Undergraduate Algebraic Topology.

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November 28, 2019

Contents

1 Euler Characteristic and Triangulations. -Tuesday, 4.2.2018

1.1 Announcements.

1.) We use Massey. There are two versions.
   (a.) We use *Algebraic Topology: An Introduction.*
2.) See webpage. math.uchicago.edu/m̃bergeron/math263.html
3.) Grades:
   (a.) Weekly hw: 25%
   (b.) Midterm: 30%
   (c.) Final: 40%
   (d.) Participation: 5%; i.e. come to class
4.) Piazza: won't answer questions that are pertinent to ≥ 2 people. (Participation points can be earned from Piazza.) Password for Piazza is relue (i.e. Euler backwords.)
5.) Homework
   (a.) Write in LaTeX

(b.) *Submit in groups of 2.*
  i.) See Piazza.
  ii.) Cannot submit more than half the assignments by same person, and they cannot be in a row.
(c.) Diagrams can be by hand.
6.) OH: TBD.

1.2 What is Algebraic Topology?

Algebraic topology is about distinguishing the qualitative features of spaces and maps. In algebra and analysis, they are more rigid. Here, things are a lot more flexible, but there is a trade off. In some sense, it is understanding the "world" in relation to "relationships."

1.3 Euler Characteristic.

Consider a planar disc, *up to homeomorphism*. How can we describe this qualitatively?

We start with: how can we break this up into triangles? There are many ways to triangulate a disc. The simplest is when we just have a disc which is a topological triangle. We also have:

Figure 1: Simplest triangulation.

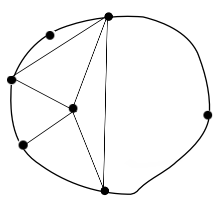


Figure 2: Triangulation 2.

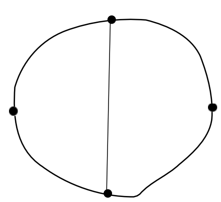


Figure 3: Triangulation 3.

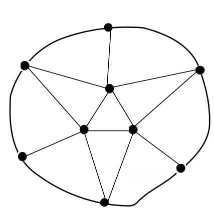
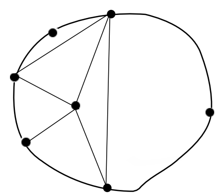


Figure 4: Triangulation 4.



We can then count, V, E, F.

Table 1: Number of faces, vertices, and edges in the three triangulations.

Triangulations.	V	E	F	V-E+F
1	3	3	1	1
2	7	12	6	1
3	9	18	10	1

Proposition 1 (Euler). $V - E + F = 1$ for a disc, regardless of triangulation.

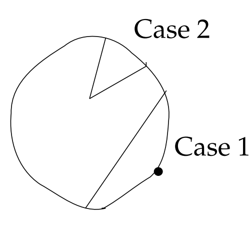
Remark 2. What's remarkable about this is that in the RHS, there are no choices whereas in the LHS, there are many choices.

Since we have not defined anything rigorously, the proof is inherently hand wavy.

PROOF 3. We do induction on the number of faces. The base case is obvious.

For the induction step: want to remove a triangle in a controlled way so take one with a boundary edge.

Figure 5: The two possible cases.



What happens when we remove a boundary triangle? Then we still get a triangulated disc with one less face (call it D'). So from the induction hypothesis,

$$V(D') - E(D') + F(D') = 1 \tag{1}$$

What changes when we add this triangle back? Case 1: We then get $V = V' + 1, E = E' + 2, F = F' + 1$. For case 2, we can do likewise.

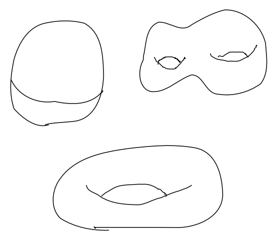
$\square$

1.4 Closed Surfaces.

Another interesting example of a topological space is a closed surface, i.e. a compact, connected, oriented, topological manifolds.

Definition 4. Σ_g is a closed surface of genus g.

Figure 6: Closed surfaces. The genus is just the number of holes. We attach handles (via surgery) to $\mathbb{S}^2$ to obtain these.



Proposition 5 (Euler). For all triangulations of Σ_g, we have

$$V - E + F = 2 - 2g \tag{2}$$

Remark 6. Note that RHS is *global* whereas LHS is *local*, so this is already a very deep fact.

We can see from the picture the following.

Example 7. For $\mathbb{S}^2$, we get 2.

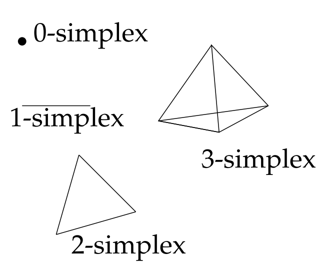
Example 8. For $\mathbb{T}^2$, we have 0.

PROOF 9 (Sphere.). Homework. Hint: split $\mathbb{S}^2$ into north and south hemispheres. $\square$

Remark 10. A big theme in algebraic topology is the interplay between local and global information.

Proposition 11. Let X be a compact [1] space triangulated using n-simplices (n=0,1,2,3,...).

Figure 7: n-simplices.



[1] We assume Hausdorff throughout.

Let C_i be the number of i-simplices. Then

$$\chi(X) = \sum_i (-1)^i C_i \tag{3}$$

and this does not triangulation.

Remark 12. Compactness gives finite triangulation.

Remark 13. We can think of alternating sum as a generalization of the counting principal that accounts for overcount.

1.5 Application: Hairy Ball and Poincaré-Hopf.

Embed Σ_g in $\mathbb{R}^3$ (e.g. a sphere). Define a smooth vector field V on Σ_g. For instance, for $\mathbb{S}^2$ we can take

$$V(x, y, z) := (-y, x, 0) \tag{4}$$

This vanishes at the north and south pole. This is a vortex. We can also have sources, sinks, and saddles.

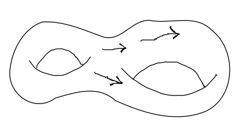
Question. Which Σ_g admits a nonvanishing field?

On a 2-torus, we can just go around the donut. (Writing the torus as a square in the usual way, we can just have the arrows pointing in the horizontal or vertical directions.)

Proposition 14 (Poincaré-Hopf). Σ_g has a nonvanishing vector field iff $g = 1$ iff $\chi(\Sigma_g) = 0$.

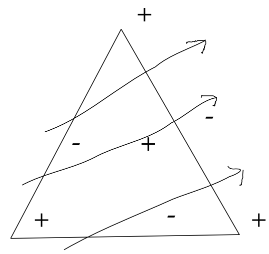
PROOF 15 (Thurston.). One direction is trivial, as we observed above. Given a nonzero field on the surface, we compute the Euler characteristic.

Figure 8: Vector field on genus 2 surface.



Use the smoothness of the vector field to take neighborhoods on which the vector fields mostly points in the same direction. Take triangulations contained in these neighborhoods. At each vertex, put a "plus charge," and a "minus charge" on the edge. The Euler characteristic is then the total charge. We then use the flow given by the vector field to move the charges. Then the charges in each triangle is 0, so the Euler characteristic is 0.

Figure 9: Charges after flowing for infinitesimal time.



(Of course, there is some technicalities we are brushing aside. For instance, if the vector field is in the direction of the edge, then we need to use the genericness of smoothness, and then you perturb as in differential topology.)

□

2 -Thursday, 4.4.2018

We want to obtain new spaces by operations on old spaces, namely gluing and collapsing.

2.1 Quotient Topology.

Example 16 (Torus.).

Figure 10: Identifying two ends of a cylinder gives a 2-torus.

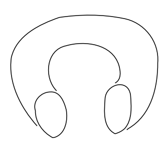
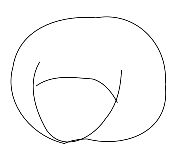


Figure 11: Collapsing the circle down to a point gives this.



Definition 17. Let X be a space and $\sim$ some equivalence relation. Then the **quotient space** $Y := X/\sim$ is set of equivalence classes with topology given by the quotient map p, i,e, $U \subset Y$ is a open set iff $p^{-1}(U) \subseteq X$ which thus gives the topology.

Remark 18. When a group $G \curvearrowright X$, we can take $x_1 \sim x_2$ iff $g \cdot x_1 = x_2$.

Example 19. For the torus, if we took a neighborhood on the torus, it could either map back to the boundary circles or interior point of the cylinder as in figure 2.1.

Example 20. $\mathbb{Z} \leq \mathbb{R}$ where we view $\mathbb{R}$ as a group under addition. Here, $x \sim x + n$. (Here $\mathbb{R}$ is the **fundamental domain.**) We have

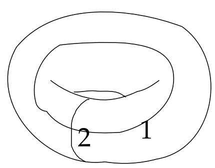
$$\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1 \tag{5}$$

Example 21 (Pacman.). $\mathbb{Z}^2 \leq \mathbb{R}^2$. Then modding out $\mathbb{R}^2$ by the translation action gives a 2-torus

$$\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 \tag{6}$$

Details: if we had a point in $\mathbb{R}^2$, we can translate to the unit square. Now for the boundaries, the opposites are identified. More intuitively, roll up the unit square, make a cylinder, and identify each end.

Figure 12: Circle 1 and 2 correspond to the boundaries in the unit square.



Remark 22. The $\mathbb{R}$ and $\mathbb{R}^2$ are examples of **covering spaces** of the resulting quotient space.

2.2 Connected Sum.

Example 23. Poke a hole in a genus 1 and genus 2 surfaces, and then identify the boundaries of the holes via some homeomorphism of the disc. When we smoothly stretch the boundary of the discs and connected the two, then we can ensure the resulting thing is smooth.
Then we get a genus 3 surface. More generally,

$$\Sigma_{g_1} \# \Sigma_{g_2} = \Sigma_{g_1+g_2} \tag{7}$$

Figure 13:

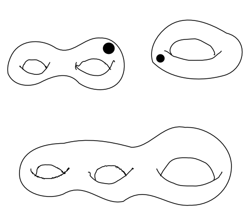
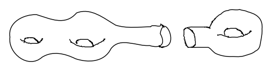


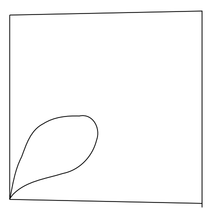
Figure 14:



Another perspective: Let's understand the connected sum in terms of quotient operation from before.

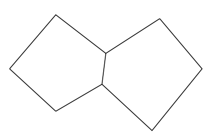
Example 24. [2]
Using a homeomorphism, we can make a hole on a torus look like the following:

Figure 15:



Deleting the interior of the disc and then using a homeomorphism, we get a pentagon.

Figure 16:



Identifying opposite ends of the respective pentagons, we get a genus 2 surface. We can see from this construction that all the vertices of the octagon are identified. (Just look at which points are the same when we glued the two pentagons together.)

Proposition 25 (Classification of Surfaces.). Any compact oriented surface is homeomorphic to a sphere or a connected sum of tori.

Remark 26. Note that we can obtain each of these by identifying edges of 2-dimensional things. A sphere can be obtained by identifying the north and south hemispheres.

[2] This whole construction is in Massey.

2.3 Orientability.

Definition 27. A surface is **orientable** if there exists a locally consistent assignment of clockwise/counterclockwise choice ("twirly") at each point.

Example 28 (Annulus). An annulus is orientable.

Example 29 (Möbius Band.).

Example 30 (Kelin Bottle.). The strip down the middle of the square is a Möbius band hence not orientable.

Example 31 ($\mathbb{R}P^2$.). Same as Klein bottle except you have two strips, vertical and horizontal.

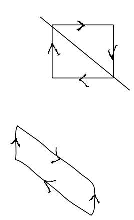
We have

$$\mathbb{R}P^2 := \mathbb{S}^2/Z_2 \tag{8}$$

i.e. mod the Z_2 action, or just the antipodal map.

What happens when we take connected sums of $\mathbb{R}P^2$? From a similar construction as for before, we see that we get a Klein bottle. The key here is cutting and pasting.

Figure 17: Cut down the diagonal.



3 Classification of Surfaces. -**Tuesday, 4.8.2018**

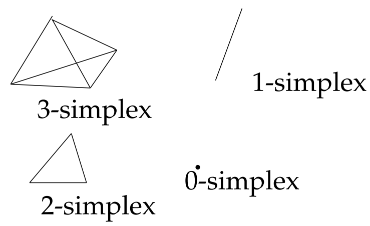
3.1 Simplicial Complex.

We are moving towards the proof of the classification of surfaces. For this, we need to understand triangulations of surfaces Σ, i.e. a homeomorphism between Σ and a collection of triangles.

Definition 32. A **simplicial subset of** X is a topological space obtained from a collection of simplicies by identifying faces.

Simplicial complex is a choice, in the same way as a triangulation of a surface.

Figure 18: Examples of simplices.



Remark 33. Give it the topology as in that of $\mathbb{R}^n$.

We can go in the other direction, i.e. decompose a space into simplices. Concretely, subspace of $\mathbb{E}^n$ (Euclidean n-space) that is a union of simplices subject to the condition that the intersection of two simplices is a simplex. We are not allowed to identify faces of a simplex with itself.

Remark 34. A graph is a 1-dimensional simplicial complex.

Figure 19: We can decompose this graph into six 1-simplices.

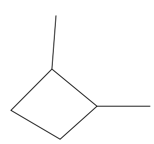


Figure 20: This graph can be decomposed into three 1-simplices.

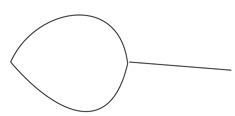


Figure 21:

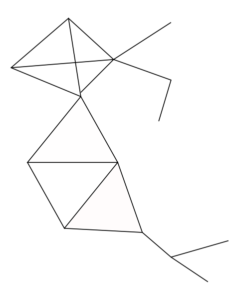
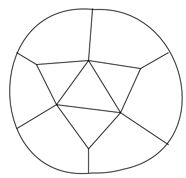
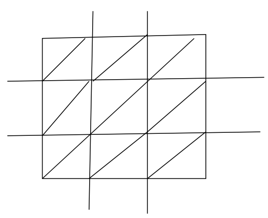


Figure 22:



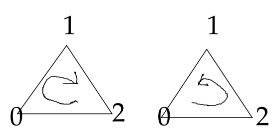
Definition 35. A **triangulation of a space** is a homeomorphism with a simplicial subset of X.

Figure 23: Triangulation of a torus. Note that it is not sufficient to just take the diagonal of the square.



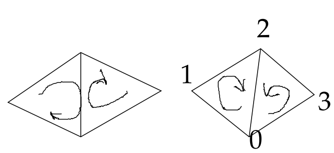
We can now make orientability precise by looking at the permutation of the vertices of triangles. We can generalize this to higher dimensions.

Figure 24: Orientation of a 2-simplex. The left has ordering $[0, 1, 2]$ (hence positive orientation) and the right has $[0, 2, 1]$ which has negative orientation.



Definition 36. **Triangulated surface is orientable** if can orient all the 2-simplices compatibly.

Figure 25: The left is a compatible, and the right is incompatible. The point is that the 2-simplices induces an orientation on the 1-simplices on the boundaries. Look at the uniquely defined labels on the vertices. Algebraically, we need to look at an even permutation that gives one to the other.



Definition 37. The **euler characteristic of a simplical complex** K is given by

$$\chi(K) = \sum (-1)^i \, |i - \text{simplices}| \tag{9}$$

We still need to show that this is homeomorphism invariant.

Let K be a surface. What can we say about $\chi(K)$? We first prove that $\chi(K) \leq 2$. The idea: since K is triangulated, we can just consider the graph given by the triangulation, i.e. the subset consisting of 1-simplices (This is the **1-skeleton**, or more generally, an **i-skeleton** is looking at the simplices of dimension i or lower).

For instance, for a sphere, it is just a tetrahedron. Let T be a spanning tree, and T^* be its "dual graph."

Figure 26: Spanning tree of the given vertex and its dual.

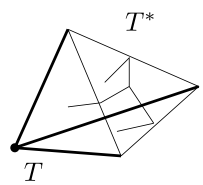
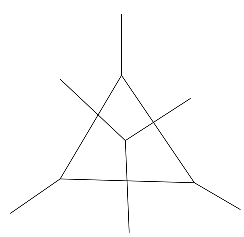


Figure 27: Graph.



Remark 38 (Barycentric Subdivision.). Puncture the triangle in the middle and break it up. This is the rigorous way of doing it. Let K' be the first barycentric subdivision.

T^* has vertices in center of each face and edges whenever faces not separated by Γ.

We see in the above that

$$\chi(K) = \chi(T) + \chi(T^*) \leq 2 \tag{10}$$

This just follows from observation. We get the inequality from homework 1. In particular, this proves Euler's formula for the plane because the above gives a sphere.
The idea for the classification of surfaces is to work inductively upwards, i.e. transform a surface in a controlled way to get a sphere.

Proposition 39 (Detecting a sphere.). Let K be a surface.The following are equivalent.

1.) Every simple closed curve in K' separates K.
2.) $\chi(K) = 2$.
3.) $K \simeq \mathbb{S}^2$.

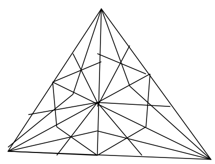
PROOF 40. We show $1 \implies 2 \implies 3 \implies 1$.

(1 $\implies$ 2.) Choose a maximal tree T in K, and let T^* be its dual (which lives in K'). In this case, T^* is a tree, for if not, then we can find a simple curve that separates.

We claim that T^* is a tree by 1, which gives $\chi(K) = 2$. $\square$

(2 $\implies$ 3.) If $\chi(K) = 2$, then $\chi(T) = \chi(T^*) = 1$. If we now thicken the two graphs T, T^* to $N(T), N(T^*)$, then we can write the sphere as the union of the two.

Figure 28: The "thickening" of this is to expand the edges to the respective boundarutes if the triangles.



Now since $N(T) \simeq N(T^*) \simeq D^2$ and so,

$$K = N(T) \cup_\phi N(T^*) \simeq \mathbb{S}^2 \tag{11}$$

where ϕ is the identification of the boundaries of the two disks.

