Introduction to Differentiable Manifolds

Tom Sawada

February 17, 2019

These course notes are based on MATH 27400 Introduction to Differentiable Manifolds and Integration on Manifolds taught by Professor Wilhelm Schlag in Spring, 2018. We used Barden and Thomas, *Introduction to Differential Manifolds* for the first three weeks. We then switched to Shifrin, *Differential Geometry on Curves and Surfaces* and Do Carmo, *Differential Forms and Applications*. There were weekly homeworks, one midterm, and one final. Note that these notes have not faithfully followed the lecture content, and that in particular, the order in which the material has been presented has been altered. The TA Nat Mayer and Professor Schlag has been extremely helpful in answering questions pertaining to these notes.

Contents

Week 1.

Inverse Function Theorem and Consequences. -Tuesday, 3.27.2018

We start with the deepest result in multivariable calculus.

Proposition 1 (Inverse Function Theorem). Let f be a C^1 map from $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Additionally, let $\det Df(x_0) \neq 0$ at some $x_0 \in U$. Then f is locally invertible at x_0, it is an open map, and its inverse is C^1.

Additionally, if f is smooth[1], then f^{-1} is also smooth.

We provide a proof from Barden and Thomas [2].

PROOF 2. **Bijectivity.**
Claim 1: We can assume wlog that $x_0 = 0, f(0) = 0, Df(0) = 0$.

Since translations and linear bijections are both diffeomorphisms, we can prove the theorem instead for the function

$$Df(0) \circ (f(x) - f(0)) \tag{1}$$

Applying the chain rule shows that this function meets the third condition. The second condition is immediate. The first is from the translation. This shows the claim.

Claim 2: The function

$$g_y(x) := x + y - f(x) := y + g_0(x) \tag{2}$$

[1] We say smooth to mean C^∞.
[2] p.176

is a contraction map on some closed ball around the origin, and therefore, by the contraction mapping principle, it has some fixed point.

Since g_y is C^1, there is some $r > 0$ such that in the ball $B(0, r)$, $\|Dg_y(x)\| \leq \frac{1}{2}$. Therefore, by the mean value theorem,

$$\|g_y(x') - g_y(x)\| \leq \frac{1}{2} \|x' - x\| \tag{3}$$

Thus, this is a contraction mapping.

We can thus take small neighborhoods to obtain the open sets.

□

We can state the above result even more concisely by introducing a new definition.

Definition 3. A C^1 map $f : U \to V$ (for open sets $U, V \subset \mathbb{R}^n$) is a C^1 diffeomorphism if f has a C^1 inverse $f^{-1} : W \to V$.

Thus, the inverse function theorem says the following:

Proposition 4 (Inverse Function Theorem). A C^1 function with a nonsingular Jacobian at some point x is a local diffeomorphism at x.

Here are some elementary, but very important examples.

Example 5 (Monotonically increasing C^1 functions in $\mathbb{R}$.). For $n = 1$, monotonically increasing, C^1 functions satisfy the conditions for the inverse function theorem (and therefore, they are local diffeomorphisms from the domain to their range). Note that the inverse function theorem is a *local* theorem. For instance, for functions such as $f(x) = x^2$, we can apply the inverse function theorem for nonzero x, but when the derivative vanishes (i.e. a critical point), the function is very clearly not one-to-one (hence not a diffeomorphism).

Example 6 (A C^1 homeomorphism is not a diffeomorphism.). Consider the function $f(x) = x^3$. The function is a differentiable (hence continuous) bijection, hence invertible, and the inverse is continuous, so it is a homeomorphism. On the other hand, the inverse function is definitely not differentiable at the origin since the derivative blows up.

Example 7 (Polar coordinates.). Polar coordinates defined by the map

$$\Phi : (r, \theta) \mapsto (r \cos\theta, r \sin\theta) \tag{4}$$

is a diffeomorphism from $(0, \infty) \times (-\pi, \pi) \to \mathbb{R}^2 \setminus \mathbb{R}_{<0}$. This is a smooth , surjective map. We can compute the inverse to be

$$\Phi^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \theta(x, y)\right) \tag{5}$$

where $\theta(x, y) = \arctan\left(\frac{y}{x}\right)$.

We can also consider the 1-form $d\theta$ defined on the punctured plane. Just by the definition of the differential, we have

$$d\theta = \frac{-ydx + xdy}{x^2 + y^2} \tag{6}$$

This is smooth (since the coefficients of dx, dy are smooth functions), closed (because $d^2\theta = 0$), but not exact (since integrating around the origin gives a nonzero value).

Example 8 (Spherical coordinates.). The natural analogue of the previous example in $\mathbb{R}^3$ is spherical coordinates:

$$\Phi : (r, \theta, \phi) \mapsto (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \tag{7}$$

which is a diffeormorphism from $(0, \infty) \times (0, 2\pi) \times (0, \pi)$ to $\mathbb{R}^3 \setminus \{x \geq 0, x_2 = 0\}$.

Example 9 (A holomorphic function.). Here is an example from complex analysis. Take $f(z) = z^2$, then $f'(z) = 2z$. Then

$$\det Df(z) = 4\,|z|^2 \neq 0 \tag{8}$$

for $z \neq 0$. Restricting this shows that it is a diffeomorphism from say the (open) upper half plane to $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

A very important corollary of the inverse function theorem is the implicit function theorem. This is best illustrated by looking at the nonlinear case first.

Suppose A is an $n \times m$ matrix of rank n (where $m > n$). Then it has submatrices A', A'' such that A'' is an $n \times n$ matrix with full rank, and

$$A = [A', A''] \tag{9}$$

(We will keep using these notations to denote submatrices.) If $x = (x', x'')$ where x'' is an n vector, then we can solve the system of linear equations

$$Ax = 0 \tag{10}$$

by observing that

$$Ax = A'x' + A''x'' = 0 \tag{11}$$

Thus,

$$x'' = -(A'')^{-1} A'x' \tag{12}$$

Note here that we wrote x'' as a function of x'. This is precisely the same idea as the nonlinear case:

Proposition 10 (Implicit Function Theorem). Let $F \in C^\infty(U, \mathbb{R}^n)$ for an open set $U \subset \mathbb{R}^m$, and suppose there is $x_0 \in U$ (the "seed value") such that $F(x_0) = 0$ and $\det\left(\frac{\partial F}{\partial x''}\right) = n$. Then locally near x_0 we can solve uniquely $F(x) = 0$ in the form

$$F(x', x''(x')) = 0 \tag{13}$$

Why do we need the seed value x_0? The point is that one cannot create a solution from thin air, e.g.

$$x_1^2 + x_2^2 + 1 = 0 \tag{14}$$

obviously does not have a solution, but it satisfies all of the other conditions of the theorem.

Definition 11. A smooth k-dimensional submanifold $M \subset \mathbb{R}^m$ (denoted M^k) is a set such that for all $p \in M$, there exists a local diffeomorphism $\Phi : U \ni p \to \mathbb{R}^k$ (called the "straightening diffeomorphism") such that for all $B_p(r) \subset \mathbb{R}^m$, we have

$$\Phi(B_p(r) \cap M) = \Phi(B_p(r)) \cap \mathbb{R}^k \times \{0_{\mathbb{R}^{m-k}}\} \tag{15}$$

The point is that we want to make the neighborhoods on the manifold $M \subset \mathbb{R}^n$ (with the subspace topology) have neighborhoods that look like neighborhoods on $\mathbb{R}^n$.

(Note: we use both M and M^k to denote the k-dimensional submanifold. The superscript simply is a reminder of the dimension of the manifold, and should not be confused with Cartesian products of sets.)

Here are equivalent characterizations of a manifold.

