed, "s"] & (\Delta^1)^\sharp
\end{tikzcd}$$

One shows that the left vertical map is marked anodyne, which gives a lift s as pictured. We can then take $f := s_0$ to be the associated functor.

References

[GR04] Alexander Grothendieck and Michele Raynaud. Revêtements étales et groupe fondamental (sga 1), 2004.

[Har] Yonatan Harpaz. (co)cartesian fibrations.

[HHR21] Fabian Hebestreit, Gijs Heuts, and Jaco Ruit. A short proof of the straightening theorem. 2021.

[HP15] Yonatan Harpaz and Matan Prasma. The grothendieck construction for model categories. 2015.

[JY20] Niles Johnson and Donald Yau. 2-dimensional categories. 2020.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009.

[Rui20] Jaco Ruit. Grothendieck constructions for higher categories. Master's thesis, Utrecht University, August 2020.

[Vis07] Angelo Vistoli. Notes on grothendieck topologies, fibered categories and descent theory, 2007.