ON THE CONVERGENCE OF REGRET MINIMIZATION DYNAMICS IN CONCAVE GAMES

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Abstract

In this survey we are going to define "socially" concave games, where by using no-external regret (no-regret as presented in class) algorithm the average actions of the players converge to Nash equilibrium.

We'll see 2 such games which presented in the paper and I'll add 1 more.

1 Introduction

Nash equilibrium is where a game arrives to a steady state- each player doesn't have incentive to change her action given the other players actions. One issue is: how to get to this steady state?

Socially concave games are sub-class of concave games (the payoff of each player is concave in her action) where the payoff of each player is convex in the actions of the other players and there exists a linear combination of the payoffs s.t. it's concave in the actions of all the players.

The paper [5] shows that following a regret minimization algorithm in socially concave games leads the average actions to be $\mathcal{O}(\frac{n}{\sqrt{T}})$ close to Nash equilibrium in T steps (n is the number of players). 3 such games are shown in the paper, from which 2 will be discussed here.

Using online learning in games is intuitive- players need to make sequential decisions to maximize their payoff / minimize their regret. No regret algorithms are rational and distributed- meaning they correspond to the benefit of the player and require small information about the other players, so by employing one as a decision making tool the player is playing to her own benefit.

In this survey I'll give the problem setup, the main results from the paper and will show that 3 games are socially concave. The proofs for lemmas and theorems will be in the appendix.

2 Problem setup

2.1 Game model

$G = (N, \{S_i\}_{i=1}^n, \{u_i\}_{i=1}^n)$ is a n-person game

- $N = \{1, 2, ..., n\}$ - set of n players
- S_i - set of possible actions for player i
- $S = S_1 \times S_2 \cdots S_n$ - joint actions set and $s \in S$ is called joint actions vector where
  1. s_i is the i-th entry (which corresponds to the action of player i)
  2. s_{-i} is the actions of the all players except player i

  S is assumed to be closed, convex and bounded
- $u_i : S \to \mathbb{R}$ - player i's payoff function when joint actions vector $s \in S$ is played
  (note: $u_i(s) = u_i(s_i, s_{-i})$)
  u_i is assumed to be twice differentiable and bounded from above by 1

Definition 2.1 (ϵ-best response)**.** Action $x_i \in S_i$ is called ϵ-best response for s_{-i} if:
$\forall y_i \in S_i : u_i(x_i, s_{-i}) \geq u_i(y_i, s_{-i}) - \epsilon$

Definition 2.2 (ϵ-Nash equilibrium)**.** $s \in S$ is ϵ-Nash equilibrium if:
$\forall i \in N : s_i$ is ϵ-best response to s_{-i}

Definition 2.3 (Concave Game)**.** A game is called concave if:
$\forall i \in N : u_i(s) = u_i(s_i, s_{-i})$ continuous in s and concave in s_i

Definition 2.4 (Socially Concave Game)**.** A game is called socially concave if:

1. $\exists \lambda \in \Delta^n$ and $\forall i \in N : \lambda_i > 0$ s.t. $g(x) = \sum_{i \in N} \lambda_i u_i(x)$ concave in x
2. $\forall i \in N : u_i(x_i, x_{-i})$ convex in x_{-i} (the actions of the other players)

Lemma 2.5. *Socially concave game is also a concave game.*

Theorem 1 in [1] states that every concave game has a Nash equilibrium in S. Therefore, every socially concave game has a Nash equilibrium in S.

2.2 Regret minimization

For convex loss functions the regret of player i defined as:

$$Reg_i\left(\left\{x^t\right\}_{t=1}^T\right) = \sum_{t=1}^T f_t(x^t) - \min_{x \in \mathcal{K}} \sum_{t=1}^T f_t(x)$$

We would like to minimize the regret so the cumulative losses are less or equal to the best fixed action in hindsight, in the paper it's called no-external regret.
Because we're given payoffs and not losses we'll use the above definition where $u_i(x_i, x_{-i}) = -f_t(x)$:

$$Reg_i\left(\left\{x^t\right\}_{t=1}^T\right) = -\sum_{t=1}^T \left(-f_t(x^t)\right) + \max_{x \in \mathcal{K}} \sum_{t=1}^T \left(-f_t(x)\right) = \max_{y_i \in S_i} \sum_{t=1}^T u_i(y_i, x_{-i}^t) - \sum_{t=1}^T u_i(x^t)$$

And define $R_i(T) = \max\limits_{\{x^t\}_{t=1}^T} Reg_i\left(\left\{x^t\right\}_{t=1}^T\right)$ as the upper bound on $Reg_i(\cdot)$.
Note that the upper bound on the regret is always non-negative i.e. $R_i(T) \geq 0$.

