Grothendieck's homotopy theory

From proper functors towards a theory of proper 2-functors

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2024

Contents

Acknowledgements

I would like to express my gratitude to Dimitri Ara, who accepted to supervise this thesis, as well as the PhD that follows, this is an incredible learning experience for me and I'm eager to start (or rather keep going), I feel very lucky to have been given that chance. I'm also grateful for the people I have supporting me, friends and family alike. The long math-related talks (and non-math too) at the cafeteria in Jussieu will be missed.

Introduction

For a few decades, the work of algebraic topologists has aimed at replacing topological spaces with more combinatorial objects, such as simplicial sets. It was Quillen's idea to replace homotopy types by (small) categories. This naturally led him and others to investigate the homotopy theory of small categories. The effort to study the homotopy theory of small categories is partly due to none other than Grothendieck himself in Pursing Stacks [Gro]. He proposed to use the notion of small category not as a mere replacement of homotopy types but as the foundation of homotopy theory while generalising the previous work of algebraic topologists on the subject.

Since then, it has become clear that while categories can replace homotopy types in a precise sense, this is not passing from the latter to the former is not always extremely natural. This observation leads us to consider higher categories, for more natural incarnations of homotopy types. As such, there is a need to investigate the homotopy theory of higher categories. By higher categories we mean n-categories (for $n \in \mathbb{N} \cup \{\infty\}$), that is, categories with objects, (1-)morphisms between those and also 2-morphisms between the 1-morphisms and so on up to n.

The theory is well studied for $n = 2$, amongst other things : Quillen's theorems A and B have been proven, the various geometric realisation functors for 2-categories have been described and compared, the Grothendieck construction and its homotopical properties have been described as well. There is only one last piece of the theory that has not been generalised to 2-categories : proper functors and their duals, smooth functors. Introduced by Grothendieck, a functor $u : A \to B$ is called proper if for any cartesian square

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

the "homotopical" base change morphism $u'_!w^* \to v^*u_!$ is an isomorphism, and this remains true after any base change. For instance, given any functor $F : I \to \mathbf{Cat}$ the projection $\int_I F \to I$ is a classical example of a proper functor. Grothendieck was able to find several equivalent characterisations for proper functors, some of which are easy to check hence making them effective tools for computation. We have the following equivalences :

Theorem 0.0.1. *Let $u : A \to B$ be a morphism of $\mathbf{Cat}$, then the following are equivalent :*

(a) u is proper,

(b) for any object b of B, the canonical functor $i_b : A_b \to A/b$ is a aspheric (the subscript $_b$ denotes the fiber over b),

(c) for any object b of B, the fibers of the induced map $a\backslash A \to u(a)\backslash B$ are aspheric,

(d) for any map $\Delta^1 \to B$, forming the pullback

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ \Delta^1 \arrow[r, "v"'] & B \end{tikzcd}$$

the inclusion of the fiber of $A' \to \Delta^1$ over 1 into A' is coaspheric,

(e) the fibers of v, the morphism $\underline{\mathrm{Hom}}(\Delta^1, A) \xrightarrow{v} A \times_B \underline{\mathrm{Hom}}(\Delta^1, B)$ induced by the square

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\Delta^1, A) \arrow[r] \arrow[d, "ev_0"] & \underline{\mathrm{Hom}}(\Delta^1, B) \arrow[d, "ev_0"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

are aspheric,

(f) for any diagram of cartesian squares

$$\begin{tikzcd} A'' \arrow[r, "v"'] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ B'' \arrow[r, "w"'] & B' \arrow[r] & B \end{tikzcd}$$

if v is coaspheric, then so is w.

This master's thesis is meant to be a first step towards comparing all of those characterisation of proper functors for 2-categories. As such the first chapter is dedicated to non-trivial prerequisite and recollection of basic categorical constructions that will be important throughout this thesis. We begin by recalling the basics about Kan extensions, because their homotopical counterparts will play a key role in the theory of proper functors. In the second part we discuss localisations to state a useful lemma that'll be used multiple times. In the third part, we briefly mention segments and the homotopy relation in a category with finite products, in order to prove a lemma that'll be used much later on. The fourth and fifth part discuss the Grothendieck construction and the basics about (pre)cofibrations. The Grothendieck construction is a cornerstone of the homotopy theory of (small) categories and even higher categories, it will be used and studied continuously throughout this text.

The second chapter is a concise and self-sufficient exposition of the theory of basic localisers, which provides the appropriate context to talk about proper functors. In the first part we describe the necessary elements of the theory of basics localisers and asphericity. In the second part we introduce the homotopical counterpart to Kan extensions which heavily relies on the grothendieck construction. This is one of the main tool to study proper functors. In the third part we finally delve into the theory of proper functors, and prove the various equivalences stated above.

In the third chapter, in order to eventually discuss properness of 2-functors, we set up the required context : the elementary theory of strict 2-categories. In the first part we introduce the first definitions, what is a 2-category, a 2-functor and so on. In the second part, we discuss the various generalisation of slice/under/over categories. While for 1-categories there's only two kinds of slices either A/a or $a\backslash A$, the situation is trickier for 2-categories, because in this context, our 2-cells are not necessarily invertible, hence there are more options regarding their direction. We then discuss in the following part one way to generalise the notion of adjoint functor and that (pre)fibrations to our 2-categorical setting. Finally in the fourth part, we introduce the Grothendieck construction for 2-categories, which is again one of the main tool to study the homotopy theory of 2-categories.

Finally in the fourth and final, we lay the foundations to study proper functor for 2-categories. In the first part, we introduce the theory of basic localisers for 2-categories which behaves very similarly as in the 1-categorical case. In the second part, we construct homotopical Kan extensions in a very similar fashion as in the previous chapter.

At this point, the main tools have been set up to study those functors, but more work is required to get compare the various notions of properness one can define. For instance, it is true that if we take our definition of properness to be a generalised version of (b) in the theorem above, then for any strict 2-functor $F : I \to 2\text{-}\mathbf{Cat}$, the canonical projection $\int_I F \to I$ is proper. As such we hope to be able to compare that definition to the first one given but this is still an open problem. As it stands, it's clear that some of the proofs and techniques used to reach the desired theorem in the 1-categorical case break down in higher dimension. It is even clear why it breaks down. For instance, the proof for (b) $\iff$ (c) $\iff$ (d) essentially hinges on the following observation : given $u : A \to B$ a morphism of $\mathbf{Cat}$, $b \in B$ and $(a, p) \in A/b$ we have an equality $(a, p)\backslash A_b = (a\backslash A)_{(b,p)}$. This is no longer true for 2-categories and the proof breaks down. There is however a natural functor to compare the two 2-categories at play, but it does not appear yet that this can be seen to be a weak equivalence and it is certainly not something stronger (equivalence or isomorphism).

Notation & terminology

We record some notation that will be used throughout this paper :

- For $n \geq 0$ we denote $[n]$ the category associated to the ordered set $\{0 \leq 1 \leq \cdots \leq n\}$. We denote Δ^n simplicial set $\mathrm{Hom}_\Delta(-,[n]) : \Delta^{\mathrm{op}} \to \mathbf{Set}$. We might sometimes make the abuse of denoting Δ^n the ordered set $[n]$ when no confusion is possible.

- For a category M, we denote $\mathrm{Mor}(M)$ the set or class of morphism and $\mathrm{Ob}(M)$ its set or class of morphism. We use the notation $\mathrm{Arr}(\mathrm{M})$ for the category of arrows of M defined as the functor category $\mathrm{Fun}([1], M)$. We regularly commit the abuse of writing $a \in A$ to say that a is an object of A.

- We tend to use standard uppercase letters for categories (M, C, A, B ...) and bold uppercase calligraphic/standard letters for 2-categories ($\boldsymbol{A}$, $\mathbf{A}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{B}$, $\mathbf{B}$, $\boldsymbol{\mathcal{B}}$ ...). We try to stick to these convention as much as humanly possible. The letter W / $\mathcal{W}$ / $\boldsymbol{\mathcal{W}}$ is generally reserved for a class of morphims / functors.

- We denote e the category with a unique object and a unique morphism.

- We denote $\mathbf{Cat}$, $\mathbf{Set}$ the categories of (small) categories and of sets respectively.

- Given a functor $u : A \to B$, $a \in A$, $b \in B$ we denote :

  (a) $A/^u b$ (or simply A/b) to be the category whose objects are pairs (a, p) with $a \in A$ and $p : u(a) \to b$ is a morphism of B. A morphism $(a, p) \to (a', p')$ is given by an arrow $f : a \to a'$ of A, such that $p = p' \circ u(f)$.

  (b) A^u_b (or simply A_b) to be the category, called the fiber of u over b, whose objects are the objects $a \in A$ such that $u(a) = b$. The morphisms are the maps $f : a \to a'$ of A such that $u(f) = \mathrm{id}_b$.

  (c) A/a is defined to be $A/^{\mathrm{id}_A} a$.

  (d) $b \backslash^u A$ (or simply $b\backslash A$) is the category whose objects are pairs (a, p) with $a \in A$ and $p : b \to u(a)$ in B. A map $(a, p) \to (a', p')$ is given by the data of $f : a \to a'$ such that $u(f) \circ^= p'$.

  (e) We record the case where $u = \mathrm{id}_b$ in which case $b\backslash^{\mathrm{id}_B} B$ is denoted $b\backslash B$.

- Given two categories A, B we denote $\mathrm{Hom}(A, B)$ the set of functors $A \to B$, and $\underline{\mathrm{Hom}}(A, B)$ the functor category whose objects are functors and morphisms natural transformations.

1 Category theory recollection

We use this section to state a bunch of definitions and results which will either be used as examples, motivations, or key steps of proofs. This part may be skipped at first, one may only read the part they do not know as it is cited later on.

1.1 Kan extensions

We first recall the basic to know about Kan extensions.

1.1.1 — Let $w : J \to I$ be a functor, and given any other category A, write w^* the induced functor

$$\begin{aligned} \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})\colon\ \underline{\mathrm{Hom}}(I, A) &\longrightarrow \underline{\mathrm{Hom}}(J, A) \\ F &\longmapsto F \circ w \\ \eta : F \Rightarrow G &\longmapsto \bar{\eta} : Fw \Rightarrow Gw \end{aligned}$$

where $\bar{\eta}_j = \eta_{w(j)} : F(w(j)) \to G(w(j))$.

Definition 1.1.2 (Global Kan extension)**.** If the functor w^* has a left adjoint, denoted $w_! : \underline{\mathrm{Hom}}(J, A) \to \underline{\mathrm{Hom}}(I, A)$, we say $w_!$ is the operation of **left Kan extension** along w. Given $f : J \to A$, we call $w_!(f)$ the left Kan extension of f along w.

If it has a right adjoint, denoted $w_* : \underline{\mathrm{Hom}}(J, A) \to \underline{\mathrm{Hom}}(I, A)$, we say w_* is the operation of **right Kan extension** along w. Given $f : J \to A$, we call $w_*(f)$ the right Kan extension of f along w.

We sometimes denote $w_!$ by Lan_w and w_* by Ran_w. Sometimes, a global kan extension may not exist, i.e. w^* may not have a left/right adjoint, and yet there still might be an extension existing for some functors. This is the local Kan extension.

Definition 1.1.3 (Local Kan extension)**.** Given $f : J \to A$, the **local left Kan extension** of f along w is a functor $\mathrm{Lan}_w(f) : I \to A$ (if it exists), such that we have the following natural isomorphism :

$$\mathrm{Hom}_{[J,A]}(f, w^*(-)) \simeq \mathrm{Hom}_{[I,A]}(\mathrm{Lan}_w(f), -).$$

Equivalently, a local left Kan extension may be described as the data of a functor $\mathrm{Lan}_w(f) : I \to A$ and a natural transformation $\eta_f : f \to w^* \circ \mathrm{Lan}_w$

$$\begin{tikzcd} J \arrow[rr, "f"] \arrow[dr, "w"'] & \arrow[d, phantom, "\Downarrow \eta_f"] & A \\ & I \arrow[ur, "\mathrm{Lan}_w(f)"'] & \end{tikzcd}$$

such that for any other such pair $g : I \to A$ and $\eta_g : f \to w^* \circ g$, the transformation η_g factors through uniquely η_f. That is

$$\begin{tikzcd} J \arrow[rr, "f"] \arrow[dr, "w"'] & \arrow[d, phantom, "\Downarrow \eta_g"] & A \\ & I \arrow[ur, "g"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} J \arrow[rr, "f"] \arrow[dr, "w"'] & \arrow[d, phantom, "\eta_f \Downarrow"] & A \\ & I \arrow[ur, "\mathrm{Lan}_w(f)"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, bend right=20, "\exists!"] & \end{tikzcd}$$

We have a similar characterisation of local right Kan extensions, the direction of the natural transformations is simply inverted.

We often drop the adjective "global" and "local", the context should make obvious what we mean.

Proposition 1.1.4. *Keeping the notations of 1.1.1, if I is the terminal category i.e. $w : J \to e$, and A is any other category then*

$$w_!(F) \simeq \mathrm{colim}_J\, F$$
$$w_*(F) \simeq \lim_J F$$

As such, Kan extensions can be used to compute limits and colimits, a feature we will also capture with their homotopical counterparts.

1.2 Localisations

One thing that will be of interest to us is to consider certain classes of equivalences to be isomorphisms, and this is done formally by considering localisations of categories.

Definition 1.2.1. Let M be a category and $W \subset \mathrm{Mor}(M)$. The **localisation** of M at W, is the data of a category $W^{-1}M$ (sometimes denoted $M[W^{-1}]$) and a functor $\gamma : M \to W^{-1}M$ such that γ maps elements of W to isomorphisms and such that the pair $(W^{-1}M, \gamma)$ is universal with this property. That is, for any other functor $f : M \to M'$ such that F maps elements of W to isomorphisms, then there exists a unique $\tilde{F}$ making the diagram commute :

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & M' \\ W^{-1}M \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

Remark 1.2.2. Given a pair (M, W) as in the definition above, the localisation may not exist, but this is mostly for set-theoretic reasons. There is a construction of $W^{-1}M$, but in general the results might have a proper class of morphisms between any two objets. We will not particularly care about the set theoretic issues, we could resolve the main problem here by working with Grothendieck universes and keeping track of the size of our categories, we will not do so.

Definition 1.2.3. We call a class of morphism **strongly saturated** if $W = \gamma^{-1}(\mathrm{Iso}_{W^{-1}M})$

For instance, in a model category, the weak equivalences are strongly saturated. We record a lemma that will be useful later on regarding adjunctions and localisations.

Lemma 1.2.4. *Let $(i \dashv j)$ be an adjunction $i : M \to M'$, $j : M' \to M$, and denote $\eta : 1_M \to ji$ and $\varepsilon : ij \to 1_{M'}$ the unit and counit respectively. Furhtermore, let $W \subset \mathrm{Mor}(M)$ and $W \subset \mathrm{Mor}(M')$ be two weakly saturated sets of morphisms. Then the following are equivalent :*

1. $W = i^{-1}(W')$ *and for all* $a' \in M'$, $\varepsilon_{a'} \in W'$ *;*
2. $W' = j^{-1}(W)$ *and for all* $a \in M$, $\eta_a \in W$ *;*

Furthermore, those conditions imply the following assertion :

- $i(W) \subset W'$, $j(W') \subset W$ *and*

$$\bar{i} : W^{-1}M \to W'^{-1}M', \quad \bar{j} : W'^{-1}M \to W^{-1}M$$

are quasi-inverses equivalences of categories.

Proof. Assume (1), and take $a \in M$. Since i, j are adjoint, they satisfy the triangle identity, as such we have the following diagram :

$$\begin{tikzcd} i(a) \arrow[r, "i(\eta_a)"] \arrow[dr, "1_{i(a)}"'] & iji(a) \arrow[d, "\varepsilon_{i(a)}"] \\ & i(a) \end{tikzcd}$$

By hypothesis $\varepsilon_{i(a)} \in W'$ hence by 2-out-of-3 so is $i(\eta_a)$, but by assumption again $W = i^{-1}(W')$ as such $\eta_a \in W$ which proves half of (2). Given $f : a' \to b' \in M'$, by naturality one can form the square :

$$\begin{tikzcd} ij(a') \arrow[r, "\varepsilon_{a'}"] \arrow[d, "ij(f)"'] & a' \arrow[d, "f"] \\ ij(b') \arrow[r, "\varepsilon_{b'}"'] & b' \end{tikzcd}$$

By assumption, the two horizontal arrows are in W', hence by 2-out-of-3 $f \in W'$ if and only if $ij(f)$ is too. The claim then follows since $i^{-1}(W') = W$.

The implication (2) $\implies$ (1) is stricly similar to prove, or just follows by duality.

It would remain to show that the induced adjunction is indeed an equivalence, but this is obvious since the unit and counit become isomorphisms in the localisation by hypothesis. ■

1.3 Segments and homotopism

The purpose of this subsection is to introduce a lemma that will be very useful later on, despite it seeming quite innocuous.

1.3.1 — Let C be a category with a final object e_C, a **segment** is a triplet $\mathbb{I} = (I, d_0, d_1)$ where I is an object of C and $d_0, d_1 : e_C \to I$ are morphisms of C. Further assume C has finite products, given two morphisms $f, g : X \rightrightarrows Y$ of C and a segment $\mathbb{I}$, we say f is $\mathbb{I}$-**elementarily homotopic** to g is there exists a morphism $h : I \times X \to Y$ such that the diagram

$$\begin{tikzcd}
 & I \times X \arrow[dd, "h"] & \\
X \arrow[ur, "d_0 \times 1_X"] \arrow[dr, "f"'] & & X \arrow[ul, "d_1 \times 1_X"'] \arrow[dl, "g"] \\
 & Y &
\end{tikzcd}$$

commutes. We call h an $\mathbb{I}$-homotopy (from f to g). Given a collection $\mathcal{I}$ of segments of C, we define the equivalence relation of $\mathcal{I}$-**homotopy** to be the one generated by "there exists a segment $\mathbb{I}$ in $\mathcal{I}$ such that f is $\mathbb{I}$-elementarily homotopic to g", we call two morphisms $\mathcal{I}$-homotopic. It follows from the definition that the relation of $\mathcal{I}$-homotopy is compatible with product and composition.

Lemma 1.3.2. *Let C be a category with finite products, W a weakly saturated class of morphism of C, $\mathcal{I}$ a set of segments of C. Assume that for each segment $\mathbb{I} = (I, d_0, d_1)$ of $\mathcal{I}$ the canonical morphism $p_I : I \to e_C$ is universally in W. If $f, g : X \rightrightarrows Y$ are $\mathcal{I}$-homotopic morphisms, then :*

(a) Denote $\gamma : C \to W^{-1}W$ the localisation functor, then $\gamma(f) = \gamma(g)$;

(b) f is in W if and only if g is ;

Proof. We can assume there is a segment $\mathbb{I} = (I, d_0, d_1)$ in $\mathcal{I}$ and a homotopy $h : I \times X \to Y$ between f and g such that $f = h(d_0 \times 1_X)$ and $g = h(d_1 \times 1_X)$. Since p_I is universally in W, it follows $pr_2 : I \times X \to X$ is in W since we have the cartesian square :

$$\begin{tikzcd}
I \times X \arrow[r] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "p_I"] \\
X \arrow[r, "p_X"'] & e_C
\end{tikzcd}$$

Furhtermore, since

$$pr_2 \circ (d_0 \times 1_X) = 1_X = pr_2 \circ (d_1 \times 1_X)$$

we have

$$\gamma(pr_2)\gamma(d_0 \times 1_X) = 1_{\gamma(X)} = \gamma(pr_2)\gamma(d_1 \times 1_X)$$

whence

$$\gamma(d_0 \times 1_X) = \gamma(d_1 \times 1_X)$$

since $\gamma(pr_2)$ is an isomorphism. Finally (a) follows because

$$\gamma(f) = \gamma(h)\gamma(d_0 \times 1_X) = \gamma(h)\gamma(d_1 \times X) = \gamma(g).$$

Consider again the equality

$$pr_2 \circ (d_0 \times 1_X) = 1_X = pr_2 \circ (d_1 \times 1_X)$$

by 2-out-of-3, it follows both $d_0 \times 1_X$ and $d_1 \times 1_X$ are in W whence by weak saturation f is in W if and only if h is, if and only if g is.

■

1.4 Grothendieck construction

1.4.1 — Given a functor $F : I \to \mathbf{Cat}$, we can obtain a category denoted $\int_I F$ or simply $\int F$, called the Grothendieck construction of F, equipped with a map $\int F \to I$. As the name suggests this is due to Grothendieck, we recall the construction.

Let $\int_I F$ be the category whose objects are pairs (i, a) with i an object of i and a an object of $F(i)$. A morphism $(i, a) \to (i', a')$ is given by a pair (k, f) such that $k : i \to i'$ is a morphism in I and $f : F(k)(a) \to a'$ is a morphism in $F(i')$. Given two composable morphisms (k, f) and (k', f') we define their composition with the following formula :

$$(k', f') \circ (k, f) := (k'k, f' \circ F(k')(f))$$

This comes with a projection defined by

$$\begin{aligned} \theta_F \colon \int_I F &\longrightarrow I \\ (i, a) &\longmapsto i \\ (k, f) &\longmapsto k \end{aligned}$$

Remark 1.4.2. Notice that for any $i \in I$, the fiber of θ_F over i i.e. $(\int_I F)_i$ is canonically equivalent (in fact even isomorphic) to $F(i)$. Indeed its objects are those of $\int_I F$ mapping to i, so this is the set $\{(i, a),\ a \in F(i)\}$ which is clearly isomorphic to $\mathrm{Ob}(F(i))$. Given $(i, a), (i, a')$ in a the fiber, morphisms $(i, a) \to (i, a')$ are morphisms in $\int_I F$ mapping to identities so they're pair $(1_i, f)$ with $f : F(1_i)(a) = a \to a'$, projecting $(1_i, f)$ to f gives the desired isomorphism. This is clearly functorial.

We will later show this functor to be proper.

1.4.3 — (Functoriality of integration) We can promote the Grothendieck construction to a functor

$$\begin{aligned} \int_I \colon \underline{\mathrm{Hom}}(I, \mathbf{Cat}) &\longrightarrow \mathbf{Cat}/I \\ F &\longmapsto \left(\int_I F, \theta_F \right) \\ \eta : F \Rightarrow G &\longmapsto \int_I \eta \end{aligned}$$

by defining $\int_I \eta$ as follows

$$\begin{aligned} \int_I \eta \colon \int_I F &\longrightarrow \int_I G \\ (i, x) &\longmapsto (i, \eta_i(x)) \\ (k : i \to i', f : F(k)(x) \to x') &\longmapsto (k, \eta_{i'}(f)) \end{aligned}$$

Notice that since naturality of η gives that $\eta_{i'}(f) : \eta_{i'}(F(k)(x)) = G(k)\eta_i(x) \to \eta_{i'}(x)$ as required. The functoriality is clear and follows from naturality of η and functoriality of each η_i.

Example 1.4.4. Let A be a category, given $a \in A$ one can look at the composition $F : A^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,a)} \mathbf{Set} \hookrightarrow \mathbf{Cat}$, by integrating we get $\int F$ which can be identified with A/a.

1.5 Cocartesian morphisms and (pre)cofibration

Definition 1.5.1. Let $u : A \to B$ be a functor, and $w : a \to a' \in A$. We say w is u-**cocartesian** (or simply cocartesian) if for any morphism $f : a \to a''$ such that $u(f) = u(w)$, there exists a unique morphism $g : a' \to a''$ with $u(g) = 1_{u(a')}$ and $f = gw$.

Warning 1.5.2. Here we made a somewhat nonstandard choice of terminology. In the main reference for this thesis ([Mal05]), the term "cocartesian morphism" is defined as we just did, but in the literature, this is rather called *locally cocartesian*. Usually the term "cocartesian morphism" is used for a stronger condition, which implies locally cartesian, we will call a morphism satisfying this *strongly cocartesian* or *globally cocartesian*.

Definition 1.5.3. Let $u : A \to B$ be a functor, and $w : a \to a' \in A$. We say w is **strongly** (or globally) u-**cocartesian** if given any map $f : a \to a''$ in A, and any map $p : u(a') \to u(a'')$ making the triangle commute, there exists a unique lift of p, i.e. a map $l : a' \to a''$ such that $u(l) = p$ and $lw = f$. We can view the situation diagramatically as follows :

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "w"'] & a'' & \longmapsfrom & u(a) \arrow[r, "u(f)"] \arrow[d, "u(w)"'] & u(a'') \\ a' \arrow[ur, dashed, "\exists! l"'] & & & u(a') \arrow[ur, dashed, "\forall p"'] & \end{tikzcd}$$

Proposition 1.5.4. *Let $u : A \to B$ be a functor, and $w : a \to a'$. If w is strongly cocartesian, it is cocartesian.*

Proof. Assume $w : a \to a'$ is strongly cocartesian. Let f be any map $a \to a''$ s.t. $u(w) = u(f)$. We can take $p = 1_{u(a')}$, since $u(w) = u(f)$, they have the same codomain, i.e. $u(a'') = u(a')$ and its clear that $u(f) = 1_{u(a')}u(w)$. Hence since w is strongly cocartesian, there exists a unique map $l : a'' \to a'$ making the relevant triangle commute and such that $u(l) = 1_{u(a')}$, so w is cocartesian. ∎

Heuristic 1.5.5. Let's motivate the various naming convention. In the case we're given $u : A \to B$ and $w : a \to a'$ as in definition 1.5.1, we can form the following pullback :

$$\begin{tikzcd} {[1] \times_B A} \arrow[r] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ {[1]} \arrow[r, "u(w)"'] & B \end{tikzcd}$$

Now, the morphism w is u-cocartesian (in the sense of 1.5.1) if and only if it is strongly u'-cocartesian in $[1] \times_B A$. Considering that $[1] \times_B A$ is smaller than A in some sense, this justifies the usual convention of calling such morphism"locally" cocartesian. In general, the converse of 1.5.4 is not true, however, in a very specific case of interest, it is, namely for the map $\theta_F : \int_I F \to I$.

Definition 1.5.6. A functor $u : A \to B$ is called a **precofibration** if given any morphism $g : b \to b'$ of B, and any lift of b in A, that is, an object $a \in A$ with $u(a) = b$, there exists a cocartesian morphism $w : a \to a'$ such that $u(w) = g$, so given any morphism of B and any lift of the source, there exists a cocartesian lift. Furhter, u is said to be a **cofibration** if it is a precofibration and that the composition of u-cocartesian morphism is again cocartesian.

As mentioned previously, we'll eventually show that the functor $\int_I F \to I$ is proper, and this will turn out to be an easy corollary of the following two propositions.

Proposition 1.5.7. *Given $F : I \to$ **Cat**, the projection $\theta_F : \int_I F \to I$ is a cofibration.*

Proof. The projection being a precofibration follows from the following stronger statement : given any $f : i \to i'$ in I, and any lift of i, there exists a strongly cocartesian lift of f. Since strongly cocartesian implies cocartesian, this is sufficient. Given $f : i \to i'$ in I and (i, x) a lift of i, we ought to provide a cocartesian morphism $(g, \varphi) : (i, x) \to (j, z)$ in $\int_I F$ with $\pi((g, \varphi)) = f$. Taking $(j, z) := (i', F(f)(x))$, $g = f$ and $\varphi = \mathrm{id}_{F(f)(x)}$. We get a morphism $(f, \mathrm{id}_{F(f)(x)}) : (i, x) \to (i', F(f)(x))$, we're left to check it is strongly cocartesian. We show that a morphism of $(k, \varphi) : (i, x) \to (i', x')$ of $\int_I F$ with φ an isomorphism is strongly cocartesian, this will imply that the morphism we provided is strongly cocartesian. Given any map $(h, \psi) : (i, x) \to (i'', x'')$, and filler of the dotted arrow in the following diagram :

$$\begin{tikzcd} i \arrow[r, "h"] \arrow[d, "k"'] & i'' \\ i' \arrow[ur, dashed, "p"'] & \end{tikzcd}$$

We can lift p by defining $(l, \eta) : (i', x') \to (i'', x'')$ to be $p, \psi \circ F(p)(\phi)^{-1})$. We've proved θ_F to be a precofibration.

We're left to show that θ_F-cocartesian maps are stable under composition, to do so, we first show that for a given map (k, φ) being θ_F-cocartesian is equivalent to φ being an isomorphism. One implication is clear, if φ is an isomorphism, by what precedes (k, φ) is strongly cocartesian hence cocartesian.

Assume $(k, \varphi) : (i, x) \to (i', x')$ is cocartesian, define a morphism in $\int_I F$ by

$$(k, 1_{F(k)(x)}) : (i, x) \to (i', F(k)(x))$$

By cocartesianness of (k, φ) we get a filler of the dotted arrow here :

$$\begin{tikzcd} F(k)(x) \arrow[r, "1"] \arrow[d, "\varphi"'] & F(k)(x) \\ x' \arrow[ur, dashed, "\exists!\, \overline{\varphi}"'] & \end{tikzcd}$$

So φ has a left inverse, let's prove it is also a right inverse. From the span

$$(i', x') \xleftarrow{(k,\varphi)} (i, x) \xrightarrow{(k,\varphi)} (i', x')$$

we get the diagram

$$\begin{tikzcd} F(k)(x) \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & x' \\ x' \arrow[ur, dashed] & \end{tikzcd}$$

which admits a unique filler since (k, φ) is cocartesian, but we have to possible fillers $1_{x'}$ and $\varphi\overline{\varphi}$ as such they are equal, so that $\overline{\varphi}$ is also a right inverse.

As such we have an easy characterisation of θ_F cocartesian morphisms, and given two composable such morphisms $(k, f), (k', f')$ the composite is $(k'k, f'F(k')(f))$ but since f', f are isomorphisms and $F(k')$ is a functor, $f'F(k')(f)$ is also an isomorphism, hence we're done. ∎

Remark 1.5.8. In the proof above, we saw that a morphism (k, φ) of $\int_I F$ where φ is an isomorphism is always strongly cocartesian, this is in fact an equivalence, if (k, φ) is strongly cocartesian, then one can also show φ to be an isomorphism. So in this case we have the chain of equivalences (k, φ) is strongly cartesian $\iff$ φ is an isomorphism $\iff$ (k, φ) is cocartesian. This is of course not true in general.

Lemma 1.5.9. *A functor $u : A \to B$ is a precofibration if and only if for any $b \in B$ the canonical functor*

$$\begin{aligned} i_b \colon A_b &\longrightarrow A/b \\ a &\longmapsto (a, 1_b) \\ f &\longmapsto f \end{aligned}$$

has a left adjoint $j_b : A/b \to A_b$.