Proposition 12. The following are equivalent:

- (Local Charts.) M is a k-dimensional submanifold of $\mathbb{R}^n$ (i.e. M looks locally like a ball in $\mathbb{R}^k$)
- (Local Graph.) For all points $p \in M$, there is a neighborhood $p \in U \subset M$ such that U is a graph of a smooth function $f : V \subset \mathbb{R}^n \to U \subset M$
- (Local Vanishing Set.) For all points $p \in M$, there is a neighborhood V around p which are the set of points $x \in \mathbb{R}^n$ such that $F(x) = 0$ for a smooth function $F : V \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ and the linearization $DF(x)$ has full rank on all of V.

Note that the second definition says that the manifold is a graph of a function whose domain is in the coordinate subspace.

PROOF 13. we don't need to worry to much about locality since all three statements are local statements, and therefore, we can keep working in the same coordinate neighborhoods.

($1 \implies 3$.) Since the straightening diffeomorphism maps neighborhoods on the manifolds to a copy of $\mathbb{R}^n$ inside $\mathbb{R}^N$, we can just project on to the last $N - n$ coordinate on which the coordinates vanish, i.e. the coordinate neighborhood is the vanishing set of $F := \pi_2 \circ \Phi$ where $\pi_2 : \mathbb{R}^N \to \mathbb{R}^{N-n}$ is the projection onto the last $N - n$ coordinates. $\square$

($3 \implies 2$.) By implicit function theorem, we can solve for the last $N - n$ coordinates in terms of the first n. Indeed, we have just the right hypothesis to do this. $\square$

($2 \implies 1$.) Since what we have is the equation, $x'' = h(x')$, we can just construct the straightening diffeomorphism by modifying this:

$$\Phi(x', x'') = (x', x'' - h(x')) \tag{16}$$

We can look at the derivative of this. Since we do not do anything to the coordinate x', we get an identity for the $n \times n$ diagonal block. We also get an identity for the remaining $N - n \times N - n$ diagonal block since the second coordinate is $x'' - h(x')$. We also know that the block above the $N - n$ block is all 0 since the first coordinate does not depend on x''. This alone would ensure the maximality of the rank of the derivative:

$$D\Phi(p) = \begin{pmatrix} \mathbf{Id}_{n\times n} & 0 \\ -Dh(p) & \mathbf{Id}_{(N-n)\times(N-n)} \end{pmatrix} \tag{17}$$

$\square$

As an addendum: here are some very important examples:

1.) k**-sphere.** $\mathbb{S}^k$ is a k-dimensional submanifold of $\mathbb{R}^{k+1}$ since it is defined as the vanishing set of the smooth map $F : \mathbb{R}^{k+1} \to \mathbb{R}$

$$F(x_1, ..., x_k + 1) = \sum_{i=1}^{k+1} x_i^2 - 1 \tag{18}$$

which has rank 1 since we never have to evaluate ∇F at the origin.

2.) **Euclidean space.** $\mathbb{R}^n$ is an n-dimensional submanifold of $\mathbb{R}^n$.

3.) **Graphs of functions.** By the second characterization of manifolds, this is immediate.

4.) **Smooth hypersurfaces.** This is usually the set on which we perform Lagrange multipliers on. Recall that it is the set of points defined by $F(x_1, ..., x_n) = k$ for some $k \in \mathbb{R}$ and $\nabla F \neq 0$ (i.e. full rank). This satisfies the third condition we saw above.

(a.) A subexample of this is $\mathbb{S}^n$.

(b.) **Kernels of linear maps with full rank** are also submanifolds.

(c.) Other hypersurfaces include: **paraboloids**, **hyperboloids**, **ellipsoids**, etc.

5.) **Special linear group.** $SL(n, \mathbb{R})$ since it is defined by $\det A = 1$ which we can check has full rank (see pset 1).

Here are examples of sets that are not submanifolds of $\mathbb{R}^n$.

1.) The graph of $f(x) = \pm x$. This has a cross at the origin which we cannot straighten out.

Tangent Spaces. -**Thursday, 3.29.2018**

Notation: Unless specified otherwise,

- All of the maps are C^∞
- $m := N - n$

Last time, we defined what a manifold is, and characterized it in three different ways. We now do some calculus to linearly approximate what we defined to be a manifold. Speifically, we want to define what a tangent space is. In this lecture, we will describe how the tangent space can be characterized based on the three definitions we provided last time.

What is a tangent space? It is the collection of all tangent vectors, given a point on the manifold.

Definition 14. Let $\mathcal{M}$ be an n-dimensional submanifold in $\mathbb{R}^N$. The tangent space $T_p\mathcal{M}$ at the point $p \in \mathcal{M}$ is defined to be

$$T_p\mathcal{M} := \{\dot{\gamma(0)} : \gamma \text{ is a curve going through } p\} \subset \mathbb{R}^N \tag{19}$$

We sometimes insist that the RHS has a $p+$ on it, i.e. the tangent vectors are attached at the point p.

Here is a very important charterization that is not immediately obvious from the definition we just gave. This follows the definition of a manifolds (i.e. the first characterization of manifold).

Proposition 15. Let Φ be a straightening diffeomorphism of $\mathcal{M}$ near p, and suppose that p maps to 0 under this map. Then the tangent space at p satisfies

$$T_p\mathcal{M} = (D\Phi(0))^{-1} (\mathbb{R}^n \times \{0\}) \tag{20}$$

and it is a linear space of dimension n.

PROOF 16. This realy comes down to unraveling definitions. Since we defined the tangent space in terms of the curves going through the given point p, we just need to look at what these curves look like. Although it is hard to visualize what the curves in $\mathcal{M}$ look like, any of the curves can be *flattened* into $\mathbb{R}^n$ using the straightening diffeomorphism. In other words, any of the curves on the manifold is a preimage (under the straightening diffeomorphism) of the straightened curve:

$$\gamma(t) = \Phi^{-1}(\eta(t)) \tag{21}$$

where $\eta(t)$ lives in the copy of $\mathbb{R}^n$ sitting in $\mathbb{R}^N$, so

$$\eta(t) = (\eta_1(t), \eta_2(t), ..., \eta_n(t), 0..., 0) \in \mathbb{R}^N \tag{22}$$

We must also require that $\eta(0) = 0$ so to have

$$\Phi^{-1}(\eta(0)) = \Phi^{-1}(0) = p = \gamma(0) \tag{23}$$

as we required as part of the definition of γ and Φ.

Now, we are done because if we use the chain rule,

$$\cdot\gamma(0) = (D\Phi)^{-1}(0)\dot{\eta}(0) \tag{24}$$

Here, we can simply take

$$\eta(t) = (tv, 0, ..., 0) \qquad v \in \mathbb{R}^n \tag{25}$$

for any given $\cdot\gamma(0)$. This makes $T_p\mathcal{M}$ into a linear subspace since it is simply the image of vectors in $\mathbb{R}^n$ under a linear map.

□

We now want to move on to the second characterization of a manifold.

Proposition 17. If $\mathcal{M}$ is given locally near $p \in \mathcal{M}$ as a graph $(x, h(x))$ for $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (so, $p = (x_0, h(x_0))$ for some $x_0 \in U \subset \mathbb{R}^n$), then

$$T_p\mathcal{M} = \{(v, T_p\mathcal{M}) : v \in \mathbb{R}^n\} \tag{26}$$

PROOF 18. The strategy is the same as the last proposition: we ask what the possible curves look like. If we had any curve $\gamma(t)$ on the manifold, we can project it down to $\mathbb{R}^n$ to get a curve $c(t)$, and indeed, this uniquely determines $\gamma(t)$ since $\mathcal{M}$ is (locally) a graph. So, we can just do what we did last time:

$$\gamma(t) = (c(t), h(c(t))) \tag{27}$$

so,

$$\dot{\gamma}(0) = (\dot{c}(0), \dot{h}(c(0))\dot{c}(0)) \tag{28}$$

But we can take $\dot{c}(0)$ to be any vector $v \in \mathbb{R}^n$, so we are done.