3 Main results

For given t define:

- $\hat{x}^t = \frac{1}{t} \sum_{\tau=1}^t x^\tau$ - average of the joint actions vectors.
- $\hat{u}_i^t = \frac{1}{t} \sum_{\tau=1}^t u_i(x^\tau)$ - average payoff for player i.

Theorem 3.1. *Let $G = (N, \{S_i\}_{i=1}^n, \{u_i\}_{i=1}^n)$ be a socially concave game. If each player plays according to a procedure with external regret bound $R_i(t)$, then at time t:*

(i) $\hat{x}^t$ *is an* ϵ^t*-Nash equilibrium, where* $\epsilon^t = \frac{1}{\lambda_{min} t} \sum_{i=1}^{n} \lambda_i R_i(t)$

(ii) Player i's average payoff is $\epsilon_i^t = \frac{1}{\lambda_i t} \sum_{j=1}^{n} \lambda_j R_j(t)$ *close to the payoff of the average of the joint actions vectors, i.e.*

$$|\hat{u}_i^t - u_i(\hat{x}^t)| \leq \epsilon_i^t$$

Proof. We'll use 4 inequalities for the proof:

1. Using the upper bound definition we get

$$\begin{aligned}
\hat{u}_i^t &= \frac{1}{t} \sum_{\tau=1}^{t} u_i(x^\tau) \\
&\geq \max_{y_i \in S_i} \frac{1}{t} \sum_{\tau=1}^{t} u_i\left(y_i, x_{-i}^\tau\right) - \frac{1}{t} R_i(t) \\
&\underset{\text{max definition}}{\geq} \frac{1}{t} \sum_{\tau=1}^{t} u_i\left(BR_i\left(\hat{x}_{-i}^t\right), x_{-i}^\tau\right) - \frac{1}{t} R_i(t)
\end{aligned}$$

   Where $BR_i\left(\hat{x}_{-i}^t\right)$ is player i's best response action to $\hat{x}_{-i}^t$.

2. By Definition 2.4.2 $u_i(y_i, x_{-i})$ is convex in x_{-i}, so for fixed action $y_i \in S_i$

$$\frac{1}{t} \sum_{\tau=1}^{t} u_i(y_i, x_{-i}^\tau) \geq u_i(y_i, \frac{1}{t} \sum_{\tau=1}^{t} x_{-i}^\tau) = u_i(y_i, \hat{x}_{-i}^t)$$

3. By definition of best response, $\forall y_i \in S_i$

$$u_i(BR_i(x_{-i}), x_{-i}) \geq u_i(y_i, x_{-i})$$

4. Using Definition 2.4.1, $\exists \lambda \in \Delta^n$ and $\forall i \in N : \lambda_i > 0$ s.t. $\sum_{i=1}^{n} \lambda_i u_i(x)$ is concave in x

$$\begin{aligned}
\Rightarrow \sum_{i=1}^{n} \lambda_i u_i(\hat{x}^t) &= \sum_{i=1}^{n} \lambda_i u_i \left(\frac{1}{t} \sum_{\tau=1}^{t} x^\tau \right) \\
&\underset{\text{concavity}}{\geq} \frac{1}{t} \sum_{\tau=1}^{t} \sum_{i=1}^{n} \lambda_i u_i(x^\tau) \\
&\underset{\text{linearity}}{=} \sum_{i=1}^{n} \lambda_i \left(\frac{1}{t} \sum_{\tau=1}^{t} u_i(x^\tau) \right) \\
&= \sum_{i=1}^{n} \lambda_i \hat{u}_i^t
\end{aligned}$$