(3 $\implies$ 1.) Follows from point set topology via Jordan Curve-type statement. $\square$

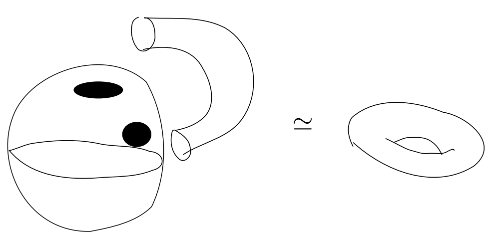
3.2 Proof of Classification of Surfaces.

To prove the classification theorem, we will work inductively on the Euler characteristic χ. To reduce any surface to $\mathbb{S}^2$ via surgeries.

Proposition 41. Any closed orientable surface is homeomorphic to $\mathbb{S}^2$ or $\mathbb{S}^2$ with handles (i.e. a genus g surface Σ_g for $g \in \mathbb{N}_0$).

Adding handles is similar to taking connected sums. It is essentially taking a connected sum with a cylinder.

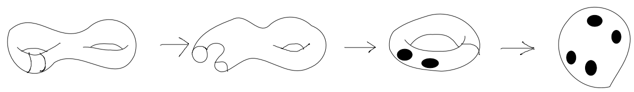
Figure 29: Examples of simplices.



PROOF 42. Given K, we know that $\chi(K) \leq 2$ with equality iff $K = \mathbb{S}^2$, so convert K to $\mathbb{S}^2$ with using χ-increasing surgeries.

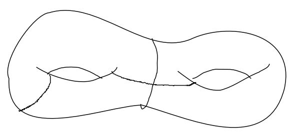
Surgeries are just cutting open with nonseparating open curve.

Figure 30: Proof idea: keep cutting open to get sphere. Process terminates because (need to show:) Euler characteristic increases with each step.



We need to say something more formal. Let L be a nonseparating simple closed curve.

Figure 31: Proof idea: keep cutting open to get sphere. Process terminates because (need to show:) Euler characteristic increases with each step.



We can now thicken L to get $N(L)$ in K'' which is homeomorphic to a cylinder. Now, $K = N \cup M$ where M us a surface with boundary consisting of L_1, L_2.

When we cut open, we take the cone to "cap up" the hole:

$$K_* = M \cup CL_1 \cup CL_2 \tag{12}$$

where C denotes the coning operation. Now K_* is obtained from K by surgery along L.

The claim:

$$\chi(K_*) > \chi(K) \tag{13}$$

Proof of Claim. This is almost entirely by observation. We then get

$$\begin{aligned}\chi(K_*) &= \chi(M) + \chi(CL_1) + \chi(CL_2) - \chi(L_1) - \chi(L_2) \\ &= \chi(M) + 1 + 1 - 0 - 0 \\ &= \chi(M) + 2\end{aligned}$$

(See HW2 for the inclusion exclusion principle for Euler characteristic.)

But now,

$$\begin{aligned}\chi(K) &= \chi(M) + \chi(N) - \chi(M \cap N) \\ &= \chi(M) + 0 - 0 = \chi(M)\end{aligned}$$

So indeed we increase Euler characteristic in each step of the surgery.

Here $K = N(L) \cup M$ where M is the complement with intersection.

□

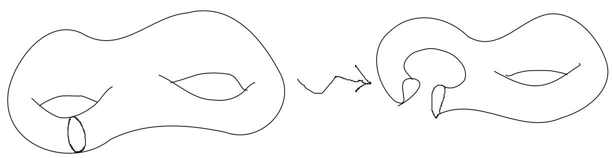
4 . -Thursday, 4.11.2018

Come talk to Bergeron if class is too slow.

4.1

Last time: surgery along nonseparating simple closed curves increases Euler characteristic.

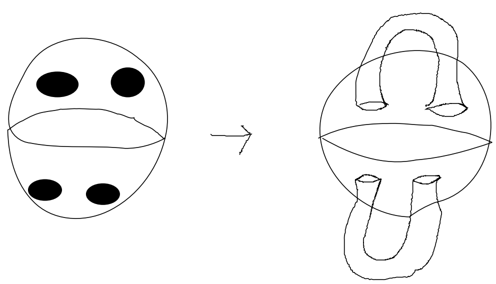
Figure 32: Decrease genus of Σ_2.



Since $\chi(\Sigma) = 2$ iff $\Sigma \simeq \mathbb{S}^2$, and so can change any surface into $\mathbb{S}^2$ using finitely many surgeries because if $\chi = 2$, then there exists a simple closed curve.

Result is a $\mathbb{S}^2$ with marked disks, and this allows us to realize any (oriented) surace as an $\mathbb{S}^2$ with handles.

Figure 33: $\mathbb{S}^2$ with marked disks.



Orientability is required to get a cylinder (as opposed to Mobius band).

Note that this is different from Massey's proof (which is more combinatorial). Our proof has a more algebraic topology flavor.

4.2 Euler Characteristic.

Euler characteristic is a generalization to counting which is good for understanding weird space. We know how to deal with numbers, so this is great.

The next thing is the notion of distance which gives geometry. This is like throwing strings into space and checking their lengths.

For both of the above, we convert spaces to numbers using a "probe": the fundamental group. This essentially looks at how we can wrap loops in our spaces.

We first look at what these objects are invariant under. Since we are losing rigidity, we need a notion less rigid than homeomorphism.

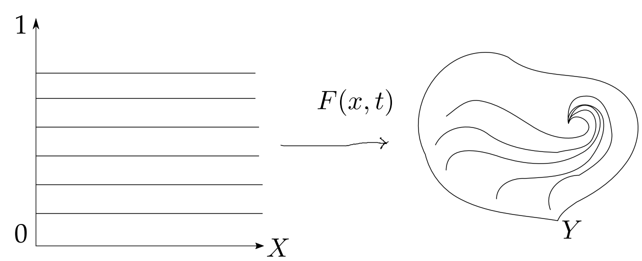
Definition 43. **Two maps $f : X \to Y$ between topological spaces are homotopic** if there exists a continuous map

$$F : X \times [0,1] \to Y \tag{14}$$

satisfying $F(x, 0) = f(x), F(x, 1) = f(x)$.

Let's look at what this means geometrically.

Figure 34: Homotopy of Maps.

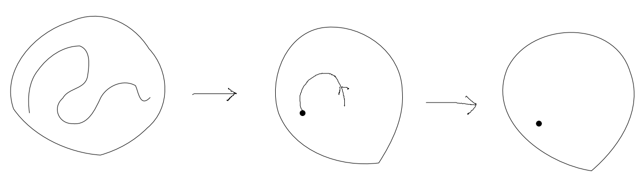


Remark 44. Note that F must be continuous as a map from the *product space* rather than on each individual variable. Otherwise, much of algebraic topology becomes trivial. See paper on Piazza: "The Fundamental Group of the Circle is Trivial." (The statement in the title is obviously false...)

Definition 45. If f is homotopic to the constant map, then it is **null homotopic**.

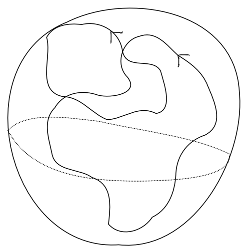
Example 46. All maps from $[0,1]$ to *any* topological space Y are null homotopic. This is because $[0,1]$ is null homotopic.

Figure 35: Just "suck in the string" into the starting point.



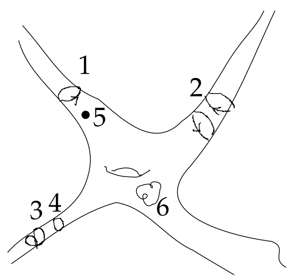
Example 47. Take a path $\mathbb{S}^1$ to $\mathbb{S}^2$. They are *all* null homotopic. Unlike the previous example, $\mathbb{S}^1$ is not null homotopic.

Figure 36: $\mathbb{S}^2$ is null homotopic.



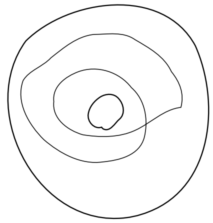
Example 48. Another example of a space for which the fundamental group is nontrivial:

Figure 37: Here, 1 and 2 are not homotopic because you "get stuck." In 3, the curve loops twice, so it is not homotopic to 4. 5 and 6 are homotopic because we can squish 6 into a point and then translate.



Example 49 (Annulus.). Now consider maps from $\mathbb{S}^1$ to an annulus Y (which we of course know is homeomorphic to $\mathbb{C}^*$). We see a group isomorphism between the homotopy classes of maps and $(\mathbb{Z}, +)$.

Figure 38: The group isomorphism is given by looking at the winding number of the paths (details nontrivial).

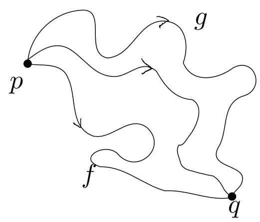


Definition 50. **Two paths** $f, g : [0,1] \to X$ **from points** p **to** q **are path homotopic** denoted $f \simeq_p g$ if there exists a continuous map

$$F : [0,1]^2 \to X \tag{15}$$

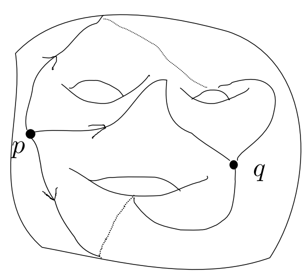
such that they keep the end points, i.e. $F(0,t) = p, F(1,t) = q$ for all $t \in [0,1]$, and (usual condition for homotopy:) $F(s,0) = f(s)$ and $F(s,1) = g(s)$ for all s.

Figure 39: Picture for path homotopy. Note that all intermediate steps has the same start and end point.



So why care about paths? Well, there is a natural operation between two paths sharing same end point, namely concatenation.

Figure 40: The three paths are not path homotopic.

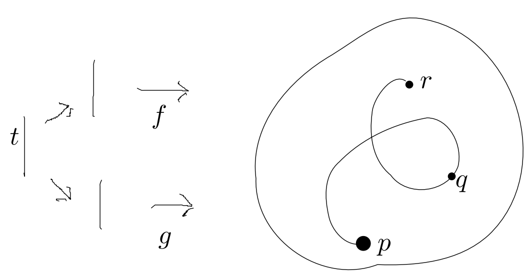


Definition 51. If f, g are paths, then the concatenation is given by

$$(f * g)(t) = \begin{cases} f(2t) & t \in [0, 1/2] \\ g(1-2t) & t \in [1/2, 1] \end{cases} \tag{16}$$

i.e., move at twice the speed and go through both paths.

Figure 41: Concatenation of paths.



Proposition 52 (Key Proposition.). The operation $*$ is well-defined for the equivalence class of path-homotopic paths.

PROOF 53. Homotope the individual paths.

□

Definition / Proposition 54. Fix the end point $p \in X$. Let $[f]$ denote homotopy class and

$$[f] * [g] := [f * g] \tag{17}$$

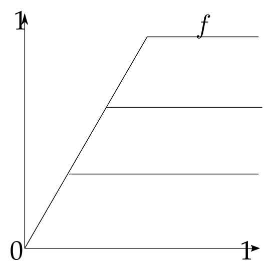
has identity (class of null homotopic maps), inverses, and the operation is associative.

The proof is a routine exercise in the diagram which is discussed in Massey.

PROOF 55. **1.)** Identity element 1_p:

$$[1_p] * [f] = [f] \iff 1_p * f \simeq_p f \tag{18}$$

Figure 42: Homotopy for identity element.



Details left as exercise or just read Massey. Continuity needs some work.

2.) Define the map $\overline{f}(t) = f(1-t)$, i.e. go backwards

$$[f][\overline{f}] = [1_p] \tag{19}$$

Figure 43: Inverse element case.

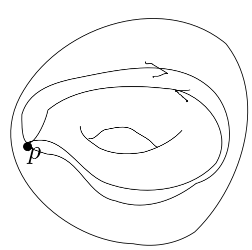
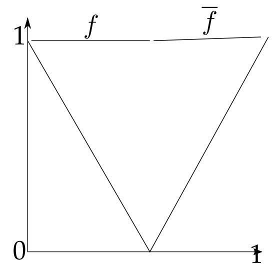


Figure 44: Diagram for the inverse case.



3.) For associativity, observe

$$([f][g])[h] = [f * g][h] \tag{20}$$

$$([f * g])[h] = [f][g * h] \tag{21}$$

and check the diagram.

□

Definition 56. This **fundamental group** (i.e. the set of the equivalence classes of paths above and the concatenation operation defined on these classes) is denoted $\pi_1(X, x)$ whee x is the base point.

Example 57 (Annulus.). Under this notation,

$$(\pi_1(\mathbb{C}^*, 1), *) \simeq (\mathbb{Z}, +) \simeq (\pi_1(\mathbb{S}^1, 1), *) \tag{22}$$

where the isomorphism is of course in the sense of groups.

Proposition 58. For a path connected space, the base point does not matter, i.e. the resulting fundamental group is isomorphic:

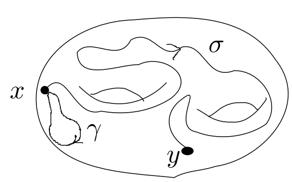
$$\pi_1(X, x) \simeq \pi_1(X, y) \tag{23}$$

for all $x, y \in X$.

PROOF 59. Isomorphism is given by

$$[\gamma] \mapsto [\sigma]^{-1}[\gamma][\sigma] \tag{24}$$

Figure 45: Isomorphic of fundamental groups are given by conjugation with the class of σ.



□

Remark 60. π_1 is a **functor** (in the sense of category theory), i.e. eats one thing, spits out another, and preserves structure. Here, π_1 functor maps from the category of topological spaces to the category of groups.

Remark 61. If $f : (X, x) \to (Y, y)$ (this notation indicates a **base point preserving map**, i.e. map $f : X \to Y$ such that $y = f(x)$), then

$$f_* : \pi_1(X, x) \to \pi_1(Y, y) \tag{25}$$

is given by pushforward, i.e.

$$[\gamma] \mapsto [f \circ \gamma] \tag{26}$$

and thus, f defines a homomorphism. If we additionally had a map $g : (Y, y) \to (Z, z)$, then

$$(f \circ g)_* = f_* \circ g_* \tag{27}$$

5 . -Tuesday, 4.16.2018

Recall that last time, we introduced the fundamental group $\pi(X, x)$ which is the collection of the homotopy classes of the loops starting and ending at x in the space X for which the group operation is concatenation of paths. $\pi_1(\cdot, \cdot)$ is an example of a **functor** in the language of category theory.

Also recall that we noted that

$$(\pi_1(\mathbb{S}^1, x), \cdot) \simeq (\mathbb{Z}, +) \tag{28}$$

We also said that

$$(\pi_1(\mathbb{S}^2, x), \cdot) \simeq \{e\} \tag{29}$$

5.1 Continuous Maps and Homotopy on Fundamental Groups.

We also noted that this machinery works well with continuous maps. This is where we start.

Take continuous maps

$$(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z) \tag{30}$$

which induces the group homomorphisms

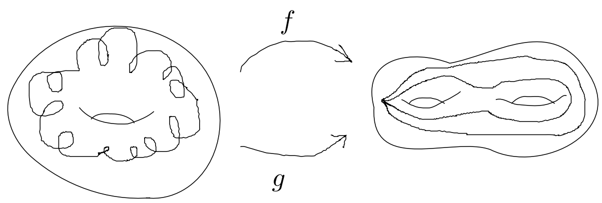
$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, y) \xrightarrow{g_*} \pi_1(Z, z) \tag{31}$$

given by $f_*[\sigma] = [f \circ \sigma]$. The nice property is that this $*$ behaves nicely with composition:

$$(f \circ g)_* = g_* \circ f_* \tag{32}$$

If $f \simeq f'$, then $f_* = f'_*$.

Figure 46: The path on the left is the same as a nice smooth loop around the hole. Two homotopic maps f, g give homotopic paths in the image.



Definition 62. A **retraction** $r : X \to X$ is a map such that $r \circ r = r$. $r(X)$ is a **retract** of X.

Example 63. r fixes points in its image.

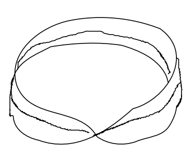
Example 64 (Constant Map.).

Remark 65. Contractible spaces have trivial fundamental group (e.g. $\mathbb{R}^n$), but the converse is not true (e.g. $\mathbb{S}^2$ is not contractible but has trivial fundamental group).

Example 66 (Projection.). Consider the map $p : X \times Y \to X \times Y$, given by

$$p : (x, y) \mapsto (x, y_0) \tag{33}$$

Figure 47: There are two canonical circles on the Möbius band: the central circle and the boundary circle. We can retract the whole thing onto the central circle, but not to the boundary circle. (The latter is nontrivial.)



Example 67 (Möbius Band.).

If $A \subseteq X$ is a retract, then the inclusion map $i : A \hookrightarrow X$ has a left inverse, i.e. $i \circ r = \mathbf{Id}_A$. This is a triviality. Passing this through the functor π_1 gives you something interesting:

Proposition 68. i_* is injective, and r_* is surjective.

PROOF 69. Proof is just notation. Just observe that

$$r_* \circ i_* = (r \circ i)_* = (\mathbf{Id}_A)_* = \mathbf{Id}_{\pi_1(A)} \tag{34}$$

It goes without saying that we are tacitly taking the base point for $\pi_1(A)$ to be a point in A. $\square$

5.2 Covering Space.

We have established some notation and built machinery, but $\pi_1(X)$ in general is hard to compute. However, we can use an approach analogous to classifying surfaces to compute nontrivial fundamental groups. In other words, we reduce a nontrivial fundamental group to the trivial fundamental group by some general procedure. [3]

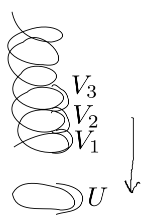
We start with an example. The idea is to unwrap looks.

