1 Introduction

Definition 1.1. Let $\mathbb{Z} \ni p \geq 0$. A *generalized Hilbert matrix* is a matrix $H \in M_n(\mathbb{R})$ whose entries are defined: $H_{ij} = \frac{1}{i+j-1+p}$

Our problem is to find $a \in \mathbb{R}^N$ such that

$$a^T H a \tag{1.1}$$

is minimized, where H is a generalized Hilbert matrix (in our case, p=2), subject to the constraint:

$$B^T a = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.2}$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \in M_{N \times 2}(\mathbb{R})$$

Our goal is to prove that such an a has coefficients that alternate in sign. Precisely:

Theorem 1.2. *Let H be a generalized Hilbert matrix of size N and $B \in M_{N \times 2}$ is as above. Furthermore, let*

$$\hat{a} = \text{argmin}_{a \in \mathbb{R}^N} \qquad a^T H a : \quad B^T a = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then for odd i, $\hat{a}(i) > 0$ and for even i $\hat{a}(i) < 0$.

We can obtain expressions for $\hat{a}$ via the method of Lagrange multipliers. Our problem amounts to minimizing

$$\frac{1}{2} a^T H a - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} (B^T a - \begin{pmatrix} 1 \\ -1 \end{pmatrix})$$

Taking the derivative with respect to a, we see that we must have

$$a = H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = H^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_2 \end{pmatrix}$$

and the constraint of $B^T a$ results in :

$$B^T H^{-1} B \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\implies \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{}$$