Proof. First assume $u : A \to B$ is a precofibration. Fix $b \in B$, we are going to define a functor $j_b : A/b \to A_b$ as follows :

1. On objects given $(a, p : u(a) \to b)$, since u is a precofibration, we can find a lift of p to some map $\overline{p} : a \to d$ such that $u(\overline{p}) = p$, then we can map $(a,p) \mapsto d$ which does belong to A_b.
2. On morphisms given $f : (a,p) \to (a',p')$, repeating the procedure above, we get two lifts of p and p' respectively given by maps $\overline{p} : a \to d = j_b(a,p)$ and $\overline{p}' : a' \to d' = j_b(a',p')$, we can form the following diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\overline{p}"'] & a' \arrow[r, "\overline{p}'"] & d' \\ d \arrow[urr, dashed, "j_b(f)"'] & & \end{tikzcd}$$

   which admits a unique filler since $\overline{p}$ is cocartesian and $u(\overline{p}'f) = p'u(f) = p$ since f is a morphism in A/b, furthermore this filler does lie in A_b since $u(j_b(f)) = 1_b$.

Before going any further we ought to check this is well defined, indeed the map on objects depends on a choice of cocartesian lift so there might be issues, fear not, there are in fact none. Given $(a,p) \in A/b$ assume we have to cocartesian lifts of p, say $p_1 : a \to d$ and $p_2 : a \to d'$. Then since p_1 and p_2 are cocartesian lifts of the same p we can find a uniques f filling the diagram below :

$$\begin{tikzcd} a \arrow[r, "p_2"] \arrow[d, "p_1"'] & d \\ d' \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

and such that $u(f) = 1_b$. But then, again by cocartesianness of p_1 we can fill uniquely the following diagram :

$$\begin{tikzcd} a \arrow[r, "p_1"] \arrow[d, "p_1"'] & d \\ d \arrow[ur, dashed] & \end{tikzcd}$$

but we have two candidates for a lift, i.e. both 1_d and f make the triangle commute, as such they must be equal, so that $p_1 = p_2$, whence $d = d'$ so the map does not depend on the choice of lift.

We're left to show functoriality of the maps so defined as well as its being a left adjoint to i_b. Functoriality is very easy, it follows from the uniqueness of the fillers in the definition of cocartesian morphism, so the following diagram commutes and the composite of the two lifts is equal to the lift of the composite :

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[d, "1"'] & & a' \arrow[r, "g"] \arrow[d, "\overline{p}'"] & a'' \arrow[r, "\overline{p}''"] & j_b(a'',p'') \\ a \arrow[r, "f"] \arrow[d, "\overline{p}"'] & a' \arrow[r, "\overline{p}'"] & j_b(a',p') \arrow[urr, dashed, "j_b(g)"'] & & \\ j_b(a,p) \arrow[urr, dashed, "j_b(f)"'] \arrow[uurrrr, dashed, bend right, "j_b(gf)"'] & & & & \end{tikzcd}$$

Proving j_b preserves identities is the same argument.

First simply writing the definition we get

$$\mathrm{Hom}_{A/b}((a,p), i_b(a')) = \{f : a \to a',\ u(f) = p\}\,.$$

Second, if given a map $g : j_b(a,p) \to a'$ in A_b, we can consider the lift $\overline{p} : a \to j_b(a,p)$ of p, composing with g we get a map $f = g\overline{p} : a \to a'$ such that $u(f) = u(g)u(\overline{p}) = 1_b p = p$. So we've define a map $\mathrm{Hom}_{A_b}(j_b(a,p), a') \to \mathrm{Hom}_{A/b}((a,p), i_b(a'))$. This map is a (natural) bijection due to the existence and uniqueness of fillers for cocartesian morphisms, this is the same style of arguments done repeatedly before, we do not wish to reiterate, hoping it is clear enough by now. We're finally done with proving one implication, it remains to see that if such a j_b exists, u is a precofibration.

Assume for all $b \in B$ the functor i_b has a left adjoit j_b. Given $g : b \to b' \in B$ and $a \in A_b$, we obtain an object $(a,g) \in A/b'$, considering the unit of the adjunction $\eta : 1_{A/b'} \Rightarrow i_{b'}j_{b'}$ we get a map $(a,g) \to i_{b'}j_{b'}(a,g) \in A/b'$ which is the data of a map $f : a \to j(a,g)$ such that $u(f) = g$, if we prove this map to be u-cocartesian we're done. Given any map $h : a \to a'$ such that $u(h) = u(f)(= g)$ we need to find a unique filler in the following diagram :

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & a' \\ j_{b'}(a,g) \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

The data of h is precisely what we need to get a map $(a,g) \to (a', 1'_b) = i_{b'}(a')$ which we still denote by h. Using the adjunction we get the unique desired map $h^* : j_{b'}(a,g) \to a'$ with $u(h^*) = 1_{b'}$ by definition. The commutativity of the triangle follows from the naturality of η. ∎

2 Theory of proper and smooth functors

In this section, we give an elementary introduction to the theory of proper and smooth functors, as was done extensively in [Mal05]. The main point is the series of theorem giving a number of equivalent conditions for a functor $u : A \to B$ to be proper (cf. 2.3.22, 2.3.13, 2.3.24, 2.3.4). Before we can get to that, we need to introduce a series of notions relevant to eventually be able to talk about proper functors.

2.1 Weak basic localisers and asphericity

We first begin by introducing weak basic localisers, as some class of functors, these will be the maps we consider to be equivalences.

Definition 2.1.1. Let M be a category and $W \subset \mathrm{Mor}(M)$. We say that W is **weakly saturated** if it satisfies the following conditions :

1. W contains id_X for all object X.
2. W satisfies 2-out-of-3.
3. Given two morphisms $X' \xrightarrow{i} X \xrightarrow{r} X'$ such that $ri = \mathrm{id}_{X'}$, if $ir \in W$ then $r \in W$ (and $i \in W$ by 1 and 2).

Definition 2.1.2. Given M a category, $W \subset \mathrm{Mor}(M)$ and $u : A \to B$ a morphism of M, we say u is **universally in** W if the following are satisfied :

1. for any $v : B' \to B$ the pullback $A \times_B B'$ exists,

2. For any cartesian square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r] & B \end{tikzcd}$$

u' is in W.

Remark 2.1.3. u is universally in W means that it is in W and remains so after any base change.

Remark 2.1.4. Given $u : A \to B$ a functor and $b \in B$, we have an induced functor $u/b : A/b \to B/b$ defined by $(a, p) \mapsto (u(a), p)$ on objects and $f \mapsto u(f)$ on morphism. In the same way, we obtain $b\backslash u : b\backslash A \to b\backslash B$.

More generally, given a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

we get a functor $u/c : A/c \to B/c$ for any $c \in C$ given by $(a, p : v(a) \to c) \mapsto (u(a), u(p))$ and mapping a morphism f to $u(f)$.

Definition 2.1.5 (Weak basic localiser)**.** Let $\mathcal{W} \subset \mathrm{Mor}(\mathbf{Cat})$. We say $\mathcal{W}$ is a **weak basic localiser** if it satisfies the following :

(La) $\mathcal{W}$ is weakly saturated.

(Lb) If A is a category with a terminal object, then $A \to e$ is in $\mathcal{W}$.

(Lc) For any functor $u : A \to B$, if $\forall b \in B$ the induced functor $A/b \to B/b$ is in $\mathcal{W}$, then $u \in \mathcal{W}$.

It is called a **basic localiser** if condition (Lc) is replaced with the following stronger condition :

(Lc') If

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

is a commutative triangle in $\mathbf{Cat}$, and for any object $c \in C$ the functor $u/c : A/c \to B/c$ is in $\mathcal{W}$, then $u \in \mathcal{W}$.

Remark 2.1.6. Notice that a basic localiser is in particular a weak basic localiser because one gets condition (Lc) from (Lc') by putting $w = 1_B$.

Heuristic 2.1.7. Let us make sense of the axioms of a basic localiser. Being weakly saturated is simply a reasonable thing to ask whenever we want to have a notion of equivalence akin to isomorphism but weaker, and more generally a class of maps that is somewhat well behaved. Axiom (Lb) makes sense homotopically : If a category has a terminal object, it is "topologically contractible" in the sense that its nerve (see 2.1.26 (4)) is contractible (in the sense of simplicial sets) or equivalently its geometric realisation is contractible (in the usual, topological sense). If we are going to define a homotopy theory of small categories it makes sense to requires that a category which is "topologically contractible" (homotopy equivalent to $*$) is "weakly equivalent" to a point, this is precisely (Lb), a contractible category is weakly equivalent to a point.

For the axiom (Lc') one can make sense of it as follows : Given a small category A, the slices A/a should give us all we need to understand its homotopy type, that is, upon localisation of $\mathbf{Cat}$ at $\mathcal{W}$ the canonical morphism

$$\mathrm{hocolim}_a\, A/a \to \mathrm{colim}_a\, A/a = A$$

should become an isomorphism and this should remain true after a change of base so that given any $A' \to A$ the induced morphism

$$\mathrm{hocolim}_a A'/a \to \mathrm{colim}_a A'/a = A'$$

should be an isomorphism in the localisation. From this requirement one can obtain (Lc') : given a triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

if u/c is an isomorphism for all c in the localisation, said differently, if u is an isomorphism locally over C in the localisation, then u is an isomorphism in the localisation (seen by applying the second isomorphism twice).

Definition 2.1.8. Given a weak basic localiser $\mathcal{W}$ we denote $\mathbf{Hot}_{\mathcal{W}}$ the category $\mathcal{W}^{-1}\mathbf{Cat}$.

The data of a weak basic localiser is the necessary context for us to define asphérical functors, and then proper and smooth functors.

Throughout the rest of this section, $\mathcal{W}$ denotes a weak basic localiser, we call elements of $\mathcal{W}$ weak equivalences.

Definition 2.1.9. Let $u : A \to B$ be a functor. We say u is $\mathcal{W}$-**aspheric** (or simply aspheric) if for all $b \in B$ the induced functor $u/b : A/b \to B/b$ is in $\mathcal{W}$. It is called $\mathcal{W}$-**coaspheric** (or simply coaspheric) is u^{op} is aspheric.

We call a category A $\mathcal{W}$-aspheric (or simply aspheric) if $A \to e$ is in aspheric.

Remark 2.1.10. The condition (Lc) of definition 2.1.5 says that aspheric functors are weak equivalences. The condition (Lb) implies that categories with a terminal object are aspheric.

We record the following proposition, listing some of the basic properties of the notions we've just introduced.

Proposition 2.1.11. *Let $u : A \to B$ and $v : B \to C$ be morphisms in $\mathbf{Cat}$.*

(a) $A \to e$ is in $\mathcal{W}$ if and only if $A \to e$ is aspheric if and only if $A \to e$ is universally in $\mathcal{W}$.

(b) u is aspheric if and only if for all $b \in B$ the category A/b is aspheric.

(c) If u is universally in $\mathcal{W}$ then u is aspheric.

(d) If u is a left adjoint (i.e. has a right adjoint) then u is aspheric.

(e) If A and B are aspheric then so is $A \times B$. If u, u' are aspheric functors, then so is $u \times u'$.

(f) If u is aspheric, the for any $c \in C$ the map $u/c : A/c \to B/c$ is aspheric.

(g) If u is aspheric, then vu is aspheric if and only if v is aspheric.

Proof. (a) Remark that since e has a single object, denote it $*$, the condition for being aspheric is : $A \to e$ is aspheric if $A/* \to e/*$ is aspheric. Since e has a single morphism and a single object, $e/*$ is isomorphic to e, and similarly $A/*$ is isomorphic to A. We have the commutative square :

$$\begin{tikzcd} A/* \arrow[r] \arrow[d, "\cong"] & e/* \arrow[d, "\cong"'] \\ A \arrow[r] & e \end{tikzcd}$$

From the fact that $\mathcal{W}$ is weakly saturated, it contains all isomorphisms, and by 2-out-of-3, $A/e \to e/*$ is in $\mathcal{W}$ if and only if $A \to e$ is in $\mathcal{W}$, which proves the first part of the claim.

For the second part, clearly if $A \to e$ is universally in $\mathcal{W}$, it is in $\mathcal{W}$. It remains to show that if $A \to e$ is in $\mathcal{W}$, then it is universally in $\mathcal{W}$. Notice that given $B \to e$ the pullback $A \times_e B$ is isomorphic to the product $A \times B$.

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r] & e \end{tikzcd}$$

So we're left to show that the projection $A \times B \to B$ is aspheric when A is. Given $b \in B$ we may form the following commutative triangle :

$$\begin{tikzcd} (A \times B)/b \arrow[rr] \arrow[dr] & & B/b \arrow[dl, "\in \mathcal{W}"] \\ & e & \end{tikzcd}$$

where $B/b \to e$ is in $\mathcal{W}$ because B/b has a terminal object. By 2-out-of-3 we're left to see that $(A \times B)/b \to e$ is in $\mathcal{W}$. But remark that $(A \times B)/b$ is isomorphic to $A \times B/b$, hence by 2-out-of-3 and since $A \to e \in \mathcal{W}$, this is equivalent to $A \times B/b \to A$ in $\mathcal{W}$. Note that for all $a \in A$, $(A \times B/b)/a \cong A/a \times B/b$, which has a terminal object, as such by 2-out-of-3 in the following diagram :

$$\begin{tikzcd} (A \times B/b)/a \arrow[rr] \arrow[dr, "\in \mathcal{W}"'] & & A/a \arrow[dl, "\in \mathcal{W}"] \\ & e & \end{tikzcd}$$

the projection $A \times B/b \to A$ is aspheric, hence in $\mathcal{W}$.

(b) Given $b \in B$, we may form the following commutative diagram :

$$\begin{tikzcd} A/b \arrow[rr, "u/b"] \arrow[dr] & & B/b \arrow[dl] \\ & e & \end{tikzcd}$$

Notice that B/b has a terminal object given by $(b, 1_b)$, as such $B/b \to e$ is in $\mathcal{W}$. Hence using the 2-out-of-3 property, we see that u/b is in $\mathcal{W}$ if and only if $A/b \to e$ is in $\mathcal{W}$, which using (a) proves the claim.

(c) Assume u is universally in $\mathcal{W}$, the claim follows from recognising that the square

$$\begin{tikzcd} A/b \arrow[r] \arrow[d, "u/b"'] & A \arrow[d, "u"] \\ B/b \arrow[r] & B \end{tikzcd}$$

is cartesian.

(d) Assume $u : A \to B$ has a right adjoint denoted $v : B \to A$. We have the natural bijection $\mathrm{Hom}_A(a, v(b)) \simeq \mathrm{Hom}_B(u(a), b)$. As such, the category A/b is isomorphic to $A/v(b)$, which has a terminal object. Hence by 2-out-of-3 $A/b \to B/b$ is in $\mathcal{W}$.

(e) Assume that A and B is aspheric and consider the pullback

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ B \arrow[r] & e \end{tikzcd}$$

By (a), the second projection $A \times B \to B$ is in $\mathcal{W}$, since $B \to e$ is as well, it follows that $A \times B \to e$ is, thus proving the claim.

Assume $u : A \to B$ and $u' : A' \to B'$ are aspheric. By (b), we need to prove that for any $(b, b') \in B \times B'$ the category $(A \times A')/(b, b')$ is aspheric. This category is isomorphic to $A/b \times A'/b'$ which is aspheric since both A/b and A'/b' are by hypothesis.

(f) Assume u is aspheric, and remark that for any $(b, p) \in B/c$ the categories $(A/c)/(b, p)$ and A/b are canonically isomorphic, whence it follows that u/c is aspheric since u is.

(g) Form the commutative triangle

$$\begin{tikzcd} A/c \arrow[rr, "u/c"] \arrow[dr, "vu/c"'] & & B/c \arrow[dl, "v/c"] \\ & C/c & \end{tikzcd}$$

where by the previous point u/c is aspheric, hence a weak equivalence. By 2-out-of-3 v/c is a weak equivalence if and only if vu/c is, hence the claim follows. ■

The proposition above has a few notable corollaries, which we record here.

Corollary 2.1.12. *Equivalences of categories are aspheric.*

Proof. An equivalence is in particular a left adjoint. ■

Corollary 2.1.13. *If A has an initial object, then A is aspheric.*

Corollary 2.1.14. *Aspheric functors are stable under composition.*

Proof. This is one of the implication of (g). ■

Remark 2.1.15. By considering dual forms of some of the points in 2.1.11, one obtains similar properties for coaspheric functors, for instance given a pair of composable morphisms $A \xrightarrow{u} B \xrightarrow{v} C$ where u is coaspheric, then v is coaspheric if and only if vu is too. We can deduce from that stability of coaspheric morphism under composition.

Proof. We have an adjunction (i, u) where u is the unique map to the final object $A \to e$, and $i : e \to A$ picks out the initial object of A. As such i is aspheric, and since, in particular it is in $\mathcal{W}$. Since $ui = 1_e$, u is in $\mathcal{W}$. ■

Proposition 2.1.16. *Let $u : A \to B$ be a morphism of* **Cat**, *and assume it is a precofibration. Then u is aspheric if and only if for any $b \in B$ the fiber A_b over u is aspheric.*

Proof. Since it is a precofibration, the functor $i_b : A_b \to A/b$ has a left adjoint

$$j_b : A/b \to A_b.$$

It follows from 2.1.11 (d) that j_b is a weak equivalence, hence by 2-out-of-3 u is aspheric if and only if A_b is. ■

2.1.17 — Let A be a category, denote $S(A)$ the category defined as follows :

- It's objects are given by $\mathrm{Ob}(S(A)) := \mathrm{Mor}(A)$,

- given a pair of morphisms of A, $f : a \to b$ and $f' : a' \to b'$ the set $\mathrm{Hom}_{S(A)}(f, f')$ is given by pairs (g, h) making the following diagram

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[d, "h"] \\ a' \arrow[u, "g"] \arrow[r, "f'"] & b' \end{tikzcd}$$

commute. Identity of an object is given by $(1_a, 1_b)$ and composition of arrows is done by pasting squares vertically such that $(g', h') \circ (g, h) = (g \circ g', h' \circ h)$.

We can define a pair of functors

$$A^{\mathrm{op}} \xleftarrow{s_A} S(A) \xrightarrow{t_A} A$$

where

$$\begin{aligned} s_A(f) &= \text{ source of } f, \quad s_A(g,h) = g \\ t_A(f) &= \text{ target of } f, \quad t_A(g,h) = h. \end{aligned}$$

Finally, we define an isomorphism

$$\begin{aligned} I\colon S(A) &\longrightarrow S(A^{\mathrm{op}}) \\ f &\longmapsto f \\ (g,h) &\longmapsto (h,g). \end{aligned}$$

It is clear that $s_A \circ I = t_A$ and $t_A \circ I = s_A$.

Lemma 2.1.18. *The functors s_A and t_A are precofibrations.*

Proof. Let us show s_A is a cofibration. We first ought to show that given any $h : b \to b'$ in A^{op} and any lift of b in $S(A)$ (so any morphism in A with source B) there is a cocartesian morphism w of $S(A)$ such that $s_A(w) = h$. So let $h : b \to b'$ in A^{op} and $f : b \to b''$ an object of $S(A)$ be given. Denote $h_*(f) = fh \in S(A)$, we have a morphism $w(h,f) : f \to h_*(f)$ in $S(A)$ given by $(h, 1_{b''})$. Let us check it is cocartesian : let $(p,q) : f \to f''$ be another morphism in $S(A)$ such that $s_A(p,q) = s_A(w(h,f))$, i.e. $p = h$ then we can define a morphism $h_*(f) \to f''$ by $(1_{b'}, q)$, it's clear such a morphism is unique and $s_A(1_{b'}, q) = 1_{b'}$ and $(p,q) = (1_{b'}, q)w(h,f)$. Hence s_A is a precofibration.

Since $s_A \circ I = t_A$ with I an isomorphism, it follows t_A is a precofibration as well. ∎

Lemma 2.1.19. *The functors t_A and s_A are aspheric.*

Proof. t_A being a precofibration, it suffices to check its fibers are aspheric. But direct calculations shows that given $a \in A$ the fiber of t_A over a is given by $S(A)_a \cong (A/a)^{\mathrm{op}}$ which has an initial object, hence is aspheric. A similar argument works for s_A. ∎

2.1.20 — Given $u : A \to B$, we can define

$$\begin{aligned} S(u)\colon S(A) &\longrightarrow S(B) \\ f &\longmapsto u(f) \\ (g,h) &\longmapsto (u(g), u(h)). \end{aligned}$$

A direct calculation shows the following diagram

$$\begin{tikzcd} A^{\mathrm{op}} \arrow[r, "s_A"] \arrow[d, "u^{\mathrm{op}}"'] & S(A) \arrow[r, "t_A"] \arrow[d, "S(u)"] & A \arrow[d, "u"] \\ B^{\mathrm{op}} \arrow[r, "s_B"'] & S(B) \arrow[r, "t_B"'] & B \end{tikzcd}$$

to be commutative.

Proposition 2.1.21. *Let $u : A \to B$ be a morphism of* $\mathbf{Cat}$*. Then u is a weak equivalence if and only if u^{op} is a weak equivalence.*

Proof. Consider the diagram in 2.1.20, the horizontal arrows are weak equivalences by 2.1.19, hence by 2-out-of-3 u is a weak equivalence if and only if $S(u)$ is if and only if u^{op} is one as well. ■

Corollary 2.1.22. *Coaspheric morphism are in particular weak equivalences.*

Proof. If $u : A \to B$ is coaaspheric, then by definition u^{op} is aspheric, hence u^{op} is a weak equivalence, whence it follows u is as well. ■

Corollary 2.1.23. *A morphism $u : A \to B$ of* $\mathbf{Cat}$ *is coaspheric if and only if for any $b \in B$ the morphism $b\backslash A \to b\backslash B$ induced by u is a weak equivalence.*

Proof. We have a canonical isomorphism $A^{\mathrm{op}}/b \cong (b\backslash A)^{\mathrm{op}}$ through which the functor

$$b\backslash u : b\backslash A \to b\backslash B$$

is canonically identified with

$$(u^{\mathrm{op}}/b)^{\mathrm{op}} : (A^{\mathrm{op}}/b)^{\mathrm{op}} \to (B^{\mathrm{op}}/b)^{\mathrm{op}}.$$

As such, u is coaspheric if and only if u^{op} is aspheric, which by definition is equivalent to u^{op}/b being a weak equivalence for every $b \in B$. By 2.1.21 this is equivalent to $(u^{\mathrm{op}}/b)^{\mathrm{op}}$ being a weak equivalence for all $b \in B$. By the identification above, this is equivalent to $b\backslash u$ being a weak equivalence for all $b \in B$. ■

For the remainder of this section, assume $\mathcal{W}$ is a basic localiser.

Proposition 2.1.24. *Let*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

be a commutative triangle of $\mathbf{Cat}$*. Assume that for every $c \in C$, the induced morphism $u_c : A_c \to B_c$ is a weak equivalence. Then if u and w are precofibrations, then u is a weak equivalence locally over C. If they are both prefibrations, then u is a weak equivalence colocally over C. In particular, in both cases, u is a weak equivalence.*

Proof. For any $c \in C$ we have the commutative diagram

$$\begin{tikzcd} A_c \arrow[r, "u_c"] \arrow[d, "i_c"'] & B_c \arrow[d, "j_c"] \\ A/c \arrow[r, "u/c"'] & B/c \end{tikzcd}$$

and if u, v are precofibrations, by 1.5.9 i_c and j_c both admit a left adjoint, hence they're weak equivalences (because they are coaspheric). It follows from our assumption that u/c is a weak equivalence too. ■

Proposition 2.1.25. *Let $F, F' : I \to$* $\mathbf{Cat}$ *be a pair of functors, and $\alpha : F \to F'$ a natural transformation. Assume α is a weak equivalence argument by argument, so that each functor $\alpha_i : F(i) \to F'(i)$ is a weak equivalence, then $\int \alpha : \int F \to \int F'$ is a weak equivalence locally over I and in particular a weak equivalence.*

Proof. We saw in 1.5.7 that the projection from the integral of a $\mathbf{Cat}$-valued functor is a precofibration, hence it follows from the previous proposition and the remark that the fibers over $i \in I$ of $\int F$ and $\int F'$ are isomorphic to $F(i)$ and $F'(i)$ respectively and that this identifies $\left(\int \alpha\right)_i$ with α_i. ■

We conclude this section with a few examples of weak basic localisers, establishing the existence of such objects.

Example 2.1.26. 1. Denote $\mathcal{W}_{tr} := \mathrm{Mor}(\mathbf{Cat})$, this is a weak basic localiser, it is the only one for which $\emptyset$, we call it the trivial weak basic localiser. We have $\mathbf{Hot}_{\mathcal{W}_{tr}} = e$.

2. Denote $\mathcal{W}_{gr} \subset \mathrm{Mor}(\mathbf{Cat})$ such that $\mathrm{id}_{\emptyset} \in \mathcal{W}_{gr}$ and any functor $u : A \to B$ with $A \neq \emptyset$ is in $\mathcal{W}_{gr}$. This is a weak basic localiser, called "coarse" (grossier in French). We have $\mathbf{Hot}_{\mathcal{W}_{gr}} = \Delta^1 = \{0 \to 1\}$.

3. Recall the functor $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ mapping a category A to $\mathrm{Ob}(A)/\sim$ where $\sim$ is the equivalence relation generated by $a \sim a' \iff \mathrm{Hom}_A(a, a') \neq \emptyset$. This also gives a well defined map on morphisms, sending a functor $u : A \to B$ to $\pi_0(u) : \pi_0(A) \to \pi_0(B)$ defined by $[a] \mapsto [u(a)]$. We can obtain a weak basic localiser by taking $\mathcal{W}_0 := \{u \in \mathrm{Mor}(\mathbf{Cat}),\ \pi_0(u) \text{ is a bijection}\}$. This is fairly easy to see. We can get a feel for asphericity in this context : A category is aspheric if $A \to e$ is in $\mathcal{W}_0$, i.e. if there is a bijection $\pi_0(A) \to \{*\}$, so if A is non empty and connected. A functor $u : A \to B$ will be aspheric if and only if it is initial. Indeed by proposition 2.1.11 (b), asphericity is equivalent to A/b being aspheric for all b, which by our previous remark is equivalent to u being initial. Dually, a functor will be coaspheric if and only if it is final. Here, we have $\mathbf{Hot}_{\mathcal{W}_0} = \mathbf{Set}$ and the localisation functor can be identified with π_0.

4. From a category A we can get its classifying space as the geometric realisation of its nerve. Recall the functor $N : \mathbf{Cat} \to \mathbf{Set}_{\Delta}$ mapping a category to a simplicial set called its nerve, defined by the following property : $\mathrm{Hom}_{\mathbf{Set}_{\Delta}}(\Delta^n, N(A)) = \mathrm{Hom}_{\mathbf{Cat}}([n], A)$. We define the geometric realisation $|\cdot| : \mathbf{Set}_{\Delta} \to \mathbf{Top}$ as the left Kan extension of the functor $\Delta \to \mathbf{Top}$ mapping Δ^n to the standard geometric simplex. Putting this all together we get the classifying space of a category as $\mathcal{B}A := |N(A)|$. We can define a weak basic localiser as follows: $\mathcal{W}_{\infty} := \{u \in \mathrm{Mor}(\mathbf{Cat}), \mathcal{B}(u) \text{ is a homotopy equivalence}\}$, this follows from Quillen's theorem A. In this case, we can make a similar remark : a category A is $\mathcal{W}_{\infty}$-aspheric if and only if it has the homotopy type of a point. Here, we recover the usual homotopy theory of spaces, since $\mathbf{Hot}_{\mathcal{W}_{\infty}} = \mathbf{CW}$.

5. There is a whole family of weak basic localiser between $\mathcal{W}_0$ and $\mathcal{W}_{\infty}$, taking $\mathcal{W}_n$ to be functors in $\mathcal{W}_0$ inducing isomorphism on homotopy groups up degree n.

In fact, all of the above examples are actually basic localiser, not simply weak ones.

Example 2.1.27. Given any $S \subset \mathrm{Mor}(\mathbf{Cat})$ we can define a basic localiser $\mathcal{W}(S)$ as the intersection of all basic localisers containing S. Taking $S = \emptyset$ we get the *minimal basic localiser*, which turns out to be $\mathcal{W}_{\infty}$ (this was conjectured by Grothendieck and proved by Cisinski, see [Cis06]). Taking $\mathcal{W}(\{\emptyset \to e\})$ we recover $\mathcal{W}_{tr}$ and taking $\mathcal{W}(\{e \coprod e \to e\})$ we obtain $\mathcal{W}_{gr}$.

2.2 Homotopical Kan extension and Grothendieck construction

For the remainder of this section, $\mathcal{W}$ denotes a weak basic localiser.

2.2.1 — We define three functors, which will give us an adjunction, eventually leading to homotopical Kan extension.

1. Given a category $I \in \mathbf{Cat}$, we can form the category $\mathbf{Cat}/I$. Given $w : J \to I$, this induces a functor $\mathbf{Cat}/w : \mathbf{Cat}/J \to \mathbf{Cat}/I$ mapping $(A, A \xrightarrow{v} J)$ to $(A, A \xrightarrow{wv} I)$, and acting as the identity on morphisms.

2. We also define a functor

$$\begin{aligned} \Theta_I \colon \mathbf{Cat}/I &\longrightarrow \underline{\mathrm{Hom}}(I, \mathbf{Cat}) \\ (A, A \xrightarrow{p} I) &\longmapsto (i \mapsto A/i) \\ ((A,p) \xrightarrow{v} (A',p')) &\longmapsto \eta_v : \Theta_I(A,p) \to \Theta_I(A',p') \end{aligned}$$

The morphism η_v is defined by $\eta_v(i) : A/i \to A'/i$ which on objects maps $(a, p(a) \xrightarrow{l} i)$ to $(v(a), v'(p(a)) \xrightarrow{l} i)$ and on morphisms :

$$\begin{tikzcd} p(a) \arrow[rr, "p(f)"] \arrow[dr, "l"'] & & p(a') \arrow[dl, "m"] \\ & i & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} p'(v(a)) \arrow[rr, "p'(v(f))"] \arrow[dr, "l"'] & & p'(v(a')) \arrow[dl, "m"] \\ & i & \end{tikzcd}$$

3. Finally we take the functor Θ'_I to be $\int_I$, which was defined in 1.4.3 as follows :

$$\begin{aligned} \Theta'_I \colon \ \underline{\mathrm{Hom}}(I, \mathbf{Cat}) &\longrightarrow \mathbf{Cat}/I \\ F &\longmapsto \left(\int F, \int F \to I \right) \\ \eta : F \Rightarrow G &\longmapsto \Theta'_I(\eta) \end{aligned}$$

where $\Theta'_I(\eta) \in \mathrm{Hom}_{\mathbf{Cat}/I}\left(\int F, \int G\right)$ is defined by

$$\begin{aligned} \Theta'_I(\eta) \colon \ \int_I F &\longrightarrow \int_I G \\ (i, x) &\longmapsto (i, \eta_i(x)) \\ (k : i \to i', f : F(k)(x) \to x') &\longmapsto (k, \eta_{i'}(f)) \end{aligned}$$

Theorem 2.2.2. *Let $w : J \to I$ be a morphism of $\mathbf{Cat}$, then the pair of functors*

$$\Theta_I \circ \mathbf{Cat}/w : \mathbf{Cat}/J \leftrightarrows \underline{\mathrm{Hom}}(I, \mathbf{Cat}) : \Theta'_J \circ \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})$$

is an adjunction.