Example 70 (Key Example.). Consider the map

$$\rho : \mathbb{R} \to \mathbb{S}^1 \subseteq \mathbb{C} \tag{35}$$

$$\theta \mapsto e^{i\theta} \tag{36}$$

Figure 48: $\mathbb{R}$ is a covering space of $\mathbb{S}^1$



Definition 71. Let $\rho : E \to B$ be a continuous map. An open set $U \subseteq B$ is **evenly covered** by ρ if

$$\rho^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha \tag{37}$$

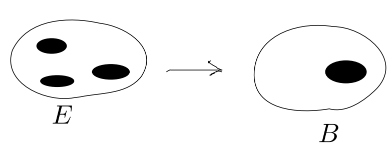
where the union is a disjoint union, and such that $\rho : V_\alpha \to U$ is a homeomorphism.

Example 72. Homeomorphism is a trivial covering map.

Remark 73. The intuition is "stack of pancakes."

Remark 74. ρ is a covering if for all $b \in B$, there exists an evenly covered neighborhood of b.

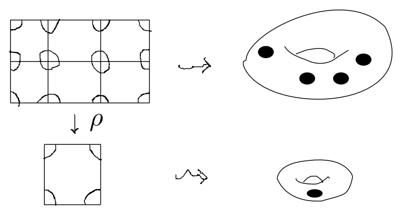
Figure 49:



[3] Note that we are not following Massey's book linearly.

Example 75. Here is the first nontrivial example:

Figure 50: One should think of the big rectangle wrapping the small torus. A neighborhood in the small torus corresponds to one circle in the small square, and then there are many copies in the big rectangle which gives bunch of circles in the big torus.



The big rectangle is a covering space of the small torus.

Figure 51: The top and bottom of the right diagram maps to the left circle in the bouquet. The left and right loops correspond to the right circle in the bouquet. They respect the orientations.

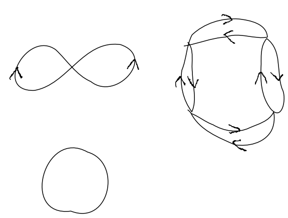


Figure 52: This also covers the bouquet of two circles. Every other circle maps to the left circle, and the others map to the right circle in the bouquet. Note that we need to be careful of the "spider point," i.e. where the circles meet.

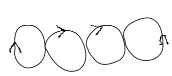
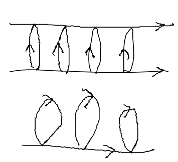
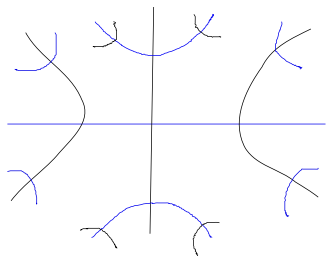


Figure 53: Unlike the previous ones, these are infinite covering spaces.



Lines map to right circle in the bouquet,
The loops map to the left circle in the bouquet.

Figure 54: Universal Covering Space of Bouquet of Two Circles: Blue maps to right circle of bouquet, others map to the left. Note that being a hyperbola is not important; we can just take a fractal thing with short line segments. Note that this is like the splitting fields in Galois theory!!! It's like adjoining roots of unity until you get the nice thing. (In fact, we can make this analogy precise!)



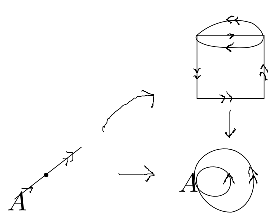
Example 76 (Bouquet of Two Circles.). Observation: Note that this last picture is contractible, and so, the fundamental group is trivial.

Definition 77. A **lift of** $X \xrightarrow{f} B$ **to** $E \xrightarrow{\rho} B$, is a map $\widehat{f}$ so that the commutative diagram commutes.

$$\begin{tikzcd} & E \arrow[d, "\rho"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Insert north east arrow from X to E called $\hat{f}$

Figure 55: A lifts to one of the four corners of the square, and after that, follow the arrows to lift the path.



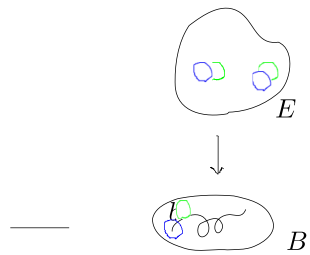
The formal statement of the above is this.

Proposition 78. If $\rho : E \to B$ be a cover. Let $\sigma : I \to B$ be a path $\sigma(0) = b \in B$. For each $C \in \rho^{-1}(b)$, there exists a unique lift $\widehat{\sigma} : I \to E$ such that $\widehat{\sigma}(0) = e$.

PROOF 79. The idea is the same as in the previous example. Let $\{U_i\}$ be a covering of B by evenly covered open sets. The set $\left\{\sigma^{-1}(U_i)\right\}$ is an open cover of $[0, 1]$. Let δ be the **Lebesgue number** (i.e., every subset of the interval of diameter less than $\delta := \frac{1}{n}$ is contained in some $\sigma^{-1}(U_i)$).

Use the evenly covered neighborhoods to lift the path upstairs.

Figure 56: Paths lift nicely for covering spaces. Use the evenly covered neighborhoods to lift the path upstairs.



□

Here is the second part of the theorem.

Proposition 80. If σ, σ' are path homotopic into B, then the lift $\widehat{\sigma}, \widehat{\sigma}'$ is unique lifts starting at C, then $\widehat{\sigma} \simeq_p \widehat{\sigma}'$.

The proof is an exercise using the Lebesgue lemma, but now using the unit square.

In particular, we have:

Proposition 81. $\rho_* : \pi_1(E) \to \pi_1(B)$ is injective.

PROOF 82. We prove the kernel is trivial. Take

$$\rho_*([\widehat{\sigma}]) = \mathbf{Id}_{\pi_1(X,x)} \tag{38}$$

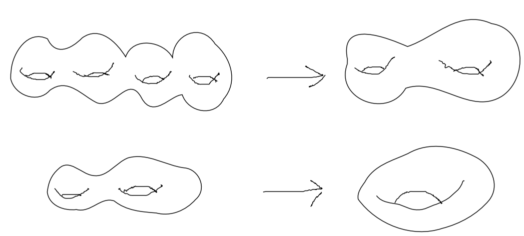
where $\widehat{\sigma}$ is a loop upstairs. Now $\rho \circ \widehat{\sigma} \simeq_p \mathbf{Id}_b$. But from the previous proposition, the lifts are homotopic, we must have $\widehat{\sigma} \simeq_p \mathbf{Id}_e$.

□

6 . -Thursday, 4.18.2018

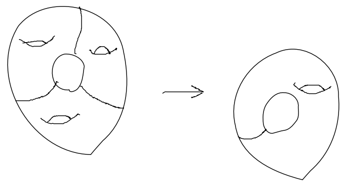
True or false: orientable surface of genus $g \geq 2$ is a a cover of genus 2. Is a genus 2 surface a covering of a torus.

Figure 57:



Answer: we have a *branched* cover, but not a cover. This is transparent from the following picture (which is homeomorphic to the first pair):

Figure 58:



We can do likewise for the second pair.

Path lifting $\rho : E \to B$ cover. In the commutative diagram, for a map σ into B induces $\widehat{\sigma}$ mapping into E. Also $\sigma \simeq_p \sigma'$ then $\widehat{\sigma} \simeq_p \widehat{\sigma}'$ here the lifts are based at the same point.

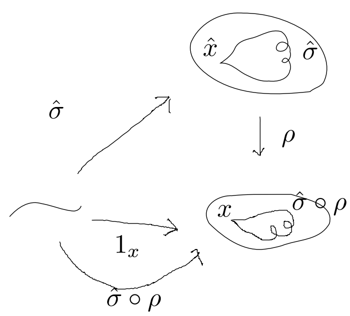
Here is a corollary.

Proposition 83. Consider the map $\rho : \widehat{X} \to X$ given by $\rho(\widehat{x}) = x$. Then

$$\rho_* : \pi_1(\widehat{X}, \widehat{x}) \to \pi_1(X, x) \tag{39}$$

is injective.

Figure 59:



PROOF 84. Consider a loop $\widehat{s}$ in $\widehat{X}$ upstairs. Then composing with ρ i.e. taking the map $\rho \circ \widehat{\sigma}$ downstairs gives a loop downstairs. Thus,

$$\rho_*([\widehat{\sigma}]) = 1_{\pi_1}(X, x) \tag{40}$$

So $\rho \circ \widehat{\sigma} \simeq_p 1_x$. Thus $\widehat{\sigma} \simeq_p 1_{\widehat{x}}$.

Insert ne arrow from Interval to $\widehat{X}$ called $\widehat{\sigma}$

Figure 60:

$$\begin{tikzcd} & \hat{X} \arrow[d, "\rho"] \\ \text{Interval} \arrow[ur, "\widehat{\sigma}"] \arrow[r, "\rho\circ\widehat{\sigma}"] & X \end{tikzcd}$$

□

Proposition 85. Take a map $\rho : (\widehat{X}, \widehat{x}) \to (X, x)$. Then there exists a map

$$\phi : \pi_1(X, x) \to \rho^{-1}(x) \tag{41}$$

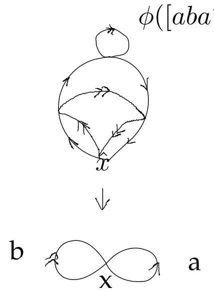
sending $[\sigma]$ to an end point of lift based at $\widehat{x}$. This induces an group action of $\pi_1(X, x) \curvearrowright \rho^{-1}(x)$ called the **monodromy**. Note that the action is on the fibers.

Remark 86. This is like the phase factor of the complex logarithm from complex analysis.

Example 87. Consider:

Figure 61:

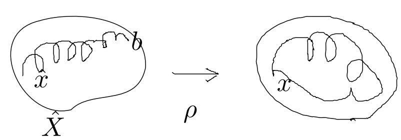
$\phi([aba^{-1}]) = \phi([aa])$



Some basic properties:

1.) The map ϕ is surjective.

Figure 62:



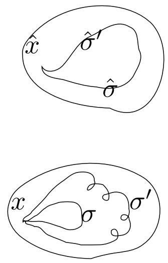
2.) ϕ is injective if $\widehat{X}$ is simply connected. Thus we get a bijection.

If

$$\phi([\sigma]) = \phi([\sigma']) \tag{42}$$

then we want to show $[\sigma] = [\sigma']$.

Figure 63:



PROOF 88.

$$\begin{aligned}[\widehat{\sigma}] &= [\widehat{\sigma}][\overline{\widehat{\sigma}} * \widehat{\sigma}'] \\ &= [\widehat{\sigma} * \overline{\widehat{\sigma}}][\widehat{\sigma}'] \\ &= [\widehat{\sigma}']\end{aligned}$$

where we used simple connectedness when we said $[\overline{\widehat{\sigma}} * \widehat{\sigma}'] = [1_{\widehat{x}}]$.

Thus, $[\sigma] = [\sigma']$. ☐

Corollary of the above is the following (which we never proved before):

Proposition 89. There is a bijection between $\pi_1(\mathbb{S}^1)$ and $\mathbb{Z}$.

We can see from the above because we have countably many fibers. The group structure needs more work.

We now discuss the morphisms in the category of fundamental group.

Definition 90. Let $\widehat{X}_1, \widehat{X}_2$ be covering spaces of X, and let the covering maps be ρ_1, ρ_2. A **morphism from** ρ_1 **to** ρ_2 is a map

$$f : \widehat{X}_1 \to \widehat{X_2} \tag{43}$$

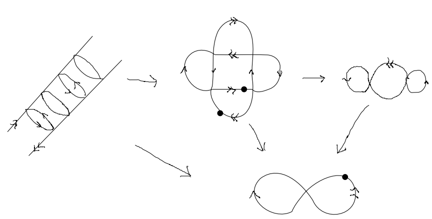
such that the diagram commutes.

Figure 64:

$$\begin{array}{ccc} \hat{X}_1 & \xrightarrow{f} & \hat{X}_2 \\ & & \\ & X & \end{array}$$

Arrow from $\hat{X}_1$ to X called ρ. Likewise for the other covering space.

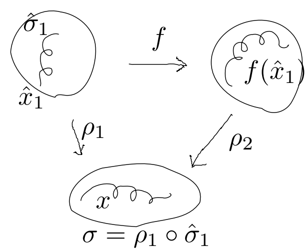
Figure 65:



Here is a key fact.

Proposition 91. Morphism $\widehat{X}_1 \to \widehat{X}_2$ uniquely determined by image of any point because there exists a unique path lifting.

Figure 66: $\widehat{\sigma}_2$ is the unique lift of σ at $f(\widehat{x}_1)$.



Now that we have morphisms, we look at automorphisms.

Definition 92. The **group of deck transformations** (or the **Galois group of transformations**) is the group $\mathrm{Aut}(X \to \widehat{X})$ which are all the isomorphisms as covers.

The silly example is the trivial automorphism group for X covering itself.

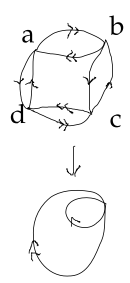
Remark 93. The analogy with Galois theory: base space is like the base field. The covering space is like the field extensions. The universal cover is like the splitting field.

Example 94. We claim that the Klein 4-group is the automorphism group.

Observe that any automorphism induces an automorphism of each fiber. Thus $\mathrm{Aut}(\widehat{X}) \hookrightarrow \mathrm{Aut}(\rho^{-1}(x))$.

Intuitively, the D_8 is the symmetry of the square, but now we need to preserve the orientations of the lines in this diagram, so we do not get the full dihedral group.

Figure 67: There are two intuitively clear candidates for the automorphism group: $\mathbb{Z}_2 \times \mathbb{Z}_2$ and D_8.



We can just consider the permutations

$$(a \mapsto a) = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} \tag{44}$$

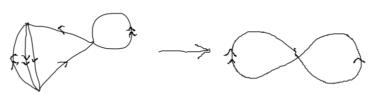
$$(a \mapsto b) = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix} \tag{45}$$

$$(a \mapsto c) = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} \tag{46}$$

Definition 95. The covering $\widehat{X} \to X$ is a **regular cover** (or **Galois cover**) if for all $\widehat{x}_1, \widehat{x}_2 \in \rho^{-1}(x)$ such that there exists $f : \widehat{X} \to \widehat{X}$ such that $f(\widehat{x}_1) = \widehat{x}_2$.

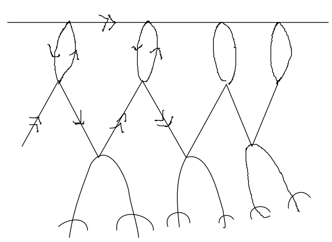
Example 96 (Nonexample). Consider:

Figure 68:



Here is an (infinite) covering space that is not regular:

Figure 69:



Proposition 97. A cover $\widehat{X} \to X$ is regular iff $\rho_*(\pi_1(\widehat{X})) \trianglelefteq \pi_1(X)$.

7 Path Lifting and Seifert Van Kampen. -Tuesday, 4.23.2019

7.1 Lifting of Paths.

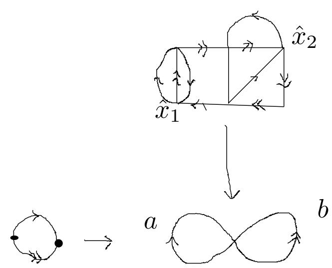
Question: when can you life paths to the covering space?

Proposition 98 (Keystone Lifting Lemma.). There exists a lift $\widehat{f} : (Y, y) \to (\hat{X}, \hat{x})$ at $\hat{x}$ iff $f_*(\pi_1(Y, y)) \hookrightarrow \rho_*(\pi_1(\widehat{X}, \widehat{x}))$.

$$\begin{tikzcd} & (\hat{X}, \hat{x}) \arrow[d, "\rho"] \\ (Y, y) \arrow[ur, "\hat{f}"] \arrow[r, "f"] & (X, x) \end{tikzcd}$$

Remark 99. No problem when $Y = [0, 1]$.

Figure 70:



We need to ask if $f_*(\pi_1(Y, y))$ is embedded in $\rho_*(\pi_1(\widehat{X}, \widehat{x}_1))$ and $\rho_*(\pi_1(\widehat{X}, \widehat{x}_2))$. But we know that this is not the case since ab^{-1} is not a loop based at $\widehat{x}_1, \widehat{x}_2$. There is also no consistent choice of lifting the points: depending on the choice of paths, there at least two points to which z is send in the top right diagram.

Remark 100. Now a purely topological question is answered algebraically in the case $Y = \mathbb{S}^2$, i.e. paths always lift.

PROOF 101 (Sketch). .
($\Longrightarrow$.) Functoriality.

($\Longleftarrow$.) Build a lifting of point by point using paths for which the lifting is well defined.

□

We come back to this topic next lecture.

7.2 Free Groups.

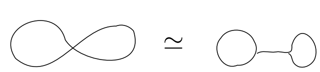
We develop algebraic machinery to compute fundamental groups explicitly. First question: when do two spaces have the same fundamental groups? This leads to the following definition:

Definition 102. X, Y have the same homotopy type if there exists maps $f : X \to Y,\ g : Y \to X$ for which each composition is the identity on X, Y respectively. The map f is a **homotopy equivalence.**

Immediate consequence of the definition:

Proposition 103. If two spaces have the same homotopy types, then they have the same fundamental groups.

Figure 71:



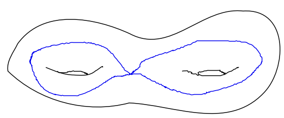
Example 104.

Example 105. $\mathbb{S}^1$, $\mathbb{S}^1 \times I$, and Möbius band. The map from $\mathbb{S}^1$ to the product space is just the inclusion map. The other direction is the obvious deformation retract.

Notation. From now on, $\simeq$ is homotopy equivalence and $\cong$ is homeomorphism.

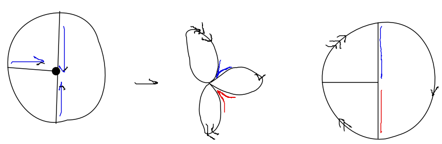
Example 106. Example of when retract which is not hootopy equivalence.

Figure 72: We can pinch the joining part together to a point, so they are not retract but not homotopy equivalence because we can't expand curve into the surface.



Example 107. Connected graphs have homotopy types of the bouquet of circles $B_r := \bigvee_{i=1}^{r} \mathbb{S}^1$. Here, the V means to take two points from the two spaces and stick them together there.