Using these 4 inequalities one by one we get

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i \hat{u}_i^t &\geq \sum_{i=1}^{n} \lambda_i \left[\frac{1}{t} \sum_{\tau=1}^{t} u_i(BR_i\left(\hat{x}_{-i}^t\right), x_{-i}^{\tau}) - \frac{1}{t} R_i(t) \right] \\
&\geq \sum_{i=1}^{n} \lambda_i \left[u_i(BR_i\left(\hat{x}_{-i}^t\right), \hat{x}_{-i}^t) - \frac{1}{t} R_i(t) \right] \\
&\geq \sum_{i=1}^{n} \lambda_i \left[u_i(\hat{x}_i^t, \hat{x}_{-i}^t) - \frac{1}{t} R_i(t) \right] \\
&= \sum_{i=1}^{n} \lambda_i \left[u_i(\hat{x}^t) - \frac{1}{t} R_i(t) \right] \\
&\geq \sum_{i=1}^{n} \lambda_i \left[\hat{u}_i^t - \frac{1}{t} R_i(t) \right] \\
&= \sum_{i=1}^{n} \lambda_i \hat{u}_i^t - \frac{1}{t} \sum_{i=1}^{n} \lambda_i R_i(t)
\end{aligned}$$

We got $a_1 \geq a_2 \geq a_3 \geq a_4 = a_5 \geq a_6 = a_7$.

$$a_3 - a_4 = \sum_{i=1}^{n} \lambda_i \left[u_i(BR_i\left(\hat{x}_{-i}^t\right), \hat{x}_{-i}^t) - u_i(\hat{x}_i^t, \hat{x}_{-i}^t) \right]$$

$$a_1 - a_7 = \frac{1}{t} \sum_{i=1}^{n} \lambda_i R_i(t)$$

By definition of player i's best response, $\forall y_i \in S_i$

$$u_i(BR_i\left(\hat{x}_{-i}^t\right), \hat{x}_{-i}^t) \geq u_i(\hat{x}_i^t, \hat{x}_{-i}^t)$$

and $\lambda_i > 0$, so $a_3 - a_4$ is sum of non-negative elements.

$$\forall j \in N : a_3 - a_4 \geq \lambda_j \left[u_j(BR_j\left(\hat{x}_{-j}^t\right), \hat{x}_{-j}^t) - u_j(\hat{x}^t) \right]$$

(the sum of non-negative elements is greater or equal than one element)

$a_1 \geq a_3$ and $a_7 \leq a_4 \Rightarrow a_3 - a_4 \leq a_1 - a_7$

$$\Rightarrow \lambda_j \left[u_j(BR_j\left(\hat{x}_{-j}^t\right), \hat{x}_{-j}^t) - u_j(\hat{x}^t) \right] \leq \frac{1}{t} \sum_{i=1}^{n} \lambda_i R_i(t)$$

$$\Rightarrow u_j(BR_j\left(\hat{x}_{-j}^t\right), \hat{x}_{-j}^t) - u_j(\hat{x}^t) \leq \frac{1}{\lambda_j t} \sum_{i=1}^{n} \lambda_i R_i(t) \leq \frac{1}{\lambda_{min} t} \sum_{i=1}^{n} \lambda_i R_i(t)$$

$$\lambda_{min} = \min_{j \in N} \lambda_j$$

Lemma 3.2. *a_i is ϵ-best response to s_{-i}* $\iff u_i(BR_i(s_{-i}), s_{-i}) - u_i(a_i, s_{-i}) \leq \epsilon$

Using Lemma 3.2: $\forall j \in N$: $\hat{x}_j^t$ is ϵ^t-best response
$\Rightarrow \hat{x}^t$ is ϵ^t-Nash equilibrium

First 3 inequalities gives

$$\hat{u}_i^t - u_i(\hat{x}^t) \geq -\frac{1}{t} R_i(t) \tag{1}$$

Inequality 4 gives

$$\begin{aligned}
&\sum_{i=1}^{n} \lambda_i \left[\hat{u}_i^t - u_i(\hat{x}^t)\right] \leq 0 \\
&\quad \Rightarrow \hat{u}_j^t - u_j(\hat{x}^t) \leq \frac{1}{\lambda_j} \sum_{i \neq j} \lambda_i \left[u_i(\hat{x}^t) - \hat{u}_i^t\right] \\
&\qquad\qquad\qquad\quad \underset{\text{Equation (1)}}{\leq} \frac{1}{\lambda_j t} \sum_{i \neq j} \lambda_i R_i(t) \\
&\qquad\qquad\qquad\quad \underset{R_i(t) \geq 0}{\leq} \frac{1}{\lambda_j t} \sum_{i=1}^{n} \lambda_i R_i(t)
\end{aligned} \tag{2}$$