Proof. We define the unit and counit of the adjunction.

1. We first define $\varepsilon : \Theta_I \circ \mathbf{Cat}/w \circ \Theta'_J \circ \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}}) \longrightarrow 1_{\underline{\mathrm{Hom}}(I,\mathbf{Cat})}$. Let $F : I \to \mathbf{Cat}$ be given, we define a functor for all $i \in I$

$$\varepsilon_{F,i} : \left(\int_J Fw \right) /i \longrightarrow F(i)$$

To do so, we first fully describe the source category as follows :

   (a) Objects are given by triples $(j, a, p : w(j) \to i)$ with $j \in J$, $a \in Fw(j)$ and $p \in \mathrm{Mor}(I)$.

   (b) A morphism between (j, a, p) and (j', a', p') is given by a pair (l, f) with $l : j \to j'$ in J and $f : Fw(l)(a) \to a'$ in $Fw(j')$ such that $p = p' \circ w(l)$.

We take

$$\begin{aligned} \varepsilon_{F,i}(j, a, p) &:= F(p)(a), \\ \varepsilon_{F,i}(l, f) &:= F(p')(f) \end{aligned}$$

2. We now define $\eta : 1_{\mathbf{Cat}/J} \longrightarrow \Theta'_J \circ \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}}) \circ \Theta_I \circ \mathbf{Cat}/w$, so given an object $(A, A \xrightarrow{v} J)$, we ought to define a functor above J

$$\eta_{(A,v)} : (A, v) \to \left(\int G, \theta_G \right)$$

where $G : J \to \mathbf{Cat}$ maps $j \in J$ to $A/^{wv}w(j)$. We begin first by describing the target category.

(a) Objects of $\int G$ are triples $(j, a, p : wv(a) \to w(j))$ with $j \in j$, $a \in A$, $p \in \mathrm{Mor}(I)$.

(b) A morphism $(j, a, p) \to (j', a', p')$ is given by a a pair (l, f) with $l : j \to j'$ in J and $f : a \to a'$ in A, such that $p' \circ wv(f) = w(l) \circ p$.

We then take :

$$\eta_{(A,v)}(a) := (v(a), a, 1_{wv(a)})$$
$$\eta_{(A,v)}(f) := (v(f), f)$$

We now ought to check the triangle identities, which is a tedious exercise in rewriting, as such we choose not to write it here, the interested reader may choose to carry out the calculation. A very similar calculation for 2-categories is performed later, which specialises to this exact one if the 2-categories in question are actually 1-categories. ■

Corollary 2.2.3. *In particular,* $\Theta_I : \mathbf{Cat}/I \leftrightarrows \underline{\mathrm{Hom}}(I, \mathbf{Cat}) : \Theta'_I$ *is an adjunction.*

Proof. Take $w = 1_I$ in the previous theorem. ■

Definition 2.2.4. For a small category I, denote $\mathcal{W}_I \subset \mathrm{Mor}(\underline{\mathrm{Hom}}(I, \mathbf{Cat}))$ the subset of natural transformations η such that $\eta_i \in \mathcal{W}$ for all $i \in I$. We also define $\mathcal{W}'_I \subset \mathrm{Mor}(\mathbf{Cat}/I)$ the weak equivalences locally over I, that is the maps $u : (A, v : A \to I) \longrightarrow (A', v' : A' \to I)$ such that for all $i \in I$ the map $u/i : A/i \to A'/i$ is in $\mathcal{W}$.

Theorem 2.2.5. *Keeping the notations from above, we have*

- $\mathcal{W}'_I = \Theta_I^{-1}(\mathcal{W}_I)$
- $\mathcal{W}_I = \Theta'^{-1}_I(\mathcal{W}'_I)$
- $\overline{\Theta}_I : \mathcal{W}'^{-1}_I \mathbf{Cat}/I \leftrightarrows \mathcal{W}_I^{-1}\, \underline{\mathrm{Hom}}(I, \mathbf{Cat}) : \overline{\Theta}'_I$ *is an equivalence of categories.*

Proof. By definition of Θ_I, $\mathcal{W}_I$ and $\mathcal{W}'_I$ it follows that the first point is true. By lemma 1.2.4 the second point is also true, and we're left to show that the counit ε_F is an equivalence argument by argument, i.e. that $\varepsilon_{F,i} \in \mathcal{W}$ for all i. Recall the proof of the previous theorem, we had a explicit description of $\varepsilon_{F,i}$, for $F : I \to \mathbf{Cat}$:

$$\begin{aligned}
\varepsilon_{F,i} \colon \Theta_I \Theta'_I(F)(i) = \left(\int F\right)/i &\longrightarrow F(i) \\
(i', a, p : i' \to i) &\longmapsto F(p)(a) \\
(i', a, p : i' \to i) \xrightarrow{(k,l)} (i'', a', p' : i'' \to i) &\longmapsto F(p')(l)
\end{aligned}$$

The key point is that we can construct a right adjoint to this functor, hence showing by proposition 2.1.11 it is aspheric and hence a weak equivalence. We define a functor as follows :

$$\begin{aligned}
\alpha_{F,i} \colon F(i) &\longrightarrow \left(\int F\right)/i \\
a &\longmapsto (i, a, 1_i) \\
p : a \to a' &\longmapsto (1_i, p)
\end{aligned}$$

We show the following natural isomorphism for all $(i', a, p) \in \left(\int F\right)/i$ and $a' \in F(i)$:

$$\mathrm{Hom}_{F(i)}(\varepsilon_{F,i}(i', a, p), a') \simeq \mathrm{Hom}_{(\int F)/i}((i', a, p), \alpha_{F,i}(a)).$$

Recalling the definition of morphisms in $\left(\int F\right)/i$ we get that $\mathrm{Hom}_{(\int F)/i}((i', a, p), \alpha_{F,i}(a))$ is given by the set of pairs (l, f) with $l : i' \to i$ and $f : F(l)(a) \to a'$ such that $p = l$. As such we have a canonical bijection with $\mathrm{Hom}_{F(i)}(F(p)(a), a')$. This is readily seen to be exactly $\mathrm{Hom}_{F(i)}(\varepsilon_{F,i}(i', a, p), a')$. ■

Notation. We introduce the following notation : given $I \in \mathbf{Cat}$ we will use the notation $\mathbf{Hot}(I)$ (or $\mathbf{Hot}_{\mathcal{W}}(I)$) for the category $\mathcal{W}_I^{-1}\,\underline{\mathrm{Hom}}(I, \mathbf{Cat})$. Sometimes this is written $\mathbb{D}(I)$ (or $\mathbb{D}_{\mathcal{W}}(I)$) to emphasize the links with derivators.

Remark 2.2.6. *From this point onward, we assume $\mathcal{W}$ is a basic localiser.*

To arrive at the desired homotopical Kan extension, we need a final lemma.

Lemma 2.2.7. *Given $w : J \to I \in \mathbf{Cat}$, and keeping the previous notations :*

1. *The functor $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}}) : \underline{\mathrm{Hom}}(I, \mathbf{Cat}) \to \underline{\mathrm{Hom}}(J, \mathbf{Cat})$ induces a functor*

$$w^* : \mathbf{Hot}(I) \to \mathbf{Hot}(J)$$

2. *The functor $\mathbf{Cat}/w : \mathbf{Cat}/J \to \mathbf{Cat}/I$ induces a functor*

$$\overline{\mathbf{Cat}/w} : {\mathcal{W}'_J}^{-1}\mathbf{Cat}/J \to {\mathcal{W}'_I}^{-1}\mathbf{Cat}/I$$

Proof. It suffices to check that $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})(\mathcal{W}_I) \subset \mathcal{W}_J$ and $(\mathbf{Cat}/w)(\mathcal{W}'_J) \subset \mathcal{W}'_I$. The first is point is clear : given $\eta \in W_I$, that is $\eta : F \Rightarrow G$ such that $\eta_i \in \mathcal{W}$ for all i, $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})(\eta)$ is the natural transformation given by $Fw(j) \xrightarrow{\eta_{w(j)}} Gw(j)$, but by hypothesis $\eta_{w(j)} \in \mathcal{W}$, hence $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})(\eta)$ lies in $\mathcal{W}_J$.

The second point is slightly trickier. Given a morphism in $\mathcal{W}'_J$, i.e. $u : (A, v) \to (B, v')$ such that $v'u = v$, we ought to show that $u/i : A/i \to B/i$ is in $\mathcal{W}$ for all $i \in I$. We have the commutative triangle :

$$\begin{tikzcd} A/i \arrow[rr, "u/i"] \arrow[dr, "v/i"'] & & B/i \arrow[dl, "v'/i"] \\ & J/i & \end{tikzcd}$$

because already $v'u = v$. Furthermore, given any object of J/i, say $(j, p : w(j) \to i)$, we have canonical isomorphisms $(A/i)/(j,k) \simeq A/j$ and $(B/i)/(j,k) \simeq B/j$ and $(u/i)/(j,k)$ is identified with u/j which is in $\mathcal{W}$ by hypothesis. As such, u/i is a weak equivalence over J/i, since $\mathcal{W}$ is a basic localiser it is a weak equivalence, which was what we needed to show.

As such the functor $\gamma \circ \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})$ (γ denotes the localisation $\underline{\mathrm{Hom}}(J, \mathbf{Cat}) \to \mathbf{Hot}(J)$) maps $\mathcal{W}_I$ to isomorphisms in $\mathbf{Hot}(J)$ hence factors through $\mathbf{Hot}(I)$. The same argument applies to the second functor.

■

Definition 2.2.8. Keeping all of the above notations, we define a functor

$$w_! : \mathbf{Hot}(J) \to \mathbf{Hot}(I)$$

by taking $w_! := \overline{\Theta}_I \circ \overline{\mathbf{Cat}/w} \circ \overline{\Theta}'_J$.

Theorem 2.2.9. *For $w : J \to I \in \mathbf{Cat}$, the pair of functors*

$$w_! : \mathbf{Hot}(J) \leftrightarrows \mathbf{Hot}(I) : w^*$$

is an adjunction.

Proof. By 2.2.2 we have the adjunction $\Theta_I \circ \mathbf{Cat}/w : \mathbf{Cat}/J \leftrightarrows \underline{\mathrm{Hom}}(I, \mathbf{Cat}) : \Theta'_J \circ \underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})$, using 1.2.4 and 2.2.7 and 2.2.5 we obtain an induced adjunction with left adjoint the composition :

$$L : {\mathcal{W}'_J}^{-1}\mathbf{Cat}/J \xrightarrow{\overline{\mathbf{Cat}/w}} {\mathcal{W}'_I}^{-1}\mathbf{Cat}/I \xrightarrow{\overline{\Theta}_I} \mathbf{Hot}(I)$$

and with right adjoint the composite

$$R : \mathbf{Hot}(I) \xrightarrow{w^*} \mathbf{Hot}(J) \xrightarrow{\overline{\Theta}'_J} \mathcal{W}'_J{}^{-1}\mathbf{Cat}/J$$

But by 2.2.5, $\overline{\Theta}'_J$ and $\overline{\Theta}_J$ are quasi inverses to each other, and as such $w^* \simeq \overline{\Theta}_J \circ \overline{\Theta}'_J \circ w^* = \overline{\Theta}_J \circ L$. Finally we ought to check the desired adjunction :

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Hot}(J)}(w^* a, b) &\simeq \mathrm{Hom}_{\mathbf{Hot}(J)}(\overline{\Theta}_J \circ La, b) \\
&\simeq \mathrm{Hom}_{\mathcal{W}'_J{}^{-1}\mathbf{Cat}/J}(\overline{\Theta}'_J \overline{\Theta}_J \circ La, \overline{\Theta}'_J b) \\
&\simeq \mathrm{Hom}_{\mathcal{W}'_J{}^{-1}\mathbf{Cat}/J}(La, \overline{\Theta}'_J b) \\
&\simeq \mathrm{Hom}_{\mathbf{Hot}(I)}(a, R\overline{\Theta}'_J b) \\
&= \mathrm{Hom}_{\mathbf{Hot}(I)}(a, w_! b)
\end{aligned}$$

■

The functor $w_!$ is the so-called "homotopical left Kan extension", indeed the regular left Kan extension is the left adjoint to $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})$, and here $w_!$ is the left adjoint to the functor induced by $\underline{\mathrm{Hom}}(w, 1_{\mathbf{Cat}})$ whence the name.

Example 2.2.10. If $\mathcal{W} = \mathcal{W}_0$, the functor $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ induces an identification between $\mathbf{Hot}(I)$ and $\underline{\mathrm{Hom}}(I, \mathbf{Set})$, and given $w : J \to I$, the functor w^* can be identified with

$$\begin{aligned}
\circ w \colon \underline{\mathrm{Hom}}(I, \mathbf{Set}) &\longrightarrow \underline{\mathrm{Hom}}(J, \mathbf{Set}) \\
F &\longmapsto F \circ w
\end{aligned}$$

and $w_!$ is its left adjoint, which is precisely the classical left Kan extension operation.

2.2.11 — Using the previous constructions, in the case where $w = p_I : I \to e$ we get

$$(p_I)_! : \mathbf{Hot}(I) \to \mathbf{Hot}_{\mathcal{W}},$$

which is sometimes denoted $\mathbf{L}\,\mathrm{colim}_I$ or $\mathbf{L}p_I$, because this is the homotopical or derived analogue of colim_I, since it the left homotopical Kan extension along the constant functor to the terminal category. We can give a nice description of $(p_I)_!(F)$, indeed given $F : I \to \mathbf{Cat}$, by definition

$$\Theta_e \circ \mathbf{Cat}/p_I \circ \Theta'_I(F) = \int_I F$$

By what precedes the integration functor $F \mapsto \int_I F$ induces precisely $(p_I)_!$ when localising, we can record this in the following proposition.

Proposition 2.2.12. *Given $F : I \to \mathbf{Cat}$, we have*

$$(p_I)_!(F) = \int_I F$$

in this sense, the integration functor, or grothendieck construction computes the "homotopy colimit" or "derived colimit" of a functor $F : I \to \mathbf{Cat}$.

2.3 Proper and smooth morphisms

Throughout this section, we fix $\mathcal{W}$ a weak basic localiser.

We can finally introduce the notion of proper and smooth functors, we give one of the more elementary definition, we will later give equivalent ones.

Definition 2.3.1. Let $u : A \to B$ we say that u is $\mathcal{W}$-**proper** (or simply proper) if for any $b \in B$ the morphism

$$\begin{aligned} i_b \colon A_b &\longrightarrow A/b \\ a &\longmapsto (a, 1_b) \\ (f : a \to a') &\longmapsto f \end{aligned}$$

is coaspheric. Dually, u is said to be $\mathcal{W}$-**smooth** (or simply smooth) if u^{op} is proper (equivalently if the canonical functor $j_b : A_b \to b\backslash A$ is aspheric).

With just the definition, it is also now clear that $\int_I F \to I$ is proper, we record this fact in the following proposition.

Proposition 2.3.2. *Let $F : I \to \mathbf{Cat}$ be a functor. Then the projection $\theta_F : \int_I F \to I$ is proper.*

Proof. We already saw in 1.5.7 that θ_F is a cofibration, so in particular it is a precofibration. This is equivalent by 1.5.9 to i_b admitting a left adjoint, which by the dual of 2.1.11(d) implies i_b is coaspheric, hence θ_F proper. ■

Corollary 2.3.3 (of the proof above)**.** *Any precofibration is proper. In particular, any cofibration is proper.*

We will mostly work with proper functors, but our results will of course have dual statements for smooth functors, we do not always state the dual version. Already with only the definition given, we can see some easy equivalences that don't require a lot of work.

Proposition 2.3.4. *Given $u : A \to B$ the following are equivalent :*

(a) u is proper ;

(b) For any object $a \in A$ the fibers of the induced map $a\backslash A \to u(a)\backslash B$ are aspheric ;

(c) For any map $\Delta^1 \to B$, form the pullback

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ \Delta^1 \arrow[r] & B \end{tikzcd}$$

then the inclusion of the fiber over 1 into A' is coaspheric, i.e. $A'_1 \hookrightarrow A'$ is coaspheric ;

(d) For any morphism $f_0 : b_0 \to b_1$ in B and any $a_0 \in A_{b_0}$ the category we denote $A(a_0, f_0)$ is aspheric.

(e) The fibers of v, the morphism $\underline{\mathrm{Hom}}(\Delta^1, A) \to A \times_B \underline{\mathrm{Hom}}(\Delta^1, B)$ induced by

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\Delta^1, A) \arrow[r] \arrow[d, "ev_0"] & \underline{\mathrm{Hom}}(\Delta^1, B) \arrow[d, "ev_0"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

are aspheric.

Remark 2.3.5. The category $A(a_0, f_0)$ above has objects the maps $f : a_0 \to a$ such that $u(f) = f_0$. A morphism between f, f' is given by $g \in A_{b_1}$ such that $gf = f'$.

Proof. This is mostly an exercise in rewriting the definitions.

We first prove (a) $\iff$ (b). By the dual of 2.1.11 (b), i_b is coaspheric if and only if for any b and any $(a, p : u(a) \to b) \in A/b$ the category $(a,p)\backslash A_b$ is aspheric. But it turns out that this category $(a,p)\backslash A_b$ is precisely the fiber of $a\backslash A \to u(a)\backslash B$. To justify that last claim we need to see that $(a,p)\backslash A_b = (a\backslash A)_{(b,p)}$.

First, by definition, objects of $(a,p)\backslash A_b$ are pairs (a', f) with $a' \in A_b$ and $f : (a,p) \to i_b(a') \in A/b$. Unravelling this, the objects are pairs (a', f) with $u(a) = b$ and $f : a \to a'$ with $u(f) = p$. Second, a morphism $(a', f) \to (a'', g)$ ought to be given by a morphism $\gamma : a' \to a'' \in A_b$ such that $i_b(\gamma) \circ f = g$, i.e. such that $\gamma \circ f = g$. Now, an object of $(a\backslash A)_{(b,p)}$ is given by an object (a', f) of $a\backslash A$ such that $(a\backslash u)(a', f) = (b, p)$. Unravelling again the definitions, this is a pair (a', f) with $a' \in A$, $f : a \to a'$ such that $u(a') = b$ and $u(f) = p$. A morphism $(a', f) \to (a'', g)$ is then given by a map $\gamma \in a\backslash A$ such that $(a\backslash u)(\gamma) = 1_{(b,p)}$, which amounts to asking that $\gamma \circ f = g$ and $u(\gamma) = 1_b$, i.e. that $\gamma \in A_b$. It is now clear that the two categories are in fact the same and as such one if aspheric if and only if the other is which proves the desired equivalence.

The proof above also gives us the equivalence of (a) / (b) and (d), since $A(a_0, f_0) = (a_0, f_0)\backslash A_{b_0}$.

We're left to see that these are also equivalent to (c) and (e). The data of an arrow $f_0 : b_0 \to b_1$ in B is equivalent to that of a functor $\Delta^1 \to B$ and the map $A'_1 \hookrightarrow A'$ is coaspheric if and only if for any $a' \in A'$ the category $a'\backslash A'_1$ is aspheric. Objects in A' can be partitioned into two sets, those from mapped to $0 \in \Delta^1$ and those mapped to $1 \in \Delta^1$. If $a' \in A'$ is among the latter set of objects, then it can be canonically identified with some object of A'_1 and as such $a'\backslash A'_1$ has an initial object, whence asphericity of $a'\backslash A'_1$ follows without any further condition on u. In the other case, when a' lives in the fiber over 0, the pullback pasting lemma shows a' can be canonically identified with some a_0 in the fiber over b_0 and it turns out that $a'\backslash A'_1 \cong A(a_0, f_0)$ which proves the desired equivalence.

Finally, we prove the equivalence with (e). An object of $A \times_B \underline{\mathrm{Hom}}(\Delta^1, B)$ is given by a pair (a, f) where $a \in A$ and $f : b \to b'$ is a morphism of B such that $u(a) = b$. The fiber of v over (a, f) is given by the category $A(a, f)$ of (d), whence the equivalence follows. ∎

With the proposition 2.3.4 we can already get a first property of proper functors which is non trivial when looking at the definition we first stated.

Proposition 2.3.6. *Proper functors are stable under base change.*

Proof. We ought to prove that given $u : A \to B$ a proper functor, and any map $B' \to B$ the functor u' :

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ B' \arrow[r, "b"] & B \end{tikzcd}$$

is proper. Using the equivalent formulation (c) we need to see that for any $\Delta^1 \to B'$, forming the pullbacks

$$\begin{tikzcd} A''_1 \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A'' \arrow[r, "a'"] \arrow[d, "c"] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "u'"] \\ e \arrow[r, "1"] & \Delta^1 \arrow[r, "b'"] & B' \end{tikzcd}$$

the map v is coaspheric. By the pullback pasting lemma, the square

$$\begin{tikzcd} A'' \arrow[r, "aa'"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\ \Delta^1 \arrow[r, "bb'"] & B \end{tikzcd}$$

is again cartesian, as such pulling back c along $e \xrightarrow{1} \Delta_1$ since u is proper gives that v is coaspheric. ∎

We want to state some more stabilities properties of proper functors, for instance regarding composition, but these require some more work.

Proposition 2.3.7. *Let $u : A \to B$ be a proper functor. The following are equivalent :*

(a) the fibers of u are aspheric ;

(b) u is aspheric ;

(c) u is universally aspheric ;

(d) u is universally in $\mathcal{W}$.

Proof. Since u is proper, $i_b : A_b \to A/b$ is coaspheric, in particular it is a weak equivalence as such we get the equivalence of (a) and (b). Having aspheric fibers and being proper are both stable under base change, hence (a) and (c) are also equivalent. Further since an aspheric functor is a weak equivalent, then (c) implies (d) and it's clear that (d) implies (a). ■

Lemma 2.3.8. *Given $u : A \to B$ a morphism of* $\mathbf{Cat}$*, if u is a proper, then the induced morphism $a\backslash A \to u(a)\backslash B$ is also proper.*

Proof. Denote $a\backslash u$ the induced map $a\backslash A \to u(a)\backslash B$, and consider an object $(a', p) \in a\backslash A$, we show the fibers of the induced morphism $(a,' p)\backslash\,(a\backslash A) \to (u(a'), u(p))\backslash\,(u(a)\backslash B)$ are aspheric. We have the following diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (a', p)\backslash(a\backslash A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & a\backslash A \arrow[d] \\ * \arrow[r] & (u(a'), u(p))\backslash(u(a)\backslash B) \arrow[r] & u(a)\backslash A \end{tikzcd}$$

hence it follows that the fibers of the middle vertical map are aspheric since that of the rightmost map are aspheric. ■

Lemma 2.3.9. *Let $u : A \to B$ be a proper morphism and $a \in A$ and $b = u(a)$, then the induced morphism $a\backslash A \to b\backslash B$ is universally in $\mathcal{W}$.*

Proof. By 2.3.4 (b) the fibers of the map $a\backslash A \to b\backslash B$ are aspheric, and by 2.3.8 it is proper, the proposition follows from 2.3.7. ■

Proposition 2.3.10. *The class of proper functors is stable under composition.*

Proof.

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & a\backslash A \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ B' \arrow[r] \arrow[d] & u(a)\backslash B \arrow[r] \arrow[d] & B \arrow[d, "v"] \\ \{(c', f)\} \arrow[r, hook] & vu(a)\backslash C \arrow[r] & C \end{tikzcd}$$

Form the above diagram where the leftmost squares are cartesian and (c', f) is any object of $vu(a)\backslash C$. By properness of v it follows from 2.3.4 (b) that B' is aspheric. Since u is proper, it follows from 2.3.9 that the morphism $a\backslash A \to u(a)\backslash B$ is universally in $\mathcal{W}$ whence $A' \to B'$ is a weak equivalence. It follows from 2-out-of-3 that A' is aspheric. It follows from 2.3.4 (b) that vu is proper. ■

Proposition 2.3.11. *Let C be a category, and u a morphism of* $\mathbf{Cat}/C$

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

Given a morphism $C' \to C$ of $\mathbf{Cat}$, we may form the change of base diagram in which every square is cartesian

$$\begin{tikzcd}
A' \arrow[rr] \arrow[dr, "u'"] \arrow[dd] & & A \arrow[dr, "u"] \arrow[dd] & \\
& B' \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
C' \arrow[rr] & & C &
\end{tikzcd}$$

Assume that the morphism $C' \to C$ is proper. If for all $c \in C$ the morphism $c\backslash u : c\backslash A \to c\backslash B$ is a weak equivalence , then the same is true for u' over C'.

Proof. Let $c' \in C'$ and denote c its image in C, we get an induced diagram

$$\begin{tikzcd}
c'\backslash A' \arrow[rr] \arrow[dr] \arrow[dd] & & c\backslash A \arrow[dr] \arrow[dd] & \\
& c'\backslash B' \arrow[rr] \arrow[dl] & & c\backslash B \arrow[dl] \\
c'\backslash C' \arrow[rr] & & c\backslash C &
\end{tikzcd}$$

where the two squares containing the map $c'\backslash C' \to c\backslash C$ are cartesian because the corresponding squares in the previous diagram were cartesian. It follows from the pullback pasting lemma that the third square is a cartesian as well.

Since $C' \to C$ is proper it follows from 2.3.9 that $c'\backslash C' \to c\backslash C$ is universally in $\mathcal{W}$ whence $c'\backslash A' \to c\backslash A$ and $c'\backslash B' \to c\backslash B$ are weak equivalences as well. If u is a weak equivalence colocally over C (i.e. $c\backslash u$ is a weak equivalence for any $c \in C$) it follows by 2-out-of-3 that $c'\backslash A' \to c'\backslash B'$ is as well. ■

Corollary 2.3.12. *The pullback of a coaspheric morphism along a proper morphism is again coaspheric.*

Proof. It follows from the previous proposition applying it to the case where $C = B$ and the map $A \to C$ is also u and the map $B \to C$ is 1_B. ■

We can now state a new theorem which gives another equivalent characterisation of proper functors.

Theorem 2.3.13. *Let $u : A \to B$ be a morphism of $\mathbf{Cat}$. The following are equivalent :*

(a) u *is proper ;*

(b) For any diagram of cartesian squares

$$\begin{tikzcd}
A'' \arrow[r, "w"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\
B'' \arrow[r, "v"] & B' \arrow[r] & B
\end{tikzcd}$$

if v is coaspheric, then so is w.

Spelling out in words, this means that u being proper is equivalent to the statement that coaspheric morphisms are stable under pullback along u and that this remains true after any base change.

Proof. We first prove (a) implies (b). If u is proper, then u' is as well by 2.3.6. The result follows because it is precisely the content of 2.3.12.

Let's turn to (b) implies (a). Given $b \in B$, we can form the diagram of cartesian squares

$$\begin{tikzcd} A_b \arrow[r, "i_b"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A/b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ e \arrow[r] & B/b \arrow[r] & B \end{tikzcd}$$

The bottom left functor $\{b\} \to B/b$ is given by $* \mapsto (b, 1_b)$, i.e. its image is the terminal object B/b, this in particular implies it has a left adjoint. Having a left adjoint, it is coaspheric by the dual of 2.1.11(d). The hypothesis implies i_b is coaspheric as desired. ■

2.3.14 — Let $u : A \to B$ be a functor. Recall the notation of theorem 2.2.9, we have an adjunction

$$u_! : \mathbf{Hot}(A) \rightleftarrows \mathbf{Hot}(B) : u^*.$$

We make the abuse of keeping the same notations for the functors

$$u_! \colon \underline{\mathrm{Hom}}(A, \mathbf{Cat}) \longrightarrow \underline{\mathrm{Hom}}(B, \mathbf{Cat}), \qquad u^* \colon \underline{\mathrm{Hom}}(B, \mathbf{Cat}) \longrightarrow \underline{\mathrm{Hom}}(A, \mathbf{Cat})$$

$$F \longmapsto \left(b \mapsto \left(\int_A F \right) /b \right) \qquad G \longmapsto Gu$$

In particular $u_! = \Theta_B \circ \mathbf{Cat}/u \circ \Theta'_A$, $u^* = \underline{\mathrm{Hom}}(u, 1_{\mathbf{Cat}})$. This way we have the following commutative squares

$$\begin{tikzcd} \underline{\mathrm{Hom}}(A, \mathbf{Cat}) \arrow[r, "u_!"'] \arrow[d, "\gamma_A"'] & \underline{\mathrm{Hom}}(B, \mathbf{Cat}) \arrow[d, "\gamma_B"'] \\ \mathbf{Hot}(A) \arrow[r, "u_!"'] & \mathbf{Hot}(B) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{Hom}}(B, \mathbf{Cat}) \arrow[r, "u^*"'] \arrow[d, "\gamma_B"] & \underline{\mathrm{Hom}}(A, \mathbf{Cat}) \arrow[d, "\gamma_A"] \\ \mathbf{Hot}(B) \arrow[r, "u^*"'] & \mathbf{Hot}(A) \end{tikzcd}$$

where the vertical maps are the localisation functors. One needs to be careful, for $u_!$ to exists, we need $\mathcal{W}$ to be a basic localiser, not simply a weak one.

2.3.15 — Let $w : J \to I$ be a morphism of $\mathbf{Cat}$ and $F : I \to \mathbf{Cat}$ be a functor. From this data, we can obtain a functor

$$\begin{aligned} \tilde{w} \colon \int_J Fw &\longrightarrow \int_I F \\ (j, a) &\longmapsto (w(j), a) \\ (k, f) &\longmapsto (w(k), f) \end{aligned}$$

Lemma 2.3.16. *Let $w : J \to I$ be a functor, and $F : I \to \mathbf{Cat}$ be a functor. The square*

$$\begin{tikzcd} \int_J Fw \arrow[r, "\tilde{w}"] \arrow[d, "\theta_{Fw}"'] & \int_I F \arrow[d, "\theta_F"] \\ J \arrow[r, "w"'] & I \end{tikzcd}$$

is cartesian.