□

Now, for the third characterization. But first, a definition.

Definition 19. For a function $F : U \to V$ for open subsets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$, a point p in the image V is a **regular value** if the linearization of the map has full rank at all preimages of p under F. More precisely, for all $p \in U$ such that $F(q) = p$, we have rank$DF(q) = m$ (i.e., maximal rank).

Proposition 20 (Regular Value Theorem.). *The (nonempty) preimage of a regular value is a submanifold*. More precisely, for a function $F : U \to V$ for open subsets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$, let $p \in V$ be a regular value, and let there be some $q \in U$ mapping to p. Then the preimage of p

$$F^{-1}(p) = \{q \in U : F(q) = p\} \tag{29}$$

is an m-codimensional (i.e. $m - n$-dimensional) submanifold of $\mathbb{R}^n$.

Here are some examples.

Example 21. Take $U = \mathbb{R}^N, V = \mathbb{R}$ and $F(x) = \|x\|^2$. Take $p = r^2$. This is a regular value because

$$\nabla F(x) = 2x \neq 0 \qquad x \neq 0 \tag{30}$$

Now, geometrically, it is obvious that

$$F^{-1}(r^2) = r\mathbb{S}^{N-1} \tag{31}$$

Here are some other examples. (See problem set 1.)

Example 22 ($SL(n,\mathbb{R})$). The map

$$\det : SL(n,\mathbb{R}) \to \mathbb{R} \tag{32}$$

defines an $n^2 - 1$ dimensional manifold in $\mathbb{R}^{n^2}$ since $\det A = 1$ for $A \in SL(n,\mathbb{R})$.

Example 23 ($O(n,\mathbb{R})$). The map $g : O(n,\mathbb{R}) \to M_n(\mathbb{R})$ defined by

$$g : A \mapsto A^t A \tag{33}$$

is a submanifold of $\mathbb{R}^{n^2}$.

Before we prove the proposition, we prove the following useful fact (which assumes the above proposition):

Proposition 24. For a function $F : U \to V$ for open subsets $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$, let $p \in V$ be a regular value, and let there be some $q \in U$ mapping to p. For all $q \in F^{-1}(p) = \mathcal{M}$,

$$T_q\mathcal{M} = \ker DF(q) \tag{34}$$

PROOF 25. Fix $q \in F^{-1}(p)$. We assume for now that there is a local diffeomorphism Φ near q such that $(F \circ \Phi)$ is a projection onto the first $N - n$ coordinates.[3]

If we assume this, then we can assume wlog that $\Phi(0) = q$ and $p = 0$ since translation is a diffeomorphism. Therefore, by our earlier characterization of tangent spaces in terms of the straightening diffeomorphism, $T_q\mathcal{M}$ is generated by vector in $\mathbb{R}^N$ whose first $N - n$ coordinates are 0.

Now taking a curve $\gamma \subset F^{-1}(q)$ such that $\gamma(0) = q$, we have

$$F(\gamma(t)) = p \tag{35}$$

just by choice of γ. Therefore, by chain rule,

$$DF(q) \cdot \dot{\gamma(0)} = 0 \tag{36}$$

Therefore,

$$T_q\mathcal{M} \subset \ker DF(q) \tag{37}$$

But since we already know that $T_q\mathcal{M}$ has dimension n, and $\dim \ker DF(q) = N - (N - n) = n$, by rank-nullity theorem. (We use the fact that p is a regular value, in particular that the map is full rank here). Therefore, we have equality.

□

[3] This is a special case of the constant rank theorem which states that for a map $F : U \to V$ for $U \subset \mathbb{R}^N, V \subset \mathbb{R}^n$ with rank$DF = k$ and $F(p) = q$, then there is a diffeomorphism Φ such that $\Phi \circ F \circ \Phi = (x_1, ..., x_k, 0, ..., 0)$. Here, we assume wlog that the diffeomorphism acting on the image of F acts as the identity map.

We can compute the tangent spaces of matrix groups such as the $SL(n, \mathbb{R})$ and $O(n, \mathbb{R})$ using this theorem! (See problem set 1.)

For the proposition, we need the following lemma (which is an easy special case of the constant rank theorem).

Proposition 26. Suppose $F : U \subset \mathbb{R}^N \to \mathbb{R}^{N-n}$ is a smooth map such that rank$DF(q) = N - n$. Then there exists a diffeomorphism $\Phi : \tilde{U} \to U$ such that

$$(F \circ \Phi)(x) = (x_1, ..., x_{N-n}) \tag{38}$$

i.e. F is a projection after a (nonlinear) change of variables in the domain.

PROOF 27. We now prove the proposition. First, wlog, we can write $DF(q)$ as

$$DF(q) = [A|B] \tag{39}$$

where B is $m \times m$ invertible matrix (let $m := N - n$). Then consider the map $G : U \subset \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$G : y \mapsto (F(y), y_1, ..., y_n) \tag{40}$$

whose matrix we can compute to be

$$DG(q) = \begin{pmatrix} A & B \\ \mathbf{Id}_{n \times n} & 0 \end{pmatrix} \tag{41}$$

just by how we defined G. This is invertible (as one can compute the determinant directly [4]). By the inverse function theorem, this makes G into a local diffeomorphism, hence it is invertible. Therefore, we can take $\Phi := G^{-1}$ and taking $x = G(y)$, we get

$$(F \circ \Phi)(x) = F(\Phi \circ G(y)) = F(y) = (x_1, ..., x_{N-n}) \tag{43}$$

which is a projection map.

□

Finally, we go to the proof of the regular value theorem.

PROOF 28. We now prove that $F^{-1}(p)$ is indeed a submanifold.

□

Now that we talked about the preimage of a map, we want to ask when the image of a map is a submanifold. More specifically, given $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^N$, when is $g(U)$ a submanifold of dimension n? For instance, if $g : \mathbb{R} \to \mathbb{S}^1$ is the map

$$g : \phi \mapsto (\cos \phi, \sin \phi) \tag{44}$$

then $\mathbb{S}^1$ is a 1-dimensional submanifold of $\mathbb{R}^2$. What nice conditions does g satisfy to make this happen?

Here is one property that is necessary:

Definition 29. A smooth map $g : U \subset \mathbb{R}^n \to \mathbb{R}^N$ is an **immersion** if

- g is a bijection

[4] For instance, we can apply elementary row operations and use the identity matrix $\mathbf{Id}_{n \times n}$ to row eliminate all of A. Notice that this will not affect columns $n + 1$ through N because columns $n + 1$ through N for rows m through N is all 0s. This shows that

$$\det DG(q) = \begin{pmatrix} 0 & B \\ \mathbf{Id}_{n \times n} & 0 \end{pmatrix} = -\det B \neq 0 \tag{42}$$

and so we do indeed get invertibility.

- g has full rank, i.e. rank$Dg(x) = n$ for all $x \in U$.

Is this condition enough? The answer to this is no; consider the case where $U = \mathbb{R}_{<0}$ and g bends U so that the 0 touches -10 in $\mathbb{R}^2$. $g(U)$ is not locally homeomorphic to $\mathbb{R}$ at the self interesection point. The open set of $g(U)$ (given by the subspace topology inherited from $\mathbb{R}^2$) are of the form $(-10-\epsilon, 10+\epsilon) \cup (-\delta, 0)$. But these are not all of the open sets in $\mathbb{R}$. The map g is an immersion since the description makes it very clearly a bijection, and $\nabla g \neq 0$, so rank$\nabla g = 1$. Problem set 1.6 gives another pathological example of this (the dense lines on a torus.)