Combining Equation (1) and Equation (2)

$$|\hat{u}_j^t - u_j(\hat{x}^t)| \leq \frac{1}{\lambda_j t} \sum_{i=1}^{n} \lambda_i R_i(t)$$

$$\left(\text{because } \frac{1}{\lambda_j} \sum_{i=1}^{n} \lambda_i R_i(t) \geq R_j(t)\right)$$

□

3.1 Consequences

- If all players play according to a no-regret algorithm, the average joint actions vector converges to Nash equilibrium and the payoff for each player converges to the payoff at that Nash equilibrium.
- If the no-regret algorithm is online gradient ascent [2], which has regret of $\mathcal{O}(\sqrt{t})$, then after t steps the average joint actions vector is $\mathcal{O}(\frac{n}{\sqrt{t}})$-Nash equilibrium.

3.1.1 Cournot Competition

Economic model used to describe competition between n firms that produce identical product. Each firm chooses the quantity of product it produces independently of each other and no cooperation is allowed. Its Nash equilibrium is called oligopoly (monopoly in when n=1), for the customers it's better than monopoly but worse than perfect competition (which can be achieved via Bertrand competition where companies choose the price and not the quantity).

$G^C = (N, \{S_i\}_{i=1}^n, \{c_i\}_{i=1}^n, p, \{u_i\}_{i=1}^n)$

- $N = \{1, 2, ..., n\}$
- $S_i = \mathbb{R}_+$ - production quantity
- $c_i : S_i \to \mathbb{R}_+$ - production cost
  Assumed convex

- $p : S \to \mathbb{R}_+$ - product price
  In Linear Cournot $p(s) = a - b\sum_{i=1}^{n} s_i$

- $u_i(s) = p(s)s_i - c_i(s_i)$ - payoff

Theorem 3.3. *Linear Cournot is socially concave game.*

3.1.2 Resource Allocation

n users competing for a share of resource with capacity $C > 0$ using bidding, no cooperation is allowed. The users know the allocation process for each them (and hence their payoff) depends on the bids of all users, what makes the process a game between n players [3].

$G^R = (N, \{W_i\}_{i=1}^{n}, C, \{M_i\}_{i=1}^{n}, \{\psi_i\}_{i=1}^{n}, \{u_i\}_{i=1}^{n})$

- $N = \{1, 2, ..., n\}$

- $W_i = [a, b]\,, a \leq b$ - bid space

- $C \in \mathbb{R}_+$ - resource capacity
  Assume $C = 1$ for simplicity

- $M_i : W \to [0, C]$ - resource allocation function
  $\forall s \in W : \sum_{i=1}^{n} M_i(w) = C = 1$
  In Linear Resource Allocation $M_i(w) = \frac{w_i^c}{\sum_{j=1}^{n} w_j^c}, c \in (0, 1]$

- $\psi_i : [0, C] \to \mathbb{R}_+$ - (monetary) value assigned to given allocation
  In Linear Resource Allocation $\psi_i(M_i(w)) = \alpha_i M_i(w), \alpha_i > 0$

- $u_i(w) = \psi_i\left(M_i(w)\right) - w_i$ - payoff
  In Linear Resource Allocation $u_i(w) = \alpha_i M_i(w) - w_i$

Theorem 3.4. *Linear Resource Allocation is socially concave game.*

3.1.3 Supply Function Bidding

n firms compete to meet infinitely divisible but inelastic demand for a product (when change in price incurs relative small change in demand). Supply function bidding means that the firms declare the amount they would supply / produce at any positive price, then a clearing house (a mediator) chooses a price so supply equals demand. The payoff of each firm depends on price, hence depends on the bids of the other firms, what makes it a game [4].