Proof. Suppose we're given the solid commutative diagram

$$\begin{tikzcd}
C \arrow[dr, dashed] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& \int_J Fw \arrow[r, "\tilde{w}"] \arrow[d, "\theta_{Fw}"'] & \int_I F \arrow[d, "\theta_F"] \\
& J \arrow[r, "w"'] & I
\end{tikzcd}$$

where for c an object of C we denote $q(c) := (i_c, a_c)$ and given $k : c \to c'$ we denote

$$q(k) := (q(k)_1 : a_c \to a_{c'}, q(k)_2 : F(q(k)_1)(a_c) \to a_{c'}).$$

We ought to find a filler for the diagonal map and show that such a filler is unique. We define a functor

$$\begin{aligned} G\colon C &\longrightarrow \int_J Fw \\ c &\longmapsto (p(c), a_c) \\ (k : c \to c') &\longmapsto (p(k), q(k)_2) \end{aligned}$$

G is well defined, indeed the non immediate point is that for $k : c \to c'$, $G(k)$ is indeed a morphism of $\int_J Fw$ but this is clear since $p(k) : p(c) \to p(c')$ as desired and $q(k)_2 : F(q(k)_1)(a_c) \to a_{c'}$ but

$$\begin{aligned} F(q(k)_1)(a_c) &:= F(\theta_F q(k))(a_c) \\ &= F(wp(k))(a_c) \quad \text{commutativity of the diagram.} \end{aligned}$$

The commutativity of the two triangles i.e. $\tilde{w}G = q$ and $\theta_{Fw}G = p$ is an easy and direct computation. As such we have the existence of the filler. Now assume another such G' is given. Then one must have $\theta_{Fw}G'(c) = p(c)$ as such on objects G and G' agree on the first coordinates. Furthermore, since $\tilde{w}G'(c) = q(c)$ we must have $(wG'(c)_1, G'(c)_2) = q(c) = (i_c, a_c)$ hence G and G' also agree on their second coordinate, hence they have the same action on objects. Similar considerations leads us to conclude we indeed must have $G' = G$, whence it follows the square is cartesian. ■

Lemma 2.3.17. *Let $u : A \to B$ be a morphism of $\mathbf{Cat}$, and $F : A \to \mathbf{Cat}$ be a functor. Given $b \in B$ denote $F|A/b$ the composition $A/b \to A \xrightarrow{F} \mathbf{Cat}$ where the first arrow is the canonical projection. Then we have a canonical isomorphism*

$$\left(\int F\right)/b \cong \int F|A/b.$$

Proof. By 2.3.16, the diagram

$$\begin{tikzcd}
\int F|A/b \arrow[r] \arrow[d] & \int F \arrow[d] \\
A/b \arrow[r] & A
\end{tikzcd}$$

is cartesian, whence the large rectangle is also cartesian

$$\begin{tikzcd}
\int F|A/b \arrow[r] \arrow[d] & \int F \arrow[d] \\
A/b \arrow[r] \arrow[d] & A \arrow[d, "u"] \\
B/b \arrow[r] & B
\end{tikzcd}$$

Since

$$\begin{tikzcd}
\left(\int F\right)/b \arrow[r] \arrow[d] & \int F \arrow[d, "u"] \\
B/b \arrow[r] & B
\end{tikzcd}$$

is also cartesian the assertion follows. ■

Proposition 2.3.18. *Let $w : J \to I$ be a morphism of* $\mathbf{Cat}$*. The following are equivalent :*

(a) w is coaspheric ;

(b) For any $F : I \to \mathbf{Cat}$, the morphism $\tilde{w} : \int Fw \to \int F$ is coaspheric ;

(c) For any $F : I \to \mathbf{Cat}$, the morphism $\tilde{w} : \int Fw \to \int F$ is a weak equivalence.

Proof. By lemma 2.3.16 we have the cartesian square

$$\begin{tikzcd}
\int_J Fw \arrow[r, "\tilde{w}"] \arrow[d, "\theta_{Fw}"'] & \int_I F \arrow[d, "\theta_F"] \\
J \arrow[r, "w"'] & I
\end{tikzcd}$$

where θ_F is the canonical projection, which we've already seen in 2.3.2 is proper. As such it follows from 2.3.12 that $\tilde{w}$ is proper if w is, hence (a) $\implies$ (b). The implication (b) $\implies$ (c) follows from 2.1.22. Finally, assume (c) and let $i \in I$ be given. Define $F : I \to \mathbf{Cat}$ to be the composite

$$I \xrightarrow{\mathrm{Hom}(i,-)} \mathbf{Set} \hookrightarrow \mathbf{Cat}$$

where the last inclusion maps a set to the discrete category given by that set. Then we have

$$\int_I F \simeq i\backslash I, \quad \int_J Fw \simeq i\backslash J$$

and with those identifications, the map $\tilde{w}$ gets identified with $i\backslash w$. By (c) it follows that for each $i \in I$ these $i\backslash w$ are weak equivalences, hence by 2.1.23 w is coaspheric. ■

2.3.19 — Let

$$\mathcal{D} = \quad \begin{tikzcd}
A' \arrow[r, "w"] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

be a cartesian square in $\mathbf{Cat}$. By what precedes, we obtain a diagram of cartesian squares

$$\begin{tikzcd}
\int Fw \arrow[r, "\tilde{w}"] \arrow[d, "\theta_{Fw}"] \arrow[dr, phantom, "\lrcorner", very near start] & \int F \arrow[d, "\theta_F"] \\
A' \arrow[r, "w"] \arrow[d, "u'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "u"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

For any object $b' \in B'$, the morphism $\tilde{w}$ induces a functor

$$\left(\int Fw\right)/b' \longrightarrow \left(\int F\right)/v(b').$$

Further, recalling the notations from 2.3.14 we remark the following equalities :

$$\left(\int Fw\right)/b' = ((u'_! w^*)(F))(b'), \quad \left(\int F\right)/v(b') = ((v^* u_!)(F))(b').$$

As such, this gives us a morphism

$$\kappa_{\mathcal{D}} : u'_! w^* \longrightarrow v^* u_!$$

in the category $\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A, \mathbf{Cat}), \underline{\mathrm{Hom}}(B', \mathbf{Cat}))$. We call $\kappa_{\mathcal{D}}$ the *change of base morphism associated to* $\mathcal{D}$. The point is that this morphism will allow us to detect proper functors. This is the content of the next two propositions and theorems.

Proposition 2.3.20. *Let*

$$\mathcal{D} = \quad \begin{array}{ccc} A' & \xrightarrow{w} & A \\ \downarrow{\scriptstyle u'} & \lrcorner & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{v} & B \end{array}$$

be a cartesian square in which u is proper. Then the change of base morphism $\kappa_{\mathcal{D}} : u'_! w^* \to v^* u_!$ is coaspheric argument by argument, i.e. for any $F : A \to \mathbf{Cat}$ and $b' \in B'$

$$\kappa_{\mathcal{D}} : u'_! w^*(F)(b') \to v^* u_!(F)(b')$$

is coaspheric (and in particular, a weak equivalence).

Proof. Since u is proper, by 2.3.6 u' is as well. Hence for any $b' \in B'$, in the following commutative square the vertical arrows are coaspheric, and the top horizontal arrow is an isomorphism since $\mathcal{D}$ is cartesian

$$\begin{array}{ccc} A'_{b'} & \xrightarrow{\cong} & A_{v(b')} \\ \downarrow{\scriptstyle i_{b'}} & & \downarrow \\ A'/b' & \longrightarrow & A/v(b') \end{array}$$

We deduce from the square above the following commutative diagram

$$\begin{array}{ccc} \int Fw|A'_{b'} & \xrightarrow{\cong} & \int F|A_{v(b')} \\ \downarrow & & \downarrow \\ \int Fw|A'/b' & \longrightarrow & \int F|A/v(b') \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \left(\int Fw\right)/b' & \longrightarrow & \left(\int F\right)/v(b') \\ \| & & \| \\ u'_! w^*(F)(b') & \longrightarrow & v^* u_!(F)(b) \end{array}$$

where the middle vertical part is due to 2.3.17 and the bottom part stems from the definitions in 2.3.14. For clarity, the top left arrow for instance is obtained as follows : we're given $i'_b : A'_{b'} \to A'/b'$ and $A'/b' \xrightarrow{\pi} A' \xrightarrow{w} A \xrightarrow{F} \mathbf{Cat}$, then by 2.3.15 we can obtain the desired functor

$$\tilde{i}_{b'} : \int Fw\pi i_b := \int Fw|A'_{b'} \longrightarrow \int Fw\pi := \int Fw|A'/b'$$

where the first equality holds because the diagram

$$\begin{array}{ccc} A'_{b'} & \xrightarrow{\pi} & A' \\ {\scriptstyle i_{b'}}\downarrow & & \| \\ A'/b' & \xrightarrow{\pi} & A' \end{array}$$

commutes (the vertical maps are the canonical projections, we make the abuse of denoting both π). The other 3 maps in the diagram above are obtained similarly, and by 2.3.18 it follows that the vertical maps are coaspheric. Finally it follows from 2.1.11 (g, dual statement) that in the diagram below

$$\begin{tikzcd}
\int Fw|A'_{b'} \arrow[r, "\cong"] \arrow[d] & \int F|A_{v(b')} \arrow[d] \\
u'_!w^*(F)(b') \arrow[r] & v^*u_!(F)(b')
\end{tikzcd}$$

since the composite of the top and right arrow is coaspheric, and the left arrow is coaspheric, the the bottom one must be too. ■

2.3.21 — Let $u : A \to B$ be a morphism of $\mathbf{Cat}$, denote $\int_B A/b$ the integral of the functor

$$\begin{aligned} F_u \colon B &\longrightarrow \mathbf{Cat} \\ b &\longmapsto A/b \\ (f : b \to b') &\longmapsto F_u(f) \end{aligned}$$

where

$$\begin{aligned} F_u(f) \colon A/b &\longrightarrow A/b' \\ (a,p) &\longmapsto (a, fp) \\ g : (a,p) \to (a',p') &\longmapsto g : (a, fp) \to (a', fp'). \end{aligned}$$

We can describe this category as follows : Its objects are given by triples (b, a, f) with $b \in B$, $a \in A$ and $f : u(a) \to B$ an arrow of B. Morphisms from (b, a, f) to (b', a', f') are given by pairs $(h : b \to b', g : a :\to a')$ such that the diagram

$$\begin{tikzcd}
u(a) \arrow[r, "u(g)"] \arrow[d, "f"'] & u(a') \arrow[d, "f'"] \\
b \arrow[r, "h"] & b'
\end{tikzcd}$$

commutes. One can define a pair of functors

$$\begin{aligned} i_u \colon A &\longrightarrow \int_B A/b \\ a &\longmapsto (u(a), a, 1_{u(a)}) \\ g &\longmapsto (u(g), g) \end{aligned} \qquad\qquad \begin{aligned} r_u \colon \int_B A/b &\longrightarrow A \\ (b, a, f) &\longmapsto a \\ (h, g) &\longmapsto g. \end{aligned}$$

One readily sees that $r_u i_u = 1_A$ and we can define a natural transformation $\alpha_u : i_u r_u \to 1_{\int_B A/b}$ by taking

$$(\alpha_u)_{(b,a,f)} := (f, 1_a) : (u(a), a, 1_{u(a)}) \to (b, a, f).$$

Taking $\mathcal{I} = \{(\Delta^1, d_0, d_1)\}$ as a set of segment, it's clear that a natural transformation $f \Rightarrow g$ is equivalent to a $\mathcal{I}$-homotopy from f to g. Furthermore since Δ^1 has a final object, by 2.1.11(a), p_{Δ^1} is universally in $\mathcal{W}$. Hence applying 1.3.2 it follows $i_u r_u$ is in $\mathcal{W}$. Hence by weak saturation i_u and r_u are weak equivalences and one can notice that u factors as $\theta_{F_u} i_u$ where θ_{F_u} denotes the canonical projection $\int F_u \to B$. Further, given a commutative square of $\mathbf{Cat}$:

$$\begin{tikzcd}
A' \arrow[r, "w"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B' \arrow[r, "v"'] & B
\end{tikzcd}$$

one can define a functor

$$\begin{aligned} s\colon \int_{B'} A'/b' &\longrightarrow \int_B A/b \\ (b', a', f') &\longmapsto (v(b'), w(a'), v(f')) \\ (h, g) &\longmapsto (v(h), w(g)). \end{aligned}$$

The commutativity of the given squares then directly implies the commutativity of the following squares :

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "i_{u'}"] & A \arrow[d, "i_u"] \\ \int_{B'} A'/b' \arrow[r, "s"'] & \int_B A/b \end{tikzcd} \qquad \begin{tikzcd} \int_{B'} A'/b' \arrow[r, "s"] \arrow[d, "r_{u'}"] & \int_B A/b \arrow[d, "r_u"] \\ A' \arrow[r, "w"'] & A \end{tikzcd} \qquad \begin{tikzcd} \int_{B'} A'/b' \arrow[r, "s"] \arrow[d, "\theta_{F_{u'}}"] & \int_B A/b \arrow[d, "\theta_{F_u}"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

as a simple calculation shows.

For the remainder of this section, assume $\mathcal{W}$ is a basic localiser.

Theorem 2.3.22. *Let $u : A \to B$ be a functor, then the following are equivalent :*

(a) u is proper ;

(b) for any diagram of cartesian squares

$$\begin{tikzcd} A'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}'"] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}"] & A \arrow[d, "u"] \\ B'' \arrow[r] & B' \arrow[r] & B \end{tikzcd}$$

the change of base morphism $\kappa_{\mathcal{D}'}$ associated to the leftmost square is a weak equivalence argument by argument.

Proof. Let us first show the easy part, that (a) implies (b). If u is proper, so is u' by 2.3.6. It then follows by 2.3.20 that the base change morphism associated to the leftmost cartesian square is a weak equivalence argument by argument.

Now, let's prove that (b) implies (a). Given a diagram of cartesian squares,

$$\begin{tikzcd} A'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}'"] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}"] & A \arrow[d, "u"] \\ B'' \arrow[r] & B' \arrow[r] & B \end{tikzcd}$$

by 2.3.13 it suffices to show that if $B'' \to B'$ is coaspheric, then so is $A'' \to A'$. Hence, assuming (b), we only have to prove that if we're given a cartesian square

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "u'"] \arrow[dr, phantom, "\mathcal{D}"] & A \arrow[d, "u"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

where the base change morphism $\kappa_{\mathcal{D}}$ is a weak equivalence argument by argument and v is coaspheric, then so is w. By assumption, for any $F : A \to \mathbf{Cat}$ and any $b' \in B'$, the functor

$$\kappa_{\mathcal{D},F} : (u'_! w^*(F))(b') \to (v^* u_!)(F)(b')$$

is a weak equivalence. Hence by 2.1.25 it follows that the functor

$$\int \kappa_{\mathcal{D},F} : \int_{B'} u'_! w^*(F) \longrightarrow \int_{B'} v^* u_!(F)$$

is a weak equivalence. Notice that we've assumed v to be coaspheric, and by 2.3.18 applied to $u_!(F)$ this implies that

$$\tilde{v} : \int u_!(F)v := \int v^* u_!(F) \longrightarrow \int u_!(F)$$

is also a weak equivalence. Weak equivalences being stable under composition we deduce that

$$\tilde{v} \int \kappa_{\mathcal{D},F} : \int u'_! w^* \longrightarrow \int u_!(F)$$

is a weak equivalence. Now, recall 2.3.19, from the given cartesian square, we can obtain the cartesian square

$$\begin{tikzcd} \int Fw \arrow[r, "\tilde{w}"] \arrow[d, "u'\theta_{Fw}"'] & \int F \arrow[d, "u\theta_F"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

By 2.3.21, this gives a square

$$\begin{tikzcd} \int Fw \arrow[r, "\tilde{w}"] \arrow[d, "i_{u'\theta_{Fw}}"'] & \int F \arrow[d, "i_{u\theta_F}"] \\ \int_{B'} \left(\int Fw\right)/b' \arrow[r, "s"'] & \int_B \left(\int F\right)/b \end{tikzcd}$$

and a careful observation of the definitions leads to noticing that

$$\int_{B'} \left(\int Fw\right)/b' := \int u'_! w^*(F), \quad \int_B \left(\int F\right)/b := \int u_!(F), \quad s = \tilde{v} \int \kappa_{\mathcal{D},F}.$$

Hence we have a square

$$\begin{tikzcd} \int Fw \arrow[r, "\tilde{w}"] \arrow[d, "i_{u'\theta_{Fw}}"'] & \int F \arrow[d, "i_{u\theta_F}"] \\ \int u'_! w^*(F) \arrow[r, "\tilde{v} \int \kappa_{\mathcal{D},F}"'] & \int u_!(F) \end{tikzcd}$$

where the vertical arrows are weak equivalences (cf. 2.3.21) and the bottom arrow is too as shown above. We conclude $\tilde{w}$ is a weak equivalence, and by 2.3.18 it follows w is coaspheric and we're done. ■

For the remainder of this section, we further assume $\mathcal{W}$ to be strongly saturared.

2.3.23 — Keeping with the notations introduced in the previous paragraphs, given a cartesian square in $\mathbf{Cat}$

$$\mathcal{D} = \quad \begin{tikzcd} A' \arrow[r, "w"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

we obtain a commutative square (not necessarily cartesian)

$$\begin{tikzcd} \mathbf{Hot}(A') & \mathbf{Hot}(A) \arrow[l, "w^*"'] \\ \mathbf{Hot}(B') \arrow[u, "u'^*"] & \mathbf{Hot}(B) \arrow[l, "v^*"] \arrow[u, "u^*"'] \end{tikzcd}$$

Recall again that both u^* and u'^* have left adjoints

$$u_! : \mathbf{Hot}(A) \to \mathbf{Hot}(B), \quad u'_! : \mathbf{Hot}(A') \to \mathbf{Hot}(B').$$

As such we have the unit and counit $1_{\mathbf{Hot}(A)} \to u^* u_!$ and $u'_! u'^* \to 1_{\mathbf{Hot}(B')}$. We call the *homotopical base change morphism associated to* $\mathcal{D}$ the morphism $c_{\mathcal{D}} : u'_! w^* \to v^* u_!$ in $\underline{\mathrm{Hom}}(\mathbf{Hot}(A), \mathbf{Hot}(B'))$ obtained as

$$u'_! w^* \longrightarrow u'_! w^* u^* u_! = u'_! u'^* v^* u_! \longrightarrow v^* u_!$$

where both arrows are given by the unit and counit respectively. In fact, $c_{\mathcal{D}}$ is induced by $\kappa_{\mathcal{D}}$ in the sense that for any functor $F : A \to \mathbf{Cat}$

$$c_{\mathcal{D},\gamma_A(F)} = \gamma_{B'}(\kappa_{\mathcal{D},F})$$

where the $\gamma_\bullet$ denotes the localisations functors.

Theorem 2.3.24. *Given $u : A \to B$ a morphism of* $\mathbf{Cat}$*, the following are equivalent :*

(a) u is proper ;

(b) for any diagram of cartesian squares

$$\begin{tikzcd} A'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}'"] & A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \mathcal{D}"] & A \arrow[d, "u"] \\ B'' \arrow[r] & B' \arrow[r] & B \end{tikzcd}$$

the homotopical change of base morphism $c_{\mathcal{D}'}$ associated to the leftmost square is an isomorphism.

Proof. This follows from 2.3.22, the strong saturation of $\mathcal{W}$ and the fact that$c_{\mathcal{D}'}$ is induced by $\kappa_{\mathcal{D}'}$. Indeed if u is proper, by 2.3.22 $\kappa_{\mathcal{D}'}$ is a weak equivalence argument by argument and as such $c_{\mathcal{D}'}$ is an isomorphism, and since $\mathcal{W}$ is strongly saturated, if $c_{\mathcal{D}'}$ is an isomorphism then $\kappa_{\mathcal{D}'}$ was a weak equivalence argument by argument, the theorem follows. ■

3 Elementary 2-category theory

Before being able to formulate some of the theory we developed for $\mathbf{Cat}$ to 2-$\mathbf{Cat}$ we need to introduce the basics of the theory of 2-categories. The context we will be working in is that of strict (2,2)-categories, i.e. 2-categories where there are non invertible morphisms in every dimension.

3.1 Fundamentals

2-categories, 2-functors and transformations

Definition 3.1.1. A 2-category $\mathcal{A}$ is the data of :

- A collection of objects denoted $\mathrm{Ob}(\mathcal{A})$;
- For each pair of objects (a, a') of $\mathrm{Ob}(\mathcal{A})$, a category $\underline{\mathrm{Hom}}_{\mathcal{A}}$. We call the objects of $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')$ the 1-cells, and the morphisms the 2-cells, composition of 2-cells is denoted by juxtaposition ;
- For each object $a \in \mathrm{Ob}(\mathcal{A})$ a unit functor :

$$1_a : e \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a) ;$$

- For each triple of objects (a, a', a'') of $\mathrm{Ob}(\mathcal{A})$, a composition functor :

$$\begin{aligned} c_{a'',a',a} : \underline{\mathrm{Hom}}_{\mathcal{A}}(a', a'') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') &\longrightarrow \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a'') \\ (g, f) &\longmapsto gf \\ (\beta, \alpha) &\longmapsto \beta \circ \alpha. \end{aligned}$$

This data is subject to the following coherence condition :

- For any quadruple of objects (a, a', a'', a''') of $\mathrm{Ob}(\mathcal{A})$ the square

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{A}}(a'', a''') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a', a'') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \arrow[r, "1\times c_{a'',a',a}"] \arrow[d, "c_{a''',a'',a'}\times 1"] & \underline{\mathrm{Hom}}_{\mathcal{A}}(a'', a''') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a'') \arrow[d, "c_{a''',a'',a}"] \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(a', a''') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \arrow[r, "c_{a''',a',a}"] & \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a''')
\end{tikzcd}$$

  commutative ;

- For any pair (a, a') of objects of $\mathcal{A}$ the following triangles :

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \times e \arrow[d, "1\times 1_a"'] \arrow[dr, "\simeq"] & \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a) \arrow[r, "c_{a',a,a}"'] & \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')
\end{tikzcd}$$

  and

$$\begin{tikzcd}
e \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \arrow[d, "1_a\times 1"'] \arrow[dr, "\simeq"] & \\
\underline{\mathrm{Hom}}_{\mathcal{A}}(a', a') \times \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a') \arrow[r, "c_{a',a',a}"'] & \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')
\end{tikzcd}$$

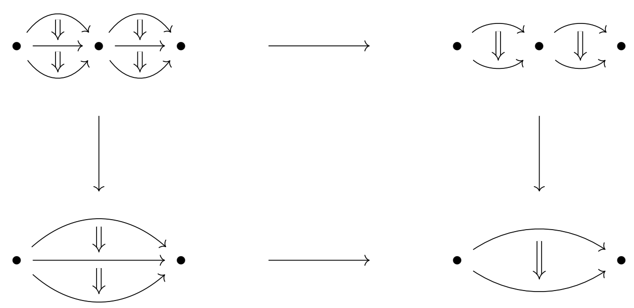
  are commutative.

Remark 3.1.2. A 2-category is a $\mathbf{Cat}$-enriched category.

3.1.3 — In a 2-category, there are two ways to compose 2-cells : Given $\alpha : f \Rightarrow f', \beta : f' \Rightarrow f''$ morphisms in the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')$, we have the vertical composition which is the composition internal to the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')$, we denote it $\beta\alpha : f \Rightarrow f''$. Furthermore, if now we're given two 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g \to g'$ such that the target of f, f' coincides with the source of g, g' we have the composition denoted by $\beta \circ \alpha$ which is the image of (β, α) by the action on morphism of the composition functor, we call this one the horizontal composition. Functoriality of composition is express by the exchange law

$$(\alpha'\alpha) \circ (\beta'\beta) = (\beta \circ \alpha)(\beta' \circ \alpha')$$

which can also be expressed with requiring the diagram



to be commutative.

Notation. Given a 1-cell $f \in \mathrm{Hom}_{\mathbb{A}}(a, a')$ we denote 1_f its identity, but we will often also write f for 1_f meaning that for a given 2-cell α the notation $f \circ \alpha$ (resp. $\alpha \circ f$) stands for $1_f \circ \alpha$ (resp. $\alpha \circ 1_f$). We use e to denote the 2-category with a unique object $*$, a unique 1-cell 1_* and a unique 2-cell 1_{1_*}.

Definition 3.1.4. A **strict 2-functor** $u : \mathcal{A} \to \mathcal{B}$ between 2-categories $\mathcal{A}$ and $\mathcal{B}$ is the data of :

- For any object a of $\mathcal{A}$ an object $u(a)$ of $\mathcal{B}$;
- For any 1-cell $f : a \to a'$ of $\mathcal{A}$, a 1-cell $u(f) : u(a) \to u(a')$ of $\mathcal{B}$;
- For any 2-cell $\alpha : f \Rightarrow f'$ of $\mathcal{A}$, a 2-cell $u(\alpha) : u(f) \Rightarrow u(f')$ of $\mathcal{B}$;

This data is subject to the following conditions :

(St1) For any pair of 1-cells f, f' of $\mathcal{A}$ such that the composition $f'f$ makes sense, we require,

$$u(f'f) = u(f')u(f) \ \ ;$$

(St2) For any object a,

$$1_{u(a)} = u(1_a) \ \ ;$$

(F1) For any 1-cell f of $\mathcal{A}$,

$$1_{u(f)} = u(1_f) \ \ ;$$

(F2) For any pair α, α' of 2-cells of $\mathcal{A}$ such that the composite $\alpha'\alpha$ is defined,

$$u(\alpha'\alpha) = u(\alpha')u(\alpha) \ \ ;$$

(sNat) For any pair of 2-cells $\alpha : f \Rightarrow g$ and $\alpha' : f' \Rightarrow g'$ such that $\alpha' \circ \alpha$ is defined, we require

$$u(\alpha' \circ \alpha) = u(\alpha') \circ u(\alpha) \ \ ;$$

Remark 3.1.5. The conditions (Sti) are the reason the 2-functor is called strict, hence the *St*. The conditions (Fi) record the functoriality of u with respect to 2-cells, and (sNat) the strict naturality of the composition 2-cells. In fact, we emphasize strictness in our definition, because there is also a very relevant notion of lax 2-functor. It is essentially a weakening of the definition of 2-functors to make better use of the data of the 2-cells inherent to 2-category theory. We no longer require a 2-functor to strictly respect composition and identities of 1-cells, it respects it only up to a 2-cell.

3.1.6 — Given $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ two strict 2-functors, we define their composite $vu : \mathcal{A} \to \mathcal{B}$ as follows :

- on objects $vu(a) := v(u(a))$;
- on 1-cells $vu(f) = v(u(f))$;
- on 2-cells $vu(\alpha) = v(u(\alpha))$;

One easily checks this defines a strict 2-functor and that composition is associative. Further, we have the existence of units, i.e. for any 2-category $\mathcal{A}$ a strict 2-functor $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ such that for any other strict 2-functor such that the composite makes sense $1_{\mathcal{A}}u = u$, $u1_{\mathcal{A}} = u$.

This data can already be arranged into a category.

Definition 3.1.7. We denote 2-$\mathbf{Cat}$ the category whose objects are 2-categories and whose morphisms are stricts 2-functors.

Remark 3.1.8. Recall the embedding $\mathbf{Set} \hookrightarrow \mathbf{Cat}$ given by taking a set to the discrete category it defines. This induces a natural embedding $\mathbf{Cat} \hookrightarrow 2$-$\mathbf{Cat}$ mapping a category M to the 2-category with the same objects and whose category of morphisms between any two objects a, b is the discrete category associated to the set $\mathrm{Hom}_M(a, b)$.

Given strict 2-functors $u, v : \mathcal{A} \to \mathcal{B}$ one wants to define a transformation $u \Rightarrow v$ to generalise natural transformations of functors.

Definition 3.1.9. Let u, v be a pair of strict 2-functors $\mathcal{A} \to \mathcal{B}$, a **transformation** σ from u to v, denoted $\sigma : u \Rightarrow v$ is the following data.

- For every object $a \in \mathcal{A}$, a 1-cell of $\mathcal{B}$

$$\sigma_a : u(a) \to v(a) \ ;$$

- For any 1-cell $f : a \to a'$ of $\mathcal{A}$, a 2-cell of $\mathcal{B}$

$$\sigma_f : \sigma_{a'} u(f) \Rightarrow v(f)\sigma_a,$$

  this is the diagram to keep in mind

$$\begin{tikzcd} u(a) \arrow[r, "u(f)"] \arrow[d, "\sigma_a"'] & u(a') \arrow[d, "\sigma_{a'}"] \arrow[dl, phantom, "\swarrow \sigma_f"] \\ v(a) \arrow[r, "v(f)"'] & v(a') \end{tikzcd}$$

This data is subject to the following coherence condition :

- For any pair $f, g : a \to a'$ of 1-cells of $\mathcal{A}$, and for every 2-cell $\alpha : f \Rightarrow g$ of $\mathcal{A}$, the diagram

$$\begin{tikzcd} \sigma_{a'} u(f) \arrow[r, Rightarrow, "\sigma_f"] \arrow[d, Rightarrow, "\sigma_{a'} \circ u(\alpha)"'] & v(f)\sigma_a \arrow[d, Rightarrow, "v(\alpha)\circ\sigma_a"] \\ \sigma_{a'} u(g) \arrow[r, Rightarrow, "\sigma_g"'] & v(g)\sigma_a \end{tikzcd}$$

  is commutative ;

- For any pair $f : a \to a'$, $f' : a' \to a''$ of composable 1-cells of $\mathcal{A}$, the following diagram

$$\begin{tikzcd} \sigma_{a''} u(f) u(f') \arrow[r, Rightarrow, "\sigma_{f'} \circ u(f)"] \arrow[d, equal] & v(f')\sigma_{a'} u(f) \arrow[r, Rightarrow, "v(f')\circ\sigma_f"] & v(f')v(f)\sigma_a \arrow[d, equal] \\ \sigma_{a''} u(f'f) \arrow[rr, Rightarrow, "\sigma_{f'f}"'] & & v(f'f)\sigma_a \end{tikzcd}$$

  is commutative ;

- For any object $a \in \mathcal{A}$ the diagram

$$\begin{tikzcd} \sigma_a 1_{u(a)} \arrow[r, equal] \arrow[d, equal] & 1_{v(a)}\sigma_a \arrow[d, equal] \\ \sigma_a u(1_a) \arrow[r, Rightarrow, "\sigma_{1_a}"'] & \sigma_a v(1_a) \end{tikzcd}$$

  is commutative, which is just saying that σ_{1_a} is an identity.

We say σ is a **strict transformation** if for any 1-cell f of $\mathcal{A}$, the 2-cell σ_f is an identity.

3.1.10 — Given a strict 2-functor $u : \mathcal{A} \to \mathcal{B}$, we define a strict transformation $1_u : u \Rightarrow u$ given by

$$(1_u)_a = 1_{u(a)}.$$

Further, given u, v, w three 2-functors $\mathcal{A} \to \mathcal{B}$ and two strict transformations $\sigma : u \Rightarrow v$ and $\tau : v \Rightarrow w$, we can define their composite $\tau\sigma$ by

$$(\tau\sigma)_a := \tau_a \sigma_a.$$

We thus obtain a vertical composition

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "u"] \arrow[r, "v" description] \arrow[r, bend right=50, "w"'] \arrow[r, phantom, shift left=3, "\Downarrow\sigma"] \arrow[r, phantom, shift right=3, "\Downarrow\tau"] & \mathcal{B} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "u"] \arrow[r, bend right, "w"'] \arrow[r, phantom, "\Downarrow\tau\sigma"] & \mathcal{B} \end{tikzcd}$$

for strict transformations.

