1 Formal definition of a Turing machine

A Turing machine is a 7-tuple (Q, Σ, Γ, δ, q_0, q_{accept}, q_{reject}):

- Q is the set of states
- Σ is the input alphabet *not* containing special blank symbol $\sqcup$
- Γ is the tape alphabet satisfying $\Sigma \subset \Gamma$ and $\sqcup \in \Gamma$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function
- $q_0 \in Q$ is the start state
- $q_{accept} \in Q$ is the accept state
- $q_{reject} \in Q$ is the reject state, $q_{reject} \neq q_{accept}$

The input alphabet Σ never contains $\sqcup$, so $\Sigma \neq \Gamma$ is always true.
A Turing machine can never contain a single state, as any machine must have distinct states q_{accept} and q_{reject}.

2 Turing machine computation

Tape content is unbounded but always finite, and the first (leftmost) blank symbol marks the end of tape content.

A configuration C_1 yields the configuration C_2 if the Turing machine can legally go from C_1 to C_2 in a single step. A **configuration** consists of:

- The current state
- The tape content
- The head location

The head can be in the same location in two successive steps if the machine attempts to move its head off the left-hand end – we assume, by definition, that it just stays in the same cell rather than throwing an error.

The **start configuration** on an input $w \in \Sigma^*$ consists of start state q_0, w as the tape content, and the head location is the first (leftmost) position of the tape.

A configuration is **accepting** if its state is q_{accept}. A configuration is **rejecting** if its state is q_{reject}.

Accepting and rejecting configurations are **halting** configurations.

A Turing machine M **accepts** an input w if there is a sequence of configurations $C_1, C_2, ..., C_k$ such that:

1. C_1 is the start configuration of M on input w

2. C_i yields C_{i+1} for $1 \leq i \leq k-1$
3. C_k is an accepting configuration

The **language** of M, denoted $L(M)$, is the set of strings accepted by M.

3 Turing-recognisable languages

A language L is **Turing-recognisable** if there is a Turing machine M that recognises it, i.e. L is the language of M.
Turing-recognisable means the same thing as **semi-decidable** and **recursively enumerable**.

If M recognises L, it may or may not halt on words not in L.

3.1 Closure under operations

Example. The collection of Turing-recognisable languages is closed under union.
Proof. Let L_1 and L_2 be Turing-recognisable languages and M_1 and M_2 be Turing machines that recognise them. Construct a Turing machine M' that recognises the union of L_1 and L_2. On input w:

- Run M_1 and M_2 alternatively on w, step-by-step. If either accept, accept. If both halt and reject, reject.

If either M_1 and M_2 accept w, M' accepts w and the accepting Turing machine (either M_1 or M_2) arrives to its accepting state after a finite number of steps.

If both M_1 and M_2 reject and either does so by looping, M' will loop.
The solution for Turing-decidable languages would not work here as Turing machines can loop. If M_1 is looping, the construction used for Turing-decidable languages will loop even if M_2 accepts w, and thus, w is the union of L_1 and L_2.

3.2 Examples

Example 1. Show that $\bar{E} = \{\langle M \rangle | M \text{ is a TM and } L(M) \neq \emptyset\}$ (the complement of E) is Turing-recognisable.

Proof. Run M in parallel on all (countably infinitely many) possible inputs.
This can be done in stages $0, 1, ...$: at stage k, run M on each of the first k inputs for k steps each.
If a simulation accepts, terminate and accept.

Example 2. Show that $E = \{\langle M \rangle | M \text{ is a TM and } L(M) = \emptyset\}$ is not Turing-recognisable.

As $\bar{E}$ is Turing-recognisable, if E itself were also Turing-recognisable, it would also be decidable. However, this is not the case (section 4.2 below), so this is a contradiction.

4 Turing-decidable languages

A language L is **Turing-decidable** if there is a Turing machine M that accepts every $w \in L$ and rejects every $w \notin L$.
Turing-decidable means the same thing as **recursive**.

If M decides L, it always halts.

4.1 Closure under operations

Example. The collection of decidable languages is closed under union.
Proof. For any two decidable languages L_1 and L_2, let M_1 and M_2 be the Turing machines that decide them. Construct a Turing machine M' that decides the union of L_1 and L_2. On input w:

1. Run M_1 on w. If it accepts, accept.
2. Run M_2 on w. If it accepts, accept. Otherwise, reject.

M' accepts w if either M_1 or M_2 accepts it. If both reject, then M' rejects.

