1   Introduction

We are often interested in pairs of filters $f_e, f_o : \mathbb{R}^3 \to \mathbb{R}$, such that f_e is even, f_o is odd, both are steerable and separable in the radial and angular component, and are positive quasi-conjugates of each other. That is, we would like the behavior of f_o, on, say, the upper half-sphere to resemble the behavior of f_e on the same domain, and both to be positive on that domain.

It is known that f_e and f_o are exactly steerable with respect to the rotational group if the angular components are polynomials of the angle. However, this ensures that they will not be exactly conjugate to each other. It is possible, however, to demand that they be as approximately conjugate to each other as possible for a certain fixed maximum degree.

Let p_e, p_o be the corresponding polynomials for the angular parts of f_e, f_o, respectively. Then our problem amounts to minimizing $\int (p_e(\theta) - p_o(\theta))^2 d\theta$ as theta varies over the upper half sphere. With the same requirement that this maximization takes place normalized so that the values at the poles are 1, and 0 at the equator, we obtain the result that such a maximization always results in positive polynomials. In particular, such optimal polynomials also contain only positive coefficients.

In the succeeding propositions, we define this more precisely and subsequently, show that these properties are closely related to the properties of the generalized Hilbert matrix.

1.1   The Problem

We are interested in quasiconjugate, radial-angular separable, exactly steerable functions of $\mathbb{R}^3$. That is, we consider functions of the type $f_e = W(r)p_e(\cos\theta)$ and $f_o = W(r)p_o(\cos\theta)$ where p_e, p_o are axially symmetric about the north pole and positive and approximately identical on the upper half sphere.

We can think of p_e, p_o as functions of $K/G/K$ where $G = SO(3)$ and K is an embedding of $SO(2)$. In particular, they are functions of the altitude on the sphere and we can thus consider them as functions of $\cos\theta$. When p_e, p_o are polynomials, the functions f_e, f_o are steerable with respect to $SO(3)$ and moreover the steering coefficients and closed forms of integrals over a shell are well-known.

We desire that both p_e and p_o are zero at the equator and 1 at the poles and are positive. Other desirable qualities (that we do not precisely demand) are monotonicity and concentration.

Fix some even integer N. We will require $\deg p_e \leq N$. For quasi-conjugacy we seek to minimize:

$$\int_0^{\pi/2} (p_e(\cos\theta) - p_o(\cos\theta))^2 \sin\theta d\theta \tag{1.1}$$

$$= \int_0^1 (p_e(z) - p_o(z))^2 dz \tag{1.2}$$

Let $p(z) = a_N z^N + a_{N-1} z^{N-1} + ... + a_0 = p_e(z) - p_o(z)$. Note that the coefficients of p correspond in a straightforward way to p_e, p_o, with the mere modification of sign in p_o. We require $p(0) = 0$, $p(1) = 1$. Write $a \in \mathbb{R}^N$ with $a(i) = a_i$ in p. Then our problem can be rewritten as:

$$\text{argmin}_{a \in \mathbb{R}^N p(0)=0, p(1)=1} \int_0^1 p(z)^2 dz, \qquad p(0) = 0, p(1) = 1$$

$$= \text{argmin}_{a \in \mathbb{R}^N} \int_0^1 \sum_{i,j=1}^N a_i a_j z^{i+j} dz, \qquad p(0) = 0, p(1) = 1$$

$$= \text{argmin}_{a \in \mathbb{R}^N} \sum_{i,j=1}^N a_i a_j \int_0^1 z^{i+j} dz, \qquad p(0) = 0, p(1) = 1$$

$$= \text{argmin}_{a \in \mathbb{R}^N} \sum_{i,j=1}^N a_i a_j \frac{1}{i+j+1} dz, \qquad p(0) = 0, p(1) = 1$$

$$= \text{argmin}_{a \in \mathbb{R}^N} a^T H a, \qquad p(0) = 0, p(1) = 1$$

where $H_{ij} = \frac{1}{i+j+1}$.