Finally given $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ three 2-categories and u, v 2-functors $\mathcal{A} \to \mathcal{B}$ and u', v' 2-functors $\mathcal{B} \to \mathcal{C}$, and strict transformations $\sigma : u \Rightarrow u'$, $\sigma' : v \Rightarrow v'$, the following diagram is commutative by naturality of σ' for any $a \in \mathcal{A}$.

$$\begin{tikzcd} u'(u(a)) \arrow[r, "u'(\sigma_a)"] \arrow[d, "\sigma'_{u(a)}"'] & u'(v(a)) \arrow[d, "\sigma'_{v(a)}"] \\ v'(u(a)) \arrow[r, "v'(\sigma_a)"'] & v'(v(a)) \end{tikzcd}$$

As such, we can define a (horizontal) composite $\sigma' \circ \sigma$ by

$$(\sigma' \circ \sigma)_a := \sigma'_{v(a)} u'(\sigma_a) = v'(\sigma_a) \sigma'_{u(a)}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathcal{B} \arrow[r, bend left, "u'"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\sigma'"] & \mathcal{C} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "u'u"] \arrow[r, bend right, "v'v"'] \arrow[r, phantom, "\Downarrow\sigma'\circ\sigma"] & \mathcal{C} \end{tikzcd}$$

Definition 3.1.11. We denote $\underline{\textbf{2-Cat}}$ the 2-category whose objects are 2-categories, 1-cells are strict 2-functors, and 2-cells are strict transformations between them. Given $\mathcal{A}, \mathcal{B}$, two 2-categories, we will denote the category of morphisms between them by $\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$.

Dualities

Definition 3.1.12. Given a 2-category $\mathcal{A}$, we define the 2-category $\mathcal{A}^{\mathrm{op}}$ which is obtained by reversing the direction of the 1-cells, that is, it is given by

- $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$;
- for any $a, a' \in \mathcal{A}$, $\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(a, a') := \underline{\mathrm{Hom}}_{\mathcal{A}}(a', a)$;

Definition 3.1.13. Given a 2-category $\mathcal{A}$, we define the 2-category $\mathcal{A}^{\mathrm{co}}$ which is obtained by reversing the direction of the 2-cells, that is, it is given by

- $\mathrm{Ob}(\mathcal{A}^{\mathrm{co}}) = \mathrm{Ob}(\mathcal{A})$;
- for any $a, a' \in \mathcal{A}$, $\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{co}}}(a, a') := \underline{\mathrm{Hom}}_{\mathcal{A}}(a, a')^{\mathrm{op}}$;

Definition 3.1.14. Given a 2-category $\mathcal{A}$, we define the 2-category $\mathcal{A}^{\mathrm{coop}}$ which is obtained by reversing the direction of the 1-cells and the 2-cells, that is, it is given by

$$\mathcal{A}^{\mathrm{coop}} := (\mathcal{A}^{\mathrm{co}})^{\mathrm{op}} = (\mathcal{A}^{\mathrm{op}})^{\mathrm{co}}$$

which is described by

- $\mathrm{Ob}(\mathcal{A}^{\mathrm{coop}}) = \mathrm{Ob}(\mathcal{A})$;
- for any $a, a' \in \mathcal{A}$, $\underline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{coop}}}(a, a') := \underline{\mathrm{Hom}}_{\mathcal{A}}(a', a)^{\mathrm{op}}$;

3.1.15 — Given a 2-functor $u : \mathcal{A} \to \mathcal{B}$, we can obtain a functor $u^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$. We define $u^{\mathrm{op}}(a) = u(a)$ for each objects $a \in \mathcal{A}$, given $f : a \to a'$ a 1-cell of $\mathcal{A}^{\mathrm{op}}$, we define $u^{\mathrm{op}}(f) = u(f)$, and given a 2-cell α of $\mathcal{A}^{\mathrm{op}}$, we take $u^{\mathrm{op}}(\alpha) = u(\alpha)$.

Similarly, a strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ defines a functor $u^{\mathrm{co}} : \mathcal{A}^{\mathrm{co}} \to \mathcal{B}^{\mathrm{co}}$. Combining the two construction, we get a strict 2-functor $u^{\mathrm{coop}} : \mathcal{A}^{\mathrm{coop}} \to \mathcal{B}^{\mathrm{coop}}$.

Moreover, the data of strict transformation $\sigma : u \Rightarrow v$ gives rise to a strict transformation $\sigma^{\mathrm{op}} : v^{\mathrm{op}} \Rightarrow u^{\mathrm{op}}$, as well as one $\sigma^{\mathrm{co}} : u^{\mathrm{co}} \Rightarrow v^{\mathrm{co}}$ and as such we also get a strict transformation $\sigma^{\mathrm{coop}} : u^{\mathrm{coop}} \Rightarrow v^{\mathrm{coop}}$.

3.1.16 — The above paragraphs give strict 2-functoriality of taking $^{\mathrm{op}}, ^{\mathrm{co}}, ^{\mathrm{coop}}$. We get three strict 2-functors

$$\underline{2\text{-}\mathbf{Cat}} \xrightarrow{\_^{\mathrm{op}}} \underline{2\text{-}\mathbf{Cat}}^{\mathrm{co}} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, bend right=50, "u"'] \arrow[d, bend left=50, "v"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma}"] \\ \mathcal{B} \end{tikzcd} \quad \begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[d, bend right=50, "u^{\mathrm{op}}"'] \arrow[d, bend left=50, "v^{\mathrm{op}}"] \arrow[d, phantom, "\overset{\Leftarrow}{\sigma^{\mathrm{op}}}"] \\ \mathcal{B}^{\mathrm{op}} \end{tikzcd}$$

$$\underline{2\text{-}\mathbf{Cat}} \xrightarrow{\_^{\mathrm{co}}} \underline{2\text{-}\mathbf{Cat}} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, bend right=50, "u"'] \arrow[d, bend left=50, "v"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma}"] \\ \mathcal{B} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{A}^{\mathrm{co}} \arrow[d, bend right=50, "u^{\mathrm{co}}"'] \arrow[d, bend left=50, "v^{\mathrm{co}}"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma^{\mathrm{co}}}"] \\ \mathcal{B}^{\mathrm{co}} \end{tikzcd}$$

$$\underline{2\text{-}\mathbf{Cat}} \xrightarrow{\_^{\mathrm{coop}}} \underline{2\text{-}\mathbf{Cat}}^{\mathrm{co}} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, bend right=50, "u"'] \arrow[d, bend left=50, "v"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma}"] \\ \mathcal{B} \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{A}^{\mathrm{coop}} \arrow[d, bend right=50, "u^{\mathrm{coop}}"'] \arrow[d, bend left=50, "v^{\mathrm{coop}}"] \arrow[d, phantom, "\overset{\Leftarrow}{\sigma^{\mathrm{coop}}}"] \\ \mathcal{B}^{\mathrm{coop}} \end{tikzcd}$$

Remark 3.1.17. The name dualities stems from the fact that $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$, $(u^{\mathrm{op}})^{\mathrm{op}} = u$ and the same holds true for the other dualities, co/coop.

Final objects

In the definition of basic localiser, one uses the notion of final object, requesting that for any category A with a final object, the morphism $A \to e$ is a weak equivalence. As such one needs to choose a relevant generalisation of final objects for $\mathbf{Cat}$-enriched categories. The most obvious generalisation -that a final object a_0 of a 2-category $\mathcal{A}$ is one such that every $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, a_0)$ is the terminal category- does not yield the correct notion, being very restrictive. Instead we have the following definition.

Definition 3.1.18. We say an object x of a 2-category $\mathcal{A}$ has a **final object** if for every object $a \in \mathcal{A}$ the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, x)$ has a final object. We say x has an initial object if it has a final object in $\mathcal{A}^{\mathrm{co}}$, i.e. if for every $a \in \mathcal{A}$, the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a, x)$ has a initial object.

Remark 3.1.19. An object having a final object is also called a local final object in the litterature. We use the terms "final" and "terminal" interchangeably.

Proposition 3.1.20. *An object of the 2-category* $\mathbf{Cat}$ *has a final (resp. initial) object in the sense of definition 3.1.18 if and only if it is a category with a final (resp. initial) object in the usual sense.*

Proof. Assume $C \in \mathbf{Cat}$ has a final object in the sense of 3.1.18, then for every category C' the category $\mathbf{Fun}(C', C)$ has a final object. In particular, for $C' = e$, we get a final object of $\mathbf{Fun}(e, C) \cong C$. Conversely, assuming C has a final object c_* in the usual sense, for any other category C', define $F_{C'} : C' \to C$ by

$$\begin{aligned} F_{C'} : C' &\longrightarrow C \\ x &\longmapsto c_* \\ f &\longmapsto 1_{c_*}. \end{aligned}$$

This is a final object of $\mathbf{Fun}(C', C)$. The other claim follows by duality. ∎

Remark 3.1.21. We will say a 2-category op-has (resp. co-has, coop-has) some property if $\mathcal{A}^{\mathrm{op}}$ (resp. $\mathcal{A}^{\mathrm{co}}$, $\mathcal{A}^{\mathrm{coop}}$) has that property. For instance we might say $\mathcal{A}$ coop-has an object which has final object to say $\mathcal{A}^{\mathrm{coop}}$ has an object which has a final object. We might change the verb "has" for "admits" every now and then.

3.2 Slice 2-category and other variations

We use this chapter to introduce the 2-categorical analogues of slices and coslices.

Definition 3.2.1. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor and $b \in \mathcal{B}$. We define the **lax-slice** of $\mathcal{A}$ over b with respect to u to be the 2-category denoted by $\mathcal{A}//^u_l b$, specified by :

- The objects are pairs (a, p) with a an object $\mathcal{A}$ and $p : u(a) \to b$ a 1-cell $\mathcal{B}$;
- The 1-cells $(a, p) \to (a', p')$ are given by pairs (f, α) with $f : a \to a'$ a 1-cell of $\mathcal{A}$ and $\alpha : p \Rightarrow p'u(f)$ a 2-cell of $\mathcal{B}$

$$\begin{tikzcd} u(a) \arrow[rr, "u(f)"] \arrow[dr, "p"'] & \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & u(a') \arrow[dl, "p'"] \\ & b & \end{tikzcd}$$

- Given $(f, \alpha), (f', \alpha') : (a, p) \rightrightarrows (a', p')$, a 2-cell $(f, \alpha) \Rightarrow (f', \alpha')$ is the data of a 2-cell $\beta : f \Rightarrow f'$ of $\mathcal{A}$ such that $(p' \circ u(\beta))\alpha = \alpha'$;
- Composition of 1-cells $(a, p) \xrightarrow{(f,\alpha)} (a', p') \xrightarrow{(f',\alpha')} (a'', p'')$ if given by

$$(f'f, (\alpha' \circ u(f))\alpha);$$

- Composition of 2-cells is that of $\mathcal{A}$;
- Identity of a 1-cell (f, α) is given by 1_f ;
- Identity of an object (a, p) is given by $(1_a, 1_p)$.

Definition 3.2.2. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor and $b \in \mathcal{B}$. We define the **lax-opslice** of $\mathcal{A}$ over b with respect to u to be the 2-category denoted by $b\backslash\backslash^u_l \mathcal{A}$ defined by

$$b\backslash\backslash^u_l \mathcal{A} := (\mathcal{A}^{\mathrm{op}}//^{u^{\mathrm{op}}}_l b)^{\mathrm{op}}$$

which can be described as follows :

- The objects are pairs (a, p) with a an object $\mathcal{A}$ and $p : b \to u(a)$ a 1-cell $\mathcal{B}$;

- The 1-cells $(a, p) \to (a', p')$ are given by pairs (f, α) with $f : a \to a'$ a 1-cell of $\mathcal{A}$ and $\alpha : p' \Rightarrow u(f)p$ a 2-cell of $\mathcal{B}$

$$\begin{tikzcd}
& b \arrow[dl, "p"'] \arrow[dr, "p'"] \arrow[d, phantom, "\overset{\Leftarrow}{\alpha}"] & \\
u(a) \arrow[rr, "u(f)"'] & & u(a')
\end{tikzcd}$$

- Given $(f, \alpha), (f', \alpha') : (a, p) \rightrightarrows (a', p')$, a 2-cell $(f, \alpha) \Rightarrow (f', \alpha')$ is the data of a 2-cell $\beta : f \Rightarrow f'$ of $\mathcal{A}$ such that $(u(\beta) \circ p)\alpha = \alpha'$;

- Composition of 1-cells $(a, p) \xrightarrow{(f,\alpha)} (a', p') \xrightarrow{(f',\alpha')} (a'', p'')$ if given by

$$(f'f, (u(f') \circ \alpha)\alpha');$$

- Composition of 2-cells is that of $\mathcal{A}$;

- Identity of a 1-cell (f, α) is given by 1_f ;

- Identity of an object (a, p) is given by $(1_a, 1_p)$.

Definition 3.2.3. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor and $b \in \mathcal{B}$. We define the **colax-slice** of $\mathcal{A}$ over b with respect to u to be the 2-category denoted by $\mathcal{A}//^u_c b$ defined by

$$\mathcal{A}//^u_c b := (\mathcal{A}^{\mathrm{co}}//^{u^{\mathrm{co}}}_l b)^{\mathrm{co}}$$

which can be described as follows :

- the objects are pairs (a, p) with a an object of $\mathcal{A}$ and $p : u(a) \to b$ a 1-cell of $\mathcal{B}$;

- the 1-cells are pairs (f, α) with $f : a \to a'$ a 1-cell of $\mathcal{A}$ and $\alpha : p'u(f) \Rightarrow p$ is a 2-cell of $\mathcal{B}$

$$\begin{tikzcd}
u(a) \arrow[rr, "u(f)"] \arrow[dr, "p"'] & \arrow[d, phantom, "\underset{\alpha}{\Leftarrow}"] & u(a') \arrow[dl, "p'"] \\
& b &
\end{tikzcd}$$

- the 2-cells $(f, \alpha), (f', \alpha') : (a, p) \to (a', p')$ are given by 2-cells $\beta : f \Rightarrow f'$ of $\mathcal{A}$ such that $\alpha'(p' \circ u(\beta)) = \alpha$.

Definition 3.2.4. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor and $b \in \mathcal{B}$. We define the **colax-opslice** of $\mathcal{A}$ over b with respect to u to be the 2-category denoted by $b\backslash\backslash^u_l \mathcal{A}$ defined by

$$b\backslash\backslash^u_c \mathcal{A} := (\mathcal{A}^{\mathrm{coop}}//^{u^{\mathrm{coop}}}_l b)^{\mathrm{coop}}$$

which can be described as follows :

- the objects are pairs (a, p) with a an object $\mathcal{A}$ and $p : b \to u(a)$ a 1-cell $\mathcal{B}$;

- the 1-cells $(a, p) \to (a', p')$ are given by pairs (f, α) with $f : a \to a'$ a 1-cell of $\mathcal{A}$ and $\alpha : u(f)p \Rightarrow p'$ a 2-cell of $\mathcal{B}$

$$\begin{tikzcd}
& b \arrow[dl, "p"'] \arrow[dr, "p'"] \arrow[d, phantom, "\overset{\Rightarrow}{\alpha}"] & \\
u(a) \arrow[rr, "u(f)"'] & & u(a')
\end{tikzcd}$$

- the 2-cells $(f, \alpha), (f', \alpha') : (a, p) \to (a', p')$ are given by 2-cells $\beta : f \Rightarrow f'$ of $\mathcal{A}$ such that $\alpha'(u(\beta) \circ p) = \alpha$.

Remark 3.2.5. We will usually write $\mathcal{A}//_l a$ (resp. $\mathcal{A}//_c a$, $a\backslash\backslash_l\mathcal{A}$, $a\backslash\backslash_c\mathcal{A}$) for $\mathcal{A}//_l^{1_\mathcal{A}} a$ (resp. $\mathcal{A}//_c^{1_\mathcal{A}} a$, $a\backslash\backslash_l^{1_\mathcal{A}}\mathcal{A}$, $a\backslash\backslash_c^{1_\mathcal{A}}\mathcal{A}$).

3.2.6 — Assume now we're given a commutative triangle of 2-$\mathbf{Cat}$ as follows.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "w"'] & & \mathcal{B} \arrow[dl, "v"] \\ & \mathcal{C} & \end{tikzcd}$$

With this data, for all $c \in \mathcal{C}$, we can define a strict 2-functor

$$u//_l c : \mathcal{A}//_l^w c \to \mathcal{B}//_l^v c.$$

We proceed as follows (a verifier que c'est strict):

1. On objects, $u//_l c$ maps $(a, p : w(a) \to c)$ to $(u(a), p : vu(a) = w(a) \to c)$;
2. On 1-cells, a morphism $(f : a \to a', \alpha : p \Rightarrow p'w(f))$, is mapped by $u//_l c$ to $(u(f), \alpha : p \Rightarrow p'vu(f) = p'w(f))$;
3. On 2-cells, given $(f, \alpha) \Rightarrow (f', \alpha')$, itself given by $\beta : f \Rightarrow f'$, then $u//_l c$ maps this to $u(\beta)$.

3.2.7 — With the same context as in 3.2.6, we can also define for all $c \in \mathcal{C}$ a strict 2-functor :

$$c\backslash\backslash_l u : c\backslash\backslash_l^w \mathcal{A} \to c\backslash\backslash_l^v \mathcal{B}.$$

The given commutative triangle induces the triangle :

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[rr, "u^{\mathrm{op}}"] \arrow[dr, "w^{\mathrm{op}}"'] & & \mathcal{B}^{\mathrm{op}} \arrow[dl, "v^{\mathrm{op}}"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

which in turns by 3.2.6 gives us a strict 2-functor

$$u^{\mathrm{op}}//_l c : \mathcal{A}^{\mathrm{op}}//_l^{w^{\mathrm{op}}} c \to \mathcal{B}^{\mathrm{op}}//_l^{v^{\mathrm{op}}} c$$

we take

$$c\backslash\backslash_l u := (u^{\mathrm{op}}//_l c)^{\mathrm{op}}.$$

3.2.8 — The same construction as in 3.2.7 with $-^{\mathrm{co}}$ instead of $-^{\mathrm{op}}$ gives us, for any commutative triangle of 2-$\mathbf{Cat}$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "w"'] & & \mathcal{B} \arrow[dl, "v"] \\ & \mathcal{C} & \end{tikzcd}$$

a strict 2-functor

$$u//_c c : \mathcal{A}//_c^w c \to \mathcal{B}//_c^v c$$

defined by

$$u//_c c := (u^{\mathrm{co}}//_l c)^{\mathrm{co}}.$$

3.2.9 — Last but not least, given a triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "w"'] & & \mathcal{B} \arrow[dl, "v"] \\ & \mathcal{C} & \end{tikzcd}$$

we again obtain a strict 2-functor

$$c\backslash\backslash_c u : c\backslash\backslash_c^w \mathcal{A} \to c\backslash\backslash_c^v \mathcal{B}$$

defined by

$$c\backslash\backslash_c u := (u^{\text{coop}} //_l c)^{\text{coop}}.$$

Proposition 3.2.10. *For any 2-category $\mathcal{A}$ and any object $a \in \mathcal{A}$, $\mathcal{A}//_c a$ admits an object admitting a final object. Dually, $\mathcal{A}//_l a$ admits an object admitting an initial object, $a\backslash\backslash_c \mathcal{A}$ op-admits an object admitting an final object, $a\backslash\backslash_l \mathcal{A}$ op-admits an object admitting an initial objects.*

Proof. Consider the object $(a, 1_a)$ of $\mathcal{A}//_c a$, then for any other (a', p) the object $(p, 1_p)$ is a final object of $\underline{\text{Hom}}_{\mathcal{A}//_c a}((a', p), (a, 1_a))$. The other statements follow naturally. ■

Example 3.2.11. Given $\mathcal{A}$ a 2-category and a_0 an object of $\mathcal{A}$, one can define a strict 2-functor

$$\begin{aligned} q\colon e &\longrightarrow \mathcal{A} \\ * &\longmapsto a_0 \\ 1_* &\longmapsto 1_{a_0} \\ 1_{1_*} &\longmapsto 1_{1_{a_0}}. \end{aligned}$$

Given another object $a \in \mathcal{A}$, unravelling the definitions one finds $e//_c^q a$ to be the 2-category associated to the 1-category $\underline{\text{Hom}}_{\mathcal{A}}(a_0, a)^{\text{op}}$ under the embedding $\mathbf{Cat} \hookrightarrow 2\text{-}\mathbf{Cat}$ defined in 3.1.8.

3.2.12 — Given a strict 2-functor $u : \mathcal{A} \to \mathcal{B}$, one can define a strict 2-functor $\mathcal{A}//_l - : \mathcal{B} \to \underline{2\text{-}\mathbf{Cat}}$ as follows. (pleins de vÃĺrifications Ãă faire)

- An object $b \in \mathcal{B}$ is mapped to the 2-category $A//_l^u b$;
- given a 1-cell $f : b \to b'$, we get a strict 2-functor

$$\begin{aligned} \mathcal{A}//_l^u(f)\colon \mathcal{A}//_l^u b &\longrightarrow \mathcal{A}//_l^u b' \\ (a, p : u(a) \to b) &\longmapsto (a, fp : u(a) \to b') \\ (g : a \to a', \alpha : p \Rightarrow p'u(g)) &\longmapsto (g, f \circ \alpha : fp \Rightarrow fp'u(g)) \\ \beta &\longmapsto \beta \ ; \end{aligned}$$

- Given a pair of 1-cells $f, f' : b \to b'$ and a 2-cell between them $\gamma : f \Rightarrow f'$, we obtain a strict transformation $\mathcal{A}//_l^u(\gamma) : \mathcal{A}//_l^u(f) \Rightarrow \mathcal{A}//_l^u(f')$ such that

$$\mathcal{A}//_l^u(\gamma)_{(a,p)} := (1_a, \gamma \circ p)$$

  for any $(a, p) \in \mathcal{A}//_l^u b$.

3.3 Preadjoints and prefibrations

Preadjoints

We again owe our terminology regarding 2-categorical notions to Jonathan Chiche's PhD thesis [Chi14], as such we feel obliged to extend our gratitude to him.

3.3.1 — Given a morphism of $\mathbf{Cat}$ $u : A \to B$, it is a left adjoint if and only if for all $b \in B$ the category A/b has a final object. Dually, it is a right adjoint if and only if $b\backslash A$ has an initial object for all $b \in B$. We will use this equivalent formulation to get a notion of left/right adjoint 2-functor.

Definition 3.3.2. Given a 2-functor $u : \mathcal{A} \to \mathcal{B}$, we say it is a **left colax preadjoint** if for any $b \in \mathcal{B}$ the 2-category $\mathcal{A}//^u_c b$ has an object admitting a final object.

We say it is a **left lax preadjoint** if for any $b \in \mathcal{B}$ the 2-category $\mathcal{A}//^u_l b$ has an object admitting an initial object (which is equivalent to u^{co} being a left colax preadjoint).

We say it is a **right colax preadjoint** if for any $b \in \mathcal{B}$ the 2-category $b\backslash\backslash^u_c\mathcal{A}$ op-has an object admitting a final object (which is equivalent to u^{op} being a left colax preadjoint).

We say it is a **right lax preadjoint** if for any $b \in \mathcal{B}$ the 2-category $b\backslash\backslash^u_l\mathcal{A}$ op-has an object admitting an initial object (which is equivalent to u^{coop} being a left colax preadjoint).

Remark 3.3.3. The data of a strict functor which is a left lax preadjoint $u : \mathcal{A} \to \mathcal{B}$ enables us to construct a 2-functor $v : \mathcal{B} \to \mathcal{A}$ as well as a transformation $uv \Rightarrow 1_{\mathcal{B}}$ which further justifies the terminology, but in general v fails to be any sort of preadjoint, it even fails to be strict. The construction of v and the construction is fairly involved to say the least, the interested reader can check 1.6.16 of [Chi14].

Prefibrations

3.3.4 — Let $u : A \to B$ be a morphism of $\mathbf{Cat}$. Recalling lemma 1.5.9, we were able to characterise precofibration using the functors $i_b : A_b \to A/b$, indeed u is a precofibration as soon as i_b has a left adjoint for each $b \in B$. Using the definitions we've stated so far, we can introduce the corresponding notion of precofibration for 2-categories.

Definition 3.3.5. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. Given $b \in \mathcal{B}$, define the 2-category we shall call the **fiber of** u **over** b, denoted $\mathcal{A}^u_b$ (or simply $\mathcal{A}_b$) by the following data.

- The objects are those a of $\mathcal{A}$ such that $u(a) = b$;
- the 1-cells $f : a \to a'$ are given by 1-cells of $\mathcal{A}$ such that $u(f) = 1_a$;
- the 2-cells $\alpha : f \Rightarrow f'$ are given by the 2-cells of $\mathcal{A}$ such that $u(\alpha) = 1_{1_a}$.
- the various compositions and identities being induced by $\mathcal{A}$.

Proposition 3.3.6. *Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. We have a canonical identification between $(\mathcal{A}^{\mathrm{op}})^{u^{\mathrm{op}}}_b$ (resp. $(\mathcal{A}^{\mathrm{co}})^{u^{\mathrm{co}}}_b$, $(\mathcal{A}^{\mathrm{coop}})^{u^{\mathrm{coop}}}_b$) and $(\mathcal{A}^u_b)^{\mathrm{op}}$ (resp. $(\mathcal{A}^u_b)^{\mathrm{co}}$, $(\mathcal{A}^u_b)^{\mathrm{coop}}$).*

Proof. This is a simple case of unravelling definitions. ∎

3.3.7 — Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor, we construct the 2-categorical analogues of the functors i_b and j_b from 2.3.1, 1.5.9. We define the strict 2-functor

$$\begin{aligned} I_b \colon \mathcal{A}_b &\longrightarrow \mathcal{A}//^u_l b \\ a &\longmapsto (a, 1_b : u(a) \to b) \\ f &\longmapsto (f, 1_{1_b}) \\ \beta &\longmapsto \beta. \end{aligned}$$

We can also define another strict 2-functor

$$\begin{aligned} J_b \colon \mathcal{A}_b &\longrightarrow b\backslash\backslash_c^u \mathcal{A} \\ a &\longmapsto (a, 1_b) \\ f &\longmapsto (f, 1_{1_b}) \\ \beta &\longmapsto \beta. \end{aligned}$$

There are also two other canonical strict 2-functors $\mathcal{A}_b \to \mathcal{A}//_c^u b$ and $\mathcal{A}_b \to b\backslash\backslash_l^u \mathcal{A}$ respectively.

Definition 3.3.8. Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. We say u is a **prefibration** if for any $b \in \mathcal{B}$ the strict 2-functor J_b is a left lax preadjoint.

We say u is a **precoopfibration** if for any $b \in \mathcal{B}$ the strict 2-functor I_b is a right colax preadjoint, which equivalently means u^{coop} is a prefibration.

We say u is a **preopfibration** if for any $b \in \mathcal{B}$ the strict 2-functor $\mathcal{A}_b \to \mathcal{A}//_c^u b$ is a right lax preadjoint, which equivalently means u^{op} is a prefibration.

We say u is a **precofibration** if for any $b \in \mathcal{B}$ the strict 2-functor $\mathcal{A}_b \to b\backslash\backslash_l^u \mathcal{A}$ is a left colax preadjoint, which equivalently means u^{co} is a prefibration.

Remark 3.3.9. Most of our examples of pre(co/op/coop)fibrations will come from the next section, i.e. they will come from the projection from the integral of a 2-functor.

3.4 Integration of 2-functors

As the name suggests, we generalise the construction of 1.4.1 of the integral of a functor to the settings of 2-categories.

3.4.1 — Let $F : \mathcal{I} \to \underline{\textbf{2-Cat}}$ be a strict 2-functor. We define the 2-category $\displaystyle\int_{\mathcal{I}} F$ as follows, which we call the integral of F or the Grothendieck construction for F.

- The objects are given by pairs (i, x) with $i \in \mathcal{I}$ and x an object of $F(i)$;
- the 1-cells $(i, x) \to (i', x')$ are given by pairs

  $$(k : i \to i', f : F(k)(x) \to x')$$

  where k is a 1-cell of $\mathcal{I}$, f a 1-cell of $F(i')$;
- given two 1-cells $(k : i \to i', f : F(k)(x) \to x')$ and $(h : i \to i', g : F(h)(x) \to x')$, the 2-cells between them are given by pairs

  $$(\gamma : k \Rightarrow h, \varphi : f \Rightarrow g(F(\gamma))_x)$$

  where γ is a 2-cell of $\mathcal{I}$ and φ a 2-cell of $F(i')$;
- the identity of an object (i, x) is given by $(1_i, 1_x)$;
- the identity of a 1-cell (k, f) is given by $(1_k, 1_f)$;
- given $(i, x) \xrightarrow{(k,f)} (i', x') \xrightarrow{(h,g)} (i'', x'')$ we define the composite of the 1-cells as

  $$(h, g)(k, f) := (hk, gF(h)(f)) ;$$
- given a triple of 1-cells between $(i, x) \to (i', x')$ denoted (k, f), (h, g) and (l, t), a pair of 2-cells $(\gamma, \varphi) : (k, f) \Rightarrow (h, g)$ and $(\delta, \psi) : (h, g) \Rightarrow (l, t)$, we define their vertical composition

  $$(\delta, \psi)(\gamma, \varphi) := (\delta\gamma, (\psi \circ (F(\gamma))_x)\varphi) ;$$

- Given $(k, f), (h, g)$ two 1-cells $(i, x) \to (i', x')$ and $(k', f'), (h', g')$ two 1-cells $(i', x') \to (i'', x'')$, as well as 2-cells $(\gamma, \varphi) : (k, f) \Rightarrow (h, g)$ and $(\gamma', \varphi') : (k', f') \to (h', g')$ we define their horizontal composition

$$(\gamma', \varphi') \circ (\gamma, \varphi) := (\gamma' \circ \gamma, \varphi' \circ F(k')(\varphi)) \; ;$$

Example 3.4.2. Given $\mathcal{A}, \mathcal{B}$ two 2-categories, considering the constant strict 2-functor $\mathcal{A} \xrightarrow{c_{\mathcal{B}}} 2\text{-}\mathbf{Cat}$, we obtain by the previous paragraph a 2-category $\int_{\mathcal{A}} c_{\mathcal{B}}$, which is canonically identified with $\mathcal{A} \times \mathcal{B}$.