Figure 73: Collapse spanning tree into bouquet (the map f). We can then collapse the two blue parts and two red parts down to the blue and red segments respectively (map g). We can then see that $f \circ g,\ g \circ f$ are respectively homotopic to the identity.



Proposition 108.

$$\pi_1 \left(B_r, \cdot\right) \simeq F_r \tag{47}$$

is a free group of rank r.

Remark 109. Recall: given an **alphabet** $S := \{a, b, c, ...\}$. The free group on S is just the combination of these letters and their inverses (called **words**). We mod out by the obvious equivalence classes, i.e. $aa^{-1} = 1$. i.e., $w_1 \sim w_2$ oif obtain w_1 from w_2 via insertions and deletions, for instance

$$abbc \sim abd^{-1}bc \sim ab^{-1}b^2d^{-1}dbc \sim ab^{-1}b^2bc \tag{48}$$

The group operation is concatenation (mod these equivalence classes), and the identity element is the empty word.

We now relate this to the topology. Let's start with the most elementary example.

Example 110. We claim

$$F_2 := F(a, b) \cong \pi_1(B_2, \cdot) \tag{49}$$

The obvious map to send the letter a to the loop a and the letter b to the loop b. It's obvious that it is a surjective group homomorphism. It takes work to show that it is a injection.

For injectivity, the covering is key. Take $w \simeq_p 1_x$. Then by our lifting result, the lift to $\widehat{x}$ of these two paths are again path homotopic. We then get a path in the covering space, but we get a contradiction since we do not end up with the same starting point $\widehat{x}$.(See HW 4 for details.)

Remark 111. The intuition is that the there is no relation because the only way to concatenate paths is to go through the central point. We can't cancel the loops on the right with the loops on the left.

Let's now look at bigger things.

7.3 Free Products and Baby Seifert Van Kampen.

Let C, D be groups. $C * D$ is a group consisting of alternating words in C, D and group operations given by just concatenation.

Remark 112. This is the analogue of the disjoint unions.

Example 113. If $C := Z_3 = \langle c \rangle,\ D := Z_4 := \langle d \rangle$. The typical element in $C * D$ is

$$c^2 dcd^2 c^2 d^3 \tag{50}$$

Example 114. $F_2 = \mathbb{Z} * \mathbb{Z}$. So in particular, the fundamental group of B_2 is just $\mathbb{Z} * \mathbb{Z}$.

The theorem is a general version of this.

Proposition 115 (Baby Seifert Van Kampen). If $X = U \cup V$ for open sets U, V and $U \cap V$ is simply connected, then $\pi_1(X, x) \simeq \pi_1(U, x) * \pi_1(V, x)$.

Example 116. In the example of B_2, we can take U, V to be the two copies of $\mathbb{S}^1$. The intersection of the two is the point where they are joint together.

Example 117. If we consider torus $\vee$ a circle, then we can take U, V to respectively be the torus and $\mathbb{S}^1$ from which we see that the fundamental group is just $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2 * \mathbb{Z}$. The intersection $U \cap V$ is just a disc with two antenaes.

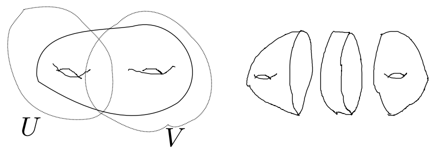
This is the full blown version:

Proposition 118 (General Seifert Van Kampen). Let $X = U \cup V$ for open sets U, V and the intersection containing x is path connected. Then

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V) / \langle \gamma_U(g)\gamma_V(g^{-1}) : g \in \pi_1(U \cap V) \rangle \tag{51}$$

Remark 119. We mod out because if we push the element to the left cover or push the element to the right cover, then we must have the same element in the fundamental group.

Figure 74:



Example 120. We can use the same argument to show that $\mathbb{S}^2$ is simply connected.

8 . -Thursday, 4.25.2019

Faidon is lecturing today and next time.

We want to answer the following: Let $(\tilde{X}, \tilde{x}_0)$ be the covering of (X, x_0). Then we have the inclusion given by the lifting of paths $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq (\pi_1(X, x_0))$. We ask the inverse question of this: if we start with a subgroup $H \subseteq \pi_1(X, x_0)$, then can we realize H as the fundamental group of a cover $\tilde{X} \to X$, $\pi_1(\tilde{X}, \tilde{x})$?

8.1 Prescribing Fundamental Groups: Trivial Group Case.

A natural place to start is the case when H is the trivial group. Equivalently, we want to construct a covering space which is simply connected.

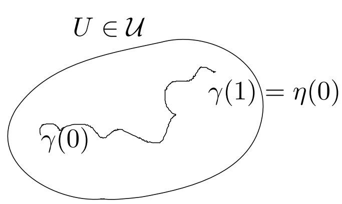
Point set topological formalisms: we assume that X is path connected, locally path connected, and semilocally path connected.

Idea: From the path connectivity of X, take $\tilde{X}$ to be the homotopy classes of paths from x_0 to $x \in X$. Then there is a correspondence between the homptopy classes of paths in $\tilde{X}$ starting at $\tilde{x}_0$ and the homotopy classes of paths in X starting at x_0.

By fixing endpoints, we can associate to each homotopy class $[\gamma]$ the points $\gamma(0), \gamma(1)$. Thus, we can define a covering map $p : \tilde{X} \to X$ given by $[\gamma] \mapsto \gamma(1)$. This map is surjective since X is path connected. Now we can define $\mathcal{U}$ be the collection of neighborhoods $U \subseteq X$ for which $\pi_1(U) \to \pi_1(X)$ is trivial. This forms a basis for the topology on X. For $U \in \mathcal{U}$, define

$$U_{[\gamma]} := \{[\gamma * \eta] : \eta \text{ path in } U \ \eta(0) = \gamma(1)\} \tag{52}$$

Figure 75:



Then $U_{[\gamma]}$ form a basis for the topology on $\tilde{X}$. Locally things are the same for $X, \tilde{X}$. Indeed, the covering map p is locally a homeomorphism and globally is continuous.

$\tilde{X}$ is path connected and simply connected. If $[\gamma]$ is a point in $\tilde{X}$, then we can define γ_t to be γ for $[0, t]$ and stationary for $[t, 1]$. This lifts γ starting at $[x_0] = \tilde{x}_0$. This is a triviality.

Remark 121. The point is that we make everything upstairs be constractible to a point.

8.2 Prescribing Fundamental Groups: General Case.

Let X_H be the covering space whose fundamental group is $H \subseteq \pi_1(X, x_0)$. We construct this. The idea is to define an equivalence relation $[\gamma] \sim [\gamma']$ in $\tilde{X}$. The two are equivalent iff $\gamma(1) = \gamma'(1)$ and $[\gamma^* \overline{\gamma}'] \in H$. Clearly, this is an equivalence relation. We can then define

$$X_H := \tilde{X}/\sim \tag{53}$$

We define the natural projection $p : X_H \to X$given by $[\gamma] \mapsto \gamma(1)$ which is a covering map.By doing what we did in the trivial group case, we get that

$$\pi_1(X_H, x_1) = H \tag{54}$$

Remark 122. If we take the fundamental group of the base space and then let it act on the universal cover, and then take its quotient, then we get what we have above.

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & X_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

Proposition 123. We can take an isomorphism $f : X_1 \to X_2$ iff

$$p_1^*(\pi_1(X, x_1)) \simeq p_2^*(\pi_1(X_2, x_2)) \tag{55}$$

PROOF 124. ($\implies$.) If f is an isomorphism, then we can write $p_1 = p_2 f,\ p_2 = p_1 f^{-1}$. The conclusion follows. □

($\impliedby$.) We use the lifting lemma. From the inclusions

$$p_1^*(\pi_1(X, x_1)) \subseteq p_2^*(\pi_1(X_2, x_2)) \tag{56}$$

$$p_1^*(\pi_1(X, x_1)) \supseteq p_2^*(\pi_1(X_2, x_2)) \tag{57}$$

we get the covering maps $\tilde{p}_2 : X_2 \to X_1$ and $\tilde{p}_1 : X_1 \to X_2$ for which we have $\tilde{p}_1\tilde{p}_2 = \mathbf{Id},\ \tilde{p}_2\tilde{p}_1 = \mathbf{Id}$. Thus, $\tilde{p}_1, \tilde{p}_2$ are inverse isomorphisms, so X_1 is homeomorphic to X_2.

□

8.3 Galois Correspondence.

Proposition 125 (Galois Correspondence). If X is sufficiently nice, we have the correspondence between the set of isomorphism classes of path connected covering spaces over (X, x_0) and the set of subgroups of $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$.

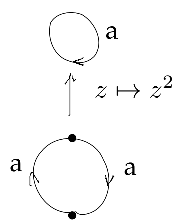
Remark 126. If we forget base points, we get the correspondence between the isomorphism classes of the covering spaces of X and the conjugacy classes of $H \subseteq \pi_1(X)$.

Remark 127. For regular covers (i.e. $H \trianglelefteq \pi_1(X, x)$), H acts on the covering space $\tilde{X}$ and then take the quotient $\tilde{X}/H$. Now the deck transformations of $\tilde{X}/H$ is just the quotient of the whole fundamental group mod H.

Ask more about this.

Example 128. Let $X = \mathbb{S}^1 (\subseteq \mathbb{C})$ and let $G := \pi_1(X) = \langle a \rangle$. Take $H = \langle a^2 \rangle$. Then $\tilde{X}$ is just $\mathbb{S}^1$ but divided into two pieces.

Figure 76:



Example 129. Let X be the bouquet of two circles, and let $G := \pi_1(X) = \langle a, b \rangle$.

Figure 77: Covering corresponding to $\langle a^2, ab, b^2 \rangle$.

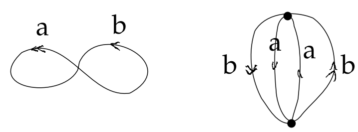
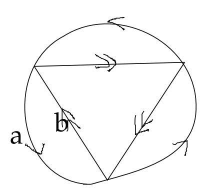


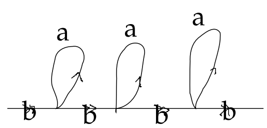
Figure 78: Covering corresponding to $\langle a^3 b^3, ab, ba \rangle$.



Think of why we get these by taking the crazy universal cover we had before and modding out by the group actions.

We can now consider $\langle b^n a b^{-n} \rangle_{n\in\mathbb{Z}}$.

Figure 79: Covering corresponding to $\langle bab^{-1} \rangle$.



9 Covering Spaces and Category Theory. -Tuesday, 4.30.2019

Faidon is lecturing again.

9.1 More Examples of Covering Spaces.

Faidon is lecturing again.

Proposition 130 (Galois Correspondence.). Let (X, x_0) be a nice space. Then

1.) There is a 1-1 correspondence between the isomorphism classes of path connected covering spaces $(\tilde{X}, \tilde{x})$ over (X, x) with covering map p and the subgroups of $\pi_1(X, x_0)$ where the isomorphism is given by the map

$$(\tilde{X}, \tilde{x}_0) \mapsto \pi_*(\pi_1(\tilde{X}, \tilde{x}_0)) \tag{58}$$

2.) If we ignore base points, then there is a 1-1 correspondence between isomorphism classes of path connected covering spaces $\tilde{X}$ over X and the conjugacy classes of subgroups $\pi_1(X, x_0)$. If we start with some subgroup $H \subseteq \pi_1(X, x_0)$, then $H = p_*(\pi_1(X, x_0))$ and X_H is unique up to isomorphism of the universal cover.

3.) We have an action of $\pi_1(X, x_0) \curvearrowright p^{-1}(x_0)$. Here we have

$$\left|p^{-1}(x_0)\right| = [\pi_1(X, x_0) : p_*(\pi_1(X_H, x_H))] \tag{59}$$

(The canonical example of this is the case $\mathbb{R}$ covering $\mathbb{S}^1$ for which the fundamental group acts by shifting between the countable copies of x_0 in the preimage.)

4.) We have the correspondence between the isomorphism classes of regular cosets $X_H \to X$ with deck group $G(X_H)$ (i.e. automorphism group of the covering space) and normal subgroups $H \trianglelefteq \pi_1(X, x_0)$ for which

$$G(X_H) \cong \pi_1(X, x_0)/H \tag{60}$$

and

$$\left|p^{-1}(x_0)\right| = [\pi_1(X, x_0) : H] = |G(X_H)| \tag{61}$$

Here, $G(X_H)$ is the deck group and X_H is a **regular cover** if $H = p_*(\pi_1(X_H, x_H)) \trianglelefteq \pi_1(X, x_0)$.

Example 131. If we are taking covering spaces, one should think of it as doing identifications on the universal cover. For instance, think of taking the universal cover $\mathbb{R}$ over $\mathbb{S}^1$ and take a "bigger" circle $\mathbb{R}/n\mathbb{Z}$ for some n. Then the covering map from the big circle over $\mathbb{S}^1$ is just $z \mapsto z^n$.

Example 132. Consider the bouquet of two circles whose fundamental group is given by $\langle a, b\rangle$.

1.) $\langle a^3, b^3, ab^{-1}, b^{-1}a\rangle$

Figure 80: Covering space corresponding to $\langle a^3, b^3, ab^{-1}, b^{-1}a\rangle$

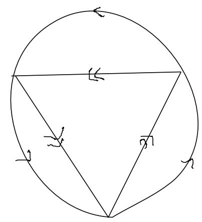
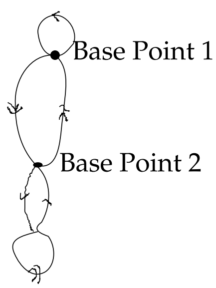


Figure 81: Covering space for both $\langle a^2, b^2, aba, bab\rangle$ and $\langle a, b^2ba^2b^{-1}, baba^{-1}b^{-1}\rangle$. In the first case, the base point is 1, and in the latter it is base point 2.



Note that the covering space for $\langle a, b^2ba^2b^{-1}, baba^{-1}b^{-1}\rangle$, the cover is not regular (since the group is not a normal subgroup). The deck group in this case is trivial.

Exercise 133. Do bunch of different generators for subgroups, and draw a covering space.

9.2 Category Theory.

Example 134. Take a continuous map $f : X \to Y$ between topological spaces. We then have the map $X \mapsto \pi_1(X)$ for which f induces the map

$$f_* : \pi_1(X) \to \pi_1(Y) \tag{62}$$

This is the **functoriality** of $\pi_1(\cdot)$.

Definition 135. A **category** $\mathscr{C}$ consists of

1.) classes of **objects**, denoted ob($\mathscr{C}$) (which are the **vertices** in diagrams)

2.) set of **morphisms** $\text{Hom}(A, B)$ for $A, B \in \text{ob}(\mathscr{C})$. (These are the **arrows** in diagrams.)

for which

1.) The composition is an operation defined for $A, B, C \in \text{ob}(\mathscr{C})$ as a map

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C) \tag{63}$$

$$(f, g) \mapsto gf \tag{64}$$

2.) Identity for composition $1_A \in \text{Hom}(A, A)$ for which for $f \in \text{Hom}(A, B)$ such that

$$f1_A = f,\ 1_B f = f \tag{65}$$

3.) Associativity for compositions:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{66}$$

for which $(hg)f = h(gf)$.

Example 136. Category of sets: the objects are the sets and the morphisms are just the functions between the spaces.

Example 137. Category of topological spaces (denoted Top): the objects are the topological spaces and the morphisms are the continuous maps.

We have Top_*(based spaces): the objects are (X, x_0) and the morphisms are the base point preserving continous maps.

Example 138. The category of groups: the objects are the groups, and the morphisms are the group homomorphisms.

Example 139. The category of homotopy: objects are the topological spaces and the morphisms are the continuous maps up to homotopy equivalence.

Definition 140. A **subcategory** $\mathscr{D}$ of $\mathscr{C}$ are such that

$$\text{ob}(\mathscr{D}) \subseteq \text{ob}(\mathscr{C}) \tag{67}$$

and for all $A, B \in \text{ob}(\mathscr{D})$,

$$\text{Hom}_{\mathscr{D}}(A, B) \subseteq \text{Hom}_{\mathscr{C}}(A, B) \tag{68}$$

We use category theory in order to get functors:

Definition 141. For categories $\mathscr{C}, \mathscr{D}$, **a map** $T : \mathscr{C} \to \mathscr{D}$ **is a (covariant) functor** for which

1.) $A \in \text{ob}(\mathscr{C})$ then $T(A) \in \text{ob}(\mathscr{D})$
2.) (**functoriality**:) If $f : A \to B$, then this induces the map $Tf : T(A) \to T(B)$

which satisfies

1.) If we have the maps

$$A \xrightarrow{f} A' \xrightarrow{g} A'' \tag{69}$$

then

$$T(A) \xrightarrow{Tf} T(A') \xrightarrow{Tg} T(A'') \tag{70}$$

2.) $T(1_A) = 1_{T(A)}$

Thus, a functor acts on both the objects and the morphisms.

Example 142 (Fundamental Groups.). We have

$$\pi_1 : \mathrm{Top}_* \to \mathrm{Groups} \tag{71}$$

$$(X, x_0) \mapsto \pi_1(X, x_0) \tag{72}$$

Example 143. Identity map.

Example 144. We have

$$\mathrm{Hom}_{\mathscr{C}}(B, \cdot) : \mathscr{C} \to \mathrm{Sets} \tag{73}$$

$$A \mapsto \mathrm{Hom}_{\mathscr{C}}(B, A) \tag{74}$$

where

$$\mathrm{Hom}_{\mathscr{C}}(B, \cdot)[A \xrightarrow{f} A'] = f_* : \mathrm{Hom}(B, A) \to \mathrm{Hom}(B, A') \tag{75}$$

for which $f_*(h) = fh$.

$$\begin{tikzcd} A \arrow[r, "f"] & A' \\ C \arrow[u, "h"] \arrow[ur, "f_*(h)"] & \end{tikzcd}$$

Remark 145. If $\mathscr{C}$ is a category, then we can consider the category of operators on this $\mathscr{C}^{\mathrm{op}}$ for which we have the dual construction

$$\mathrm{Hom}_{\mathscr{C}^{op}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A) \tag{76}$$

Example 146 (Categorical Construction of Disjoint Union.). If we are in the category of sets, then we have the two sets A, B. We can then define disjoint union as just

$$(A \times \{1\}) \cup (B \times \{0\}) \tag{77}$$

Take the maps

$$\alpha : A \to A \cup B \tag{78}$$

$$x \mapsto (x, 1) \tag{79}$$

and β defined analogously. Then we have the diagram

Figure 82: Intuition: We can think of A, B living in some plane and X is away from this plane. We can think of the disjoint union living between the two so that you cannot pass from A, B to X without going through the disjoint union.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ A \cup B \arrow[ur, "\theta"] & B \arrow[l, "\beta"] \arrow[u, "g"'] \end{tikzcd}$$

where all the unions are just disjoint unions. This is an example of a **universal construction**. We call the disjoint union construction we did above the **coproduct** of A, B.