We will in fact prove that such a solution a is always alternating in sign so that the corresponding p_e, p_o have only positive coefficients (and thus are in fact positive and monotone). To do so, we will need some theory on the generalized Hilbert matrix.

2   Formulation as Generalized Hilbert Matrix Problem

Definition 2.1. Let $\mathbb{Z} \ni p \geq 0$. A *generalized Hilbert matrix* is a matrix $H \in M_n(\mathbb{R})$ whose entries are defined: $H_{ij} = \frac{1}{i+j-1+p}$

Our problem is to find $a \in \mathbb{R}^N$ such that

$$a^T H a \tag{2.1}$$

is minimized, where H is a generalized Hilbert matrix (in our case, p=2,4), subject to the constraint:

$$B^T a = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.2}$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \in M_{N \times 2}(\mathbb{R})$$

Our goal is to prove that such an a has coefficients that alternate in sign. Precisely:

Theorem 2.2. *Let H be a generalized Hilbert matrix of size N and $B \in M_{N \times 2}$ is as above. Furthermore, let*

$$\hat{a} = \text{argmin}_{a \in \mathbb{R}^N} \qquad a^T H a : \quad B^T a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Then for odd i, $\hat{a}(i) < 0$ and for even i $\hat{a}(i) > 0$.

We can obtain expressions for $\hat{a}$ via the method of Lagrange multipliers. Consider the generalized Lagrangian:

$$L(a, \lambda) = \frac{1}{2} a^T H a + \lambda^T (B^T a - \begin{pmatrix} -1 \\ 1 \end{pmatrix})$$

Since the objective function associated with this Lagrangian is clearly convex and the equality condition is a hyperplane (affine), by KKT, a vector a^* is a minimum of $f(a) = \frac{1}{2} a^T H a$ if and only if there exists a λ^* such that together the pair a^*, λ^* satisfy $\frac{\partial L(a^*, \lambda^*)}{\partial a} = \frac{\partial L(a^*, \lambda^*)}{\partial \lambda} = 0$. It will be more convenient to work with the alternate expression

$$L'(a, \lambda) = L(a, -\lambda) = \frac{1}{2} a^T H a - \lambda^T (B^T a - \begin{pmatrix} -1 \\ 1 \end{pmatrix})$$

Note that $\frac{\partial L'(a^*, \lambda^*)}{\partial a} = \frac{\partial L(a^*, -\lambda^*)}{\partial a}$, and $\frac{\partial L'(a^*, \lambda^*)}{\partial \lambda} = -\frac{\partial L(a^*, -\lambda^*)}{\partial \lambda}$, so if both partials of L' are zero at (a^*, λ^*), then both partials of L are zero at $(a^*, -\lambda^*)$. Thus we must simply find the values of λ, a such that the partials of L' are zero.

Now, taking the derivative with respect to a, we see that we must have

$$a = H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = H^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_2 \end{pmatrix},$$

under the constraint

$$B^T a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\iff B^T H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

It will be useful now to establish additional notation. Write $G = H^{-1}$ and let G_{ij} (sometimes $G_{i,j}$ when there is ambiguity in the expression) refers to the entry in the ith row, jth column. Furthermore it will be useful to refer to sums of subsections of rows or columns of G.

Definition 2.3. Let $G = H^{-1}$ where H is a generalized Hilbert matrix of size N for constant $p > 0$. Let $G_{i,j}$ denote the entries of G (we write G_{ij} when unambiguous). Furthermore, let

$$g^{oo} = \sum_{i,j=1}^{N/2} G_{2i-1,2j-1}$$

$$g^{ee} = \sum_{i,j=1}^{N/2} G_{2i,2j}$$

$$g^{oe} = \sum_{i,j=1}^{N/2} G_{2i-1,2j}$$

$$g^{eo} = \sum_{i,j=1}^{N/2} G_{2i,2j-1}$$

be the sums of all entries of G in, for example, odd-indexed rows and odd-indexed columns. Note in particular that since H is symmetric, G is also symmetric and therefore $g^{oe} = g^{eo}$ (when G has even dimension).