Definition 3.4.3. Let $F : \mathcal{I} \to \underline{2\text{-}\mathbf{Cat}}$ be a strict 2-functor, we define a strict 2-functor, the canonical projection P_F, by the formula

$$\begin{aligned} P_F \colon \int_{\mathcal{I}} F &\longrightarrow \mathcal{I} \\ (i, x) &\longmapsto i \\ (k, f) &\longmapsto k \\ (\gamma, \varphi) &\longmapsto \gamma \end{aligned}$$

3.4.4 — Given a transformation $\sigma : F \Rightarrow G$ of strict 2-functors $F, G : \mathcal{I} \to \underline{2\text{-}\mathbf{Cat}}$, we can define a strict 2-functor as follows

$$\begin{aligned} \int_{\mathcal{I}} \sigma \colon \int_{\mathcal{I}} F &\longrightarrow \int_{\mathcal{I}} G \\ (i, x) &\longmapsto (i, \sigma_i(x)) \\ (k, f) &\longmapsto (k, \sigma_{i'}(f)) \\ (\gamma, \varphi) &\longmapsto (\gamma, \sigma_{i'}(\varphi)). \end{aligned}$$

As such, the integral can be promoted to a functor

$$\begin{aligned} \int_{\mathcal{I}} \colon \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) &\longrightarrow 2\text{-}\mathbf{Cat}/\mathcal{I} \\ F &\longmapsto \left(\int_{\mathcal{I}} F, P_F \right) \\ \sigma : F \Rightarrow G &\longmapsto \int_{\mathcal{I}} \sigma \end{aligned}$$

Proposition 3.4.5. *Let $F : \mathcal{A} \to \underline{2\text{-}\mathbf{Cat}}$ be a strict 2-functor, then the canonical projection*

$$P_F : \int_{\mathcal{A}} F \to \mathcal{A}$$

is a precoopfibration.

Proof. Let $a \in \mathcal{A}$ be given. We ought to show the functor $I_a : \left(\int_{\mathcal{A}} F\right)_a^{P_F} \to \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$ is a right colax preadjoint for all $a \in \mathcal{A}$. Recalling the definitions, we can describe the 2-category $\left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$ as follows.

- Objects are pairs $((a', x'), p : a' \to a)$;
- 1-cells $((a', x'), p) \to ((a'', x''), p')$ are given by triples

$$((f : a' \to a'', r : F(f)(x') \to x''), \sigma : p \Rightarrow p' f) \; ;$$

- 2-cells $((f : a' \to a'', r : F(f)(x') \to x''), \sigma : p \Rightarrow p'f) \Rightarrow ((g : a' \to a'', s : F(f)(x') \to x''), \sigma' : p \Rightarrow p'f)$ are given by pairs
$$(\gamma : f \Rightarrow g, \varphi : r \Rightarrow s'F(\gamma))_{x'}$$
such that $(p' \circ \gamma)\sigma = \sigma'$.

The 2-functor I_a is then described by the formula
$$\begin{aligned} I_a\colon \left(\int_{\mathcal{A}} F\right)_a^{P_F} &\longrightarrow \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a \\ (a, x) &\longmapsto ((a, x), 1_a) \\ (1_a, r) &\longmapsto ((1_a, r), 1_{1_a}) \\ (1_{1_a}, \varphi) &\longmapsto (1_{1_a}, \varphi). \end{aligned}$$

Fix $((a', x'), p : a' \to a)$ in $\left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$, we must describe the 2-category
$$((a', x'), p)\backslash\backslash_c^{I_a} \left(\int_{\mathcal{A}} F\right)_a^{P_F}$$
and show it op-admits an object admitting a final object.

- Objects are given by quadruples
$$((a, x), ((q : a' \to a, r : F(q)(x') \to x), \sigma : p \Rightarrow q))\ ;$$
- 1-cells from
$$((a, x), ((q : a' \to a, r : F(q)(x') \to x), \sigma : p \Rightarrow q))$$
to
$$((a, x''), ((q' : a' \to a, r' : F(q')(x') \to x''), \sigma' : p \Rightarrow q'))$$
are given by
$$(s : x \to x'', \gamma : q \Rightarrow q', \varphi : sr \Rightarrow r'(F(\gamma))_{x'}) \quad \text{such that } \gamma\sigma = \sigma'\ ;$$
- 2-cells from
$$(s : x \to x'', \gamma : q \Rightarrow q', \varphi : sr \Rightarrow r'(F(\gamma))_{x'}) \quad \text{such that } \gamma\sigma = \sigma'$$
to
$$(t : x \to x'', \mu : q \Rightarrow q', \psi : tr \Rightarrow r'(F(\mu))_{x'}) \quad \text{such that } \mu\sigma = \sigma'$$
are given by
$$\tau : s \Rightarrow t$$
such that $\mu = \gamma$ and $\psi(\tau \circ r) = \varphi$.

We have the object given by
$$o_0 = ((a, F(p)(x')), ((p, 1_{F(p)(x')}), 1_p)) \in ((a', x'), p)\backslash\backslash_c^{I_a} \left(\int_{\mathcal{A}} F\right)_a^{P_F}.$$
Given any other object
$$o_1 = ((a, x''), ((q' : a' \to a, r' : F(q')(x') \to x''), \sigma' : p \Rightarrow q')),$$
we have the 1-cell given by
$$(r'(F(\sigma'))_{x'}, \sigma', 1_{r'(F(\sigma'))_{x'}}) \in \mathrm{Ob}\left(\underline{\mathrm{Hom}}_{((a',x'),p)\backslash\backslash_c^{I_a}\left(\int_{\mathcal{A}} F\right)_a^{P_F}}(o_0, o_1)\right).$$

Further, given any other 1-cell $o_0 \to o_1$

$$(s : F(p)(x') \to x'', \gamma : p \Rightarrow q', \varphi : s \Rightarrow r'(F(\sigma'))_{x'}),$$

by definition, we must have $\gamma = \sigma'$. Furthermore, the 2-cells $(s, \gamma, \varphi) \Rightarrow (r'(F(\sigma'))_{x'}, \sigma', 1_{r'(F(\sigma'))_{x'}})$ are given by $\tau : s \Rightarrow r'(F(\sigma'))_{x'}$ such that $\gamma = \sigma'$ and $1_{r'(F(\sigma'))_{x'}}(\tau \circ 1_{F(p)(x')}) = \varphi$. The first condition is verified by hypothesis, the second may be rewritten as $\tau = \varphi$. As such, existence and unicity of such 2-cells are verified, which means that $((a', x'), p)\backslash\backslash_c^{I_a} \left(\int_{\mathcal{A}} F\right)_a^{P_F}$ op-has an object admitting a terminal object.

■

3.4.6 — In the above proof, we've seen that $I_a : \left(\int_{\mathcal{A}} F\right)_a^{P_F} \to \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$ is a right colax preadjoint. According to the some dual remark to 3.3.3, there exists a functor $K_a : \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a \to \left(\int_{\mathcal{A}} F\right)_a^{P_F}$. In fact in this specific case, we can define the functor K_a which turns out to be indeed a strict 2-functor. Keeping the notation from the proof above, we define

$$\begin{aligned} K_a \colon \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a &\longrightarrow \left(\int_{\mathcal{A}} F\right)_a^{P_F} \\ ((a', x'), p : a' \to a) &\longmapsto (a, F(p)(x)) \\ ((f : a' \to a'', r : F(f)(x') \to x''), \sigma : p \Rightarrow p'f) &\longmapsto (1_a, F(p')(r)(F(\sigma))_{x'}) \\ (\gamma, \varphi) &\longmapsto (1_{1_a}, F(p')(\varphi) \circ (F(\sigma))_{x'}). \end{aligned}$$

We will see this to be strict in the proof of the next proposition. It turns out K_a is also a left lax preadjoint.

Proposition 3.4.7. *Let $F : \mathcal{A} \to$ <u>2-**Cat**</u> be a strict 2-functor and $a \in \mathcal{A}$ be any object, the strict 2-functor $I_a : \left(\int_{\mathcal{A}} F\right)_a^{P_F} \to \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$ (recall it is a right colax preadjoint) has a retraction $K_a : \left(\int_{\mathcal{A}} F\right) //_l^{P_F} a \to \left(\int_{\mathcal{A}} F\right)_a^{P_F}$, i.e. $K_a I_a = 1_{\left(\int_{\mathcal{A}} F\right)_a^{P_F}}$, which is a left lax preadjoint.*

Proof. We must show a few things, first that K_a is indeed a strict 2-functor, that it is a retraction of I_a and that it is a left lax preadjoint, buckle up.

- We begin with strictness of K_a. Let $((a', x'), p)$ be an object of $\left(\int_{\mathcal{A}} F\right) //_l^{P_F} a$, then

$$\begin{aligned} K_a(1_{((a',x'),p)}) &= K_a((1'_a, 1'_x), 1_p) \\ &= (1_a, F(p)(1_{x'})(F(1_p))_{x'}) \\ &= (1_a, 1_{F(p)(x')}) \\ &= 1_{(a,F(p)(x'))} \\ &= 1_{K_a((a',x'),p)}. \end{aligned}$$

  As such K_a respects the identities 1-cells. We turn to checking it respects composition of 1-cells. Given

$$((f, r), \sigma) : ((a', x'), p) \to ((a'', x''), p') \quad \text{and} \quad ((f', r'), \sigma') : ((a'', x''), p') \to ((a''', x'''), p''),$$

  we compute

$$\begin{aligned} K_a\left(((f', r'), \sigma')((f, r), \sigma)\right) &= K_a((f'f, r'F(f')(r)), (\sigma' \circ f)\sigma) \\ &= (1_a, F(p'')(r'F(f')(r))(F((\sigma' \circ f)\sigma))_{x'}) \\ &= (1_a, F(p'')(r')F(p'')F(f')(r)(F(\sigma') \circ F(f))_{x'}(F(\sigma))_{x'}) \\ &= (1_a, F(p'')(r')F(p'')F(f')(r)(F(\sigma'))_{F(f)(x')}(F(\sigma))_{x'}). \end{aligned}$$

as well as

$$K_a(((f',r'),\sigma))K_a(((f,r),\sigma)) = (1_a, F(p'')(r')(F(\sigma'))_{x''})(1_a, F(p')(r)(F(\sigma))_{x'}).$$

The two formula can be seen to be equal due to the commutativity of the diagram

$$\begin{tikzcd}
F(p')F(f)(x') \arrow[r, "(F(\sigma'))_{F(f)(x')}"] \arrow[d, "F(p')(r)"'] & F(p'')F(f')F(f)(x') \arrow[d, "F(p'')F(f')(r)"] \\
F(p')(x'') \arrow[r, "(F(\sigma'))_{x''}"'] & F(p'')F(f)(x'')
\end{tikzcd}$$

which stems from naturality of $F(\sigma') : F(p') \Rightarrow F(p''f) = F(p'')F(f)$. We now turn to checking it respects the identities of 1-cells, so fix

$$((f,r),\sigma) : ((a',x'),p) \to ((a'',x''),p')$$

a 1-cell, we compute

$$\begin{aligned}
K_a(1_{((f,r),\sigma)}) &= K_a(1_f, 1_r) \\
&= (1_{1_a}, F(p')(1_r) \circ (F(\sigma))_{x'}) \\
&= (1_{1_a}, 1_{F(p')(r)} \circ (F(\sigma))_{x'}) \\
&= (1_{1_a}, 1_{F(p')(r)(F(\sigma))_{x'}}) \\
&= 1_{K_a((f,r),\sigma)}.
\end{aligned}$$

The final verification is regarding the composition of 2-cells, given three 1-cells $((f,r),\sigma)$, $((g,s),\tau)$ and $((h,t),\mu)$ and a pair of 2-cells

$$(\gamma,\varphi) : ((f,r),\sigma) \Rightarrow ((g,s),\tau), \quad (\delta,\psi) : ((g,s),\tau) \Rightarrow ((h,t),\mu),$$

we have

$$\begin{aligned}
K_a(\delta,\psi)K_a(\gamma,\varphi) &= (1_{1_a}, F(p')(\psi) \circ (F(\tau))_{x'})\,(1_{1_a}, F(p')(\varphi) \circ (F(\sigma))_{x'}) \\
&= (1_{1_a}, (F(p')(\psi) \circ F(p')((F(\gamma))_{x'})(F(\sigma))_{x'})(F(p')(\varphi) \circ (F(\sigma))_{x'})) \\
&= (1_{1_a}, ((F(p')(\psi) \circ F(p')((F(\gamma))_{x'}))F(p')(\varphi)) \circ (F(\sigma))_{x'})\,.
\end{aligned}$$

Further, one computes

$$\begin{aligned}
K_a\left((\delta,\psi)(\gamma,\varphi)\right) &= K_a(\delta\gamma, (\psi \circ (F(\gamma))_{x'})\varphi) \\
&= (1_{1_a}, F(p')((\psi \circ (F(\gamma))_{x'})\varphi) \circ (F(\sigma))_{x'}) \\
&= (1_{1_a}, ((F(p')(\psi) \circ F(p')((F(\gamma))_{x'}))F(p')(\varphi)) \circ (F(\sigma))_{x'})\,.
\end{aligned}$$

It follows K_a is indeed a strict 2-functor.

- We now check $K_a I_a = 1_{\left(\int_{\mathcal{A}} F\right)_a^{P_F}}$: On objects, given $(a,x) \in \left(\int_{\mathcal{A}} F\right)_a^{P_F}$

$$\begin{aligned}
K_a I_a(a,x) &= K_a((a,x),1_a) \\
&= (a,x).
\end{aligned}$$

Further, given a 1-cell $(a,x) \xrightarrow{(1_a,f)} (a,x')$, we have

$$\begin{aligned}
K_a I_a(1_a,f) &= K_a((1_a,f),1_{1_a}) \\
&= (1_a,f).
\end{aligned}$$

Finally, given another such one cell $(1_a, g) : (a, x) \to (a, x')$ and a 2-cell $(1_{1_a}, \varphi) : (1_a, f) \Rightarrow (1_a, g)$,

$$\begin{aligned} K_a I_a (1_{1_a}, \varphi) &= K_a (1_{1_a}, \varphi) \\ &= (1_{1_a}, \varphi). \end{aligned}$$

- Finally, we prove K_a is a left lax preadjoint. Fix $(a, x) \in \left(\int_{\mathcal{A}} F\right)_a^{P_F}$, we must describe the 2-category

$$\left(\left(\int_{\mathcal{A}} F\right) //_l^{P_F} a\right) //_l^{K_a} (a, x)$$

and show it has an object which has an initial object.

  - The objects are given by tuples
  $$(((a', x'), p : a' \to a), (1_a, r : F(p)(x) \to x'))\ ;$$
  - given two such objects
  $$(((a', x'), p : a' \to a), (1_a, r : F(p)(x) \to x'))$$
  $$(((a'', x''), p' : a'' \to a), (1_a, r' : F(p')(x) \to x''))$$
  the 1-cells between the two are given by
  $$(((f : a' \to a'', s : F(f)(x') \to x''), \sigma : p \Rightarrow p'f), (1_{1_a}, \varphi : r \Rightarrow r' F(p')(s)(F(\sigma))_{x'}))\ ;$$
  - the 2-cells between
  $$(((f : a' \to a'', s : F(f)(x') \to x''), \sigma : p \Rightarrow p'f), (1_{1_a}, \varphi : r \Rightarrow r' F(p')(s)(F(\sigma))_{x'}))$$
  and
  $$(((g : a' \to a'', t : F(g)(x') \to x''), \tau : p \Rightarrow p'g), (1_{1_a}, \psi : r \Rightarrow r' F(p')(t)(F(\tau))_{x'}))$$
  are given by pairs
  $$(\mu : f \Rightarrow g, \nu : s \Rightarrow t(F(\mu))_{x'}),$$
  such that $(p' \circ \mu)\sigma = \tau$ and $r' \circ F(p')(\nu) \circ (F(\sigma))_{x'} = \psi$.

We can exhibit the object specified by

$$(((a, x), 1_a), (1_a, 1_x)).$$

Given any other object

$$(((a', x'), p : a' \to a), (1_a, r : F(p)(x) \to x')),$$

we have the 1-cell given by

$$(((p, r), 1_p), (1_{1_a}, 1_r))$$

from $(((a, x), 1_a), (1_a, 1_x))$ to $(((a', x'), p, (1_a, r))$. Given any other parallel 1-cell

$$(((f : a' \to a, s : F(f)(x') \to x), \sigma : p \Rightarrow f), (1_{1_a}, \varphi : r \Rightarrow s(F(\sigma))_{x'}))$$

the two cells $(((p, r), 1_p, (1_{1_a}, 1_x)) \Rightarrow (((f, s), \sigma), (1_{1_a}, \varphi)$ are by definition pairs

$$(\mu : p \Rightarrow f, \nu : r \Rightarrow s(F(\mu))_{x'})$$

with

$$\mu = \sigma, \nu = \varphi.$$

It is clear (σ, φ) is a well defined two cell, and as such it is unique. It follows $\left(\left(\int_{\mathcal{A}} F\right) //_l^{P_F} a\right) //_l^{K_a} (a, x)$ admits an object admitting an initial object, as such K_a is a left lax preadjoint.

■

Dualities

As with all things in category theory, we have a few dual versions of what we've introduce, we will not discuss every dualities but we introduce a few we'll need.

3.4.8 — Given a functor $F : \mathcal{A}^{\text{co}} \to \underline{\textbf{2-Cat}}$, we define a 2-category

$$\int_{\mathcal{A}}^{\text{co}} F := \left(\int_{\mathcal{A}^{\text{co}}} (-)^{\text{co}} \circ F \right)^{\text{co}}.$$

This naturally comes equipped with a projection $\int_{\mathcal{A}}^{\text{co}} F \to \mathcal{A}$ which is a preopfibration (because $\int_{\mathcal{A}^{\text{co}}} (-)^{\text{co}} \circ F \to \mathcal{A}^{\text{co}}$ is a precoopfibration) whose fiber over a is $F(a)$. The objects may be described as pairs (a, x) with a an object of $\mathcal{A}$ and $x \in F(a)$. The 1-cells $(a, x) \to (a', x')$ are pairs $(f : a \to a', r : F(f)(x) \to x')$. The 2-cells between two such pairs $(f, r), (g, s)$ are given by pairs $(\gamma : f \Rightarrow g, \varphi : r(F(\gamma))_x \Rightarrow s)$.

The proposition 3.4.7 can easily be adapted for the dual version with $^{\text{co}}$, as such we get the following proposition.

Proposition 3.4.9. *Let $F : \mathcal{A}^{\text{co}} \to \underline{\textbf{2-Cat}}$ be a strict 2-functor and $a \in \mathcal{A}$ be any object, the strict 2-functor $I_a : \left(\int_{\mathcal{A}}^{\text{co}} F\right)_a^{P_F} \to \left(\int_{\mathcal{A}}^{\text{co}} F\right) /\!/_c^{P_F} a$ (which is a right lax preadjoint) has a retraction K_a, which is a left colax preadjoint.*

3.4.10 — Given a functor $F : \mathcal{A}^{\text{op}} \to \underline{\textbf{2-Cat}}$, we define a 2-category

$$\int_{\mathcal{A}}^{\text{op}} F := \left(\int_{\mathcal{A}^{\text{coop}}} ((-)^{\text{coop}} \circ F)^{\text{co}} \right)^{\text{coop}}.$$

This naturally comes equipped with a projection $\int_{\mathcal{A}}^{\text{co}} F \to \mathcal{A}$ which is a prefibration whose fiber over a is $F(a)$. The objects may be described as pairs (a, x) with a an object of $\mathcal{A}$ and $x \in F(a)$. The 1-cells $(a, x) \to (a', x')$ are pairs $(f : a \to a', r : x \to F(f)(x'))$. The 2-cells between two such pairs $(f, r), (g, s)$ are given by pairs $(\gamma : f \Rightarrow g, \varphi : (F(\gamma))_{x'} r \Rightarrow s)$.

The proposition 3.4.7 can easily be adapted for the dual version with $^{\text{op}}$, as such we get the following proposition.

Proposition 3.4.11. *Let $F : \mathcal{A}^{\text{op}} \to \underline{\textbf{2-Cat}}$ be a strict 2-functor and $a \in \mathcal{A}$ be any object, the strict 2-functor $J_a : \left(\int_{\mathcal{A}}^{\text{op}} F\right)_a^{P_F} \to a \backslash\!\backslash_c^{P_F} \left(\int_{\mathcal{A}}^{\text{op}} F\right)$ (which is a left lax preadjoint) has a retraction K_a, which is a right colax preadjoint.*

3.4.12 — Given a functor $F : \mathcal{A}^{\text{coop}} \to \underline{\textbf{2-Cat}}$, we define a 2-category

$$\int_{\mathcal{A}}^{\text{coop}} F := \left(\int_{\mathcal{A}^{\text{op}}} ((-)^{\text{op}} \circ F)^{\text{co}} \right)^{\text{op}}.$$

This naturally comes equipped with a projection $\int_{\mathcal{A}}^{\text{coop}} F \to \mathcal{A}$ which is a precofibration whose fiber over a is $F(a)$. The objects may be described as pairs (a, x) with a an object of $\mathcal{A}$ and $x \in F(a)$. The 1-cells $(a, x) \to (a', x')$ are pairs $(f : a \to a', r : x \to F(f)(x'))$. The 2-cells between two such pairs $(f, r), (g, s)$ are given by pairs $(\gamma : f \Rightarrow g, \varphi : r \Rightarrow (F(\gamma))_{x'} s)$.

The proposition 3.4.7 can easily be adapted for the dual version with $^{\text{op}}$, as such we get the following proposition.

Proposition 3.4.13. *Let $F : \mathcal{A}^{\text{coop}} \to \underline{\textbf{2-Cat}}$ be a strict 2-functor and $a \in \mathcal{A}$ be any object, the strict 2-functor $J_a : \left(\int_{\mathcal{A}}^{\text{coop}} F\right)_a^{P_F} \to a \backslash\!\backslash_l^{P_F} \left(\int_{\mathcal{A}}^{\text{coop}} F\right)$ (which is a left colax preadjoint) has a retraction K_a, which is a right lax preadjoint.*

We now turn to understanding the integral of a specific kind of strict 2-functor, understanding which will come in handy later.

3.4.14 — Let $\mathcal{A}$ be a 2-category, we can define a pair of strict 2-functor which on objects act as

$$\begin{aligned} -\backslash\!\backslash_l \mathcal{A} \colon \mathcal{A}^{\mathrm{coop}} &\longrightarrow \underline{\mathbf{2\text{-}Cat}} \\ a &\longmapsto a\backslash\!\backslash_l \mathcal{A} \end{aligned}$$

and

$$\begin{aligned} (\mathcal{A}/\!/_l -)^{\mathrm{op}} \colon \mathcal{A}^{\mathrm{co}} &\longrightarrow \underline{\mathbf{2\text{-}Cat}} \\ a &\longmapsto (\mathcal{A}/\!/_l a)^{\mathrm{op}}. \end{aligned}$$

The well-definedness and strictness of those 2-functors follows from duality from the construction described in 3.2.12, looking at the special case $u = 1_{\mathcal{A}}$. We can describe the category

$$\int_{\mathcal{A}^{\mathrm{op}}}^{\mathrm{co}} -\backslash\!\backslash_l \mathcal{A}$$

as follows.

- Objects are given by $(b, (a, b \to a))$ where $a, b \in \mathcal{A}$ and $b \to a$ is a 1-cell of $\mathcal{A}$.
- The 1-cells $(b, (a, k : b \to a) \to (b', (a', k'; b' \to a')$ are given by

  $$(f : b' \to b, (g : a \to a', \alpha : k' \Rightarrow gkf))$$

  where f, g are 1-cells of $\mathcal{A}$ and α a 2-cell of $\mathcal{A}$.
- The 2-cells

  $$(f, (g, \alpha)) \Rightarrow (f', (g', \alpha'))$$

  are given by pairs

  $$(\varphi : f \Rightarrow f', \gamma : g \Rightarrow g')$$

  such that $(\gamma \circ k \circ \varphi)\alpha = \alpha'$.

Similarly, we can describe the category

$$\int_{\mathcal{A}}^{\mathrm{co}} (\mathcal{A}/\!/_l -)^{\mathrm{op}}$$

as follows.

- Objects are given by $(a, (b, b \to a))$ where $a, b \in \mathcal{A}$ and $b \to a$ is a 1-cell of $\mathcal{A}$.
- The 1-cells $(a, (b, k : b \to a) \to (a', (b', k'; b' \to a')$ are given by

  $$(g : a' \to a, (f : b \to b', \alpha : k' \Rightarrow gkf))$$

  where f, g are 1-cells of $\mathcal{A}$ and α a 2-cell of $\mathcal{A}$.
- The 2-cells

  $$(g, (f, \alpha)) \Rightarrow (g', (f', \alpha'))$$

  are given by pairs

  $$(\gamma : g \Rightarrow g', \varphi : f \Rightarrow f')$$

  such that $(\gamma \circ k \circ \varphi)\alpha = \alpha'$.

Given the above descriptions, it's clear we have a canonical isomorphism

$$\begin{aligned}\int_{\mathcal{A}}^{\mathrm{co}} (\mathcal{A}//_l-)^{\mathrm{op}} &\longrightarrow \int_{\mathcal{A}^{\mathrm{op}}}^{\mathrm{co}} -\backslash\backslash_l\mathcal{A}\\ (a,(b,k)) &\longmapsto (b,(a,k))\\ (g,(f,\alpha)) &\longmapsto (f,(g,\alpha))\\ (\gamma,\varphi) &\longmapsto (\varphi,\gamma).\end{aligned}$$

Further, both 2-categories are isomorphic canonically to the 2-category $S_1(\mathcal{A})$ whose objects are given by the 1-cells $k : b \to a$ of $\mathcal{A}$. The 1-cells from $k : b \to a$ to $k' : b' \to a'$ are given by triples $(f : b \to b', g : a \to a', \alpha : k' \Rightarrow gkf)$. The 2-cells $(f,g,\alpha) \Rightarrow (f',g',\alpha')$ are given by pairs $(\varphi : f \Rightarrow f', \gamma : g \Rightarrow g')$. Using those isomorphism, we get a pair of strict 2-functors, defined to be the projections

$$\mathcal{A} \xleftarrow{t_1^{\mathcal{A}}} S_1(\mathcal{A}) \xrightarrow{s_1^{\mathcal{A}}} \mathcal{A}^{\mathrm{op}}.$$

Being projections from $\int^{\mathrm{co}}$, by virtue of 3.4.8 they are preopfibrations. Further, if $u : \mathcal{A} \to \mathcal{B}$ is a strict 2-functor, one can define a strict 2-functor

$$\begin{aligned}S_1(u)\colon S_1(\mathcal{A}) &\longrightarrow S_1(\mathcal{B})\\ k &\longmapsto u(k)\\ (f,g,\alpha) &\longmapsto (u(f),u(g),u(\alpha))\\ (\varphi,\gamma) &\longmapsto (u(\varphi),u(\gamma)).\end{aligned}$$

The goal of the paragraph was to be able to establish the commutativity of the following diagram

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[d, "u^{\mathrm{op}}"'] & S_1(\mathcal{A}) \arrow[l, "s_1^{\mathcal{A}}"'] \arrow[r, "t_1^{\mathcal{A}}"] \arrow[d, "S_1(u)"'] & \mathcal{A} \arrow[d, "u"] \\ \mathcal{B}^{\mathrm{op}} & S_1(\mathcal{B}) \arrow[l, "s_1^{\mathcal{B}}"] \arrow[r, "t_1^{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

of strict 2-functors.

3.4.15 — Dualising the previous paragraph, we have a pair of strict 2-functors

$$\begin{aligned}(-\backslash\backslash_l\mathcal{A})^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} &\longrightarrow \underline{2\text{-}\mathbf{Cat}}\\ a &\longmapsto (a\backslash\backslash_l\mathcal{A})^{\mathrm{op}}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{A}//_c-)^{\mathrm{co}}\colon \mathcal{A}^{\mathrm{co}} &\longrightarrow \underline{2\text{-}\mathbf{Cat}}\\ a &\longmapsto (\mathcal{A}//_c a)^{\mathrm{co}}.\end{aligned}$$

Further, we can define a 2-category $S_2(\mathcal{A})$, whose objects are 1-cells $k : b \to a$ of $\mathcal{A}$. The 1-cells from $k : b \to a$ to $k' : b' \to a'$ are triples $(f : b \to b', g : a \to a', \alpha : k'f \Rightarrow gk)$, the 2-cells from (f,g,α) to (f',g',α') are given by pairs $(\varphi : f \Rightarrow f', \gamma : g \Rightarrow g')$ such that $(\gamma \circ k)\alpha(k' \circ \varphi) = \alpha'$. We have canonical isomorphisms

$$\left(\int_{\mathcal{A}^{\mathrm{coop}}}^{\mathrm{co}} (-\backslash\backslash_l\mathcal{A})^{\mathrm{op}}\right)^{\mathrm{op}} \cong S_2(\mathcal{A}) \cong \int_{\mathcal{A}}^{\mathrm{co}} (\mathcal{A}//_c-)^{\mathrm{co}}.$$

As such we get two projections maps,

$$\mathcal{A} \xleftarrow{t_2^{\mathcal{A}}} S_2(\mathcal{A}) \xrightarrow{s_2^{\mathcal{A}}} \mathcal{A}^{\mathrm{co}}$$

where $s_2^{\mathcal{A}}$ is a prefibration and $t_2^{\mathcal{A}}$ is a preopfibration. Similarly, given $u : \mathcal{A} \to \mathcal{B}$, we get a strict 2-functor

$$\begin{aligned} S_2(u)\colon S_2(\mathcal{A}) &\longrightarrow S_2(\mathcal{B}) \\ k &\longmapsto u(k) \\ (f,g,\alpha) &\longmapsto (u(f), u(g), u(\alpha)) \\ (\varphi,\gamma) &\longmapsto (u(\varphi), u(\gamma)). \end{aligned}$$

As such we have the commutative diagram

$$\begin{tikzcd} \mathcal{A}^{\mathrm{co}} \arrow[d, "u^{\mathrm{co}}"'] & S_2(\mathcal{A}) \arrow[l, "s_2^{\mathcal{A}}"'] \arrow[r, "t_2^{\mathcal{A}}"] \arrow[d, "S_1(u)"'] & \mathcal{A} \arrow[d, "u"] \\ \mathcal{B}^{\mathrm{co}} & S_2(\mathcal{B}) \arrow[l, "s_2^{\mathcal{B}}"] \arrow[r, "t_2^{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

of strict 2-functors.

4 Towards a theory of smooth and proper 2-functors

4.1 Basic localisers of 2-categories

For this section, we must thank again Jonathan Chiche for leading the way in his PhD thesis [Chi14], establishing the fundamental definitions such as basic localisers for 2-categories.