In order to circumvent this problem, we need one more topological condition.

Definition 30. The smooth map $g : U \subset \mathbb{R}^n \to g(U) \subset \mathbb{R}^N$ is an **embedding** if it is an immersion and a homeomorphism of U onto its image.

Since an immersion is a bijection, the last condition is equivalent to g being an open (or closed) map. Note that the previous "bending the half interval" example, g is not a homeomorphism since we can remove any point (but the self intersection point) from $g(U)$ to get a connected set whereas U is disconnected if we remove any point from it.

We will now prove that this condition is sufficient to ensure that the image is a manifold.

Proposition 31. If $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is an embedding then $g(U)$ is an n-dimensional submanifold of $\mathbb{R}^N$.

This is proved in the next lecture.

Week 2.

Parametrizations and Abstract Manifolds. -Tuesday, 4.3.2018

We would like to ask when the image of an open set in $\mathbb{R}^n$ is a submanifold of dimension n (i.e. **locally parametrize** a manifold with n variables). We note that a figure eight is not a 1-submanifold of $\mathbb{R}^2$ because it has a self-intersection. Therefore, we at least need the following notion.

Definition 32. A mapping $g : U \subset \mathbb{R}^n \to \mathbb{R}^N$ is an **immersion** if it is injective and it has full rank.

Now we go back to the question: is $g(U)$ a submanifold of $\mathbb{R}^N$? The answer is no. If we consider bending a ray (whose end is open) and loop back onto itself, then the closure of the ray would intersect itself. This transformation of $\mathbb{R}_{>0}$ is an immersion from $\mathbb{R}$ to $\mathbb{R}^2$ (since clearly injective, and nonzero for all points, so full rank). However, this is not a manifold since we can remove some points from this loop, and it would still be connected, but removing any points from $\mathbb{R}$ breaks connectivity. Therefore, the $\mathbb{R}$ and this object cannot be homeomorphic.

We need an even stronger notion:

Definition 33. An immersion of g is an **embedding** if $g : U \subset \mathbb{R}^n \to g(U) \subset \mathbb{R}^N$ is a homeomorphism (or equivalently, an open map).

In the bending example above, the map is not an embedding because it is not a homeomorphism (as we exhibitted). Another example is the dense orbital on the torus (see pset 1.6).

Proposition 34. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^N$ ($n < N$) be an embedding. Then $f(U)$ is an n-dimensional smooth submanifold of $\mathbb{R}^N$.

PROOF 35. We want to construct a straightening diffeomorphism $\phi : \mathbb{R}^N \to \mathbb{R}^N$. Fix $p \in U, q = f(p)$. From the constant rank theorem, there are local diffeomorphisms $\psi : V \to \Psi(V) \subset \mathbb{R}^n, \phi : W' \subset U \to \mathbb{R}^N$ such that $(\phi \circ f \circ \psi)$ is a projection map onto the first n coordinates from $\mathbb{R}^n$ to $\mathbb{R}^N$. In particular, ϕ maps into the copy of $\mathbb{R}^n$ inside $\mathbb{R}^N$, as we would like straightening diffeomorphisms to do.

Naively, we would hope that we can simply take

$$\phi(f(U) \cap W) \subset \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N \tag{45}$$

for an open set W in the ambient space of the manifold. The problem with this is that the manifold could wrap around and intersect W again, which gives a neighborhood of the manifold which is definitely not homeomorphic to a neighborhood in $\mathbb{R}^n$. Therefore, we need a little better (which is to be expected since so far, we've only used the conditions of embeddings shared by immersions; we better use the fact that our map is a homeomorphism somewhere).

Since W is open, there is an open set $\tilde{W} \subset W$ such that[5]

$$\phi(f(\psi(V)) \cap \tilde{W}) = \phi(\tilde{W}) \cap \mathbb{R}^n \times \{0\} \tag{46}$$

Note that $\psi(V) \subset U$, and since f is an open map (by hypothesis), $f(\psi(V))$ is open. Therefore, ϕ is a straightening diffeomorphism as desired.

□

[5] This does follow from the fact that W is open, but one can look at the "bending back" case from before to convinve oneself that this is indeed the case.

The irrational orbits on a torus (pset 1.6) fails this condition. More specifically, the orbits are dense on the torus which means, as a picture, the "bending back" of the manifold (i.e., the curve) is dense on the ambient space (i.e., the torus) such that we cannot choose the good neighborhood $\tilde{W}$. On the contrary, rational orbits are fine because they are not dense, meaning we can choose the open set so that it is disjoint from the bend back.

We thus have the following characterization of local parametrizations.

Definition 36. A submanifld $\mathcal{M}^n \subset \mathbb{R}^N$ is **locally parametrizable** if for all $p \in \mathcal{M}$ there is a neighborhood U in the ambient space $\mathbb{R}^N$ such that there is a map $f : U \to \mathcal{M} \subset \mathcal{M}$ (the **parametrization**) which is a homeomorphism onto $f(U) \ni p$ and it is an immersion (i.e., an embedding of U into $\mathbb{R}^N$).

Proposition 37. $\mathcal{M} \subset \mathbb{R}^N$ is an n-dimensional submanifold iff $\mathcal{M}$ is locally parametrizable.

PROOF 38. ($\Longleftarrow$): This is the previous proposition.

($\Longrightarrow$): This is similar to when we proved that the second characterization of a manifold implied the first, i.e. if a manifold can be patched up by local graphs, then there is a flattening diffeomorphism. Here, we have a similar procedure since we must construct a map (the parametrization which is not necessarily a diffeomorpism) from the coordinate space to the manifold. If we use the local graph characterization of a manifold, we can use the usual trick of producing a map from coordinate space to manifold.

Fix $p \in \mathcal{M}$ and $U \subset \mathbb{R}^n$ such that the neighborhood $V \subset \mathcal{M}$ of p can be written as a graph of $h : U \to V$. Take $f : U \subset \mathbb{R}^n \to \mathbb{R}^N$ such that

$$f : x \mapsto (x, h(x)) \tag{47}$$

f is continuous, bijective, and full rank[6] by how we defined f. Therefore, we just need to show that it is a closed map (which is equivalent to openess for bijective maps). For this, we could simply invoke the closed graph theorem[7], or if one prefers a hard analysis argument, take a convergent sequence $x_n \to x$ in the domain. Then $f(x_n) \to (x, h(x))$, and so, the graph contains its limit points. Thus, f is an immersion homeomorphic to its image.

□

Example 39. For $\mathcal{M} := \mathbb{S}^1 \subset \mathbb{R}^2$, we have the local parametrization

$$f_1(\theta) = (\cos\theta, \sin\theta)\ 0 < \theta < 2\pi \tag{48}$$

and

$$f_2(\theta) = (\cos\theta, \sin\theta)\ -\pi < \theta < \pi \tag{49}$$

We need two because the neighborhoods we take on the circle must be homeomorphic to the coordinate space, i.e. segments on $\mathbb{R}$. We can do the same thing on $\mathbb{S}^2$ using spherical coordinates.

We now move on to the topic of abstract manifold which was postponed until now so that we can properly motivate its definition. The main idea is that if we have two coordinate neighborhoods $f(U), g(V)$ on the manifold $\mathcal{M}$, then we can pull back their intersection $\mathcal{O}$ to the respective coordinate spaces. But coordinates live in Euclidean spaces, so it would be very nice if we can directly go from the coordinate space to the left to the coordinate space on the right.

[6] $Df(x)$ must have full rank since the left $n \times n$ block consitute the identity.

[7] The moral of the story is that f inherits most of its nice properties from h which is nice to begin with since $\mathcal{M}$ is a manifold.