Now, let $1 \leq i \leq N$. Then define

$$g^o_i = \sum_{j=1}^{N/2} G_{i,2j-1}$$

$$g^e_i = \sum_{j=1}^{N/2} G_{i,2j}$$

be the sums of entries of row i in an odd-indexed (resp., even-indexed) column. Finally define

$$g^o = \sum_{i=1}^{N/2} \sum_{j=1}^{N} G_{2i-1,j}$$

$$g^e = \sum_{i=1}^{N/2} \sum_{j=1}^{N} G_{2i,j}$$

be the sum of all entries in odd-indexed or even-indexed rows.

Later we will prove results on the relative magnitudes and signs of the g^{oo}, etc., defined above. For now we will simply define the absolute sums $s^{oo} = |g^{oo}|$, etc.

We can thus rewrite the constraint of $B^T a$ earlier as:

$$B^T H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} g^{oo} & g^{oe} \\ g^{eo} & g^{ee} \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{g^{oo}g^{ee} - g^{oe}g^{eo}} \begin{pmatrix} -g^{ee} - g^{eo} \\ g^{oe} + g^{oo} \end{pmatrix}$$

$$= \frac{1}{g^{oo}g^{ee} - g^{oe}g^{eo}} \begin{pmatrix} -g^e \\ g^o \end{pmatrix}$$

We will denote the scaling factor $\frac{1}{g^{oo}g^{ee} - g^{oe}g^{eo}}$ as λ_G.

3 Properties of the Inverse Generalized Hilbert Matrix

It is clear now that to prove Theorem 2.2, we will need to study the inverse of the generalized Hilbert matrix. We inherit the notation from the previous section 1.

Proposition 3.1. *Let $G = H^{-1}$ be the inverse of a generalized Hilbert matrix of size N and constant $p > 0$. Then the entries of G are given by*

$$G_{ij} = \frac{(-1)^{i+j}}{p+i+j-1} \left(\frac{\prod_{k=0}^{N-1}(p+i+k)(p+j+k)}{(i-1)!(N-i)!(j-1)!(N-j)!} \right) \tag{3.1}$$

$$= (-1)^{i+j}(p+i+j-1)\binom{N+p+i-1}{N-j}\binom{N+p+j-1}{N-i}\binom{p+i+j-2}{i-1}\binom{p+i+j-2}{j-1} \tag{3.2}$$

Proposition 3.2. *Let $G = H^{-1}$ be the inverse of a generalized Hilbert matrix of size N and constant $p > 0$. Fix some $i \in \{1, ..., N\}$. Then the sum of the entries of row i is given by:*

$$\sum_{j=1}^{N} G_{ij} = (-1)^{N+i} \frac{\prod_{k=0}^{N-1}(p+i+k)}{(i-1)!(N-i)!} \tag{3.3}$$

$$= (-1)^{N+i} i \binom{N-1+p+i}{N}\binom{N}{i} \tag{3.4}$$

Proposition 3.3. *Let $G = H^{-1}$ be the inverse of a generalized Hilbert matrix of size N and constant $p > 0$. Then the sum of entry of the matrix is given by:*

$$\sum_{i,j=1}^{N} G_{ij} = N(p+N). \tag{3.5}$$

A note on source: The equation 3.1 is given in (Collar, "On the Reciprocation of Certain Matrices"). The equations 3.3, 3.5 are given in (Smith, "Two Theorems on Inverses of Finite Segments of the Generalized Hilbert Matrix). The alternate expressions 3.2, 3.4 are easily verified.