Example 147. In the category of groups, the coproduct is just the free product $G * H$.

Example 148. In the category of spaces with base point, the coproduct is just the wedge $(X \vee Y, z_0)$.

Seifert van Kampen follows from this:

PROOF 149 (Proof by Abstract Nonsense).

$$\pi_1(X \vee Y, x_0) = \pi_1(X, x_0) * \pi_1(X, x_0) \tag{80}$$

□

10 . -Tuesday, 5.7.2019

10.1 Where we are and where we're headed.

We're entering the second half of the course, so we'd like to look at what we've done so far.

1.) $\chi(\mathcal{M})$: The idea was to cut up and reassemble. We never showed that this is a topological invariant.
2.) π_1: Look for holes of spaces.

We now want an algebraic/algorithmic way to capture the higher dimensional analogues of these.

10.2 Higher Homotopy Groups and Homology.

Recall that π_1 was just the loops up to an equivalence, i.e. homotopy classes of maps from $(\mathbb{S}^1, \cdot)$ to $(X, \cdot)$. The natural higher dimensional versions of this is to take the following:

Definition 150. The n**th homotopy group** $\pi_m(X, x_0)$ is the set of homotopy classes $(\mathbb{S}^n, \cdot) \to (X, x_0)$.

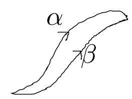
Remark 151. These measure the n-dimensional holes. Since $\mathbb{S}^0$ are two points, the 0-th homotopy group measures the number of connected components.

Remark 152. These are much harder to work with than π_1. Even the group operation is not easy to see. (The hint: for the $n = 1$ case, we used the interval. How do we generalize this for higher dimensions?)

Example 153. $\pi_k(\mathbb{S}^n)$ is an open problem.

In general, this is super hard to compute. So, what do we do? We use homology! We replace homotopy equivalence by boundaries.

Figure 83: α, β are homotopy equivalent, but $\alpha\beta^{-1}$ bounds the region inside.



The nice thing is that homology allows us to do linear algebra which is the only tool we actually have.

Recall that an n-simplex is just the convex hull of the basis vectors $e_1, ..., e_{n+1} \in \mathbb{R}^{n+1}$. We use the notation

$$\Delta^n := [v_0, ..., v_n] := \left\{ x \in \mathbb{R}^{n+1} : x = \sum_i \alpha_i v_i,\ \sum_i \alpha_i = 1 \right\} \tag{81}$$

What is the boundary of Δ^n? For this, we need to algebraicize this system, so we can write things.

Definition 154. The **boundary of the n-simplex** is

$$\partial[v_0, ..., v_n] := \sum_{i=0}^{n} (-1)^i [v_0, ..., \widehat{v}_i, ..., v_n] \tag{82}$$

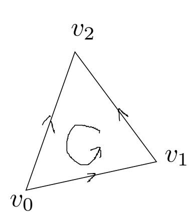
Example 155. Consider the 1-simplex Δ^1 which is a line segment from $(1, 0)$ to $(0, 1)$ sitting inside $\mathbb{R}^2$. There is a natural ordering induced on this from v_0 to v_1.

$$\partial \Delta^1 = [v_1] - [v_0] \tag{83}$$

Example 156. Δ^2 is just the triangle formed by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Then there is a natural ordering of the boundary, and

$$\partial \Delta^2 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \tag{84}$$

Figure 84: There is one edge with the "wrong" orientation.



Remark 157. Note that we define a formal sum on the simplices.

Homology is built from maps of simplices into X using the boundary map to give higher dimensional analog of holes.

10.3 Simplicial Homology.

Definition 158. A **Δ-complex structure on** X is a collection of maps

$$\{\sigma_\alpha : \Delta^n \to X\}_{\alpha \in I} \tag{85}$$

such that

1.) $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)}$ is injective (so in particular, the image cannot have self intersections, but we can have loops)
2.) $\sigma_\alpha|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]}$ is one of the $\sigma_\beta : \Delta^{n-1} \to X$
3.) $A \subseteq X$ iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all σ_α. (This rules out breaking up the space as bunch of point.)

Remark 159. This is a relaxed version of a simplicial complex structure. For constructing simplicial complexes, we only allowed gluing stuff to other stuff (so for instance, we could not get loops). But now, we can get loops because we can glue endpoints.

We then get a space built from simplicies glued along subsimplices.

Example 160. Here we have

$$\partial(a) = V - V = 0 \tag{86}$$

$$\partial(c) = \partial(b) = 0 \tag{87}$$

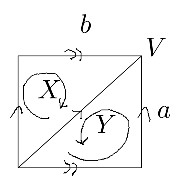
$$\partial(X) = a + b - c \tag{88}$$

$$\partial(Y) = a + b - c \tag{89}$$

$$\partial(X - Y) = 0 \tag{90}$$

Note that the orientations depend on the map when defining Δ-complexes.

Figure 85: Torus.



Definition 161. If X is a Δ-complex, then **the space of n-chains on** X $\Delta_n(X)$ is the vector space over the reals with basis given by all n-simplices in X, or formally

$$\Delta_n(X) := \sum_\alpha k_\alpha[\sigma_\alpha] \tag{91}$$

where $\sigma_\alpha : \Delta^n \to X$.

Remark 162. Most of our spaces are compact, so we take finite sums for now.

Definition 163. A **chain complex of** X is

$$\dots \xrightarrow{\partial_4} \Delta_3(X) \xrightarrow{\partial_3} \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \tag{92}$$

where this $\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ morphism is defined by

$$\partial[\sigma] := \sum_i (-1)^i [\sigma|_{[v_0,\dots,\widehat{v}_i,\dots,v_n]}] \tag{93}$$

Definition 164. An n**-cycle** is a sum

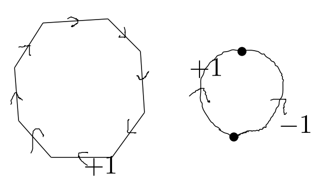
$$\sum_a k_\alpha[\sigma_\alpha] \in \Delta_n(X) \tag{94}$$

such that

$$\partial\left(\sum_a k_\alpha[\sigma_\alpha]\right) = 0 \tag{95}$$

Example 165. The boundary cancels out in both of these.

Figure 86: In the octagon, assign $+1$ on all the boundary segments.



Definition 166. An n-chain which is in the image of $\partial_{n+1} : \Delta_{n+1}(X) \to \Delta_n(X)$ is called a **boundary**.

Definition 167. The n**th homology group** is the space

$$H_n(X) := \ker(\partial_n : \Delta_n(X) \to \Delta_{n-1}(X))/\operatorname{im}(\partial_{n+1} : \Delta_{n+1}(X) \to \Delta_n(X)) \tag{96}$$

Remark 168. We call them *groups* because in full generality, we use rings instead of $\mathbb{R}$, and so we get groups. But for our purposes, they will always be vector spaces.

Remark 169. This makes sense because $\partial^2 = 0$, i.e. the image is contained in the kernel.

Example 170. Take $X = \mathbb{S}^1$. We can see that

$$\Delta_1(\mathbb{S}^1) = \mathbb{R}a \oplus \mathbb{R}b \tag{97}$$

$$\Delta_0(\mathbb{S}^1) = \mathbb{R}v \oplus \mathbb{R}w \tag{98}$$

Note that despite the above two being isomorphic as vector spaces, they are different spaces since they hold different meanings in this context.

Now in the sequence

$$0 \xrightarrow{\partial_2} \Delta_1(\mathbb{S}^1) \xrightarrow{\partial_1} \Delta_0(\mathbb{S}^1) \xrightarrow{\partial_0} 0 \tag{99}$$

Then

$$\partial_1 : ka + lb \mapsto k(w - v) + l(w - v) = (-k - l)v + (k + l)w \tag{100}$$

Thus,

$$H_1(\mathbb{S}^1) = \text{ker}(\partial_1)/\text{im}(\partial_2) = \mathbb{R} \cdot (a - b) \cong \mathbb{R} \tag{101}$$

It's also interesting to note that $a - b$ is the loop that generates the fundamental group.

We have

$$H_0(\mathbb{S}^1) = \mathbb{R}v \oplus \mathbb{R}w/\text{im}(\partial_1) \cong \mathbb{R} \oplus \mathbb{R}/(1, -1) \cong \mathbb{R} \tag{102}$$

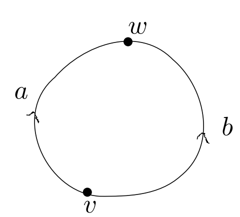
which is generated by $(1, 0)$.

Thus,

$$H_n(\mathbb{S}^1) = \begin{cases} \mathbb{R} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{103}$$

This measures one path component and one hole.

Figure 87:



Example 171. Let's look at the torus with the structure we gave it before.

$$0 \xrightarrow{\partial_3} \mathbb{R}x \oplus \mathbb{R}y \xrightarrow{\partial_2} \mathbb{R}a \oplus \mathbb{R}b \oplus \mathbb{R}c \xrightarrow{\partial_1} \mathbb{R}v \xrightarrow{\partial_0} 0 \tag{104}$$

∂_1 is the 0 map, and

$$\partial_2 : (k, n) \mapsto (k + n, k + n, -k - n) \tag{105}$$

Then

$$H^0(\mathbb{T}^2) = \text{ker}(\partial_0)/\text{im}(\partial_1) = \mathbb{R}v \tag{106}$$

$$H^2(\mathbb{T}^2) = \text{ker}(\partial_2)/\text{im}(\partial_3) = \mathbb{R}(x - y) \cong \mathbb{R} \tag{107}$$

$$H^1(\mathbb{T}^2) = \ker(\partial_1)/\mathrm{im}(\partial_2) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}/\langle(1,1,-1)\rangle \cong \mathbb{R}^2 \tag{108}$$

which is generated by a, b. Thus,

$$H_n(\mathbb{T}^2) = \begin{cases} \mathbb{R} & n = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{109}$$

The $n = 1$ corresponds to the two loops as captured by the fundamental group. The $n = 2$ corresponds to the whole torus. (Here, orientability is important. For Klein bottle, $n = 2$ gives 0.)

10.4 Midterm.

1.) Median: 14/20
2.) Mean: 14.5/20

11 . -Thursday, 5.9.2019

Missed first 40 minutes

Definition 172. The i**th Betti number** is the dimension of the $H_i(X)$.

Proposition 173. The Euler characteristic is the alternating sum of the Betti numbers.

Definition 174. Let $f : X \to Y$ be continuous maps between simplicial complexes. f is **simplicial** if it sends vertices of X to vertieces of Y and is linear on the interior of the simplices.

Fix notations.

Proposition 175 (Simplicial Approximation.). If K, L are complex simplicial complexes, and if $f : K \to L$ are continious, then there exists barycentric subdivision K' and the homotopy $f \simeq g$ such that $g : K' \to L$ is simplicial.

Proposition 176. Let $f : X \to Y$ be simplicial, then there exists induced map $f_\# : \Delta_m(X) \to \Delta_n(Y)$, so that $[\Delta^n \xrightarrow{\sigma} X] \mapsto [\Delta^n \xrightarrow{f \circ \sigma} Y]$.

Remark 177. The key is

$$\partial \circ f_\# = f_\# \circ \partial \tag{110}$$

PROOF 178. This is just a computation:

$$\begin{aligned} \partial(f_\# \sigma) &= \sum (-1)^i (f \circ \sigma)|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]} \\ &= \sum (-1)^i f(\sigma|_{[v_0, \ldots, \widehat{v}_i, \ldots, v_n]}) \\ &= f_\#(\partial \sigma) \end{aligned}$$

$\square$

We know from the remark that $f_\#$ is a homomorphism of chain complexes.

This induces $f_\# : \ker\partial / \mathrm{im}(\partial) = H_n(X) \to H_n(X)$ because $f_\#$ sends cycles to cycles and boundaries to boundaries.

Subsequence of chain complexes

Example 179. If $Z \in \Delta_n(X)$ and $\partial Z = 0$, then

$$\partial(f_\# Z) = f_\#(\partial Z) = 0 \tag{111}$$

Remark 180. It is common in algebraic topology to just let $*$ to mean induced map.

Moreover for

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{112}$$

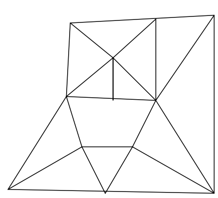
then $(g \circ f)_* = g_* \circ f_*$.

11.1 Homology Vision.

Definition 181. [4] An n**-dimensional pseudomanifold** M is a Δ-complex such that each $n-1$-simplex is a face of a 1 or 2 simpleces. ∂M is a union of $n-1$-simplices that lie in only 1 simplex.

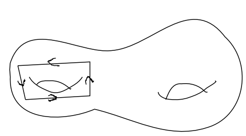
We say this is **oriented** if it is orientable in the sense of simplicial complexes.

Figure 88: Example of a pseudomanifold.



Remark 182. The heuristic: each class in $H_n(X)$ is represented by compact oriented pseudo n-manifold (without boundary).

Figure 89:



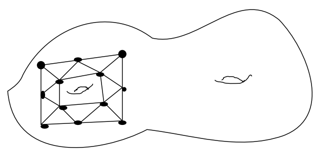
[4] This is inconsistent with the Wikipedia definition.

Example 183.

Definition 184. Two n-cycles Z_1, Z_2 represented by pseudomanifold M_1, M_2 are homologous, i.e. $Z_1 = Z_2$ modulo $\mathrm{im}\partial_{n+1}$. Equivalently, there exists a $(n+1)$-pseudomanifold M^{n+1} so that

$$\partial M^{n+1} = M_1 - M_2 \tag{113}$$

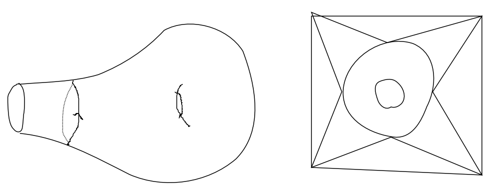
Figure 90: The inner loop and the outer loop are homologous because it is a boundary of a 2 pseudomanifold.



Example 185.

Example 186. Nontrivial element in fundamental group but trivial in homology:

Figure 91: We can look at the triangulation of everything to the left of the curve and see that the curve is homologous to a point.



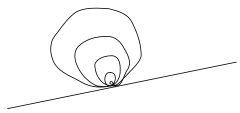
12 . -Tuesday, 5.14.2019

Last time we discussed: homology is a functor from topological spaces to vector spaces. We build tools today.

12.1 Tool 1: Long Exact Sequence of Homology.

The idea is to compute modulo a subspace. Let $A \subseteq X$ be nice, i.e. any point has a neighborhood that retracts onto it.

Figure 92: Not a nice space.



For our purposes (i.e. simplicial complexes), any subset of X is a nice space.

We want to use $\tilde{H}_n(X/A),\ \tilde{H}_n(A)$ to get $H_n(X)$. But in algebra, there is no "collapsing down to a point," so instead, we want quotient of groups. To make this work, we need the tilde:

$$\tilde{H}_n(X/A) \cong H_n(X)/H_n(A) \tag{114}$$

Example 187. Let X be connected. Then

$$H_n(X) = H_0(A) = \mathbb{R} \tag{115}$$

and so,

$$H_n(X)/H_n(A) = 0 \tag{116}$$

but

$$H_0(X/A) = \mathbb{R} \tag{117}$$

and so,

$$H_n(X/A) \ncong H_n(X)/H_n(A) \tag{118}$$

To fix this notational issue, we introduce the following:

Definition 188. Take the **augmentation map**

$$\mathcal{E} : \Delta_0(X) \to \mathbb{R} \tag{119}$$

$$\sum_i k_i[\sigma_i : \Delta^0 \to X] \mapsto \sum_i k_i \tag{120}$$

The **reduced complex** is given by

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_0} \Delta_0(X) \to \mathbb{R} \to 0 \tag{121}$$

and the **reduced homology groups** $\tilde{H}_n(X)$ are the homology groups given by this sequence.

Remark 189. In particular, if we are away from the right end, then the reduced homology coincides with the usual homology:

$$H_n(X) \cong \tilde{H}_n(X) \qquad n > 0 \tag{122}$$

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{R} \tag{123}$$

Example 190. For a single point, $\tilde{H}_n(\{\cdot\}) = 0$ for all n.

Proposition 191. If $A \subseteq X$ nice and nonempty, then

$$A \overset{i}{\hookrightarrow} X \overset{j}{\to} X/A \tag{124}$$

induces the **long exact sequence of the pair** (A, X):

$$\dots \tilde{H}_{n+1}(X/A) \overset{\delta}{\to} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \overset{\delta}{\to} \tilde{H}_{n-1}(A) \to \dots \tag{125}$$

where δ is the **connecting homomorphism**.

Remark 192. This is a degenerate version of a spectral sequence.

Remark 193. In practice, we want to get a lot of zeros in the exact sequence so that we get short exact sequences and get isomorphisms.

Example 194. Let's look at the k-sphere. Recall that

$$H_n(\mathbb{S}) = \begin{cases} \mathbb{R} & n = 0, 1 \\ 0 & n > 1 \end{cases} \tag{126}$$

Now recall that $\partial \mathbb{D}^k = \mathbb{S}^{k-1}$. But now since $\partial \mathbb{D}^k$ is homotopy equivalent to a point and also $\mathbb{D}^k / \partial \mathbb{D}^k \cong \mathbb{S}^k$, we can do an inductive procedure on $A = \mathbb{S}^{k-1},\ X = \mathbb{D}^k,\ X/A = \mathbb{S}^k$.