Proposition 40. If $f : U \subset \mathbb{R}^n \to f(U) \subset \mathcal{M} \subset \mathbb{R}^N$ and $f : V \subset \mathbb{R}^n \to g(V) \subset \mathcal{M}$ are homeomorphism and an immersion (i.e., parametrizations of $\mathcal{M}$ with coordinate spaces U, V) and $\mathcal{O} := f(U) \cap g(V) \neq \emptyset$, then the **change of coordinate map** $g^{-1}f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

PROOF 41. One would realize after staring at the proposition that this is mildly astonishing; since g is a map from n dimensions to $N \geq n$ dimensions, taking an inverse "collapses a dimension" (or "loses variables"). However, as we will se in a moment, via the straightening diffeomorphism, we can first project onto a copy of $\mathbb{R}^n$ inside $\mathbb{R}^N$ and then take compositions.

Fix $x_0 \in f^{-1}(\mathcal{O})$ and $p = f(x_0) \in \mathcal{M}$. Then by definition of a smooth submanifold, there is a ball $B_r(p)$ in the ambient space $\mathbb{R}^N$ such that there exists a straightening diffeomorphism from the ball to $\mathbb{R}^N$. Then, we can define the functions $\tilde{f}_i, \tilde{g}_i$ by

$$\Phi \circ f(x) =: (\tilde{f}_1(x), ..., \tilde{f}_n(x), 0, ..., 0),\ \Phi \circ g(x) =: (\tilde{g}_1(x), ..., \tilde{g}_n(x), 0, ..., 0) \tag{50}$$

Since f, g are immersions, they have full rank, and therefore, the compositions $\Phi \circ f, \Phi \circ g$ are local diffeomorphisms on $\mathbb{R}^{n}$[8], by the inverse function theorem. Therefore,

$$\tilde{g}^{-1} \circ f = (\Phi \circ g)^{-1} \circ (\Phi \circ f) \tag{51}$$

where $\tilde{g} : V' \subset \mathbb{R}^n \to \tilde{g}(V') \subset B_r(p)$ given by the inversion of $\Phi \circ g$.

□

Example 42 (Polar coordinates and $\mathbb{S}^1$.). Back to our favorite example, the 1-sphere. If we use the notation from before, we have

$$g^{-1} \circ f = \begin{cases} \theta & 0 < \theta < \pi \\ \theta - 2\pi & \pi < \theta < 2\pi \end{cases} \tag{52}$$

The first part is obvious since that is where $\mathcal{O}_1, \mathcal{O}_2$ overlap, and the second is also clear since that is where the domain differs by exactly one revolution.

Now we are ready to talk about abstract manifolds.

Definition 43. Let $\mathcal{M}$ be a **topological manifold**, i.e.

- it is a Hausdorff, second countable topological space,
- there exists $\{(\phi_\alpha, U_\alpha)\}_{\alpha \in \mathcal{A}}$ (i.e. an **atlas** and the pair (ϕ_α, U_α) is a **chart**) such that $\mathcal{M} \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ for open set U_α, and $\phi_\alpha \to \tilde{U}_\alpha \subset \mathbb{R}^n$ is a homeomorphism.

Additionally, $\mathcal{M}$ is a **smooth manifold** if for all α,

$$\phi_\beta \circ \phi_\alpha :^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{53}$$

is a diffeomorphism.

In other words, we defined smooth manifolds to mean topological manifolds for which change of coordinates are diffeomorphism of the space of coordinates. With the previous proposition, we see that this condition is yet another characterization of smooth manifolds.

We now would like to ask the following question: is the class of "abstract manifolds" strictly larger[9] than the class of submanifolds of <u>some</u> Eulcidean space (i.e. the "smooth submanifolds" we have been dealing

[8] Technically, the projection onto the first n-coordinates of this composition is a diffeomorphism, i.e. $\pi_{1,\dots,n} \circ \Phi \circ f, \pi_{1,\dots,n} \circ \Phi \circ g$, but since the remaining $N - n$ coordinates are filled by 0, we will leave this part alone.

[9] We know from the start that any smooth submanifold of $\mathbb{R}^N$ is second countable Hausdorff, and the existence of the straightening diffeomorphism gives charts, and so, it is an abstract manifold. Additionally, its coordinate transformations must be smooth by the theorem, and so, it is a smooth manifold. This establishes that the class of abstract manifolds are larger.

with thus far.)

The Whitney embedding theorem addresses this question. There are many versions to the theorem, and we only prove a weak version, i.e. compact abstract manifold embedding into arbitrarily large dimensions. [10]

Proposition 44 (Whitney Embedding Theorem). Let $\mathcal{M}$ be a compact "abstract manifold," then there exists $N \in \mathbb{N}$ and an embedding

$$i : \mathcal{M} \to \mathbb{R}^N \tag{54}$$

We actually don't know what this statement means since we've never defined what an embedding of an *abstract* manifold is. (We only had a notion for a submanifold.) We shall define this here:

Definition 45. Let $\mathcal{M}, \mathcal{N}$ be smooth abstract manifolds.[11] Let $f : \mathcal{M} \to \mathcal{N}$ be a continuous map. We say f is **smooth** if f is smooth in local coordinates, i.e.

$$f(U) = V,\ \psi \circ f \circ \phi^{-1} \in C^\infty(\phi(U)) \tag{55}$$

where $(\phi, U), (\psi, V)$ are charts.[12]

The map $i : \mathcal{M} \to i(\mathcal{M}) \subset \mathbb{R}^N$ is an **embedding** if it is a homeomorphism, smooth, and has full rank.

The map $f : \mathcal{M}^m \to \mathcal{N}^n,\ n \geq m$ is an **immersion** if it has full rank, i.e.

$$\text{rank} D(\psi \circ f \circ \phi^{-1}) = m \tag{56}$$

for everywhere it is well-defined.

Note that in the last definition, if we composed the map with a change of coordinates map, the rank will remain the same since the derivative of a change of coordinates matrix is invertible, and hence automatically full rank.

Also note that in the above definition, an injective immersion $i : \mathcal{M} \to \mathbb{R}^n$ is a homeomorphism onto the image, hence an embedding.

Whitney Embedding Theorem and Parititons of Unity. -Thursday, 4.5.2018

Our goal of this lecture is to prove a (version of) the Whitney embedding theorem.

Proposition 46. Let $\mathcal{M}$ be a compact[13] n-dimensional manifold. Then there exists an embedding

$$i : \mathcal{M}^n \to \mathbb{R}^N \tag{57}$$

for sufficiently large N.

Remark 47. The strong Whitney embedding theorem states that a k-dimensional manifold can be embedded into $N = 2k$.

[10] The strongest version of the theorem embeds an n dimensional manifold into $\mathbb{R}^{2n}$. We notice that this bound is sharp since we cannot embed $\mathbb{S}^1$ into $\mathbb{R}$, b ut we can embed it into $\mathbb{R}^2$.

[11] Note that these are abstract manifolds, which means we do not have an ambient space associated to them. Note that the definition of embedding includes the ambient space.

[12] Note that the smoothness of the map f do not depend on the choice of the charts since (by definition), they are diffeomorphisms.

[13] We need compactness for two reasons. One, we want to have a finite cover (hence a finite atlas). Two, it would be sufficient to show that the map is an injective immersion since an injective map from a compact space to a Hausdorff space is a homeomorphism, which is what makes an immersion an embedding.

The necessary machinery in proving this theorem is the partition of unity.

Let's first consider a very elementary, yet concrete example: the partition of unity on $\mathbb{R}$.

Example 48 (Partition of Unity on $\mathbb{R}$.). Step 0. Recall that

$$h(x) := \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{58}$$

is smooth.