Corollary 3.4. *Let N be even. Then for every k, $|g_k^o| < |g_k^e|$. Furthermore $|g^{oo}| < |g^{oe}|$ and $|g^{eo}| < |g^{ee}|$. If N is odd then the inequalities are reversed.*

Proof. This follows quickly from 3.2 and 3.1 □

Corollary 3.5. *Let N be even. Then $g^o < 0 < g^e$. If N is odd then the inequalities are reversed.*

Proposition 3.6. *λ_G as defined earlier is positive.*

Proof. First, note that H is positive definite, and therefore $G = H^{-1}$ is positive definite. Thus, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $x^T B^T G B x = (x_1 x_2 \cdots x_2) G \begin{pmatrix} x_1 \\ x_2 \\ :x_2 \end{pmatrix} > 0$ when x is not identically zero. (More to the point, B has full column rank).

Thus $B^T G B \in M_2(\mathbb{R})$ with positive determinant. Since $\lambda_G = \frac{1}{\det B^T G B} > 0$ as needed. □

4 Proof

Our goal is to prove Theorem 2.2. In this section we show an equivalent result that we will prove, as well as properties of generalized Hilbert matrices that are key elements for the proof. We inherit the notation of Section 1.

Theorem 4.1. *Let $G = H^{-1}$ be the inverse of a Hilbert matrix of dimension N even, and define $g^{oo}, etc., g_i^o, etc., s^{oo}$, etc. as above. Then for any $1 \le i \le N$*

$$\frac{s^o}{s^e} < \frac{s_i^o}{s_i^e} \tag{4.1}$$

If N is odd, then the inequality is reversed.

Proposition 4.2. *Let $G = H^{-1}$ be a matrix of dimension N, and define sums of rows, etc. as s^o as above such that 4.1 is satisfied (or if N is odd, then the reversed inequality). Then let $a = H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ where λ_1, λ_2 are defined as in Section 1. Then a is alternating in sign with the first element negative.*

Proof. From the previous section we have

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \lambda_G \begin{pmatrix} -g^e \\ g^o \end{pmatrix}.$$

Now since $B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_2 \end{pmatrix}$, the kth element of a is given by

$$\lambda_1 g_k^o + \lambda_2 g_k^e \tag{4.2}$$

First, suppose N is even and consider odd k. Expression 4.2 is negative if

$$\lambda_1 g_k^o + \lambda_2 g_k^e < 0$$
$$\iff \lambda_G(-g^e g_k^o + g^o g_k^e) < 0$$
$$\iff g^e g_k^o > g^o g_k^e$$
$$\iff \frac{g_k^o}{g_k^e} < \frac{g^o}{g^e}$$

where we take care to note that g_k^e is negative when k is odd. On the other hand, if k is even we need

$$\lambda_1 g_k^o + \lambda_2 g_k^e > 0$$
$$\iff \lambda_G(-g^e g_k^o + g^o g_k^e) > 0$$
$$\iff g^o g_k^e > g^e g_k^o$$
$$\iff \frac{g^o}{g^e} > \frac{g_k^o}{g_k^e}$$

as before, where here we used the fact that both g^e and g_k^e are positive. Now, regardless of parity, on either side of the inequality we have one term in the fraction negative. We must always have one of g_k^o, g_k^e negative, and g^o is always negative, so taking absolute values of all terms, we are multiplying both side by negative 1 and thus obtain:

$$\frac{s^o}{s^e} < \frac{s_k^o}{s_k^e}$$

$\square$

Essentially, Proposition 4.2 states that Theorem 4.1 our main Theorem 2.2.

Theorem 4.3. *Let $G = H^{-1}$ be the inverse of a Hilbert matrix of dimension N, and define $g^{oo}, g_i^o, s^o,$ etc. as previously. If N is even, let $s^+ = s^o + s^e$, $s^- = s^e - s^o$, and for any $i \in \{1, ..., N\}$, let $s_i^+ = s_i^o + s_i^e$ and $s_i^- = s_i^e - s_i^o$. Otherwise, let $s^- = s^o - s^e, s_i^- = s_i^o - s_i^e$. Then:*

$$\frac{s^+}{s^-} < \frac{s_i^+}{s_i^-} \tag{4.3}$$

Furthermore, this is equivalent to

$$\frac{s^o}{s^e} < \frac{s_k^o}{s_k^e}, \qquad \text{for } N \text{ even}$$

$$\frac{s^o}{s^e} > \frac{s_k^o}{s_k^e}, \qquad \text{for } N \text{ odd}$$

We choose to work with this ratio as opposed to the one in 4.1 because the differences between the two are magnified, giving us a larger room for error. The manipulations to achieve the inequality are thus more conveniently derived.