$G^S = (N, \{W_i\}_{i=1}^{n}, D, \{C_i\}_{i=1}^{n}, p, \{S_i\}_{i=1}^{n}, \{Q_i\}_{i=1}^{n})$

- $N = \{1, 2, ..., n\}$

- $W_i = \mathbb{R}_+$ - bid space (intuitively- revenue firm i willing to "forgo" at a given price)

- $D \in \mathbb{R}_+$ - (inelastic) demand

- $p(w) = \frac{\sum_{j=1}^{n} w_j}{(n-1)D} : W \to \mathbb{R}_+$ - product price

- $S_i(w) = D - \frac{w_i}{p(w)} : W \to \mathbb{R}$ - firm i supply
  (if negative- the supply firm i buys instead of produces)

- $c_i : S_i \to \mathbb{R}_+$ - supply cost
  Assume: continuous, $c_i(s_i) = 0$ if $s_i \leq 0$, $c_i(s_i)$ convex and strictly increasing for $s_i \geq 0$

- $Q_i(w) = p(w)S_i(w) - c_i\left(S_i(w)\right)$ - payoff

From Theorem 1 in [4] for $n > 2$ there exists Nash equilibrium $w^* \in W$ and it satisfies $\sum_{i=1}^{n} w_i^* > 0$.

Theorem 3.5. *Supply Function Bidding is socially concave game.*

4 Critical review and conclusion

The paper introduces a new sub-class of games and shows that by employing no-regret algorithm we get to Nash equilibrium. Achieving equilibrium in certain games (say resource allocation of energy) is important (say for sustainability), and if we can prove the game is social we can have an algorithm to get there.
The intuition to Definition 2.4

1. lower bound the payoff of the average actions

2. lower bound the average payoff for specific action

Assumption 2 makes sense: it is so we can work with the average of the other players' actions.
Assumption 1 is problematic because there isn't analytic way to find those λs except from guessing.
The paper also shows that 3 games are in this class, 2 are pretty simple (Cournot and resource allocation) while the 3rd (congestion control) is more complex. Finding another game in this class (supply function bidding) took me some time and is from the same authors as resource allocation, so I'm not sure how big / important is this class.

5 Open questions

The paper deals with the situation where the players are rational and non-cooperative so they employ no-regret algorithm. One way of further investigation is to attack the 'rationality'- what will happen when fraction of the players play for a goal other than maximizing their payoff? e.g. adversarial players: in the Cournot case Chinese firms that want to burden the American economy (and vise versa) or in congestion control hackers that want to slow down the network. Another way is to allow cooperation- players forming coalitions to maximize their joint payoff.

References

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6 Appendix

Proof of Lemma 2.5

$u_i(x_i, x_{-i}) = \frac{1}{\lambda_i}\left(\sum_{j=1}^{n} \lambda_j u_j(x_i, x_{-i}) + \sum_{j\neq i} \lambda_j \left(-u_j(x_i, x_{-i})\right)\right)$

First expression concave in x from Definition 2.4.1.

u_j convex in x_{-j} from Definition 2.4.2 so convex in x_i, $(i \neq j) \Rightarrow -u_j$ concave in $x_i \Rightarrow$ second expression concave in x_i.

$\Rightarrow u_i(x_i, x_{-i})$ concave in x_i.

$\square$

Proof of Lemma 3.2

1. a_i is ϵ-best response to $s_{-i} \Rightarrow \forall y_i \in S_i$, in particular $y_i = BR_i(s_{-i})$

$$u_i(BR_i\left(s_{-i}\right), s_{-i}) - u_i(a_i, s_{-i}) \leq \epsilon$$

2. $u_i(BR_i\left(s_{-i}\right), s_{-i}) - u_i(a_i, s_{-i}) \leq \epsilon$, from best response definition, $\forall y_i \in S_i$

$$u_i(y_i, s_{-i}) \leq u_i(BR_i(s_{-i}), s_{-i})$$

$$\Rightarrow u_i(y_i, s_{-i}) - u_i(a_i, s_{-i}) \leq u_i(BR_i(s_{-i}), s_{-i}) - u_i(a_i, s_{-i}) \leq \epsilon$$