Step 1. Define a hill

$$\phi_\delta(x) := h_0(x) h_0(\delta - x), \quad \delta > 0 \tag{59}$$

Note that $\phi_\delta(x) > 0$ if $0 < x < \delta$.

Step 2. We can normalize this to get the function

$$\psi_\delta(x) := \frac{\int_{-\infty}^{x} \phi_\delta(t) dt}{\int_{-\infty}^{\infty} \phi_\delta(t) dt} \tag{60}$$

Step 3. We can turn this around to get the partition of unity:

$$m_\delta(x) := \psi_\delta(1 + \delta - x)\psi_\delta(x - 1 - \delta) \tag{61}$$

Now the general case, which extends the above idea.

Lemma 49. There exists a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\psi(x) = \begin{cases} 1 & \|x\| \leq 1 \\ 0 & \|x\| \geq 2 \end{cases} \tag{62}$$

and $0 \leq \psi \leq 1$ for $1 \leq \|x\| \leq 2$.

In other words, this smooth function approximates the identity on the open ball, and outside of a larger ball, vanishes completely. We know one such example from calculus which we will use in the (constructive) proof.

PROOF 50. Define h as above. By computing the right hand derivatives and using induction, we see that $h \in C^\infty(\mathbb{R})$. Now, consider the function

$$\psi(x) = \frac{h(4 - \|x\|^2)}{h(4 - \|x\|^2) + h(\|x\|^2 - 1)} \tag{63}$$

Note that this must be smooth since the denominator never vanishes[14]. Cooking up the above function is rather straightforward once we look at the statement of the theorem.

□

Definition / Proposition 51. A **smooth partition of unity subordinate to an open cover** $\{U_\alpha\}_{\alpha \in A}$ **of the manifold** $\mathcal{M}$ is a collection of smooth non-negative functions $\{f_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha \in A}$ such that

- The support is locally finite everywhere (i.e., only finitely many functions are nonzero on a neighborhood).[15]

[14] The first term vanishes for $\|x\| \geq 2$ and second term vanishes for $\|x\| \leq 1$.

[15] We often must sum partitions of unity, and so, this allows us to consider finite sums rather than infinite sums, circumventing the issue of convergence.

- $\text{supp} f_\alpha \subseteq U_\alpha$
- $\sum_{\alpha \in A} f_\alpha(x) = 1$ for all $x \in \mathcal{M}$

In the case in which $\mathcal{M}$ is compact, the atlas is finite. We will prove that in this case, there is a partition of unity.

Most of the proof is just applying what we've shown for Euclidean space by using the chart. However, adjusting for the open cover for which the partition of unity is subordinate to requires a topological argument.

Lemma 52 (Scaling lemma.). For an abstract manifold $\mathcal{M}$ and a finite open cover $\{U_i\}$, there exists a covering $\{V_i\}$ such that $\bar{V}_i \subset U_i$.

PROOF 53. Induction on $1 \leq i \leq n$. Let $A_1 := \mathcal{M} \setminus \{U_2, ..., U_n\}$. Then take the open set V_1 such that $A_1 \subset V_1 \subset \bar{V}_1 \subset U_1$ (which must exist since A_1 is closed, and U_1 is open). Do the same with the other values of i.

□

Now for the proof of the theorem.

PROOF 54 (Existence of Partitions of Unity.). We can take wlog an atlas $\{(\phi_j, U_j)\}$ such that $\phi_j(U_j) \supseteq B(0,3)$ and $\{\phi_j^{-1}(B(0,1))\}$ is a cover of $\mathcal{M}$.[16]

Now, let ψ be the bump function on the Euclidean space $\mathcal{M}$ is locally homeomorphic to, i.e. it satisfies

$$\psi(x) = \begin{cases} 1 & x \in \overline{B(0,1)} \\ 0 & x \in \overline{B(0,2)^c} \end{cases} \tag{64}$$

and otherwise (i.e., $x \in B(0,2) \cap B(0,1)^c$), $0 \leq \psi \leq 1$. Take $V_1 := \phi_1^{-1}$. We can now define the bump function on the manifold $g_j : \mathcal{M} \to \mathbb{R}$ by

$$g_j(x) = \begin{cases} \psi(\phi(x)) & x \in U_j \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

In particuar, $g_j = 1$ on V_j and the support of g_j is contained in U_j. Now, we just normalize:

$$f_j(x) := \frac{g_j(x)}{\sum_j g_j(x)} \tag{66}$$

Note that the denominator is nonzero since V_j is a covering if the domain, and so, there is always some j for which g_j is positive.

□

Remark 55. We do not prove this here, but in general, there is a partition of unity subordinate to any open cover on a smooth manifold.

PROOF 56 (Whitney Embedding Theorem.). It suffices to show that there is an injective immersion $i : \mathcal{M} \to i(\mathcal{M}) \subset \mathbb{R}^N$ for some N. Since the map is bijective onto the image, in order to show that it is a homeomorphism, we can just show that it is a closed map.

[16] This is where we use the shrinking lemma. If we start with an arbitrary atlas $\{(\phi_j, U_j)\}$, then by the shrinking lemma, we can take a smaller covering $\{V_i\}$ such that $\bar{V}_i \subset U_i$. If $\phi^{-1}(B(0,1)) \supseteq V_i$, then the second condition we want is satisfied. For this, we just need $B(0,1) \supseteq \phi_j(V_i)$. On the other hand, we also need $\phi_j(U_j) \supseteq B(0,3)$. We can achieve this by dilating ϕ_j.

Now for a given finite[17] atlas $\{(\phi_i, U_i)\}_{i=1}^J$, let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$. We can then define the new function $i : \mathcal{M} \to \mathbb{R}^N$ by[18]

$$i(x) := (f_1(x)\phi_1(x), ..., f_J(x)\phi_J(x), f_1(x), ..., f_J(x)) \tag{67}$$

Here, $\phi_i(x)f_i(x) \in \mathbb{R}^n$, so $N = Jn + J = J(n+1)$. Observe that since ϕ_i, f_i are respectively smooth, i is also smooth.[19]

Now let's show that i is injective. Let $i(x) = i(\tilde{x})$ for $x, \tilde{x} \in \mathcal{M}$. Then $f_j(x) = f_j(\tilde{x})$ for each j. But since $\sum_{j=1}^N f_j \neq 0$ by definition of the partition of unity, there is some j such that $f_j(x) = f_j(\tilde{x}) \neq 0$. For this j, we also have $f_j(x)\phi_j(x) = f_j(\tilde{x})\phi_j(\tilde{x})$ just by comparing coordinates. Dividing through by $f_j(x), f_j(\tilde{x})$ gives $\phi_j(x) = \phi_j(\tilde{x})$, and since charts are bijections, we have $x = \tilde{x}$.[20]

Finally, to show that i has full rank, observe that $i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^N$ is a map given by

$$i \circ \phi_j^{-1} : x \mapsto (x, 0, ..., 0) \tag{68}$$

where there are $N - n$ zeros. Therefore,

$$D(i \circ \phi_j^{-1}) = [\mathbf{Id}_{n\times n}|0] \tag{69}$$

so, the map has full rank.

Therefore, i is a injective immersion. Since injective immersions from compact sets are always embeddings, i is an embedding, and so, we are done.

(Alternatively,) any closed set in $\mathcal{M}$ is compact since $\mathcal{M}$ is compact. The image of this set is then closed since continuous image of closed set is closed. Therefore, i is a closed map, hence a homeomorphism. Thus, i is an embedding which concludes the proof.

□

We now want to switch to exhibiting some examples of manifolds.