Proof. We first prove equivalence of the theorems, focusing on even N. The odd case is analagous. Note

$$\frac{s^o}{s^e} < \frac{s_k^o}{s_k^e}$$

$$\iff \frac{s^+ - s^-}{s^+ + s^-} < \frac{s_k^+ - s_k^-}{s_k^+ + s_k^-}$$

$$\iff \frac{1 - s^-/s^+}{1 + s^-/s^+} < \frac{1 - s_k^-/s_k^+}{1 + s_k^-/s_k^+}$$

$$\iff \frac{s^-}{s^+} > \frac{s_k^-}{s_k^+}$$

$$\iff \frac{s^+}{s^-} < \frac{s_i^+}{s_i^-}$$

where we have taken advantage of the fact that $0 < \frac{s^-}{s^+}, \frac{s_k^-}{s_k^+} < 1$.

Now onto the inequality. We write down two lemmas on binomial coefficients that will prove useful. Both are well-known and easily verified.

Lemma 4.4. *Let $N > j > 0$. Then*

$$j\binom{N}{j} = j\binom{N}{N-j} = N\binom{N-1}{j-1}$$

Lemma 4.5.

$$\binom{N}{m}\binom{m}{k} = \binom{N}{k}\binom{N-k}{m-k}$$

Note that in our expressions for the individual entries or sums of rows of G, there is usually a term like $(-1)^{i+j}$ dictating the sign of the value. Since we wish to compare a ratio of sums after taking an absolute value, we can in fact ignore the sign term. Thus plugging in our values from our propositions into the inequality 4.3, we see that our inequality is equivalent to

$$\frac{\sum_{i=1}^{N} i\binom{N-1+p+i}{N}\binom{N}{i}}{N(P+N)} < \frac{\sum_{j=1}^{N}(p+k+j-1)\binom{N+p+k-1}{N-j}\binom{N+p+j-1}{N-k}\binom{p+k+j-2}{k-1}^2}{k\binom{N-1+p+k}{N}\binom{N}{k}}$$
(4.4)

for every $k \in \{1, ..., N\}$.

Let us first consider the LHS. Using Proposition 4.4, we see that

$$\frac{\sum_{i=1}^{N} i\binom{N-1+p+i}{N}\binom{N}{i}}{N(P+N)} = \frac{\sum_{i=1}^{N} N\binom{N-1+p+i}{N}\binom{N-1}{i-1}}{N(P+N)}$$
(4.5)

$$= \frac{\sum_{i=1}^{N}\binom{N-1+p+i}{N}\binom{N-1}{i-1}}{P+N}$$
(4.6)

$$< \frac{\sum_{i=1}^{N}\binom{N-1+p+i}{N}\binom{N}{i}}{P+N}$$
(4.7)

Now consider the RHS. The denominator can be rewritten as

$$k\binom{N-1+p+k}{N}\binom{N}{k} = N\binom{N-1+p+k}{N}\binom{N-1}{k-1}$$

$$= (N+p+k-1)\binom{N+p+k-2}{N-1}\binom{N-1}{k-1}$$

and plugging into the RHS and expanding gives

$$\frac{\sum_{j=1}^{N}(p+k+j-1)\binom{N+p+k-1}{N-j}\binom{N+p+j-1}{N-k}\binom{p+k+j-2}{k-1}\binom{p+k+j-2}{j-1}}{(N+p+k-1)\binom{N+p+k-2}{N-1}\binom{N-1}{k-1}}$$