Then by the proposition, we get the exact sequence

$$\dots \tilde{H}_{n+1}(\mathbb{S}^{k+1}) \overset{\delta}{\to} \tilde{H}_n(\mathbb{S}^{k-1}) \xrightarrow{i_*} \tilde{H}_n(\mathbb{D}^k) \xrightarrow{j_*} \tilde{H}_n(\mathbb{S}^k) \overset{\delta}{\to} \tilde{H}_{n-1}(\mathbb{S}^{k-1}) \to \dots \tag{127}$$

But now, $\tilde{H}_n(\mathbb{D}^k) = 0$ for all n, so in fact

$$\tilde{H}_n(\mathbb{S}^k) \cong \tilde{H}_{n-1}(\mathbb{S}^{k-1}) \tag{128}$$

Thus, we know the homology of the k-sphere if we know the homology of the 1-sphere.

We first know that

$$\tilde{H}_n(\mathbb{S}^0) = \begin{cases} \mathbb{R} & n = 0 \\ 0 & n \geq 1 \end{cases} \tag{129}$$

Thus, by induction

$$\tilde{H}_n(\mathbb{S}^k) = \begin{cases} \mathbb{R} & n = k \\ 0 & n \geq 1 \end{cases} \tag{130}$$

and converting back to usual homology

$$H_n(\mathbb{S}^0) = \begin{cases} \mathbb{R} & n = 0, k \\ 0 & n \geq 1 \end{cases} \tag{131}$$

Remark 195 (General Principle.). When you have homology theory, you first figure out what happens at a single point, and then use machinery such as above to deduce facts for nontrivial spaces.

Proposition 196 (Brouwer Fixed Point Theorem.). If $f : \mathbb{D}^n \to \mathbb{D}^n$ continuous, then it has a fixed point.

Remark 197. We proved Brouwer in 2-dimensions because we could use fundamental groups. But we cannot do that for higher dimensions, so for this we use homology.

PROOF 198. Suppose not. As before, we construct the map

$$r : \mathbb{D}^n \to \partial\mathbb{D}^n \cong \mathbb{S}^{n-1} \tag{132}$$

$$x \mapsto \mathbb{S}^{n-1} \cap \text{Ray}(x, f(x)) \tag{133}$$

Then we get

$$\mathbb{S}^{n-1} \overset{i}{\hookrightarrow} \mathbb{D}^n \xrightarrow{r} \mathbb{S}^{n-1} \tag{134}$$

which then gives

$$\mathbb{R} = H_{n-1}(\mathbb{S}^{n-1}) \to 0 \to H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{R} \tag{135}$$

which is impossible. □

12.2 Homological Algebra Construction.

Proposition 199 (Short exact sequence in complexes induces long exact sequence in homology.). Take a short exact sequence

$$0 \to A. \xrightarrow{i} B. \xrightarrow{j} C. \to 0 \tag{136}$$

where $A.$ is a full complex. (This is standard notation.)

Then this induces

$$\ldots \tilde{H}_{n+1}(C.) \xrightarrow{\delta} \tilde{H}_n(A.) \xrightarrow{i_*} \tilde{H}_n(B.) \xrightarrow{j_*} \tilde{H}_n(C.) \xrightarrow{\delta} \tilde{H}_{n-1}(A.) \to \ldots \tag{137}$$

PROOF 200. The proof is by Snake Lemma.

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & \\
0 \arrow[r, "i"] & A_{k+1} \arrow[r] \arrow[d, "\partial"] & B_{k+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{k+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r, "i"] & A_k \arrow[r] \arrow[d, "\partial"] & B_k \arrow[r, "j"] \arrow[d, "\partial"] & C_k \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r, "i"] & A_{k-1} \arrow[r] \arrow[d, "\partial"] & B_{k-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{k-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We do some diagram chasing. We want to define the connecting homomorphism $\delta : H_k(C.) \to H_{k-1}(A.)$. The ideas is

1.) Take $\sigma \in C_k$ such that $\partial\sigma = 0$
2.) Choose η such that $j(\eta) = \sigma$
3.) Then $j(\partial\eta) = \partial(j(\eta)) = \partial\sigma = 0$, so there exists $\theta \in A_{k-1}$ so that $i\theta = \partial\eta$

Then define $\delta([\sigma]) := [\theta]$. □

12.3 Simplicial Homology.

Simplicial homology is computable, is a topological invariant, and it works (even though we never proved this).

There is also **singular homology.** Take X topological space. A **singular simplex on** X is a continuous map $\sigma : \Delta^n \to X$.

Now let $C_n(X)$ be the vector space whose bases are the singular simplices. There is also the boundary operator given by

$$\partial_n : C_n(X) \to C_{n-1}(X) \tag{138}$$

$$\partial_n([\sigma]) = \sum_i (-1)^i \sigma|_{[v_0,\ldots,\widehat{v}_i,\ldots,v_n]} \tag{139}$$

with

$$\ldots \to C_n \xrightarrow{\partial} \ldots \to C_1 \to C_0 \to 0 \tag{140}$$

We don't compute this because it is too crazy. Instead, we know that

$$H_n^{\text{sing}}(X) \cong H_n^{\Delta}(X) \tag{141}$$

and so, we just don't use singular homology.

12.4 Tool 2: Mayer Vietoris.

Remark 201. Mayer-Vietoris is the analogue of Seifert van Kampen for homology.

Proposition 202 (Mayer-Vietoris). Let $X = A \cup B$ be an open cover of the space. Then we have the short exact sequence

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(A + B) \to 0 \tag{142}$$

where

$$\sigma \mapsto (\sigma, \sigma) \tag{143}$$

in the first map, and in the second map

$$(\eta_1, \eta_2) \mapsto \eta_1 - \eta_2 \tag{144}$$

Clearly, these maps are linear. This induces the **Mayer-Vietoris sequence**

$$\ldots H_{n+1}(X) \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots \tag{145}$$

Remark 203. Here C_n is singular homology to make sense of singular homology for open sets. For nice things, we can deformation contract onto a nice subsimplex and think there if there is a nice subsimplex.

The first map is clearly injective. Surjective is also clear if we think of what $A + B$ is. Exactness is left as an exercise.

Example 204. We recompute the homology of the torus using this new machinery. Let B be a neighborhood around a point and A be a neighborhood in the complement of the point. Then we can deformation retract the A to the boquet of two circles. B is contractible. Also, $A \cap B \simeq \mathbb{S}^1$. We also know

$$H_n(A) \begin{cases} \mathbb{R} & n = 0 \\ \mathbb{R} \oplus \mathbb{R} & n = 1 \\ 0 & n > 1 \end{cases} \qquad H_n(B) \begin{cases} \mathbb{R} & n = 0 \\ 0 & \text{else} \end{cases} \qquad H_n(A \cap B) \begin{cases} \mathbb{R} & n = 0, 1 \\ 0 & \text{else} \end{cases} \tag{146}$$

This gives

$$0 \to H_2(\mathbb{T}^2) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(\mathbb{T}) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(\mathbb{T}) \to 0 \tag{147}$$

$$0 \to H_2(\mathbb{T}^2) \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H_1(\mathbb{T}) \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0 \tag{148}$$

Now since the third to last map must be an injection, the fourth to last map must be 0. The third map must also be 0. This implies that the fourth map is an isomorphism. We also know that the first map is surjective. We thus have

$$H_n(\mathbb{T}^2) = \begin{cases} \mathbb{R}^2 & n = 1, 0 \\ \mathbb{R} & n = 2 \end{cases} \tag{149}$$

13 . -Thursday, 5.16.2019

13.1 Homology with Different Coefficients.

We play with homology with different coefficients.

Recall that we constructed

$$\Delta^n(X) = \mathbb{R}\sigma_1 \oplus ... \oplus \mathbb{R}\sigma_n \tag{150}$$

with

$$\sigma_i : \Delta^n \to X \tag{151}$$

and $\partial : \Delta^n \to \Delta^{n-1}$ defined on σ_i and extended linearly. The key is that for arbitrary coefficients, we can once again, *extend linearly.*

We also have

$$\Delta_n(X) \xrightarrow{\partial_n} ... \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0 \tag{152}$$

and

$$H_k(X) = \ker(\partial_k)/\text{im}(\partial_{k+1}) \tag{153}$$

Remark 205. Another observation is that *we are not using* $\mathbb{R}$ *that much.* We use it for $\oplus$, kernels, and when we take quotients, but not much otherwise. We can thus replace it with any other field, and the construction still works.

Example 206. Let's use $\mathbb{Z}/2\mathbb{Z}$ instead. This is a lot more concrete. The intuition: we flip each n-simplex on or off corresponding to 1 or 0.

More explicitly,

$$\Delta_n(X) = \mathbb{Z}/2\mathbb{Z}\sigma_1 \oplus ... \oplus \mathbb{Z}/2\mathbb{Z}\sigma_d \tag{154}$$

For instance, when $X = \mathbb{T}^2$, we just get 4 things in $\Delta_2(X)$ which is a very concrete vector space.

We also have the natural boundary map $\partial : \Delta^n \to \Delta^{n-1}$ with

$$\partial([\sigma]) = \sum_i (-1)^i \sigma|_{[v_0,...,\widehat{v}_i,...,v_n]} \tag{155}$$

Remark 207. In analogy, we can think for the $\mathbb{R}$ case that we are assigning a "strength" to the simplices based on the coefficient of the simplex.

Example 208. We don't even need vector spaces. If we take $\mathbb{Z}$, we get free abelian group on the set of n-simpices:

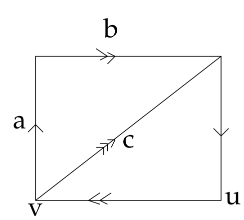
$$\Delta_n(X) = \mathbb{Z}\sigma_1 \oplus ... \oplus \mathbb{Z}\sigma_{d_n} \cong \mathbb{Z}^{d_n} \tag{156}$$

where d_n is the number of n simplices.

This is close enough to a vector space that we can play the same game: we can define ∂ by the same formula. However, now instead of linear map, we have a group homomorphism. Kernels and images make sense, and the group is abelian, so all subgroups are normal and quotients make sense. So we just do the same construction. However, since we cannot divide in rings, we do genuinely get different homologies when we change coefficients.

13.2 Homology of $\mathbb{R}P^2$ with Different Coefficients.

Example 209. Consider $\mathbb{R}P^2$:



With this structure, we get

$$\Delta_2(X) = Rx \oplus Ry \tag{157}$$

$$\Delta_1(X) = Ra \oplus Rb \oplus Rc \tag{158}$$

$$\Delta_0(X) = Ru \oplus Rv \tag{159}$$

where R is a ring. We compute homology for $R = \mathbb{R}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$.

For $R = \mathbb{R}$:

$$0 \xrightarrow{\partial_3} \mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^2 \xrightarrow{\partial_0} 0 \tag{160}$$

$$\partial_2(X) = a + b - c,\ \partial_2(Y) = -b - a - c \tag{161}$$

and so,

$$\partial_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{162}$$

Likewise,

$$\partial_1(a) = u - v,\ \partial_2(b) = v - u,\ \partial_1(c) = 0 \tag{163}$$

so

$$\partial_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{164}$$

Thus, $\dim \ker\partial_2 = 0,\ \text{rank}\partial_2 = 2$ and $\dim\ker\partial_1 = 2,\ \text{rank}\partial_1 = 1.$

Thus,

$$H_n(\mathbb{P}^2;\mathbb{R}) = \ker\partial_n/\text{im}\partial_n = \begin{cases} 0 & n = 1,2 \\ \mathbb{R} & n = 0 \end{cases} \tag{165}$$

We need to observe that $\pi_1(\mathbb{R}P^2)$ is nontrivial which suggests that there are holes. But the real homology says that this is like a *point*. So, we need to look at the other coefficients.
For $R = \mathbb{F}_2$:

$$0 \xrightarrow{\partial_3} \mathbb{F}_2^2 \xrightarrow{\partial_2} \mathbb{F}_2^3 \xrightarrow{\partial_1} \mathbb{F}_2^2 \xrightarrow{\partial_0} 0 \tag{166}$$

$$\partial_2(X) = a + b - c = a + b + c,\ \partial_2(Y) = -b - a - c = a + b + c \tag{167}$$

and so,

$$\partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{168}$$

Likewise,

$$\partial_1(a) = u - v = u + v,\ \partial_2(b) = v - u = u + v,\ \partial_1(c) = 0 \tag{169}$$

so

$$\partial_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{170}$$

Thus, $\dim\ker\partial_2 = 1,\ \text{rank}\partial_2 = 1$ and $\dim\ker\partial_1 = 2,\ \text{rank}\partial_1 = 1.$

Thus,

$$H_n(\mathbb{P}^2;\mathbb{F}_2) = \mathbb{F}_2 \qquad n = 0,1,2 \tag{171}$$

Finally, consider $R = \mathbb{Z}$:

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0 \tag{172}$$

$$\partial_2(X) = a + b - c,\ \partial_2(Y) = -b - a - c \tag{173}$$

$$\partial_1(a) = u - v,\ \partial_2(b) = v - u,\ \partial_1(c) = 0 \tag{174}$$

$$\partial_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{175}$$

Thus,

$$H_n(\mathbb{P}^2; \mathbb{Z}) = \mathrm{ker}\partial_n / \mathrm{im}\partial_n = \begin{cases} 0 & n = 2 \\ \mathbb{F}_2 & n = 1 \\ \mathbb{Z} & n = 0 \end{cases} \tag{176}$$

For example,

$$H_1(\mathbb{R}P^2 l; \mathbb{Z}) = \mathrm{ker}\partial_1 / \mathrm{im}\partial_2 = \langle a + b, c \rangle / \langle a + b - c, -a - b - c \rangle \tag{177}$$

Table 2: Homology of $\mathbb{R}P^2$ for different coefficients.

$\mathbb{R}P^2$	$\mathbb{R}$	$\mathbb{F}_2$	$\mathbb{Z}$	Meaning
H_2	0	$\mathbb{F}_2$	0	Orientability.
H_1	0	$\mathbb{F}_2$	$\mathbb{F}_2$	Fundamental Group.
H_0	$\mathbb{R}$	$\mathbb{F}_2$	$\mathbb{Z}$	Connected Components.

Remark 210 (Orientability.). $H_2(X)$ says something about the orientability of the 2-manifold X since orientability is just how we can put triangles with twirlies compatibly. For instance,

$$H_n(\mathbb{T}^2) = \begin{cases} \mathbb{R} & n = 0, 2 \\ \mathbb{R}^2 & n = 1 \end{cases},\ H_n(\mathrm{KB}) = \begin{cases} \mathbb{R} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{178}$$

Klein bottle is nonorientable, so highest one dies.

More generally: For a compact manifold, it is orientable iff its highest homology is $\mathbb{R}$.

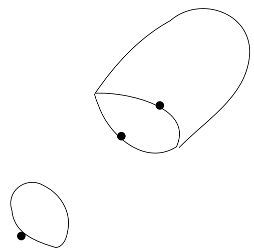
Remark 211. $\mathbb{Z}$ is hard, so you use $\mathbb{R}$ as a proxy. The only R you ever really care about is $\mathbb{Z}$ and $\mathbb{F}_2$.

Remark 212. H_1 is essentially the abelianization of fundamental group.

We can see $H_1(\mathbb{R}P^2; \mathbb{F}_2)$ immediately:

Look up the intuition for homology about blowing air into the manifold. See MathOverflow. Having intuition for these is very important.

Figure 93: Recall from Pset 5 (?) what happens when we attach discs to surfaces with a hole. To get $\mathbb{R}P^2$, we attach a disc but wrap the boundary *twice*. This is how we see that $H_1(\mathbb{R}P^2;\mathbb{F}_2)=\mathbb{F}_2$. We can also see it from $\mathbb{R}P^2=\mathbb{S}^2/(\mathbb{Z}/2\mathbb{Z})$.



In general,

$$H_1(X;\mathbb{Z})\cong\pi_1(X)^{\text{AB}} \tag{179}$$

So for instance, for the bouquet of 2 circles, we have $H_1=\mathbb{Z}\oplus\mathbb{Z}$ and $\pi_1=\mathbb{Z}*\mathbb{Z}$.

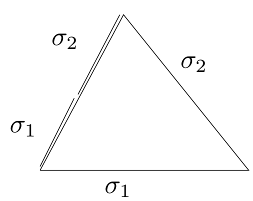
In general:

Proposition 213 (Hurewicz). For $h:\pi_1(X,x)\to H_1(X)$ is a map such that $\sigma:[0,1]\to X$ such that $\sigma(0)=\sigma(1)=x_0$ whcih can also be viewed as a singular 1-simplex in X.

h is a homomorphism means that

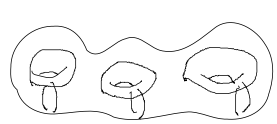
$$h(\sigma_1*\sigma_2)=h(\sigma_1)+h(\sigma_2) \tag{180}$$

i.e., $\sigma_1*\sigma_2$ is homologous to $\sigma_1+\sigma_2$. FOr this we need an element of $\Delta_2(X)$ whose boundary is $\sigma_1*\sigma_2-\sigma_1+\sigma_2$.



Example 214. For a surface, there are $2g$ generators of the fundamental group, where g is the genus.

Figure 94: These curves are the images of the generators inside the fundamental groups.



Proposition 215 (Alternative Hurewicz). The map $f : \mathbb{S}^1 \to X$ indices

$$H^1(\mathbb{S}^1) = \mathbb{Z} \xrightarrow{f_*} H_1(X) \tag{181}$$

14 . -Tuesday, 5.21.2019

14.1 Announcements.

1.) Second to last pset
2.) Pset due next Tuesday
3.) OH and psession moved

14.2 Relative Homology.

Last lecture on homology; next time we discuss cohomology.

We have

$$\tilde{H}_*(X/A) \cong H_*(X)/H_*(A) \tag{182}$$

Our goal is to understand $\tilde{H}_*$ and good pairs A, X for which we can do this.