4.2 Examples

Example 1. Let $E = \{\langle M \rangle | M \text{ is a TM and } L(M) = \emptyset\}$ be the language of all descriptions of Turing machines recognising the empty language. E is undecidable.

Proof. Given an instance of the Halting problem M and w, construct a Turing machine S that takes a single number k as input, simulates M on w for at most k steps, and accepts only if the simulation has reached a terminal configuration. Now, $L(S)$ is empty if and only if M doesn't terminate on w.
Therefore, an algorithm that decides $E(S)$ would solve the halting problem, so E is undecidable.

Example 2. Let $EQ = \{\langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ be the language of all pairs of descriptions of Turing machines recognising the same language. EQ is undecidable (by reduction from E).

Proof. Take E_0 to be a trivial Turing machine that rejects anything. This means that $EQ(M, E_0)$ would solve $E(M)$ (example 1), a contradiction.

5 Multitape Turing machines

A **Multitape Turing machine** is like a single tape Turing machine with several tapes, each with its own head. The only difference in the formal definition is the transition function, which is now:

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

where k is the number of tapes.

Theorem. Every multitape Turing machine has an equivalent single tape Turing machine.

6 Non-deterministic Turing machines

A **non-deterministic Turing machine** has a transition function:

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

Theorem. Every non-deterministic Turing machine has an equivalent deterministic Turing machine.
Proof. Consider the tree of all possible computations of the non-deterministic Turing machine. Start from the root (the start configuration) and do a breadth-first search. Accept only if an accepting configuration is found.

- DFS would not work
- Can use a multitape Turing machine to implement the BFS

7 Church-Turing thesis

Intuitive notion of an algorithm is equivalent to the mathematical concept of an algorithm defined by Turing machines (or any other formal model of computation, such as λ-calculus, Post machines, recursive functions)

8 Universal Turing machine

Every Turing machine M can be encoded as a word over a finite alphabet. Use $\langle M \rangle$ to denote the **encoding** of a Turing machine M.

Theorem. There is a Turing machine U that takes a two-part input – the encoding of a Turing machine M ($\langle M \rangle$) and a word w, and simulate M on w. U is called a **universal Turing machine**.

9 Halting problem

Halting problem: Given an encoding of a Turing machine M and a word w, does M terminate on w?

Proposition. The Halting problem is Turing-recognisable.
Proof. Run a universal Turing machine on the pair $(\langle M \rangle, w)$. Accept if the computation eventually terminates.

Proposition. The Halting problem is not Turing-decidable.

Proof. Assume, for contradiction, there is a turing machine H that decides the Halting problem.

$$H(\langle M \rangle, w) = \begin{cases} \text{accept} & \text{if } M \text{ terminates on } w \\ \text{reject} & \text{if } M \text{ does not terminate on } w \end{cases}$$

Use H as a black box to create an instance of the Halting problem, on which, H fails.
Consider a Turing machine D that takes the description of a single Turing machine M as an input and does the following:

$$D(\langle M \rangle) = \begin{cases} \text{accept} & \text{if } H(\langle M \rangle, \langle M \rangle) \text{ rejects} \\ \text{loop} & \text{if } H(\langle M \rangle, \langle M \rangle) \text{ accepts} \end{cases}$$

What happens when D runs on its own encoding, $\langle D \rangle$?

1. D terminates on $\langle D \rangle$. By the construction of D, $H(\langle D \rangle, \langle D \rangle)$ rejects, giving a wrong answer

2. D does not terminate on $\langle D \rangle$. By the construction of D, $H(\langle D \rangle, \langle D \rangle)$ accepts, giving a wrong answer.

10 Turing-recognisable vs. Turing-decidable

Theorem. A language L is Turing-decidable if and only if both L and its complement, $\bar{L}$, are Turing-recognisable.
Proof. Suppose M_1 recognises L, and M_2 recognises $\bar{L}$.
On an input w, run M_1 and M_2 in parallel – i.e. simulate alternating steps of M_1 and M_2 on a multitape Turing machine.
Either M_1 or M_2 must eventually accept – accept if M_1 accepts and reject if M_2 accepts.

11 Co-Halting problem

Given an encoding of a Turing machine M, and word w, is it the case that M doesn't terminate on w, i.e. is not Turing-recognisable.

12 Step-counter predicate

Step (M, w, k) if and only if the machine M terminates on w in no more than k steps, is Turing-decidable.