$\Rightarrow x_i$ is ϵ-best response to s_{-i}

$\square$

Proof of Theorem 3.3

1. Take $\lambda_i = \frac{1}{n}$:

$$\begin{aligned} g(s) &= \sum_{i=1}^{n} \frac{1}{n} u_i(s) \\ &= \frac{1}{n} \sum_{i=1}^{n} \left[a - b \sum_{i=1}^{n} s_i\right] s_i - c_i(s_i) \\ &= \frac{1}{n}\left[a \sum_{i=1}^{n} s_i - b\left(\sum_{i=1}^{n} s_i\right)^2 - \sum_{i=1}^{n} c_i(s_i)\right] \end{aligned}$$

$c_i(s_i)$ convex in s_i and independent on i, so the hessian is positive definite $\Rightarrow c_i(s_i)$ convex in s.
$\left(\sum_{i=1}^{n} s_i\right)^2$ convex in s (the hessian is a positive semi-definite matrix of 2s).
$\Rightarrow -c_i(s_i)$ and $-b\left(\sum_{i=1}^{n} s_i\right)^2$ concave in s.
$a\sum_{i=1}^{n} s_i$ linear in s so also concave.

$\Rightarrow g(s)$ concave in s.

2.

$$\begin{aligned} u_i(s) &= p(s)s_i - c_i(s_i) \\ &= \left(a - b\sum_{i=1}^{n} s_i\right) s_i - c_i(s_i) \\ &= as_i - bs_i\left(s_i + \sum s_{-i}\right) - c_i(s_i) \end{aligned}$$

$-bs_i\left(s_i + \sum s_{-i}\right)$ linear in $s_{-i} \Rightarrow$ convex in $s_{-i} \Rightarrow u_i$ convex in s_{-i}.

$\square$

<u>Proof of Theorem 3.4</u>
Assume $\forall i \in N, w_i \geq 0$ and $\sum_{i=1}^{n} w_i > 0$
Note that in Linear Resource Allocation $u_i(w) = \alpha_i \frac{w_i^c}{\sum_{j=1}^{n} w_j^c} - w_i$

1. Take $\lambda_i = \frac{\frac{1}{\alpha_i}}{\sum_{j=1}^{n} \frac{1}{\alpha_j}}$:

$$\begin{aligned} g(w) &= \sum_{i=1}^{n} \frac{\frac{1}{\alpha_i}}{\sum_{j=1}^{n} \frac{1}{\alpha_j}} u_i(w) \\ &= \frac{1}{\sum_{j=1}^{n} \frac{1}{\alpha_j}} \sum_{i=1}^{n} \left[\frac{w_i^c}{\sum_{j=1}^{n} w_j^c} - \frac{w_i}{\alpha_i} \right] \\ &= \frac{1}{\sum_{j=1}^{n} \frac{1}{\alpha_j}} \left(\frac{\sum_{i=1}^{n} w_i^c}{\sum_{j=1}^{n} w_j^c} - \frac{1}{\alpha_i} \sum_{i=1}^{n} w_i \right) \\ &= \frac{1}{\sum_{j=1}^{n} \frac{1}{\alpha_j}} \left(1 - \frac{1}{\alpha_i} \sum_{i=1}^{n} w_i \right) \end{aligned}$$

   $g(w)$ linear in $w \Rightarrow$ concave in w.

2. I'll prove this part with 2 different approaches (a and b):

   (a) Define

   - $w_{-i}^c = \left(w_1^c, \cdots, w_{i-1}^c, w_{i+1}^c, \cdots, w_n^c\right)^T$
   - W_{-i}^c matrix s.t. $\forall r, j \in N - \{i\}$:

$$W_{-i}^c(r, j) = \delta_{r,j} w_j^c$$

   Note W_{-i}^c is PSD (diagonal matrix with non-negative diagonal).

   $\Rightarrow u_i(w) = u_i(w_i, w_{-i}) = \alpha_i \frac{w_i^c}{w_i^c + \mathbf{1}^T w_{-i}^c} - w_i$

   We'll show $u_i(w)$ convex in w_{-i}:

$$\frac{\partial u_i(w_i, w_{-i})}{\partial w_{-i}} = -\alpha_i w_i^c c \left(w_i^c + \mathbf{1}^T w_{-i}^c\right)^{-2} w_{-i}^{c-1}$$

$$\begin{aligned} \frac{\partial^2 u_i(w_i, w_{-i})}{\partial^2 w_{-i}} =& 2\alpha_i w_i^c c \left(w_i^c + \mathbf{1}^T w_{-i}^c\right)^{-3} w_{-i}^{c-1} \left(w_{-i}^{c-1}\right)^T \\ &+ (1-c)\alpha_i w_i^c c \left(w_i^c + \mathbf{1}^T w_{-i}^c\right)^2 W_{-i}^{c-2} \end{aligned}$$