$$=\frac{(N+p+k-1)\sum_{j=1}^{N}\binom{N+p+k-2}{p+k+j-2}\binom{N+p+j-1}{N-k}\binom{p+k+j-2}{k-1}\binom{p+k+j-2}{j-1}}{(N+p+k-1)\binom{N+p+k-2}{N-1}\binom{N-1}{k-1}}$$

$$=\frac{\sum_{j=1}^{N}\frac{(N+p+k-2)!}{(p+k+j-2)!(N-j)!}\frac{(N+p+j-1)!}{(N-k)!(p+j+k-1)!}\left(\frac{(p+k+j-2)!^2}{(k-1)!(p+j-1)!(j-1)!(p+k-1)!}\right)}{\frac{(N+p+k-2)!}{(N-1)!(p+k-1)!}\frac{(N-1)!}{(k-1)!(N-k)!}}$$

$$=\sum_{j=1}^{N}\frac{(N+p+j-1)!(p+k+j-2)!}{(N-j)!(p+j+k-1)!(p+j-1)!(j-1)!}$$

$$=\sum_{j=1}^{N}\frac{(N+p+j-1)!N!j}{(p+j-1)!N!(N-j)!j!(p+j+k-1)}$$

$$=\sum_{j=1}^{N}\binom{N+p+j-1}{N}\binom{N}{j}\frac{j}{p+j+k-1}$$

which is clearly monotonically decreasing in k, so it suffices to consider

$$\sum_{j=1}^{N}\binom{N+p+j-1}{N}\binom{N}{j}\frac{j}{p+j+N-1}>\sum_{j=1}^{N}\binom{N+p+j-1}{N}\binom{N}{j}\frac{1}{p+N}$$

since the fraction $\frac{j}{p+j+N-1}$ is increasing in j because both j and $p+N-1$ positive. $\square$

5   Quadrature Loss

We now show that not only does this optimization of quasi-conjugate polynomial pairs give us well-behaved polynomials with positive coefficients, but also that our quadrature loss vanishes as the maximal degree allowed N is increased.

Theorem 5.1. *Let* $a^* = \operatorname{argmin}_{a\in\mathbb{R}^N}\frac{1}{2}a^THa$, $\quad B^Ta = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ *where B is the N by 2 matrix of alternating 1s and 0s as before. Define* $f(N) = \frac{1}{2}(a^*)^THa^*$ *the minimum for the objective function achieved for dimension N. Then $f(N) \to 0$ as $N \to \infty$.*

Proof. Abusing notation, we will simply write a as the optimal vector of coefficients. We proceed to by deriving an alternative expression for the objective function using our previous results and then prove some facts about the growth. Now, we have

$$\frac{1}{2}a^THa = \frac{1}{2}(H^{-1}B\lambda)^TH(H^{-1}B\lambda)$$

$$= \frac{1}{2}\lambda^TB^TH^{-1}B\lambda$$

$$= \frac{1}{2}\lambda^TK\lambda, \quad \text{where } K = \begin{pmatrix} g^{oo} & g^{oe} \\ g^{eo} & g^{ee} \end{pmatrix}.$$

But $\lambda = K^{-1}\begin{pmatrix}-1\\1\end{pmatrix} = \lambda_G\begin{pmatrix}-g^e\\g^o\end{pmatrix}$, where $\lambda_G = \frac{1}{g^{oo}g^{ee}-g^{eo}g^{oe}} > 0$, so noting the symmetry of K, we have

$$\frac{1}{2}a^T H a = \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}^T K^{-1} K \lambda$$

$$= \frac{\lambda_G}{2}(-1\ 1)\begin{pmatrix}-g^e\\g^o\end{pmatrix}$$

$$= \frac{g^e + g^o}{2(g^{oo}g^{ee} - g^{eo}g^{oe})}$$

$$= \frac{N(p+N)}{2(g^{oo}g^{ee} - (g^{oe})^2)}$$

Now, let $d = |g^{oe}| - |g^{oo}|$. Then since $N(p+N) = g^{ee} + g^{oo} + g^{oe} + g^{eo} = g^{oo} + g^{ee} - 2|g^{oe}|$, we may write $g^{oo} = |g^{oe}| - d$, $g^{ee} = |g^{oe}| + d + N(P+N)$. Thus