Definition 216. If $A \subseteq X$ is a subspace, take

$$C_k(A) := \text{span } \left\{\sigma : \Delta^k \to A\right\} \leq C_k(X) \tag{183}$$

Then

$$C_k(X, A) := C_k(X)/C_k(A) \tag{184}$$

(The quotient is the quotient of vector spaces, so this makes sense). We also have the induced maps

$$\overline{\partial} : C_k(X, A) \to C_{k-1}(C, A) \tag{185}$$

induced by the usual map $\partial : C_k(X) \to C_{k-1}(X)$. The **relative homology of** (X, A) is the homology given by the vector space sequence given by these maps:

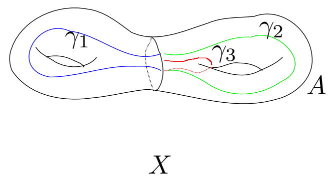
$$H_k(X, A) := \ker\overline{\partial}_k/\text{im}\partial_{k+1} \tag{186}$$

Remark 217. This took the geometric idea in equation 182 and made it purely algebraic. i.e., this is like collapsing down the space X using a space A. But of course, if A is a crazy subspace, then the topology of the quotient becomes a mess.

Remark 218. From the definition, clearly the elements of $C_k(X, A)$ are equivalence classes of chains that agree away from A.

Example 219. We have $\gamma \in C_1(X)$, but $\gamma \notin H_1(X)$. But is it an element in $H_1(X, A)$?

Figure 95:



We can extend γ_1 to A via γ_2 and γ_3 respectively, but they are still the same element in relative homology since they agree up to an element of $H_k(A)$.

Remark 220. $[\sigma] \in C_k(X, A)$ represents an element in $H_k(X, A)$. Then $\overline{\partial}[\sigma] = 0$, i.e. $\sigma \in C_k(X)$ and $\partial\sigma \in C_{k-1}(A)$.

Remark 221. We have the short exact sequence of vector spaces

$$0 \to C_k(A) \xrightarrow{i} C_k(X) \xrightarrow{j} C_k(X, A) \to 0 \tag{187}$$

which should be though of as an algebraic analogue of $A \hookrightarrow X \to X/A$. We then get the **long exact sequence to the pair** (X, A)

$$\dots H_{k+1}(X, A) \to H_k(A) \to H_k(A) \to H_k(X) \to H_k(X, A) \to H_{k-1}(A) \to \dots \tag{188}$$

For a triple, we get the short exact sequence

$$0 \to C_*(B, A) \to C_*(X, A) \to C_*(X, B) \to 0 \tag{189}$$

and the **long exact sequence of a triple** $A \subseteq B \subseteq X$ is then

$$\dots H_{k+1}(X, B) \to H_k(B, A) \to H_k(X, A) \to H_k(X, B) \to H_{k-1}(B, A) \to \dots \tag{190}$$

Or goals is to show that for $A \subseteq X$ "good," we get

$$H_k(X, A) \cong \tilde{H}_k(X/A) \tag{191}$$

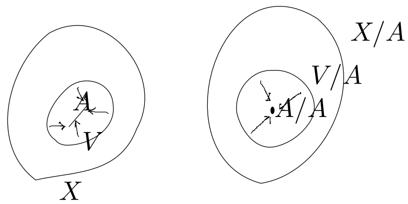
PROOF 222. Take the quotient map

$$q : (X, A) \to (X/A, A/A) \tag{192}$$

Let V be the neighborhood of A that deformation retracts onto A. Then

$$0 \cong H_*(A, A) \cong H_*(V, A) \xrightarrow{q_*} H_*(V/A, A/A) \cong H_*(A/A, A/A) \cong 0 \tag{193}$$

Figure 96:



This induces the diagram

$$\begin{array}{ccccc} H_k(X,A) & \longrightarrow & H_k(X,V) & \longleftarrow & H_k(X\setminus A, V\setminus A) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ \tilde{H}_k(X/A) \cong H_k(X/A, A/A) & \longrightarrow & H_k(X/A, V/A) & \longleftarrow & H_k(X/A \setminus A/A, V/A\setminus A/A) \end{array}$$

We get the right arrows from the inclusion $(X,A) \hookrightarrow (X,V)$ and the left arrow is $(X\setminus A, V\setminus A) \hookrightarrow (X,V)$. We claim that all the horizontal arrows are isomorphisms.

For the right arrows, we have the long exact sequence for (X,V,A) given by

$$0 = H_k(V,A) \to H_k(V,A) \to H_k(X,V) \to H_{k-1}(V,A) = 0 \tag{194}$$

So we get isomorphism. We get the rightmost vertical arrow is an isomorphism because q is homeomorphism.

The left arrow is given by excision.

□

Proposition 223 (Excision.). Given $A \subseteq B \subseteq X$ such that $\overline{A} \subseteq B$, then $(X\setminus A, B\setminus A) \hookrightarrow (X,B)$ induces

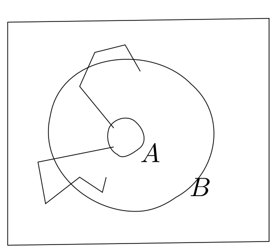
$$H_k(X\setminus A, B\setminus A) \cong H_k(X,B) \tag{195}$$

Remark 224. Excision is the axiom of homology when one does homology theory axiomatically.

PROOF 225. The idea is clear, but the details are horrendous.

Relative cycles in (X,B) are equivalent to relative cycles in $(X\setminus A, B\setminus A)$. Just baricentric subdivide infinitely many times.

Figure 97:



□

14.3 Local Homology.

We're leading up to the formal definition of orientation.

Definition 226. If $p \in X$, then the **local homology** is $H_*(X, X \setminus p)$.

The elements are just cycles with boundary in $X \setminus p$. i.e., forget everything outside an infinitesimal neighborhood of p.

Key lemma for understanding this.

Proposition 227. If V is a neighborhood of p, then

$$H_*(X, X \setminus p) \cong H_*(V, V \setminus p) \tag{196}$$

PROOF 228. Excise $X \setminus V$ from $X \setminus p$. □

Example 229. Let X be an n-manifold, i.e. for all $p \in X$ there is a neighborhood $V \cong \mathbb{R}^n$.

Then wlog $p = 0$, and

$$H_*(X, X \setminus p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus p) \cong H_*(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_*(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases} \tag{197}$$

Now there are two choices of generators of $\mathbb{Z}$: ± 1.

Definition 230. A **local orientation of a manifold** X **at a point** $p \in X$ is a choice of $+$ or $-$ as generator of $H_n(X, X \setminus p) \cong \mathbb{Z}$.

A **global orientation** is a function $p \mapsto \mu_p$ assigning a local orientation for all $p \in X$ so that if $U \subseteq X$ is a coordinate neighborhood and $B \subseteq U$ is a ball, then there exists $\mu_B \in H_n(U, U \setminus B) \cong \mathbb{Z}$ so that $\mu_B \mapsto \mu_p$ under $H_n(X, X \setminus B) \to H_n(X, X \setminus p)$ for all $p \in B$ (i.e. compatibility).

Note that this does not require a simplicial structure.

Remark 231 (Reconcile with Twirly Definition.). A disk is just a filled in n-simplex, and the above is just a nice order in on this n-simplex.

Proposition 232. If X is a compact orientable n-manifold, then $H_n(X) \cong H_n(X, X \setminus x) \cong \mathbb{Z}$ for all $x \in X$.

Moreover, there exists a class (the **fundamental class**) $[X] \in H_n(X)$ mapping to μ_x under the isomorphism for all x. THis is the "triangulation with canceling the boundaries."

Remark 233. We saw this for torus and other manifolds. We get bunch of copies of $\mathbb{Z}$.

14.4 Winding Numbers and Degrees.

Definition 234. Let $f : X \to Y$ and take the induced map $f_* : H_*(X) \to H_*(Y)$. The **degree of** $f : \mathbb{S}^k \to \mathbb{S}^k$ is the number n of $H(f) : \alpha \mapsto n\alpha$.

Remark 235. f_* should be thought of as analogues of winding numbers since they give how many times a curve winds around a point in X corresponds to how many times it winds in Y.

Indeed, just consider the case $X = \mathbb{S}^1,\ \mathbb{R}^2 \setminus p \simeq \mathbb{S}^1,\ \gamma := f$ for which $H_1(\mathbb{S}^1) = H_1(\mathbb{R}^2 \setminus p) \cong \mathbb{Z}$. A group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ must be a multiplication by some $n \in \mathbb{Z}$. This is the **winding number**.

15 . -Thursday, 5.23.2019

15.1 Duality.

We are moving towards cohomology. Cohomology is just flipping more arrows on homology, and somehow it is more interesting!

Let's start with linear algebra.

Definition 236. The **dual space of a real vector space** V is

$$V^v := \mathrm{Hom}(V, \mathbb{R}) : \{\alpha : V \to \mathbb{R} \text{ linear}\} \tag{198}$$

with the usual addition and scalar multiplication:

$$(\alpha + \beta)(v) := \alpha(v) + \beta(v) \tag{199}$$

$$(\lambda\alpha)(v) := \lambda(\alpha(v)) \tag{200}$$

We think of Hom as a functor.

Remark 237 (Dual space is genuinely new object.). We have some basic facts:
1.) $\dim V^v = \dim V$ implies $V^v \cong V$
  (a.) Isomorphism of vector spaces only make sense after fixing an origin, so this is not canonical, so indeed V^v is a new object.
2.) $(V^v)^v \cong V$ is canonical.

But now, taking a dual lets you pass to the reals, and so there is also a product structure:

$$(\alpha \cdot \beta)(v) = \alpha(v) \cdot \beta(v) \tag{201}$$

This is why most people would like to pass to the dual space.

Remark 238 (Hom as a functor.). We think of $\mathrm{Hom}(\cdot, \mathbb{R})$ as a functor from vector spaces to vector spaces. But now if $V \xrightarrow{f} W$, then we can induce the map $V^v \xrightarrow{f^v} W^v$ given by

$$f^v : \beta \mapsto \beta \circ f \tag{202}$$

Thus $\mathrm{Hom}(\cdot, \mathbb{R})$ is a contravariant functor.

The idea is

Figure 98: We cannot use f to induce a map going from V to $\mathbb{R}$, but we can induce the map $\beta \circ f$ if we have $\beta : W \to \mathbb{R}$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "\beta\circ f"'] & W \arrow[d, "\beta"] \\ & \mathbb{R} \end{tikzcd}$$

Indeed, f^v is homomorphism:

$$(f^v(\alpha + \beta))(v) = (f^v(\alpha) + f^v(\beta))(v) \tag{203}$$

just by unraveling definitions.

There is nothing special about $\mathbb{R}$ so we can also pass to $\mathbb{F}_2, \mathbb{Z}$.

Now if we have the sequence

$$V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \ldots \to V_1 \xrightarrow{f_1} V_0 \tag{204}$$

which induces

$$V_n^v \xleftarrow{f_n} V_{n-1}^v \xleftarrow{f_{n-1}} \ldots \to V_1^v \xleftarrow{f_1} V_0^v \tag{205}$$

We can now apply this to complexes!

Definition 239. Let X be a space of dimension n so that

$$\Delta(X)_n \xrightarrow{\partial_n} \Delta(X)_{n-1} \xrightarrow{\partial_{n-1}} \ldots \to \Delta(X)_1 \xrightarrow{\partial_1} \Delta(X)_0 \tag{206}$$

which induces

$$\Delta(X)_n^v \xleftarrow{\partial_n^v} \Delta(X)_{n-1}^v \xleftarrow{\partial_{n-1}^v} \ldots \to \Delta(X)_1^v \xleftarrow{\partial_1^v} \Delta(X)_0^v \tag{207}$$

The **cohomology of** X is the vector space

$$H^k(X) := \ker(\partial^v)/\mathrm{im}(\partial^v) := \ker(\partial^v : \Delta_{k-1}(X)^v \to \Delta_k(X)^v)/\mathrm{im}(\partial^v : \Delta_{k-1}(X)^v \to \Delta_k(X)^v) \tag{208}$$

(Since we are used to having ∂_n going from n to $n-1$, we just defined $d_n := \partial_n^v$.) ∂^v is the **coboundary map**.

Remark 240. This is a notational mess, so we start denoting duals with a superscript. For instance, $\Delta^1(X) := \Delta_1(X)^v$ and put "co-" on the names, e.g. elements

$$\Delta_1(X)^v := \mathrm{Hom}(\Delta_1(X) \to \mathbb{R}) \tag{209}$$

are the **1-cochains** $\varphi : \Delta_1(X) \to \mathbb{R}$.

Example 241. We should work with $\mathbb{F}_2$ to build intuition. (They are great because a simplex is either there or not there.) Consider $X = \mathbb{D}^2$.

Elements of $\Delta_1(X)$ are 1-chains like

$$c := \sum_i k_i[\sigma_i : \Delta^1 \to X] \tag{210}$$

and the elements of

$$\Delta^1(X); = \Delta_1(X)^v = \mathrm{Hom}(\Delta_1(X) \to \mathbb{R}) \tag{211}$$

are called 1-cochains.

We often write

$$\varphi(x) = \langle \varphi, c \rangle \tag{212}$$

If ℓ is the number of 1-simplices σ in c, such that $\varphi(\sigma) = 1$ then $\langle \varphi, c \rangle = 1$ iff ℓ odd.

Remark 242. Cohomology group *over a field* is isomorphic to homology group *over a field*. This does not hold over rings such as $\mathbb{Z}$. (We will not go into this.)

Example 243. Consider

$$\Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{0} \tag{213}$$

$$\Delta^2 \xleftarrow{d^1} \Delta^1 \xleftarrow{d^0} \Delta^0 \xleftarrow{0} 0 \tag{214}$$

Then from the commutative diagram we had before, $d^1(\varphi) = \varphi \circ \partial_2$ for $\varphi \in \Delta^1$. In particular,

$$\varphi(\partial_2 c) = (d^1 \varphi)(c) \tag{215}$$

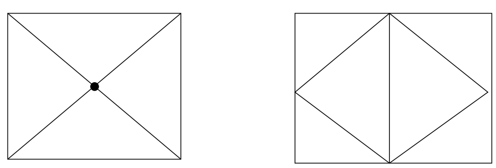
$$\langle \varphi, \partial c \rangle = \langle d\varphi, c \rangle \tag{216}$$

which in particular means d is adjoint to ∂ under this inner product $\langle \cdot, \cdot \rangle$.

If φ is the indicator function on a single p-simplex σ, then $d\varphi$ is the indicator function on the cofaces [5] of σ.

Example 244. Here is an example:

Figure 99: (Left:) The cofaces of the vertex at the center are the four edges coming out of the vertex. If we identified the center vertex is identified on the top left vertex, then the top edge and the left edge of the square also becomes a coface. But the top left diagonal will become counted twice, and it becomes 0. (Right:) the coface of the line segment in the center are the two triangles adjacent to it.

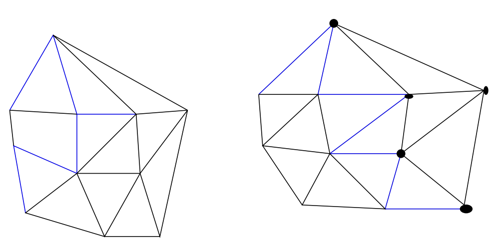


Example 245. Back to the $\mathbb{D}^2$ example. Consider a triangulation of $\mathbb{D}^2$. The 0-cochain that gives value 1 to every vertex in a 0-cocycle. This gives the kerd^0. This must be the only one because if we had even one vertex that is not 1, then we get coboundaries.

Thus,

$$H^0(\mathbb{D}^2) = \text{ker}(d^0 : \Delta^0 \to \Delta^1)/0 \cong \mathbb{F}_2 \tag{217}$$

Figure 100: Examples of 1-cochains. We can get these heuristically by just starting at one edge and "canceling out faces."



[5] **Cofaces** are simplices that have σ as a face.

We see that $d^1(\phi) = 0$ iff every triangle is adjacent to even number of $\mathscr{C}$ such that $\phi(\mathscr{C}) = 1$.

The blue cocycle on the right is d of the 0-cochain indicated by the black dots, so

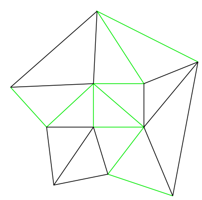
$$H^1(\mathbb{D}^2) = \text{ker}(d^1 : \Delta^1 \to \Delta^2)/\text{im}(d^0 : \Delta^0 \to \Delta^1) \cong 0 \tag{218}$$

For 2-cochains,

$$\text{ker}(d^2 : \Delta^2 \to 0) = \Delta^2 \tag{219}$$

So, we just need to understand $\text{im}(d : \Delta^1 \to \Delta^2)$.

Figure 101:



We get

$$H_2(\mathbb{D}^2) = 0 \tag{220}$$

If $f : V \to W$, then $f^v : W^v \to V^v$. Then since dualizing gives transposes,

$$[f]^t = [f^v] \tag{221}$$

So in particular,

$$[\partial]^t = [d] \tag{222}$$

Proposition 246 (Universal Coefficient Theorem). Given space X, there exist maps for all fields $F(= \mathbb{R}, \mathbb{F}_2)$

$$H^p(X; F) \overset{\cong}{\to} \text{Hom}(H_p(X); F) \overset{\cong}{\to} H_p(X; F) \tag{223}$$

where the first map is natural, but the second is not (since we need to choose a basis).

Remark 247. Instead of a field, we can take $\mathbb{Z}$, but we will need to introduce Ext and Tor functors.

Definition 248. If $f : X \to Y$, then

$$\begin{tikzcd} H^p(X) \arrow[r, "f^*"] \arrow[d, "\cong"] & H^p(Y) \arrow[d, "\cong"] \\ \text{Hom}(H_p(X), F) \arrow[r, "f_*"] & \text{Hom}(H_p(Y), F) \end{tikzcd}$$

Mayer-Vietoris etc are natural since they play well with functors.

16 . -Tuesday, 5.28.2019

16.1 Poincaré Duality: Cohomology and Hidden Symmetries.

Remark 249. Cohomology allows us to see hidden symmetries in manifolds that we cannot see otherwise. The higher dimensional cohomologies are mirror images of the lower dimensional ones.