Example 57. $\mathbb{S}^2$ is a 2-dimensional submanifold in $\mathbb{R}^3$. The straightening diffeomorphisms is the **stereographic projections**, or simply "Santa shooting lasers from north and south poles." In the neighborhood of the north pole $N := (0, 0, 1)$, we have the chart (ϕ_N, U_N) for $U_N := \mathbb{S}^2 \setminus \{N\}$ and

$$\phi_N(x, y, z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \tag{70}$$

and likewise, for $S := (0, 0, -1)$, we have the chart (ϕ_S, S_N) with $U_S := \mathbb{S}^2 \setminus \{S\}$ and

$$\phi_S(x, y, z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right) \tag{71}$$

The atlas is smooth since

$$\phi_S \circ \phi_N^{-1} = \phi_N \circ \phi_S^{-1} = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) \qquad u, v \neq 0 \tag{72}$$

[17] Note that we are using the compactness of $\mathcal{M}$ here.

[18] Note that we are abusing notation here. Strictly speaking, ϕ_i is a function defined only on the coordinate neighborhood U_i. However, since supp$f_i \subset U_i$, we can extend $\phi_i f_i$ to a function on $\mathcal{M}$ by letting it be ϕ_i inside the support of f_i and 0 elsewhere. This is what is meant by "patching together the atlas to make a global function."

[19] We've never proved the details for this, so let's verify this here. If ϕ_i, f_i are smooth, then by definition, for another chart ψ_i, $\psi_i\phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $f_i\psi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ are smooth maps in the sense of calculus on Euclidean space. But from calculus, the product of these two functions, $f_i\phi_i = (f_i\psi_i^{-1})(\psi_i\phi_i^{-1}) : \mathbb{R}^n \to \mathbb{R}$ is a smooth map. Note that this is one of the reasons why we need the partitions of unity to be a smooth map.

[20] Note here that we did indeed need the $f_1, ..., f_J$ in the original construction of i here. We needed to be able to compare coordinatewise to deduce that the $f_i(x) =_i (\tilde{x})$.

[21]

Example 58 (Real projective space.). The **real projective space** denoted $\mathbb{R}P^n$ (or $P(\mathbb{R}^{n+1})$) are simply lines through the origin in $\mathbb{R}^n$. More formally,

$$\mathbb{R}P^n := \mathbb{S}^n/\{\pm 1\} \tag{75}$$

We take a quotient since we want two lines spanned by vectors pointing in the exact opposite directions to be the same lines.

Let's look at the simplest example $\mathbb{R}P^1$. A natural coordinate system for this space is the **homogeneous coordinates**, i.e. the slopes of the lines. Formally, a neighborhood of this coordinate system takes the form

$$[x : y] := \{(cx, cy)c \in \mathbb{R}^*\} \tag{76}$$

where $x, y \in \mathbb{R}P^n$. We can cover $\mathbb{R}P^1$ with two coordinate neighborhoods:

$$U_0 := \{[x : y] : x \neq 0\}\,,\ U_1 := \{[x : y] : y \neq 0\} \tag{77}$$

or geometrically, the set of lines with $\frac{y}{x} \in (-\infty, \infty)$ and $\frac{x}{y} \in (-\infty, \infty)$. Indeed, the maps cooresponding to these neighborhoods is $\phi_0 : U_0 \to \mathbb{R}$ given by

$$\phi_0 : [x : y] \mapsto \frac{y}{x} \tag{78}$$

and $\phi_1 : U_1 \to \mathbb{R}$ given by

$$\phi_0 : [x : y] \mapsto \frac{x}{y} \tag{79}$$

This is a smooth atlas since for $x, y \neq 0$

$$\theta_{0,1} : \frac{x}{y} \mapsto \frac{y}{x} \tag{80}$$

is smooth.

The cooresponding map for $\mathbb{R}P^2$ is the stereographic projection we saw above. In the general case for arbitrary n, the coordinate neighborhoods are given by

$$U_i := \{[x_1, ..., x_{n+1}] : x_i \neq 0\} \tag{81}$$

and $\phi_i : U_i \to \mathbb{R}$ is given by

$$[x_1, ..., x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i}\right) \tag{82}$$

---

[21] This is fairly easy to verify. Let's do the first one, and the second will follow from the same computation. Let $\phi_N(x, y, z) =: (u, v)$. Then

$$x = (1 - z)u,\ y = (1 - z)v \tag{73}$$

So,

$$\phi_S \circ \phi_N^{-1} = \frac{1 - z}{1 + z}(u, v) \tag{74}$$

Now, $u^2 + v^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1+z}{1-z}$ since $x^2 + y^2 + z^2 = 1$. This gives the result.

Week 3.

Differential Forms in the Plane -Tuesday, 4.10.2018

We will review all of the basic notions for differential forms. We will specialize to open sets in Euclidean space, and later, to open sets in $\mathbb{R}^2$.

We will start with 1-forms. The motivation for this is derivatives as linear maps. Take $U \subset \mathbb{R}^n$ and $f \in C^\infty(U)$. The **directional derivative of a function f at the point p in the direction v**, as we may recall is simply defined by

$$df(p)(v) := \frac{d}{dt} f(p + tv) \tag{83}$$

$$= \nabla f(p) \cdot v \tag{84}$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) dx_j(v) \tag{85}$$

Now, $\{dx_j\}_{j=1}^n$ are not just some mystery objects from when one integrates. They are the **dual of the standard basis**, i.e.

$$dx_j(e_k) = \delta_{j,k} \tag{86}$$

and so, in particular, dx_j extracts the jth coordinate of v:

$$dx_j(v) = dx_j\left(\sum_{k=1}^{n} v_k e_k\right) = \sum_{j=1}^{n} v_k \delta j, k = v_j \tag{87}$$

But there is no reason to stop here; instead of partials, we can take any smooth function to be the coefficient of dx_j. This defines a 1-form, which is simply a *linear function that smoothly varies with p.*

Definition 59. A **1-form on** $U \subset \mathbb{R}^n$ is an expression of the form

$$\omega = \sum_{j=1}^{n} \omega_j dx_j \qquad \omega_j \in C^\infty(U) \tag{88}$$

We already know how this things behaves; we know from the discussion above that

$$\omega(p)(v) = \sum_{j=1}^{n} \omega_j(p) v_j \tag{89}$$

There is clearly something special about the 1-forms we saw at the beginning. In fact, 1-forms that can be written that way has a special name.

Definition 60. A 1-form ω is said to be **exact** if there exists a function $f \in C^\infty(U)$ such that $\omega = df$.[22]

Are there other ways of thinking what a 1-form is? In general, a linear functional can be thought of simply as a family of parallel lines. i.e., the line $\{x : \phi(x) = 1\}$ full determines the linear functional ϕ.[23] This gives us another way of interpreting 1-forms: *1-forms are just a field of lines that vary smoothly in space, i.e. a smooth*

[22] Note here that we have not defined the exterior derivative yet; we are merely denoting df to be the linear functional we saw at the very beginning. Of course, the formalism of exterior derivatives will indeed agree with the notion of derivatives we started with in the first place.

[23] Note that $\{x : \phi(x) = 0\}$ does not have as much information as $\{x : \phi(x) = 1\}$ since the former is invariant under scaling ϕ whereas the latter will move in the plane.

vector field.

When we have a differential form, we want to integrate it. In particular, this generalizes the notion of *work* from physics.

Definition 61. Given the 1-form ω in U and smooth[24], oriented curve $c : [a, b] \to U$, then

$$\int_c \omega = \int_a^b \omega(c(t))\dot{c}(t)dt \tag{90}$$

One must ask if this new notion is well-defined, i.e. do different parametrization of c give the same value for the integral? (Or, if you are a physicist, is the work the same if we used a different clock?)