$$g^{oo}g^{ee} - g^{eo}g^{oe} = (|g^{oe}| + d + N(p+N))(|g^{oe}| - d) - |g^{oe}|^2$$

$$= N(p+N)(g^{oo}) - d^2$$

We now examine the growth of g^{oo} and d^2. d is simply a sign change times the sum of the odd rows. We have

$$d = (-1)^N \sum_{i\ \text{odd}}^{N} \frac{\prod_{k=1}^{N-1}(p+i+k)}{(i-1)!(N-i)!}$$

and thus we may write

$$d^2 = \sum_{i,j\ \text{odd}}^{N} \frac{\prod_{k=0}^{N-1}(p+i+k)(p+j+k)}{(i-1)!(N-i)!(j-1)!(N-j)!}$$

On the other hand, using the fact that g^{oo} is a sum of only positive entries we can write

$$g^{oo} = \sum_{i,j\ \text{odd}}^{N} \frac{1}{p+i+j-1}\frac{\prod_{k=0}^{N-1}(p+i+k)(p+j+k)}{(i-1)!(N-i)!(j-1)!(N-j)!}$$

$\square$

So we can write the denominator of our objective function value as

$$2(g^{oo}g^{ee} - (g^{oe})^2) = 2(N(p+N)(g^{oo}) - d^2)$$

$$= \sum_{\substack{i,j \text{ odd}}}^{N} \left(\frac{2N(p+N)}{p+i+j-1} - 1 \right) \frac{\prod_{k=0}^{N-1}(p+i+k)(p+j+k)}{(i-1)!(N-i)!(j-1)!(N-j)!}$$

$$\geq \sum_{\substack{i,j \text{ odd}}}^{N} (N-1) \frac{\prod_{k=0}^{N-1}(p+i+k)(p+j+k)}{(i-1)!(N-i)!(j-1)!(N-j)!}$$

$$\geq \sum_{\substack{i,j \text{ odd}}}^{N} (N-1) \frac{(N-1)!^2}{(i-1)!(N-i)!(j-1)!(N-j)!}$$

$$= \sum_{\substack{i,j \text{ odd}}}^{N} (N-1) \binom{N-1}{i-1} \binom{N-1}{j-1}$$

$$= (N-1) \left(\sum_{i' \text{even}}^{N-1} \binom{N-1}{i'} \right) \left(\sum_{j' \text{even}}^{N-1} \binom{N-1}{j'} \right)$$

$$= (N-1)(2^{N-2})(2^{N-2}) = (N-1)2^{2N-4}$$

Since our numerator grows like $O(N^2)$, we are done, we see clearly that our fraction vanishes as $N \to \infty$.

6 Concentration and Convergence

In this section we prove a few results related to the behavior of our optimal functions as $N \to \infty$. We try to answer what might be useful questions regarding concentration.

Note that for a fixed degree, any polynomial with positive coefficients that add up to 1, the function is bounded from below by x^N for every x. With this in mind, we give the following result:

Proposition 6.1. *Let $\hat{p_N}$ be the optimal polynomial derived from the optimization problem in 1.2 of dimension N. Fix any positive $x < 1$. Then $p_N(x) \to 0$ as $N \to \infty$.*

This proposition essentially says that as we allow the maximum degree of our optimal polynomials to increase, the functions resemble a delta function at 1.

Lemma 6.2. *Let $\hat{a}(N)$ denote the optimal a obtained from the optimization problem 2.2 of dimension N, and $\hat{a}(N,k)$ its k-th coefficient. Now, fix k. Then, $\hat{a}(N,k) \to 0$ as $N \to \infty$.*

Proof. □

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