Proposition 250 ((Baby) Poincar'e Duality.). Let M be a closed manifold. Then

$$H_p(M, \mathbb{F}_2) \cong H^{n-p}(M, \mathbb{F}_2) \tag{224}$$

Remark 251. The map is a natural isomorphism.

Remark 252. Compactness and $\mathbb{F}_2$ makes this very simple. In particular, it avoids the issues about compactly supported cohomologies.

Example 253. Recall that

$$H_p(M, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & k = 0, 2 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{225}$$

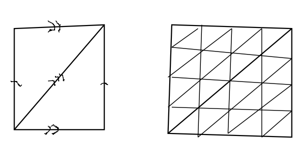
$$H_k(K) = \begin{cases} \mathbb{F}_2 & k = 0, 2 \\ \mathbb{F}_2^2 & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{226}$$

Remark 254. By Universal Coefficient Theorem,

$$H_p(M) \cong H_{n-p}(M) \tag{227}$$

Let's get a better intuition of this by looking at a concrete example: triangulated manifolds. Recall that triangulations are genuine simplicial complex structures. We need this because we want every triangle to be genuinely a triangle (without extra identifications), i.e. homeomorphic to a disc:

Figure 102: Δ-complex Structure vs. Simplicial Complex Structure on a torus.



Also recall:

Definition 255. A **simplicial complex** is a collection of simplicies K such that

1.) $\sigma \in K$ and $\tau \leq \sigma$ then $\tau \in K$
2.) $\sigma_1, \sigma_2 \in K$ then $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \leq \sigma_1, \sigma_2$

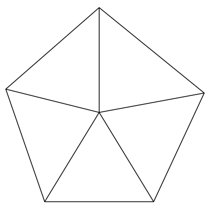
Definition 256. The **start of a simplex** is

$$\text{St}(\tau) := \{\sigma \in K : \tau \leq \sigma\} \tag{228}$$

which are the "cofaces." The **closed star** is $\overline{st}(\tau)$ is the smallest subcomplex containing $\text{st}(\tau)$.

Example 257. The star is not a subsimplicial complex. In this example, the star misses the outer points. One can think of the closed start like a closed set.

Figure 103: The star of the point in the middle are the 5 edges and faces adjacent to it.



Definition 258. The **link of a simplex** is

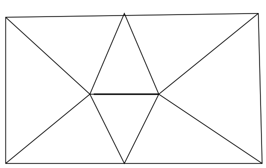
$$\text{Lk}(\tau) := \left\{\sigma \in \overline{\text{st}}(\tau) : \sigma \cap \tau = \emptyset\right\} \tag{229}$$

Think of this as intersect K with small high dimensional ball and take the boundary.

Example 259. Going back the the previous example, the link of the vertex at the center is the pentagon on the outside.

Example 260. The star of the segment consists of the top and bottom triangles along with itself. The link is just the top and bottom vertices.

Figure 104:



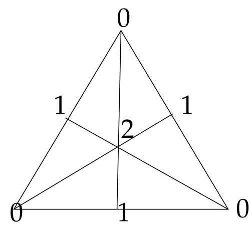
16.2 Barycentric Subdivisions.

Recall that $K^{(i)}$ is the collection of i-skeletons $\{\sigma \in K : \dim(\sigma) \leq i\}$.

Definition 261. The **barycentric subdivision of** K denoted K' is given inductively by ${K'}^{(0)} = K^{(0)}$ and for ${K'}^{(i)}$, add barycenter of every i-simplex as new vertex and connect it to simplices in its boundary.

Example 262. Start with three vertices. The barycenter of a segment is the midpoint. So we take midpoints of respective pairs and connect the with the endpoints of the segments. We then take barycenter of the three midpoints, and connect this new midpoints to all the previous vertices. There are three steps in this construction which gives a label for the vertices.

Figure 105:



Definition 263. A **combinatorial n-manifold** M is a triangulation K of M such that for all i-simplex σ, $\mathrm{Lk}(\sigma)$ is a triangulation of $\mathbb{S}^{n-i-1}$.

Remark 264. Manifolds locally look like $\mathbb{R}^n$, so neighborhoods should look like spheres. Thus, the triangulations should look like triangulations of these spheres.

Example 265. In the pentagon example, we get triangulation of $\mathbb{S}^1$. In the example with the rectangle, we got two points, i.e. $\mathbb{S}^0$.

16.3 Dual Blocks.

We take simplices to get many simplices. Subdivide K to K', and label vertices accordingly.

Remark 266. Observe that in each simplex in K', there is a unique vertex with minimal label. This is clear from the triangle examples.

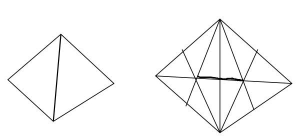
Definition 267. Let $\sigma \in K$, and let u be its barycenter. The **dual block** $\widehat{\sigma}$ is the union of all simplices in K' such that u is the vertex with the minimal label.

Example 268. Back to the example with the triangle: If σ is a σ is the big triangle, then $\widehat{\sigma}$ is the vertex in the center.

If σ is the bottom left corner of the triangle, then $\widehat{\sigma}$ is the quadrangle in the bottom left corner.

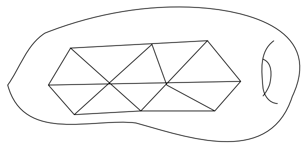
Example 269. Here is another:

Figure 106: If the segment on the left is σ, then $\widehat{\sigma}$ is the right.



Example 270. Here is another:

Figure 107: If σ is the whole thing, then $\widehat{\sigma}$ are two hexagons with edges coming out of it, where the vertices of the hexagons are the barycenters of the respective triangles.



The point is that 0 simplex goes to 2 simplex, 1 simplex to 1 simplex, 2 simplex to 0 simplex.

Remark 271. If p-simplex σ is a face of $(p+1)$ -simplex τ then $\widehat{\sigma}$ contains $\widehat{\tau}$ in its boundary. The boundary (in the sense of point set topology) of $\widehat{\sigma}$ is $\mathbb{S}^{n-p-1}$

Definition 272. A **blockchain** of dimension $q := n - p$ is a formal sum

$$\sum_i a_i \widehat{\sigma}_i \tag{230}$$

where σ_i are p-simplices and $a_i \in \mathbb{F}_2$. D_q is the vector space with basis $\widehat{\sigma}_i$'s where σ_i are p-simplices in K. We define the **boundary maps**

$$\partial_\square : D_q \mapsto D_{q-1} \tag{231}$$

$$\widehat{\sigma}_i \mapsto \sum \widehat{\tau}_i \tag{232}$$

τ_j is $(p+1)$-dimensional coface of σ_i.

Exercise 273. $\partial_\square^2 = 0$ and hence homology make sense.

Remark 274. We have the sequence

$$0 \to D_n \xrightarrow{\partial_\square} D_{n-1} \xrightarrow{\partial_\square} \dots \to D_0 \xrightarrow{\partial_\square} 0 \tag{233}$$

Now view M as K', and we get

$$0 \to \Delta_n \xrightarrow{\partial_\triangle} \Delta_{n-1} \xrightarrow{\partial_\triangle} \dots \xrightarrow{\partial_\triangle} \Delta_0 \to 0 \tag{234}$$

Observe that we have the map $T : D_q \to \Delta_q$ with $\widehat{\sigma}$ to sum of q-simplices in $\widehat{\sigma}$. We can then check that $T \circ \partial_\square = \partial_\triangle \circ T$, so this induces $T_* : H_q(D.) \xrightarrow{\cong} H_q(C.)$.

If we assume this, we can prove Poincaré duality.

PROOF 275. We've shown that

$$H_q(M) \cong H_q(K) = H_q(K') = H_q(D.) \tag{235}$$

So we need to show $H_q(D.) \cong H^p(K)$.

For any p-simplex $\sigma \in K$, let σ^* be dual p-cochain in $\Delta^p(K)$,

$$\langle \sigma^*, \tau \rangle = \begin{cases} 1 & \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \tag{236}$$

We then have a (noncanonical) map

$$\phi_q : D_q \xrightarrow{\cong} \Delta^p \tag{237}$$

$$\widehat{\sigma} \mapsto \sigma^* \tag{238}$$

which is an isomorphism because the basis of the LHS and RHS are both indexed by p-simplices. So, to show the claim, we need to show that ϕ_p commutes with boundary and coboundary.

Figure 108: $\phi_{q-1} \circ \partial(\widehat{\sigma})$ is $p+1$-cochain that evaluates to 1 only on $p+1$ dimesional cofaces of σ.

$$\begin{array}{ccc} D_q & \xrightarrow{\partial_\square} & D_{q-1} \\ \downarrow\cong & & \downarrow\cong \\ \Delta^p & \xrightarrow{d_\square} & \Delta^{p+1} \end{array}$$

And $d_\triangle \circ \phi_q(\widehat{\sigma})$ does same thing.

$\square$

17 . -Thursday, 5.30.2019

17.1 Intersection Theory.

For the rest of the course, we work mod 2.

Remark 276. In undergraduate, we work with triangulations, and so we have coordinates. But in graduate school, we don't have triangulations, and everything is coordinate free.

Recall:

Proposition 277. Let M be a combinatorial d-manifold with triangulation K. Recall that $p+q=d$ and σ is a simplex, then $\widehat{\sigma}$ is a q-dimensional union of q-simplices in K' such that the barycenter of σ has a minimal label.

We used these to define **block chain complex**

$$\ldots \to D_q \xrightarrow{\partial_\square} D_{q-1} \xrightarrow{\partial_\square} \ldots \tag{239}$$

where we defined

$$\partial_\square(\widehat{\sigma}) = \sum_{\tau_j:\ (p+1)-\text{coface of } \sigma} \widehat{\tau}_j \tag{240}$$

We then take the homology $H_q(\mathscr{D}.)$ by the usual construction.

Proposition 278 (Poincaré Duality.).

$$H_q(M) \cong H_q(\mathscr{D}.) \cong H^q(M) \tag{241}$$

The first isomorphism follows from the homework. The second follows from $D_q \cong \Delta^p$ with $\phi_q : \widehat{\sigma} \leftrightarrow \sigma^*$ respects ∂, d.

We want a better way to interpret this isomorphism.

Let σ be a p-simplex and $\widehat{\sigma}$ be a union of q-simplices. Then $\sigma \cap \widehat{\sigma}$ is a single point, and it is the barycenter of σ. On the other hand, $\sigma \cap \tilde{\tau}$ for $\tau \neq \sigma$ is just the empty set which we should think as 0 points of intersection.

We want to capture this idea algebraically which will give us the Poincaré duality. Algebraically,

$$\sigma \cdot \tilde{\tau} = \begin{cases} 1 & \sigma = \tau \\ 0 & \sigma \neq \tau \end{cases} \tag{242}$$

We can use this to count the intersections of cycles. We have

$$c = \sum q_i \sigma_i \in \ker(\partial), \; d = \sum b_j \widehat{\tau}_j \in \ker(\partial) \tag{243}$$

Definition 279. We extend linearly to get

$$c \cdot d := \sum a_i b_j (\sigma_i \cdot \widehat{\tau}_j) \tag{244}$$

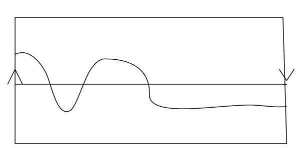
This is the **intersection number of** c, d.

In other words, this counts how many times is the summand of c dual to the summand of d.

Remark 280. $c \cdot d = 0$ if $\#(c \cap d)$ is even. $c \cdot d = 1$ otherwise.

Example 281. Consider the central circle of a Möbius band.

Figure 109: Observe that there will always be an odd number of intersection points (via intermediate value theorem). If the curve is tangent to the central circle, then we can perturb (by transversality theorem) to make it into the form below.



Proposition 282. If $c \sim c_0$ (homologous), then $c \cdot d = c_0 \cdot d$.

This is an easy exercise.

Definition 283. By the above construction, we get the map

$$\# : H_p(M) \times H_q(M) \to \mathbb{F}_2 \tag{245}$$

$$(c, d) \mapsto c \cdot d \tag{246}$$

This is well-defined, symmetric [6] , and bilinear.

Here are some definitions from linear algebra:

Definition 284. Let U, V be vector spaces (in our case over $\mathbb{F}_2$). A **bilinear pairing** $\# : U \times V \to \mathbb{F}_2$ gives us a natural isomorphism

$$\phi_{\#} : U \times V \to \mathrm{Hom}(U, \mathbb{F}_2) = U^{\times} \tag{247}$$

$$v \mapsto \#(\cdot, v) \tag{248}$$

A **pairing is perfect** if for all nonzero $u \in U$, there is $v \in V$ such taht $\#(u, v) = 1$. Likewise, for all $v \in V$, there is a $u \in U$ if $\#(u, v) = 1$

This last definition rules out any degeneracy:

Proposition 285 (Perfect Pairing Lemma.). $\# : U \times V \to \mathbb{F}_2$ is perfect iff $\phi_{\#}$ is an isomorphism and thus $V \cong U^* \cong U$ where the first map is natural but the second is not.

17.2 Cohomology.

Poincaré duality gives $\phi_q(\widehat{\sigma}) = \sigma^*$. That is, $\phi_q(\widehat{\sigma})$ is the p-cochain for which

$$\langle \sigma^*, \tau \rangle = \begin{cases} 1 & \sigma = \tau \\ 0 & \sigma \neq \tau \end{cases} \tag{249}$$

Notice that

$$\langle \phi_q(\widehat{\sigma}), \tau \rangle = \widehat{\sigma} \cdot \tau \tag{250}$$

We extend linearly to get induced maps on homology to defined

$$\langle \phi_*(\gamma), \delta \rangle = \gamma \cdot \delta \tag{251}$$

where $\gamma \in H_p, \ \delta \in H_q$.

We can rephrase Poincaré duality as follows:

Proposition 286 (Poincaré Duality: Second Version.). Let M be a combinatorial $p + q = d$ dimensional manifold. Then the map

$$\# : H_p(M) \times H_q(M) \to \mathbb{F}_2 \tag{252}$$

$$(\gamma, \delta) \mapsto \gamma \cdot \delta \tag{253}$$

is a perfect pairing.

This is a statement in terms of homology groups, but it is really about cohomology.

Remark 287. Cup product is Poinacaré dual to intersections.

[6] We get symmetric because of $\mathbb{F}_2$. In full generality, we get alternating symmetric.

Example 288. Consider the 2-torus T. Let's look at the pairing $(p, q) = (1, 1)$ which is the most interesting pair.

Let x, y be the two curves that generate $H_1(T)$. It suffices to compute $\#$ for each pair formed by x, y since they are basis elements. If we do the same construction as we did for the Möbius band, we can perturb x to get a homologous curve x_0 which allows us to compute

$$x \cdot x = x \cdot x_0 = 0 \tag{254}$$

Likewise,

$$y \cdot y = y \cdot y_0 = 0 \tag{255}$$

where all computation is in $\mathbb{F}_2$. We also have

$$x \cdot y = 1 \tag{256}$$

We thus get a matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{257}$$

which captures all the information on the intersection pairings on the torus. We note that the determinant is 1 in mod 2.

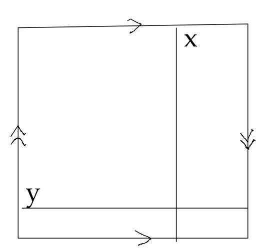
We recall that

$$H_k(T) = \begin{cases} \mathbb{F}_2 & k = 0, 2 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & k = 1 \end{cases} \tag{258}$$

Example 289. We recall that the homology group over $\mathbb{F}_2$ of the Klein bottle KB is the same as the torus, so in particular we cannot tell them apart using just homology over $\mathbb{F}_2$. We can consider the intersection pairings.

For Klein bottle, consider the usual generators.

Figure 110:



We have

$$x \cdot x = 0, \; x \cdot y = 1, \; y \cdot y = 1 \tag{259}$$

and so

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{260}$$

Now we could have chosen different basis elements, so we would also need to consider its conjugates. However, this matrix is not conjugate to the matrix for the torus (by looking at the traces of the matrices for instance). So the two space are indeed different.

17.3 Euler Characteristic and Betti Numbers.

Recall that

$$\chi(M) = \sum_p (-1)^p n_p \tag{261}$$

where n_p is the number of p-simplices.

Recall that

$$H_p(M) = \text{ker}(\partial : \Delta_p \to \Delta_{p-1})/\text{im}(\partial : \Delta_{p+1} \to \Delta_p) \tag{262}$$

where $Z_p := \dim(\text{ker}(\partial : \Delta_p \to \Delta_{p-1}))$ and $b_p := \dim(\text{im}(\partial : \Delta_{p+1} \to \Delta_p))$.

Then the Betti numbers are just $\beta_p = Z_p - b_p$ and $n_p = Z_p + b_{p-1} = \dim(\Delta_p)$. Then

$$\sum_p (-1)^p n_p = \sum (-1)^p (Z_p + b_{p-1}) = \sum (-1)^p (Z_p - b_p) = \sum (-1)^p \beta_p \tag{263}$$

Thus, the Euler characteristic is a topological invariant iff the Betti numbers are topological invariants. The latter is more intuitively believable since it is just a number of holes whereas triangulations are not obviously invariant.

Here is a very nice corollary:

Proposition 290. The Euler characteristic of an odd dimensional manifold is 0.

PROOF 291. From the above:

$$\chi(M) = \beta_0 - \beta_1 + \beta_2 - ... + \beta_{2k} - \beta_{2k+1} = 0 \tag{264}$$

where we get the cancellations from Poincaré duality.

□

For an even dimensional manifold, we have

$$\chi(M) = 2(\beta_0 - \beta_1 + ... \pm \beta_{2k-1}) \mp \beta_k \tag{265}$$

and so the Euler characteristic is even iff β_k is even.

17.4 Announcements.

Some remarks on the final exam:

1.) Mostly computational
2.) Seifert van Kampen computations
3.) No covering spaces
4.) Homology: Mayer-Vietoris, exact sequence
   (a.) Check sanity check in piazza