Definition 4.1.1. A basic localiser of $2\text{-}\mathbf{Cat}$ is a class $\boldsymbol{\mathcal{W}} \subset \mathrm{Mor}(2\text{-}\mathbf{Cat})$ such that

(LF1) $\boldsymbol{\mathcal{W}}$ is weakly saturated

(LF2) If a small 2-category $\mathcal{A}$ has an object admitting a final object, then $\mathcal{A} \to e$ is in $\boldsymbol{\mathcal{W}}$.

(LF3) Given any commutative triangle of strict 2-functors in $2\text{-}\mathbf{Cat}$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "w"'] & & \mathcal{B} \arrow[dl, "v"] \\ & \mathcal{C} & \end{tikzcd}$$

such that for any $c \in \mathcal{C}$, the strict 2-functor

$$u//_c c : \mathcal{A}//^w_c c \longrightarrow \mathcal{B}//^v_c c$$

is in $\boldsymbol{\mathcal{W}}$, then u is in $\boldsymbol{\mathcal{W}}$.

Remark 4.1.2. There is a way to obtain a basic localiser of $2\text{-}\mathbf{Cat}$ given one of $\mathbf{Cat}$, see [Chi14] , as such these objects do in fact exists. Furthermore, in his PhD thesis, Chiche proves that all of them arise this way.

Remark 4.1.3. We can naturally see $\mathbf{Cat}$ as a full subcategory of $2\text{-}\mathbf{Cat}$ by considering a 1-category to be a 2-category all of whose 2-cells are identities. In this way, a category admits a final object if and only if it admits an object admitting a final object. Furthermore, if the 2-categories in (LF3) are in fact 1-categories, the functors will be 1-functors and the map $u//_c c$ can be identified with u/c. As such, from a basic localiser of $2\text{-}\mathbf{Cat}$ one can obtain a basic localiser of $\mathbf{Cat}$ by considering $\mathcal{W} := \boldsymbol{\mathcal{W}} \cap \mathrm{Mor}(\mathbf{Cat})$. In [Chi14], Chiche proves every basic localiser of $\mathbf{Cat}$ can be obtained this way.

Throughout the remainder of this section we fix $\mathcal{W}$ a basic localiser of 2-$\mathbf{Cat}$.

Definition 4.1.4. We call the elements of $\mathcal{W}$ the weak equivalences (or $\mathcal{W}$-equivalences, or 2-weak equivalence), and a 2-category $\mathcal{A}$ such that $\mathcal{A} \to e$ is a weak equivalence is called aspheric (or $\mathcal{W}$-aspheric).

Proposition 4.1.5. *For any small 2-category $\mathcal{A}$ and $a \in \mathcal{A}$, the 2-category $\mathcal{A}//_c a$ is aspheric.*

Proof. $\mathcal{A}//_c a$ has a object admitting a final object, we recall it here: the object $(a, 1_a)$ has a final object, as given any object (a', p) the category $\underline{\mathrm{Hom}}_{\mathcal{A}//_c a}((a', p), (a, 1_a))$ admits a final object given by $(p, 1_p)$, the claim follows from (LF2). ∎

Lemma 4.1.6. *Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. Assume that for each $b \in \mathcal{B}$, the 2-category $\mathcal{A}//^u_c b$ is aspheric, then u is a weak equivalence.*

Proof. From the map

$$\mathcal{A} \xrightarrow{\quad u \quad} \mathcal{B}$$

we get the triangle

$$\begin{tikzcd} \mathcal{A}//^u_c b \arrow[rr, "u//_c b"] \arrow[dr, "\in\mathcal{W}"'] & & \mathcal{B}//_c b \arrow[dl, "\in\mathcal{W}"] \\ & e & \end{tikzcd}$$

where the leftmost map is in $\mathcal{W}$ by assumption, and the rightmost map is a weak equivalence by (LF2). Weak saturation of $\mathcal{W}$ (in particular, 2-out-of-3) implies that for each $b \in \mathcal{B}$, $u//_c b$ is a weak equivalence, hence by (LF3) u is a weak equivalence. ∎

Corollary 4.1.7. *Let $u : \mathcal{A} \to \mathcal{B}$ be a left colax preadjoint, then it is a weak equivalence.*

Proof. If $u : \mathcal{A} \to \mathcal{B}$ is a left colax preadjoint, then for each $b \in \mathcal{B}$ the category $\mathcal{A}//_c b$ has an object admitting a final object, as such by 4.1.6 u is a weak equivalence. ∎

Remark 4.1.8. In the proposition above, the situation seems asymmetric, since it seems at first only left colax preadjoint are weak equivalences. In fact we will see later that this is not the case, every kind of preadjoint is in particular a weak equivalence. In fact it is a result of Chiche [Chi14] that there are a few variations of the axioms of a basic localiser of 2-categories which are all equivalent, so the asymmetry is only apparent.

Lemma 4.1.9. *Let $\mathcal{A}$ be a small 2-category op-admitting an object admitting an initial object, then $\mathcal{A}$ is aspheric.*

Proof. Let $\mathcal{A}$ be such a category, and $a_0 \in \mathcal{A}$ an object such that for any $a \in \mathcal{A}$ the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_0, a)$ has an initial object. Consider the 2-functors $p : \mathcal{A} \to e$ and $q : e \to \mathcal{A}$ where $q(*) = a_0$, we have $pq = 1_e$, furthermore according to 3.2.11, and the hypothesis that each $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_0, a)$ has an initial object, it follows that $e//^q_c a$ has an object admitting a final object for each a, as such each $e//^q_c a$ is aspheric. It follows from lemma 4.1.6 that q is a weak equivalence. Weak saturation of weak equivalences implies that p is as well, as such $\mathcal{A}$ is aspheric. ∎

Proposition 4.1.10. *For any small 2-category $\mathcal{A}$ and $a \in \mathcal{A}$, the 2-category $a\backslash\backslash_l \mathcal{A}$ is aspheric.*

Proof. The 2-category $a\backslash\backslash_l \mathcal{A}$ op-has an object which has an initial object (see 3.2.10), the results follows from 4.1.9. ∎

Lemma 4.1.11. *Let $\mathcal{A}$ be a small 2-category op-admitting an object admitting a final object, then $\mathcal{A}$ is aspheric.*

Proof. Let $\mathcal{A}$ be such a category, and $a_0 \in \mathcal{A}$ an object such that for any $a \in \mathcal{A}$ the category $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_0, a)$ has a final object. Consider the 2-functors $p : \mathcal{A} \to e$ and $q : e \to \mathcal{A}$ where $q(*) = a_0$, we have $pq = 1_e$, furthermore according to 3.2.11, and the hypothesis that each $\underline{\mathrm{Hom}}_{\mathcal{A}}(a_0, a)$ has a final object, it follows that $e//_c^q a$ op-has an object admitting an initial object for each a, as such each $e//_c^q a$ is aspheric. It follows from lemma 4.1.6 that q is a weak equivalence. Weak saturation of weak equivalences implies that p is as well, as such $\mathcal{A}$ is aspheric. ■

Proposition 4.1.12. *For any small 2-category $\mathcal{A}$ and $a \in \mathcal{A}$, the 2-category $a\backslash\backslash_c\mathcal{A}$ is aspheric.*

Proof. Follows from 4.1.11 since $a\backslash\backslash_c\mathcal{A}$ op-has an object which has a final object. ■

Proposition 4.1.13. *Let $F : \mathcal{A}^{\mathrm{co}} \to \underline{2\text{-}\mathbf{Cat}}$ be a strict 2-functor such that each $F(a)$ is aspheric, then the projection*

$$P_F : \int_{\mathcal{A}}^{\mathrm{co}} F \longrightarrow \mathcal{A}$$

is a weak equivalence.

Proof. By 3.4.9 there is a strict 2-functor for each $a \in \mathcal{A}$

$$K_a : \left(\int_{\mathcal{A}}^{\mathrm{co}} F\right) //_c^{P_F} a \longrightarrow \left(\int_{\mathcal{A}}^{\mathrm{co}} F\right)_a^{P_F} \cong F(a)$$

which is a left colax preadjoint as such it is a weak equivalence by . Since $F(a)$ is aspheric, it follows that $\left(\int_{\mathcal{A}}^{\mathrm{co}} F\right) //_c^{P_F} a$ is aspheric, hence by 4.1.6 P_F is a weak equivalence. ■

Proposition 4.1.14. *A strict 2-functor u is a weak equivalence if and only if u^{op} is.*

Proof. Recall the construction from 3.4.14 where we obtained a commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}} \arrow[d, "u^{\mathrm{op}}"'] & S_1(\mathcal{A}) \arrow[l] \arrow[r] \arrow[d, "S_1(u)"'] & \mathcal{A} \arrow[d, "u"] \\
\mathcal{B}^{\mathrm{op}} & S_1(\mathcal{B}) \arrow[l] \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where $S_1(\mathcal{A}) \cong \int_{\mathcal{A}^{\mathrm{op}}}^{\mathrm{co}}(-\backslash\backslash_l\mathcal{A}) \cong \int_{\mathcal{A}}^{\mathrm{co}}(\mathcal{A}//_l-)^{\mathrm{op}}$ and the horizontal maps are the canonical projections. The fibers of the two left horizontal arrows are precisely $a\backslash\backslash_l\mathcal{A}$ and $b\backslash\backslash_l\mathcal{B}$ which are aspheric by 4.1.10. The fibers of the right horizontal maps are $(\mathcal{A}//_l a)^{\mathrm{op}}$ and $(\mathcal{B}//_l b)^{\mathrm{op}}$ which are aspheric because they op-admit an object admitting an initial object (see 3.2.10) thus we get the result by 4.1.9. It follows by 4.1.13 that the four vertical maps are weak equivalences, the proposition then follows by two consecutive 2-out-of-3 arguments.

■

Corollary 4.1.15. *A 2-category $\mathcal{A}$ is aspheric if and only if $\mathcal{A}^{\mathrm{op}}$ is.*

Proof. Apply 4.1.14 to $\mathcal{A} \to e$. ■

Corollary 4.1.16. *Let $\mathcal{A}$ be a small 2-category admitting an object admitting an initial object, then $\mathcal{A}$ is aspheric.*

Proof. If $\mathcal{A}$ has an object admitting an initial object, then $\mathcal{A}^{\mathrm{op}}$ op-has that same property, then by 4.1.9 $\mathcal{A}^{\mathrm{op}}$ is aspheric, the claim follows by 4.1.15. ■

Corollary 4.1.17. *Given any 2-category $\mathcal{A}$ and a an object, the 2-category $\mathcal{A}//_l a$ is aspheric.*

Proof. It admits an object admitting an initial object, as such the claim follows by 4.1.16. ■

Proposition 4.1.18. *A strict 2-functor u is a weak equivalence if and only if u^{co} is as well.*

Proof. Recall the construction from 3.4.15, we obtained a commutative diagram

$$\begin{tikzcd} \mathcal{A}^{\mathrm{co}} \arrow[d, "u^{\mathrm{co}}"'] & S_2(\mathcal{A}) \arrow[l] \arrow[r] \arrow[d, "S_2(u)"] & \mathcal{A} \arrow[d, "u"] \\ \mathcal{B}^{\mathrm{co}} & S_2(\mathcal{B}) \arrow[l] \arrow[r] & \mathcal{B} \end{tikzcd}$$

where $S_2(\mathcal{A}) \cong \int_{\mathcal{A}}^{\mathrm{co}} (\mathcal{A}//_c-)^{\mathrm{co}} \cong \left(\int_{\mathcal{A}^{\mathrm{coop}}}^{\mathrm{co}} (a\backslash\backslash_l \mathcal{A})^{\mathrm{op}}\right)^{\mathrm{op}}$ and the horizontal maps are the respective projection from the integral. The rightmost horizontal maps are weak equivalence by proposition 4.1.13 since their fibers are $(\mathcal{A}//_c a)^{\mathrm{co}} = (\mathcal{A}^{\mathrm{co}}//_l a)$ and $(\mathcal{B}//_c b)^{\mathrm{co}} = \mathcal{B}^{\mathrm{co}}//_l b$ which are aspheric by 4.1.10, hence being projection with aspheric fibers, they are indeed weak equivalences by 4.1.13.

The leftmost horizontal maps, usually denoted $s_2^{\mathcal{A}}, s_2^{\mathcal{B}}$ can be identified with

$$\left(\int_{\mathcal{A}^{\mathrm{coop}}}^{\mathrm{co}} (a\backslash\backslash_l \mathcal{A})^{\mathrm{op}} \longrightarrow \mathcal{A}^{\mathrm{coop}}\right)^{\mathrm{op}}$$

and

$$\left(\int_{\mathcal{B}^{\mathrm{coop}}}^{\mathrm{co}} (b\backslash\backslash_l \mathcal{B})^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{coop}}\right)^{\mathrm{op}}$$

respectively. By 4.1.13 and 4.1.16 these maps are weak equivalences, whence it follows by 4.1.14 that $s_2^{\mathcal{A}}, s_2^{\mathcal{B}}$ are weak equivalences as well. The proposition follows by consecutive 2-out-of-three arguments. ■

Corollary 4.1.19. *A strict 2-functor u is a weak equivalence if and only if u^{coop} is as well.*

Proof. u is a weak equivalence if and only if u^{co} is. u^{co} is a weak equivalence if and only if $(u^{\mathrm{co}})^{\mathrm{op}}$ is which is u^{coop}. ■

Corollary 4.1.20. *A 2-category $\mathcal{A}$ is aspheric if and only if $\mathcal{A}^{\mathrm{co}}$ is.*

Proof. Apply 4.1.18 to $\mathcal{A} \to e$. ■

Corollary 4.1.21. *A 2-category $\mathcal{A}$ is aspheric if and only if $\mathcal{A}^{\mathrm{coop}}$ is.*

Proof. Apply 4.1.19 to $\mathcal{A} \to e$. ■

Proposition 4.1.22. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "v"'] & & \mathcal{B} \arrow[dl, "w"] \\ & \mathcal{C} & \end{tikzcd}$$

be a commutative diagram in 2-**Cat**. *Assume that for each $c \in \mathcal{C}$ the strict 2-functor*

$$c\backslash\backslash_l u : c\backslash\backslash_l^w \mathcal{A} \to c\backslash\backslash_l^v \mathcal{B}$$

is a weak equivalence, then u is a weak equivalence.

Proof. By definition, $c\backslash\backslash_l u := (u^{\mathrm{op}}//_l c)^{\mathrm{op}} = (u^{\mathrm{coop}}//_c c)^{\mathrm{coop}}$, hence by hypothesis $(u^{\mathrm{coop}}//_c c)^{\mathrm{coop}}$ is a weak equivalence for each $c \in \mathcal{C}$. By 4.1.19 it follows $u^{\mathrm{coop}}//_c c$ is a weak equivalence for each $c \in \mathcal{C}$, hence by LF3 u^{coop} is a weak equivalence. It follows again by 4.1.19 that u is a weak equivalence. ■

Proposition 4.1.23. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "v"'] & & \mathcal{B} \arrow[dl, "w"] \\ & \mathcal{C} & \end{tikzcd}$$

be a commutative diagram in 2-**Cat**. *Assume that for each $c \in \mathcal{C}$ the strict 2-functor*

$$c\backslash\backslash_c u : c\backslash\backslash_c^w \mathcal{A} \to c\backslash\backslash_c^v \mathcal{B}$$

is a weak equivalence, then u is a weak equivalence.

Proof. By definition, $c\backslash\backslash_c u := (u^{\mathrm{coop}}//_l c)^{\mathrm{coop}} = (u^{\mathrm{op}}//_c c)^{\mathrm{op}}$, hence by hypothesis $(u^{\mathrm{op}}//_c c)^{\mathrm{op}}$ is a weak equivalence for each $c \in \mathcal{C}$. By 4.1.14 it follows $u^{\mathrm{op}}//_c c$ is a weak equivalence for each $c \in \mathcal{C}$, hence by LF3 u^{op} is a weak equivalence. It follows again by 4.1.14 that u is a weak equivalence. ■

Proposition 4.1.24. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "v"'] & & \mathcal{B} \arrow[dl, "w"] \\ & \mathcal{C} & \end{tikzcd}$$

be a commutative diagram in 2-**Cat**. *Assume that for each $c \in \mathcal{C}$ the strict 2-functor*

$$u//_l c : \mathcal{A}//^w_l c \to \mathcal{B}//^v_l c$$

is a weak equivalence, then u is a weak equivalence.

Proof. By definition $u//_l c := (u^{\mathrm{co}}//_c c)^{\mathrm{co}}$, as such by hypothesis $(u^{\mathrm{co}}//_c c)^{\mathrm{co}}$ is a weak equivalence for each $c \in \mathcal{C}$. It follows by 4.1.18 that $u^{\mathrm{co}}//_c c$ is a weak equivalence for each $c \in \mathcal{C}$. By LF3, it follows u^{co} is a weak equivalence, whence by 4.1.18 it follows that u is a weak equivalence. ■

With the previous propositions, we can now improve slightly the lemma 4.1.6 as follows.

Corollary 4.1.25. *Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. Assume that for each $b \in \mathcal{B}$ the 2-category $\mathcal{A}//^u_c b$ (resp. $\mathcal{A}//^u_l b$, $b\backslash\backslash^u_c \mathcal{A}$, $b\backslash\backslash^u_l \mathcal{A}$) is aspheric, then u is a weak equivalence.*

Proof. The first statement is already proven in 4.1.6. The others follow by the same argument thanks to the propositions 4.1.24, 4.1.23, 4.1.22 and the fact that the 2-categories $\mathcal{B}//_l b$, $b\backslash\backslash_c \mathcal{B}$, $b\backslash\backslash_l \mathcal{B}$ are aspheric. ■

Asphericity of strict 2-functors

Definition 4.1.26. Consider a commutative triangle of strict 2-functors

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "w"'] & & \mathcal{B} \arrow[dl, "v"] \\ & \mathcal{C} & \end{tikzcd}$$

we call u lax-aspheric (resp. lax-opaspheric, resp. colax-aspheric, resp. colax-opaspheric) over $\mathcal{C}$ if for every $c \in \mathcal{C}$ the 2-functor $u//_l c$ (resp. $c\backslash\backslash_l u$, resp. $u//_c c$, resp. $c\backslash\backslash_c u$) is a weak equivalence. In the case where $v = 1_{\mathcal{B}}$ we drop the "over $\mathcal{C}$".

Remark 4.1.27. It follows from the axiom (LF3) that colax-aspheric (over $\mathcal{C}$) 2-functors are weak equivalences. It follows from 4.1.24, 4.1.23, 4.1.22 that the same holds for other kind of aspheric 2-functors. Further, we can improve the corollary 4.1.25 as recorded by the next proposition.

Proposition 4.1.28. *Let $u : \mathcal{A} \to \mathcal{B}$ be a strict 2-functor. Then u is colax-aspheric (resp. lax-aspheric, colax-opaspheric, lax-opaspheric) if and only if for each $b \in \mathcal{B}$ the 2-category $\mathcal{A}//^u_c b$ (resp. $\mathcal{A}//^u_l b$, $b\backslash\backslash^u_c \mathcal{A}$, $b\backslash\backslash^u_l \mathcal{A}$) is aspheric.*

Proof. Each case follows from 2-out-of-3 and noticing that the 2-categories $\mathcal{B}//_c b, \mathcal{B}//_l b$, $b\backslash\backslash_c \mathcal{B}$, $b\backslash\backslash_l \mathcal{B}$ are aspheric. ■

As the names suggests, being op/co aspheric can be detected by u^{op} and u^{co}, this is the content of the next propositions.

Proposition 4.1.29. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is lax-opaspheric over* ***C*** *if and only if u^{op} is lax-aspheric over $\mathcal{C}^{\mathrm{op}}$.*

Proof. Assume $u : \mathcal{A} \to \mathcal{B}$ is lax-opaspheric over $\mathcal{C}$. By definition this means the functor

$$c\backslash\backslash_l u : c\backslash\backslash_l^w \mathcal{A} \to c\backslash\backslash_l^v \mathcal{B}$$

is a weak equivalence for all $b \in \mathcal{B}$. By 4.1.14 this is equivalent to $(c\backslash\backslash_l u)^{\mathrm{op}} : (c\backslash\backslash_l^w \mathcal{A})^{\mathrm{op}} \to (c\backslash\backslash_l^v \mathcal{B})^{\mathrm{op}}$ being a weak equivalence. By definition 3.2.2 $(c\backslash\backslash_l^w \mathcal{A})^{\mathrm{op}} := \mathcal{A}^{\mathrm{op}}//_l^{w^{\mathrm{op}}} c$ and under this identification, $(c\backslash\backslash_l u)^{\mathrm{op}}$ is identified with $u^{\mathrm{op}}//_l c$. Summing up, u is lax-opaspheric if and only if for all $c \in \mathcal{C}$ the functor $u^{\mathrm{op}}//_l c$ is a weak equivalence, which is saying that u^{op} is lax-aspheric over $\mathcal{C}^{\mathrm{op}}$. ■

Corollary 4.1.30. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is lax-opaspheric if and only if u^{op} is lax-aspheric.*

Proof. Take $\mathcal{C} = \mathcal{B}$ and $v = 1_{\mathcal{B}}$ in the proof above. ■

Proposition 4.1.31. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is colax-aspheric over $\mathcal{C}$ if and only if u^{co} is lax-aspheric over $\mathcal{C}^{\mathrm{co}}$.*

Proof. Follows from the same argument as 4.1.29 by using 4.1.18 and 3.2.3. ■

Corollary 4.1.32. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is colax-aspheric if and only if u^{co} is lax aspheric.*

Proof. This is the absolute case of 4.1.31. ■

Proposition 4.1.33. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is colax-opaspheric over $\mathcal{C}$ if and only if u^{coop} is lax-aspheric over $\mathcal{C}^{\mathrm{coop}}$.*

Proof. Follows from the two previous propositions. ■

Corollary 4.1.34. *A strict 2-functor $u : \mathcal{A} \to \mathcal{B}$ is colax-opaspheric if and only if u^{coop} is lax-aspheric.*

Proof. This is the absolute case of 4.1.33. ■

Proposition 4.1.35. *Let $u : \mathcal{A} \to \mathcal{B}$ be a functor.*

1. *Assume $u : \mathcal{A} \to \mathcal{B}$ is a right colax preadjoint, then u is colax-opaspheric ;*
2. *Assume $u : \mathcal{A} \to \mathcal{B}$ is a right lax preadjoint, then u is lax-opaspheric ;*
3. *Assume $u : \mathcal{A} \to \mathcal{B}$ is a left colax preadjoint, then u is colax-aspheric (in particular it is a weak equivalence) ;*
4. *Assume $u : \mathcal{A} \to \mathcal{B}$ is a left lax preadjoint, then u is lax-aspheric ;*

Proof. Assume $u : \mathcal{A} \to \mathcal{B}$ is a right colax preadjoint, by definition for every $b \in \mathcal{B}$, $b\backslash\backslash_c^u \mathcal{A}$ op-has an object admitting a final object, as such by 4.1.11 the unique map $b\backslash\backslash_c^u \mathcal{A} \to e$ is a weak equivalence. Further, by 4.1.12 the 2-category $b\backslash\backslash_c \mathcal{B}$ is aspheric. Forming the following diagram

$$\begin{tikzcd}
b\backslash\backslash_c^u \mathcal{A} \arrow[rr, "b\backslash\backslash_c u"] \arrow[dr, "\in\mathcal{W}"'] & & b\backslash\backslash_c \mathcal{B} \arrow[dl, "\in\mathcal{W}"] \\
& e &
\end{tikzcd}$$

by (LF1) we get that $b\backslash\backslash_c u$ is a weak equivalence, i.e. u is colax-opaspheric.
The other arguments are dual version of this one. ■

4.2 Higher homotopical Kan extensions

The aim of this section is to try and adapt the results from section 2.2 to the setting of 2-$\mathbf{Cat}$.

Throughout the remainder of this section, we fix $\mathcal{W}$ a basic localiser of 2-$\mathbf{Cat}$.

4.2.1 — We generalise the three functors of 2.2.1 as follows :

1. Given a 2-category $\mathcal{I}$, we can form the category 2-$\mathbf{Cat}/\mathcal{I}$, whose objects are pairs $(\mathcal{A}, v : \mathcal{A} \to \mathcal{I})$ with $\mathcal{A}$ a 2-category, and v a strict 2 functor. Morphisms $(\mathcal{A}, v) \to (\mathcal{A}', v')$ are given by strict 2-functors $f : \mathcal{A} \to \mathcal{A}'$ such that $v'f = v$. Given $w : \mathcal{J} \to \mathcal{I}$ a strict 2-functor, we can form the functor

$$\begin{aligned} 2\text{-}\mathbf{Cat}/w \colon 2\text{-}\mathbf{Cat}/\mathcal{J} &\longrightarrow 2\text{-}\mathbf{Cat}/\mathcal{I} \\ (\mathcal{A}, v) &\longmapsto (\mathcal{A}, wv) \\ f &\longmapsto f. \end{aligned}$$

2. Recalling the construction given in 3.2.12, as well as that of 3.2.6 , the second functor is taken to be

$$\begin{aligned} \Psi_{\mathcal{I}} \colon 2\text{-}\mathbf{Cat}/\mathcal{I} &\longrightarrow \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) \\ (\mathcal{A}, v : \mathcal{A} \to \mathcal{I}) &\longmapsto (\mathcal{A}//^{v}_{l}- : \mathcal{I} \to \underline{2\text{-}\mathbf{Cat}}) \\ (f : (\mathcal{A}, v) \to (\mathcal{A}', w)) &\longmapsto (f//_{l}- : \mathcal{A}//^{v}_{l}- \to \mathcal{A}'//^{w}_{l}-). \end{aligned}$$

   To be clear, given a triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "f"] \arrow[dr, "v"'] & & \mathcal{A}' \arrow[dl, "w"] \\ & \mathcal{I} & \end{tikzcd}$$

   we constructed in 3.2.6 for each $i \in \mathcal{I}$ a strict 2-functor

$$f//_{l}i : \mathcal{A}//^{v}_{l}i \to \mathcal{A}'//^{w}_{l}i,$$

   which we can assemble into a strict transformation

$$\begin{tikzcd} \mathcal{I} \arrow[r, bend left=40, "\mathcal{A}//^{v}_{l}-"{name=U}] \arrow[r, bend right=40, "\mathcal{A}'//^{w}_{l}-"'{name=D}] & \underline{2\text{-}\mathbf{Cat}} \arrow[from=U, to=D, phantom, "\Downarrow f//_{l}-"] \end{tikzcd}$$

   by taking $(f//_{l})_i := f//_{l}i$.

3. Finally, we take the third functor to be the one defined in 3.4.4, i.e.

$$\begin{aligned} \Psi'_{\mathcal{I}} \colon \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) &\longrightarrow 2\text{-}\mathbf{Cat}/\mathcal{I} \\ F &\longmapsto \left(\int_{\mathcal{I}} F, P_F \right) \\ \sigma : F \Rightarrow G &\longmapsto \int_{\mathcal{I}} \sigma \end{aligned}$$

Theorem 4.2.2. *Let $w : \mathcal{J} \to \mathcal{I}$ be a morphism of* 2-$\mathbf{Cat}$*, then the pair of strict 2-functors*

$$\Psi_{\mathcal{I}} \circ 2\text{-}\mathbf{Cat}/w : 2\text{-}\mathbf{Cat}/\mathcal{J} \leftrightarrows \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) : \Psi'_{\mathcal{J}} \circ \underline{\mathrm{Hom}}(w, 1_{\underline{2\text{-}\mathbf{Cat}}})$$

is an adjunction.

Proof. We define the unit and counit. For convenience write $L := \Psi_{\mathcal{I}} \circ 2\text{-}\mathbf{Cat}/w$ and $R := \Psi'_{\mathcal{J}} \circ \underline{\mathrm{Hom}}(w, 1_{\underline{2\text{-}\mathbf{Cat}}})$, we define the counit $\varepsilon : LR \Rightarrow 1_{\underline{\mathrm{Hom}}(\mathcal{I},\underline{2\text{-}\mathbf{Cat}})}$ For a given strict 2-functor $F : \mathcal{I} \to \underline{2\text{-}\mathbf{Cat}}$, we must define for each $i \in \mathcal{I}$ a strict 2-functor

$$\varepsilon_{F,i} : \left(\int_{\mathcal{J}} Fw\right) //_{l}^{wP_{Fw}} i \to F(i).$$

We first describe the domain of $\varepsilon_{F,i}$ as follows.

The objects are given by triples (j, a, p) where j is an object of $\mathcal{J}$, a an object of $Fw(j)$, $p : wj \to i$ a 1-cell of $\mathcal{I}$. The 1-cells $(j, a, p) \to (j', a', p')$ are given by triples (k, f, α) where $k : j \to j'$ is a 1-cell of $\mathcal{J}$, $f : (Fw)(k)(a) \to a'$ is a 1-cell of $Fw(j')$ and $\alpha : p \Rightarrow p'w(k)$ is a 2-cell of $\mathcal{I}$. Finally, a 2-cell $(k, f, \alpha) \Rightarrow (h, g, \beta)$ is given by a pair (γ, φ) where $\gamma : k \Rightarrow h$ is a 2-cell of $\mathcal{J}$ and $\varphi : f \Rightarrow g(Fw(\gamma))_a$ is a 2-cell of $Fw(j')$, such that $(p' \circ w(\gamma))\alpha = \beta$.