   - $w_{-i}^{c-1} \left(w_{-i}^{c-1}\right)^T$ is PSD $\left(\forall x \in \mathbb{R}^{n-1} : x^T y y^T x = \left\|y^T x\right\| \geq 0\right)$
   - W_{-i}^{c-2} is PSD
   - $\alpha_i > 0$
   - $w_i^c \geq 0$
   - $\left(w_i^c + \mathbf{1}^T w_{-i}^c\right) = \sum_{i=1}^{n} w_i^c > 0$
   - $c \in (0, 1]$

$\Rightarrow u_i(w)$ convex in w_{-i}.

(b) Let $k \in W_i$:
(note: $k + \sum_{j \neq i} w_j > 0$)

$g_k(x) := \sum_{j=1}^{n-1} x_j^c : \mathbb{R}_+^{n-1} \to \mathbb{R}_+$
$g_k(x)$ concave in x $\Rightarrow \forall x, y \in \mathbb{R}_+^{n-1}, \beta \in [0,1] : g_k(\beta x + (1-\beta)y) \geq \beta g_k(x) + (1-\beta) g_k(y)$

$f_k(x) := \frac{\alpha_i}{k+x} - k : \mathbb{R}_+ \to \mathbb{R}_+$
*(note $f_k(x)$ strictly decreasing as x gets bigger)
$f_k''(x) = \frac{2\alpha_i}{(k+x)^3} > 0 \Rightarrow f_k(x)$ convex in $x \Rightarrow \forall x \in \mathbb{R}_+, \gamma \in [0,1] : f_k(\gamma x + (1-\gamma)y) \leq \gamma f_k(x) + (1-\gamma) f_k(y)$

$u_k(w_{-i}) := u(k, w_{-i}) = f_k(g_k(w_{-i})) : \mathbb{R}_+^{n-1} \to \mathbb{R}_+$
We'll show $u_k(\cdot)$ convex for $\mathbb{R}_+^{n-1}$:

$$\begin{aligned}
&\forall x, y \in \mathbb{R}_+^{n-1}, \lambda \in [0,1] : \\
&\qquad u_k(\lambda x + (1-\lambda)y) = f_k(g_k(\lambda x + (1-\lambda)y)) \\
&\qquad \underset{*}{\leq} f_k(\lambda g_k(x) + (1-\lambda) g_k(y)) \\
&\qquad \leq \lambda f_k(g_k(x)) + (1-\lambda) f_k(g_k(y)) \\
&\qquad = \lambda u_k(x) + (1-\lambda) u_k(y)
\end{aligned}$$

□

Proof of Theorem 3.5
Assume $\sum_{i=1}^n w_i > 0$

$Q_i(w) = p(w)D - w_i - c_i(S_i(w)) = \frac{1}{n-1} \sum_{j=1}^n w_j - w_i - c_i(S_i(w))$

$\frac{1}{n-1} \sum_{j=1}^n w_j - w_i$ linear in w_i
$c_i(S_i(w))$ convex in $w_i \Rightarrow -c_i(S_i(w))$ concave in w_i
$\Rightarrow Q_i(w)$ concave in $w_i \Rightarrow G^S$ concave game.

1. Take $\lambda_i = \frac{1}{n}$:
   $g(w) = \sum_{i=1}^n \lambda_i Q_i(w) = \frac{1}{n} \sum_{i=1}^n Q_i(w)$

   $\frac{1}{n-1} \sum_{j=1}^n w_j - w_i$ linear in w
   $-c_i(S_i(w))$ concave in w_i for each i and independent on i $\Rightarrow$ concave in w
   $\Rightarrow g(w)$ concave in w

2. $Q_i(w) = Q_i(w_i, w_{-i}) = \frac{1}{n-1} \sum_{i \neq j} w_j + \frac{n}{n-1} w_i - c_i(S_i(w))$
   $\sum_{i \neq j} w_j$ linear in $w_{-i} \Rightarrow Q_i(w)$ convex in w_{-i}

□