On objects

$$\varepsilon_{F,i}(j, a, p) := F(p)(a) \in F(i).$$

Further, given a morphism $(k, f, \alpha) : (j, a, p) \to (j', a', p')$, we get $F(p')(f) : F(p'w(k))(a) \to F(p')(a)$, which we can precompose with $(F(\alpha))_a$ to get the desired 1-cell, i.e.

$$\varepsilon_{F,i}(k, f, \alpha) := (F(p')(f))(F(\alpha))_a.$$

Finally, given $(\gamma, \varphi) : (k, f, \alpha) \Rightarrow (h, g, \beta)$ we need to provide a 2-cell $F(p')(f)(F(\alpha))_a \Rightarrow F(p')(g)(F(\beta)_a)$ of $F(i)$. We have the following diagram

$$\begin{tikzcd}
& F(p)(a) \arrow[dl, bend right, "F(\alpha)_a"'] \arrow[dr, bend left, "F(\beta)_a"] \arrow[d, phantom, "="] & \\
F(p'w(k))(a) \arrow[rr, "(F(p'\circ w(\gamma)))_a"] \arrow[d, "F(p')(f)"'] & & F(p'w(h))(a) \arrow[d, "F(p')(g)"] \\
F(p')(a') \arrow[rr, equal] \arrow[urr, phantom, "\Nearrow F(p')(\varphi)"] & & F(p')(a')
\end{tikzcd}$$

where the upper triangle strictly commutes because by definition $(p' \circ w(\gamma))\alpha = \beta$ whence we conclude $F(\alpha)_a(F(p' \circ w(\gamma)))_a = (F((p' \circ w(\gamma))\alpha))_a = (F(\beta))_a$. As such, taking

$$\varepsilon_{F,i}(\gamma, \varphi) := F(p')(\varphi) \circ (F(\alpha))_a$$

gives the desired 2-cell

$$F(p')(f) \circ (F(\alpha))_a \Rightarrow F(p')(g)(F(p' \circ w(\gamma)))_a(F(\alpha))_a = F(p')(g)(F(\beta))_a.$$

We now turn to defining the unit $\eta : 1_{2\text{-}\mathbf{Cat}/\mathcal{J}} \Rightarrow RL$, so for each object $(\mathcal{A}, v)$ in $2\text{-}\mathbf{Cat}/\mathcal{J}$, we must define a strict 2-functor

$$\eta_{\mathcal{A},v} : \mathcal{A} \to \int_{\mathcal{J}} \mathcal{A}//_{l}^{wv} w(-)$$

such that $v = P_{\mathcal{A}//_{l}^{wv} w(-)}\eta_{\mathcal{A},v}$. We begin by describing the target of $\eta_{\mathcal{A},v}$ as follows. Its objects are given by triples (j, a, p) where j is an object of $\mathcal{J}$, $a \in \mathcal{A}$ and $p : wv(a) \to w(j)$ is a 1-cell of $\mathcal{I}$. The 1-cells $(j, a, p) \to (j', a', p')$ are given by triples (k, f, α) where $k : j \to j'$ is a 1-cell of $\mathcal{J}$, $f : a \to a'$ is a 1-cell of $\mathcal{A}$, and $\alpha : w(k)p \Rightarrow p'wv(f)$ is a 2-cell of $\mathcal{I}$. Finally the 2-cells $(k, f, \alpha) \Rightarrow (h, g, \beta)$ are given by pairs (γ, φ) where $\gamma : k \Rightarrow h$ is a 2-cell of $\mathcal{J}$ and $\varphi : f \Rightarrow g$ a 2-cell of $\mathcal{A}$ such that $(p' \circ wv(\varphi))\alpha = (p \circ w(\gamma))\beta$.

We can now take η to be defined by

$$\begin{aligned}
\eta_{\mathcal{A},v}\colon \mathcal{A} &\longrightarrow \int_{\mathcal{J}} \mathcal{A}//_l^{wv} w(-)\\
a &\longmapsto (v(a), a, 1_{wv(a)})\\
(f : a \to a') &\longmapsto (v(f), f, 1_{wv(f)})\\
(\gamma : f \Rightarrow g) &\longmapsto (v(\gamma), \gamma)
\end{aligned}$$

the various coherence condition are easily verified, further the fact that $v = \eta_{\mathcal{A},v} P_{\mathcal{A}//_l^{wv} w(-)}$ is obvious.

It only remains to check the triangle identities. We must first check that

$$(\varepsilon L) \circ (L\eta) = 1_L,$$

i.e. for any $(\mathcal{A}, v) \in 2\text{-}\mathbf{Cat}/\mathcal{J}$,

$$\varepsilon_{L(\mathcal{A},v)} \circ L(\eta_{\mathcal{A},v}) = 1_{L(\mathcal{A},v)}.$$

Unravelling the definitions, for each $i \in \mathcal{I}$

$$\begin{aligned}
L(\eta_{\mathcal{A},v})_i = \eta_{\mathcal{A},v}//_l i\colon \mathcal{A}//_l^{wv} &\longrightarrow \left(\int_{\mathcal{J}} \mathcal{A}//_l^{wv} w(-)\right) //_l^{wP_{\mathcal{A}//_l^{wv} w(-)}} i\\
(a, p : wv(a) \to i) &\longmapsto (v(a), (a, 1_{wv(a)}), p)\\
(f, \alpha) : (a, p) \to (a', p') &\longmapsto (v(f), (f, 1_{wv(f)}), \alpha)\\
\gamma : (f, \alpha) \Rightarrow (g, \beta) &\longmapsto (v(\gamma), \gamma).
\end{aligned}$$

Further,

$$\begin{aligned}
\varepsilon_{L(\mathcal{A},v),i}\colon \left(\int_{\mathcal{J}} \mathcal{A}//_l^{wv} w(-)\right) //_l^{wP_{\mathcal{A}//_l^{wv} w(-)}} i &\longrightarrow \mathcal{A}//_l^{wv} i\\
(j, (a, q : wva \to wj), p : wj \to i &\longmapsto L(\mathcal{A}, v)(p)(a, q)\\
(k : j \to j', f : L(\mathcal{A}, v)w(k)(a, q) \to (a', q'), \alpha : p \Rightarrow p'w(k)) &\longmapsto L(\mathcal{A}, v)(p')(f)(L(\mathcal{A}, v)(\alpha)_{(a,q)})\\
(\gamma, \zeta) &\longmapsto L(\mathcal{A}, v)(p')(\zeta) \circ L(\mathcal{A}, v)(\alpha)_{(a,q)}.
\end{aligned}$$

Composing the two we get on objects

$$\begin{aligned}
\varepsilon_{L(\mathcal{A},v),i} \circ L(\eta_{(\mathcal{A},v),i})(a, p) &= \varepsilon_{L(\mathcal{A},v),i}(v(a), (a, 1_{wv(a)}), p)\\
&= L(\mathcal{A}, v)(p)(a, 1_{wv(a)})\\
&= \mathcal{A}//_l^{wv}(-)(p)(a, 1_{wv(a)})\\
&= (a, p1_{wv(a)})\\
&= (a, p).
\end{aligned}$$

On 1-cells, given $(f, \alpha) : (a, p) \to (a', p')$, we get

$$\begin{aligned}
\varepsilon_{L(\mathcal{A},v),i} \circ L(\eta_{(\mathcal{A},v),i})(f, \alpha) &= \varepsilon_{L(\mathcal{A},v),i}(v(f), (f, 1_{wv(f)}), \alpha)\\
&= (f, p' \circ 1_{wv(f)})(1_a, \alpha \circ 1_{1_{wv(a)}})\\
&= (f, 1_{p'wvf})(1_a, \alpha \circ 1_{1_{wv(a)}})\\
&= (f, (1_{p'wv(f)} \circ wv(1_a))\alpha \circ 1_{wv(a)})\\
&= (f, (1_{p'wv(f)1_a})\alpha \circ 1_{1_{wv(a)}})\\
&= (f, \alpha).
\end{aligned}$$

Finally given $\gamma : (f,\alpha) \Rightarrow (g,\beta)$, where (f,α) and (g,β) are 1-cells $(a,p) \to (a',p')$:

$$\begin{aligned}\varepsilon_{L(\mathcal{A},v),i} \circ L(\eta_{(\mathcal{A},v),i})(\gamma) &= \varepsilon_{L(\mathcal{A},v),i}(v(\gamma),\gamma)\\ &= L(\mathcal{A},v)(p')(\gamma) \circ L(\mathcal{A},v)(\alpha)_{(a,p)}\\ &= \gamma \circ (1_a, \alpha \circ p)\\ &= \gamma \circ 1_{1_a}\\ &= \gamma.\end{aligned}$$

Hence we indeed have

$$(\varepsilon L) \circ (L\eta) = 1_L,$$

it remains only to see the second triangle identity. We must now check that

$$1_R = (R\varepsilon) \circ (\eta R),$$

i.e. for any $F \in \underline{\mathrm{Hom}}(\mathcal{I}, \underline{\mathbf{2\text{-}Cat}})$

$$1_{RF} = R(\varepsilon_F) \circ \eta_{RF}.$$

Painful calculations lead to the following descriptions :

The 2-category

$$\int_{\mathcal{J}} \left(\int_{\mathcal{J}} Fw \right) //_l^{wP_{Fw}} w(-)$$

can be described by

1. objects are given by $(j, ((j',a), p : w(j') \to w(j))$ where $j, j' \in \mathcal{J}$ and $a \in Fw(j')$ and p is a 1-cell of $\mathcal{I}$;
2. 1-cells $(j, ((j',a),p)) \to (i, ((i',b),q))$ are given by $(k, ((f,g),\alpha))$ where $k : j \to i$, $f : j' \to i'$ are 1-cells of $\mathcal{J}$, $g : Fw(f)(a) \to b$, and $\alpha : w(k)p \Rightarrow qw(f)$;
3. 2-cells $(k, ((f,g),\alpha)) \Rightarrow (k', ((f',g'),\alpha'))$ are given by $(\gamma, (\zeta, \varphi))$ where $\gamma : k \Rightarrow k'$, $\zeta : f \Rightarrow f'$ and $\varphi : g \Rightarrow g' Fw(\zeta)_a$.

As such, we can describe η_{RF} and $R(\varepsilon_F)$ as follows

$$\begin{aligned}\eta_{RF} := \eta_{(\int Fw, P_{Fw})} \colon \int_{\mathcal{J}} Fw &\longrightarrow \int_{\mathcal{J}} \left(\int_{\mathcal{J}} Fw \right) //_l^{wP_{Fw}} w(-)\\ (j,a) &\longmapsto (j, ((j,a), 1_{w(j)}))\\ (k,f) &\longmapsto (k, ((k,f), 1_{w(k)}))\\ (\gamma,\varphi) &\longmapsto (\gamma, (\gamma,\varphi))\end{aligned}$$

and

$$\begin{aligned}R(\varepsilon_F) := \int_{\mathcal{J}} \underline{\mathrm{Hom}}(w, 1_{\underline{\mathbf{2\text{-}Cat}}})(\varepsilon_f) \colon \int_{\mathcal{J}} \left(\int_{\mathcal{J}} Fw \right) //_l^{wP_{Fw}} w(-) &\longrightarrow \int_{\mathcal{J}} Fw\\ (j, ((j',a), p : w(j') \to w(j)) &\longmapsto (j, F(p)(a))\\ (k, ((f,g),\alpha)) &\longmapsto (k, F(q)(g) F(\alpha)_a)\\ (\gamma, (\zeta,\varphi)) &\longmapsto (\gamma, F(q)(\varphi) \circ F(\alpha)_a).\end{aligned}$$

Composing the two, we get on objects

$$\begin{aligned}R(\varepsilon_F)\eta_{RF}(j,a) &= R(\varepsilon_F)(j, ((j,a), 1_{w(j)}))\\ &= (j, F(1_{w(j)})(a))\\ &= (j,a)\end{aligned}$$

and on 1-cells

$$\begin{aligned} R(\varepsilon_F)\eta_{RF}(k,f) &= R(\varepsilon_F)(k,((k,f),1_{w(k)})) \\ &= (k, F(1_{w(j')})(f)F(1_{w(k)})_a) \\ &= (k,f) \end{aligned}$$

and finally on 2-cells

$$\begin{aligned} R(\varepsilon_F)\eta_{RF}(\gamma,\varphi) &= R(\varepsilon_F)(\gamma,(\gamma,\varphi)) \\ &= (\gamma, F(1_{w(j')})(\varphi)\circ F(1_{w(k)})_a) \\ &= (\gamma,\varphi). \end{aligned}$$

Hence

$$1_R = (R\varepsilon)\circ(\eta R),$$

and indeed (L,R,η,ε) is an adjunction. ■

Corollary 4.2.3. *The pair of functors*

$$\psi_{\mathcal{I}} : 2\text{-}\mathbf{Cat}/\mathcal{I} \rightleftarrows \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) : \Psi'_{\mathcal{I}}$$

is an adjunction.

Proof. Take $w = 1_{\mathcal{I}}$ in 4.2.2. ■

4.2.4 — We denote $\mathcal{W}_{\mathcal{I}}$ the subset of $\mathrm{Mor}(\underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}))$ of those strict 2-transformations $\eta : F \Rightarrow G$ such that for all $i \in \mathcal{I}$ $\eta_i \in \mathcal{W}$. Similarly, we denote $\mathcal{W}'_{\mathcal{I}}$ the subset of $\mathrm{Mor}(2\text{-}\mathbf{Cat}/\mathcal{I})$ of those $u : (\mathcal{A}, v) \to (\mathcal{A}', v')$ such that for any $i \in \mathcal{I}$ the induced 2-functor $u//_l i \in \mathcal{W}$.

Theorem 4.2.5. *Keeping with the previous notations, we have :*

- $\mathcal{W}'_{\mathcal{I}} = \Psi_{\mathcal{I}}^{-1}(\mathcal{W}_{\mathcal{I}})$ *;*
- $\mathcal{W}_{\mathcal{I}} = \Psi'^{-1}_{\mathcal{I}}(\mathcal{W}_{\mathcal{I}})$ *;*
- *The induced functors* $\overline{\Psi}_{\mathcal{I}} : \mathcal{W}'^{-1}_{\mathcal{I}} 2\text{-}\mathbf{Cat}/\mathcal{I} \rightleftarrows \mathcal{W}_{\mathcal{I}}^{-1}\,\underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) : \overline{\Psi}'_{\mathcal{I}}$ *are quasi-inverses, thus induce an equivalence of categories.*

Proof. The first point is essentially definitional :

$$\begin{aligned} u \in \Psi_{\mathcal{I}}^{-1}(\mathcal{W}_{\mathcal{I}}) &\iff u//_l - \in \mathcal{W}_{\mathcal{I}} \\ &\iff \forall i \in \mathcal{I},\ u//_l i \in \mathcal{W} \\ &\iff u \in \mathcal{W}'_{\mathcal{I}}. \end{aligned}$$

The rest will follow from 1.2.4 if we can show $\varepsilon_F \in \mathcal{W}_{\mathcal{I}}$ for any $F : \mathcal{I} \to \underline{2\text{-}\mathbf{Cat}}$. We show that for each $i \in \mathcal{I}$, the strict 2-functor $\varepsilon_{F,i}$ is a left lax preadjoint, as such by 4.1.35 it is lax-aspheric whence it will follow from the remark 4.1.27 that it is a weak equivalence. Recall how the counit was defined in the proof of 4.2.2, in the case where $w = 1_{\mathcal{I}}$ it's defined by

$$\begin{aligned} \varepsilon_{F,i} : \left(\int_{\mathcal{I}} F\right) //_l^{p_F} i &\longrightarrow F(i) \\ ((i',a), p : i' \to i) &\longmapsto F(p)(a) \\ ((k : i' \to i'', f : F(k)(a) \to a'), \alpha : p \Rightarrow p'k) &\longmapsto F(p')(f)((F(\alpha))_a \\ (\gamma : k \Rightarrow h, \varphi : f \Rightarrow g(F(\gamma))_a) &\longmapsto F(p')(\varphi)\circ (F(\alpha))_a. \end{aligned}$$

We now ought to describe the 2-category for any $b \in F(i)$

$$\left(\left(\int_{\mathcal{I}} F\right) //_l^{p_F} i\right) //_l^{\varepsilon_{F,i}} b.$$

1. The objects are given by $(((i', a), p : i' \to i), q : F(p)(a) \to b)$ where $i' \in \mathcal{I}$, $a \in F(i')$, p is a 1-cell of $\mathcal{I}$, q a 1-cell of $F(i)$.
2. The 1-cells from
$$(((i', a), p : i' \to i), q : F(p)(a) \to b)$$
to
$$(((i'', a'), p' : i'' \to i), q' : F(p')(a') \to b)$$
are given by
$$(((k : i \to i', f : F(k)(a) \to a'), \alpha : p \Rightarrow p'k), \beta : q \Rightarrow q'F(p')(f)(F(\alpha))_a).$$
3. The 2-cells from
$$(((k : i \to i', f : F(k)(a) \to a'), \alpha : p \Rightarrow p'k), \beta : q \Rightarrow q'F(p')(f)(F(\alpha))_a)$$
to
$$(((k', f'), \alpha'), \beta')$$
are given by pairs
$$(\gamma : k \Rightarrow k', \varphi : f \Rightarrow f'(F(\gamma))_a)$$
such that $(p' \circ \gamma)\alpha = \alpha'$ and $(q' \circ (F(p')(\varphi) \circ (F(\alpha))_a))\beta = \beta'$.

We can exhibit the following object of $\left(\left(\int_{\mathcal{I}} F\right) //_l^{p_F} i\right) //_l^{\varepsilon_{F,i}} b$
$$o = (((i, b), 1_i), 1_b)$$
which we show admits an initial object. Given any other object
$$o_1 = (((i', a), p : i' \to i), q : F(p)(a) \to b),$$
we have a 1-cell $o_1 \to o$ given by
$$c = (((p, q), 1_p), 1_q).$$
Given any other 1-cell $o_1 \to o$ specified by
$$c_1 = (((k : i' \to i, f : F(k)(a) \to b), \alpha : p \Rightarrow k), \beta : q \Rightarrow f(F(\alpha)_a)$$
the 2-cells $c \Rightarrow c_1$ are given by pairs $(\gamma : p \Rightarrow k, \varphi : q \Rightarrow f(F(\gamma))_a)$ such that $\gamma = \alpha$, $\varphi = \beta$. The existence and unicity of such a 2-cell is clear, as such we have indeed exhibited an object o such that for any other object o_1 the category of morphisms $o_1 \to o$ has an initial object, which finishes the proof. ■

Lemma 4.2.6. *Let $w : \mathcal{J} \to \mathcal{I}$ be a strict 2-functor. Then*
$$\underline{\mathrm{Hom}}(w, 1_{\underline{\mathbf{2\text{-}Cat}}})(\boldsymbol{\mathcal{W}}_{\mathcal{I}}) \subset \boldsymbol{\mathcal{W}}_{\mathcal{J}}, \quad (2\text{-}\mathbf{Cat}/w)(\boldsymbol{\mathcal{W}}'_{\mathcal{J}}) \subset \boldsymbol{\mathcal{W}}'_{\mathcal{I}}.$$

Proof. For the first part, the proof is easy, and essentially the same as the first part of 2.2.7. Indeed given $\eta : F \Rightarrow G \in \boldsymbol{\mathcal{W}}_{\mathcal{I}}$, then for any $j \in \mathcal{J}$, $\underline{\mathrm{Hom}}(w, 1_{\underline{\mathbf{2\text{-}Cat}}})(\eta)_j := \eta_{w(j)} \in \boldsymbol{\mathcal{W}}$ as needed.

For the second part, the analogous proof in 2.2.7 breaks down, hence we must be careful. Suppose we're given a diagram
$$\begin{tikzcd} \mathcal{A} \arrow[rr, "u"] \arrow[dr, "v"'] & & \mathcal{A}' \arrow[dl, "v'"] \\ & \mathcal{J} \arrow[d, "w"] & \\ & \mathcal{I} & \end{tikzcd}$$

where $u \in \mathcal{W}'_{\mathcal{J}}$, we must show u is a weak equivalence locally over $\mathfrak{I}$ i.e. that for any $i \in \mathfrak{I}$ the induced map $u//_l i$ is in $\mathcal{W}$. Consider the commutative triangle for any $i \in \mathfrak{I}$

$$\begin{tikzcd} \mathcal{A}//_l^{wv} i \arrow[rr, "u//_l i"] \arrow[dr, "v//_l i"'] & & \mathcal{A}'//_l^{wv'} i \arrow[dl, "v'//_l i"] \\ & \mathcal{J}//_l^{w} i & \end{tikzcd}$$

we show that for any $(j, k : w(j) \to i) \in \mathcal{J}//_l^w i$ the map $(u//_l i)//_l(j,k)$ is a weak equivalence, which by 4.1.24 shows $u//_l i$ to be a weak equivalence. First, we describe the 2-category $(\mathcal{A}//_l^{wv} i)\, //_l^{v//_l i}(j,k)$ as follows :

1. Its objects are given by tuples

$$((a, p : wv(a) \to i), (f : v(a) \to j, \alpha : p \Rightarrow kw(f)))$$

with $a \in \mathcal{A}$, p a 1-cell of $\mathfrak{I}$, f a 1-cell of $\mathcal{J}$ and α a 2-cell of $\mathfrak{I}$.

2. Given $((a,p),(f,\alpha))$ and $((a',p'),(f',\alpha'))$ two such objects the 1-cells from the first to the second are given by tuples

$$((g : a \to a', \beta : p \Rightarrow p'wv(g)), \eta : f \Rightarrow f'v(g))$$

where g is a 1-cell of $\mathcal{A}$, β a 2-cell of $\mathfrak{I}$ and η a 2-cell of $\mathcal{J}$, such that

$$(k \circ w(\eta))\alpha = (\alpha' \circ wv(g))\beta.$$

3. Given two such 1-cells $((g,\beta),\eta)$ and $((g',\beta'),\eta')$, the 2-cells from the first to the second are given by the 2-cells $\gamma : g \Rightarrow g'$ of $\mathcal{A}$ such that

$$\begin{cases} (f' \circ v(\gamma))\eta = \eta' \\ (p' \circ wv(\gamma))\beta = \beta' \end{cases}.$$

Furthermore we can define a strict 2-functor as follows

$$\begin{aligned} F_{\mathcal{A}} : \mathcal{A}//_l^v j &\longrightarrow (\mathcal{A}//i^{wv})\, //_l^{v//i}(j,k) \\ (a,f) &\longmapsto ((a, kw(f)), (f, 1_{kw(f)})) \\ (g,\eta) &\longmapsto ((g, k \circ w(\eta)), \eta) \\ \gamma &\longmapsto \gamma. \end{aligned}$$

Further, one checks that the following diagram commutes :

$$\begin{tikzcd} \mathcal{A}//_l^v j \arrow[r, "u//_l j"] \arrow[d, "F_{\mathcal{A}}"'] & \mathcal{A}'//_l^{v'} j \arrow[d, "F_{\mathcal{A}'}"] \\ (\mathcal{A}//_l^{wv} i)\, //_l^{v//i}(j,k) \arrow[r, "(u//_l i)//_l(j,k)"'] & \left(\mathcal{A}'//_l^{wv'} i\right) //_l^{v'//i}(j,k) \end{tikzcd}$$

But it turns out that $F_{\mathcal{A}}$ is a right lax preadjoint, as such it is a weak equivalence and by assumption $u//_l j$ is a weak equivalence for each $j \in \mathcal{J}$ hence it follows by 2-out-of-3 that each $(u//_l i)//_l(j,k)$ is a weak equivalence, which allows us to conclude that $u//_l i$ itself is a weak equivalence for each i. The only thing we're left to prove it that $F_{\mathcal{A}}$ is indeed a right lax preadjoint. To do so, we must describe for each $((a,p),(f,\alpha)) \in (\mathcal{A}//_l^{wv} i)\, //_l^{v//i}(j,k)$ the 2-category $((a,p),(f,\alpha))\backslash\backslash_l^{F_{\mathcal{A}}} (\mathcal{A}//_l^v j)$ and show it op-admits an object admitting an initial object. Let us start with the description of the structure of that 2-category :

1. Its objects are given by tuples

$$((x, q : v(x) \to j), ((g : a \to x, \beta : p \Rightarrow kw(q)wv(g)), \eta : f \Rightarrow qv(g)))$$

such that

$$(k \circ w(\eta))\alpha = (kw(q) \circ wv(g))\beta$$

where $x \in \mathcal{A}$, q is a 1-cell of $\mathcal{J}$, g a 1-cell of $\mathcal{A}$, β a 2-cell of $\mathcal{I}$ and η a 2-cell of $\mathcal{J}$.

2. The 1-cells from $((x, q), ((g, \beta), \eta))$ to $((x', q'), ((g', \beta'), \eta'))$ are given by

$$((h : x \to x', \pi : q \Rightarrow q'v(h)), \zeta : g' \Rightarrow hg)$$

such that

$$\begin{cases} (q' \circ v(\zeta))\eta' = (\pi \circ v(g))\eta \\ (kw(q') \circ wv(\zeta))\beta' = ((k \circ w(\pi)) \circ wv(g))\beta \end{cases}$$

where h is a 1-cell of $\mathcal{A}$, π a 2-cell of $\mathcal{J}$ and ζ a 2-cell of $\mathcal{A}$.

3. The 2-cells from $((h, \pi), \zeta)$ to $((h', \pi'), \zeta')$ are given by 2-cells $\varepsilon : h \to h'$ of $\mathcal{A}$ such that

$$\begin{cases} (\varepsilon \circ g)\zeta = \zeta' \\ (q' \circ v(\varepsilon))\pi = \pi'. \end{cases}$$

We must show this 2-category op-admits an object admitting an initial object, i.e. that there is an object o such that for any other object o_1 the category $\underline{\mathrm{Hom}}(o, o_1)$ has an initial object. In this 2-category we can exhibit the following object

$$((a, f), ((1_a, \alpha), 1_f)),$$

and given any other object $((x, q), ((g, \beta), \eta))$ using the descriptions above we get that the 1-cells from the first to the second are given by tuples

$$((h : a \to x, \pi : f \Rightarrow qv(h)), \zeta : g \Rightarrow h)$$

such that

$$\begin{cases} (kw(q) \circ wv(\zeta))\beta = (k \circ w(\pi))\beta \\ (q \circ v(\zeta))\eta = \pi. \end{cases}$$

Hence we can specify a 1-cell defined by

$$((g, \eta), 1_g).$$

Furthermore, given any other 1-cell $((h, \pi), \zeta)$, the 2-cells $((g, \eta), 1_g) \Rightarrow ((h, \pi), \zeta)$ are given by $\varepsilon : g \Rightarrow h$ such that $\varepsilon = \zeta$ and $(q \circ v(\zeta))\eta = \pi$. Such a 2-cell is given by ζ itself where the two conditions are readily verified by definition and unicity of such a 2-cell is obvious from the first condition, as such we're done, and $F_{\mathcal{A}}$ is indeed a right lax preadjoint.

■

Notation. We introduce the following notation : given $\mathcal{I} \in 2\text{-}\mathbf{Cat}$ we will use the notation $\mathbf{Hot}(\mathcal{I})$ (or $\mathbf{Hot}_{\mathcal{W}}(\mathcal{I})$) for the category $\mathcal{W}_{\mathcal{I}}^{-1}\,\underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}})$.

Corollary 4.2.7. *Given $w : \mathcal{J} \to \mathcal{I} \in 2\text{-}\mathbf{Cat}$, and keeping the previous notations :*

1. *The strict functor $\underline{\mathrm{Hom}}(w, 1_{2\text{-}\mathbf{Cat}}) : \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) \to \underline{\mathrm{Hom}}(\mathcal{J}, \underline{2\text{-}\mathbf{Cat}})$ induces a functor*

$$w^* : \mathbf{Hot}(\mathcal{I}) \to \mathbf{Hot}(\mathcal{J})$$

2. *The functor $2\text{-}\mathbf{Cat}/w : 2\text{-}\mathbf{Cat}/\mathcal{J} \to 2\text{-}\mathbf{Cat}/\mathcal{I}$ induces a functor*

$$\overline{2\text{-}\mathbf{Cat}/w} : \mathcal{W}'^{-1}_{\mathcal{J}}\mathbf{Cat}/\mathcal{J} \to \mathcal{W}'^{-1}_{\mathcal{I}}\mathbf{Cat}/\mathcal{I}$$

Proof. Follows from 4.2.6. ■

Definition 4.2.8. Given $w : \mathcal{J} \to \mathcal{I}$ we define

$$w_! : \mathbf{Hot}(\mathcal{J}) \to \mathbf{Hot}(\mathcal{I})$$

by taking $w_! = \overline{\Psi}_{\mathcal{I}} \circ \overline{2\text{-}\mathbf{Cat}/w} \circ \overline{\Psi}'_{\mathcal{J}}$

Theorem 4.2.9. *For $w : \mathcal{J} \to \mathcal{I}$ a morphism of $2\text{-}\mathbf{Cat}$, the pair of functors*

$$w_! : \mathbf{Hot}(\mathcal{J}) \leftrightarrows \mathbf{Hot}(\mathcal{I}) : w^*$$

is an adjunction.

Proof. By 4.2.2 we have the adjunction $\Psi_{\mathcal{I}} \circ 2\text{-}\mathbf{Cat}/w : 2\text{-}\mathbf{Cat}/J \leftrightarrows \underline{\mathrm{Hom}}(\mathcal{I}, \underline{2\text{-}\mathbf{Cat}}) : \Psi'_{\mathcal{J}} \circ \underline{\mathrm{Hom}}(w, 1_{2\text{-}\mathbf{Cat}})$, using 1.2.4 and 4.2.7 and 4.2.5 we obtain an induced adjunction with left adjoint the composition :

$$L : \mathcal{W}'^{-1}_{\mathcal{J}} 2\text{-}\mathbf{Cat}/\mathcal{J} \xrightarrow{\overline{2\text{-}\mathbf{Cat}/w}} \mathcal{W}'^{-1}_{\mathcal{I}} 2\text{-}\mathbf{Cat}/\mathcal{I} \xrightarrow{\overline{\Psi}_{\mathcal{I}}} \mathbf{Hot}(\mathcal{I})$$

and with right adjoint the composite

$$R : \mathbf{Hot}(\mathcal{I}) \xrightarrow{w^*} \mathbf{Hot}(\mathcal{J}) \xrightarrow{\overline{\Psi}'_{\mathcal{J}}} \mathcal{W}'^{-1}_{\mathcal{J}} 2\text{-}\mathbf{Cat}/\mathcal{J}$$

But by 4.2.5, $\overline{\Psi}'_{\mathcal{J}}$ and $\overline{\Psi}_{\mathcal{J}}$ are quasi inverses to each other, and as such $w^* \simeq \overline{\Psi}_{\mathcal{J}} \circ \overline{\Psi}'_{\mathcal{J}} \circ w^* = \overline{\Psi}_{\mathcal{J}} \circ L$. Finally we ought to check the desired adjunction :

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Hot}(\mathcal{J})}(w^* a, b) &\simeq \mathrm{Hom}_{\mathbf{Hot}(\mathcal{J})}(\overline{\Psi}_{\mathcal{J}} \circ La, b) \\
&\simeq \mathrm{Hom}_{\mathcal{W}'^{-1}_{\mathcal{J}} 2\text{-}\mathbf{Cat}/\mathcal{J}}(\overline{\Psi}'_{\mathcal{J}} \overline{\Psi}_{\mathcal{J}} \circ La, \overline{\Psi}'_{\mathcal{J}} b) \\
&\simeq \mathrm{Hom}_{\mathcal{W}'^{-1}_{\mathcal{J}} 2\text{-}\mathbf{Cat}/\mathcal{J}}(La, \overline{\Psi}'_{\mathcal{J}} b) \\
&\simeq \mathrm{Hom}_{\mathbf{Hot}(\mathcal{I})}(a, R\overline{\Psi}'_{\mathcal{J}} b) \\
&= \mathrm{Hom}_{\mathbf{Hot}(\mathcal{I})}(a, w_! b)
\end{aligned}$$

■

At last we've managed to generalise the section 2.2 to the setting of 2-categories, in the hope of being able to eventually use them to describe proper 2-functors the same way they're used for proper 2-functors.

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Index of terminology

Index of notations