This is indeed well defined. The idea is that the change of variables from the parametrization and the differential cancels each other out. Suppose that we have a different parametrization s so that $t(s)$ is a smooth and monotone ($t'(s) > 0$). Then

$$\int_a^b \omega(c(t)) \cdot \dot{c}(t)dt = \int_c^d \omega(\tilde{c}(s)) \cdot \left(\dot{\tilde{c}}(s) \cdot \frac{dt}{ds}\right)\left(\frac{dt}{ds}ds\right) = \int_c^d \omega(\tilde{c}(s))\dot{\tilde{c}}(s)ds \tag{91}$$

We would now like to motivate the exterior derivative. We define the exterior derivative as something that makes the Stokes theorem look like the fundamental theorem of calculus which states that

$$\int_I D_x f(x)dx = \int_I df = \int_{\partial I} f = f(b) - f(a) \tag{92}$$

where $\partial I = \{a, b\}$. Here, we are peeling off the "D_x" from the integrand and sticking it onto the region on which we are integrating over. The Stokes theorem generalizes this to differential forms. Since we have not defined the exterior derivative, we will stick to classical notation.[25]

Proposition 62 (Stokes Theorem (Classical Version).). For a triangular region $\Delta \subset U$,

$$\int_{\partial\Delta} Pdx + Qdy = \int\int_\Delta \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{93}$$

where P, Q are C^1.

PROOF 63. The proof is just the fundamental theorem of calculus. If y_0, y_1, y_2 are respectively the y-coordinates of the vertices of the triangle, then

$$\int_\Delta \frac{\partial Q}{\partial x}dxdx = \int_{y_0}^{y_1} Q(x_1(y), y) - Q(x_0(y), y)dy + \int_{y_1}^{y_2} Q(x_2(y), y)dy - Q(x_0(y), y)dy \tag{94}$$

where x_0, x_1, x_2 are respectively the x coordinates parametrized by y coordinates. Doing the same for $\frac{\partial P}{\partial y}$ and adding all the terms up gives the line integral.

□

Now let's do the same for 1-forms. (We will not prove this since this is just the same as above but in different notations.)

Proposition 64 (Stokes Theorem (for 1-forms).).

$$\int_{\partial\Delta} \omega = \int_\Delta d\omega \tag{95}$$

[24] In general, all we need is piecewise C^1

[25] Shifrin, *Differential Geometry* provides this version of the proof (under "Greens' theorem."

We can now define the exterior derivative to match this statement. Given the 1-forms

$$\omega = Pdx + Qdy \tag{96}$$

we want to have

$$d\omega = \frac{\partial P}{\partial y} dy \wedge dx = \frac{\partial Q}{\partial x} dx \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \tag{97}$$

where $\wedge$ is the **wedge product**. What exactly is this? The formalism is an algebraic idea, but geometrically, it is just a multilinear functional that gives the *signed volume spanned by the vectors it eats*. In terms of determinants,

$$(dx \wedge dy)(v, w) = \det[v, w] \tag{98}$$

Let's formalise this:

Definition 65. If ω, η are 1-forms,

$$\omega \wedge \eta := (\omega_1 \eta_2 - \omega_2 \eta_1) dx \wedge dy \tag{99}$$

We are now ready to talk about 2-forms. In particular, we will discuss the integration of two forms, and how it behaves under changes of variables. To a physicist, this is imperative since one should be allowed to change coordinate systems freely without worrying about it affecting the result of the computation. The proper formalism for this is the pullback of a differential form.

Definition 66. A **2-form in an open set** $U \subset \mathbb{R}^2$ is a differential form of the form

$$\omega = f dx \wedge dy \tag{100}$$

for $f \in C^\infty(U)$. In this case, the **integral of a 2-form** is

$$\int_\Omega \alpha = \int_\Omega f(x, y) dx dy \tag{101}$$

Note the distinction between the 2-form $dx \wedge dy$ and the differential $dxdy$ in the integrand. The former is a *functional* whereas the latter is a Lebesgue measure (or part of a Riemann integral). They are different objects.

Example 67 (Angle form.). Here is the favorite example from physics. Consider the 1-form

$$\omega_0 := \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \tag{102}$$

on $U := \mathbb{R}^2 \setminus \{0\}$ and the orientation preserving diffeomorphism $\Phi : \mathbb{R}_{>0} \times (-\pi, \pi) \to \mathbb{R}^2 \setminus \mathbb{R}_{<0}$ by

$$\Phi(r, \theta) := (r \cos\theta, r \sin\theta) \tag{103}$$

Note that this is simply the change of coordinates from polar to Cartesian coordinates. Then, we can compute the pullback of ω_0 (or more colloquially, convert ω_0 to polar coordinates)

$$\Phi^* \omega_0 = -\frac{r \sin\theta}{r^2} d(r \cos\theta) + \frac{r \cos\theta}{r^2} d(r \sin\theta) = d\theta \tag{104}$$

In physics, we often write this as $\omega_0 = d\theta$, but strictly speaking, this is an abuse of notations. Thus, the pullback of the angle form in Cartesian coordinates via the polar coordinate map is just the differential of the angle.

We played fairly loosely with the notion of pullback because we thought of it as simply a change of variables. Here is the formal definition.

Definition 68. Let ω be a 1-form on $U \subset \mathbb{R}^2$ and let $\Phi : U \to V$ be a smooth diffeomorphism of $\mathbb{R}^2$. For $p \in U$, the **pullback of** ω **under** Φ is the 1-form in V given by

$$(\Phi^*\omega)(p)(v) := \omega(\Phi(p))(D\Phi(p)v) \tag{105}$$

Likewise, the **pullback of a 2-form is** the 2-form in V given by

$$(\Phi^*\alpha)(p)(v, w) := \alpha(\Phi(p))(D\Phi(p)v, D\Phi(p)w) \tag{106}$$

We can now properly state the lemma.

Lemma 69. Let $\Phi : U \to V$ be an orientation preserving[26] diffeomorphism of $\mathbb{R}^2$ and $\Omega \subset U$. Then for a 2-form $\alpha = f dy_1 \wedge dy_2$

$$\int_{\Phi(\Omega)} \alpha = \int_\Omega \Phi^*\alpha \tag{107}$$

More loosely speaking, change of variables does not change the result of the integration.

PROOF 70. We just unravel definitions. If $x_1, x_2 \in U, (y_1, y_2) := \Phi(x_1, x_2), v, w \in U$ and denote $A := D\Phi(x_1, x_2)$, then

$$\begin{aligned}
\Phi^*\alpha(x_1, x_2)(v, w) &= \alpha(y_1, y_2)(Av, Aw) \qquad \text{(definition of pullback)} && (108)\\
&= f(y_1, y_2) dx_1 \wedge dx_2(Av, Aw) \qquad \text{(definition of } \alpha\text{)} && (109)\\
&= f(y_1, y_2) \det[Av, Aw] \qquad \text{(definition of } dx_1 \wedge dx_2\text{)} && (110)\\
&= f(y_1, y_2) \det A dx_1 \wedge dx_2(v, w) \qquad \text{(multiplicability of determinant)} && (111)\\
& && (112)
\end{aligned}$$

Thus,

$$\Phi^*\alpha = (f \circ \Phi) \det(D\Phi) dx_1 \wedge dx_2 \tag{113}$$

But now,

$$\begin{aligned}
\int_\Omega \Phi^*\alpha &= \int_\Omega (f \circ \Phi) \det(D\Phi) dx_1 \wedge dx_2 && (114)\\
&= \int_\Omega f dy_1 \wedge dy_2 && (115)\\
&= \int_\Omega \alpha && (116)\\
& && (117)
\end{aligned}$$

is just the change of variables formula for integrals (where $dy_1 \wedge dy_2 = \det(D\Phi) dx_1 \wedge dx_2$). Note that the the fact that Φ preserves orientation is crucial since we need $\det(D\Phi) = |\det(D\Phi)|$ in the above for the change of variables to work.

□

We might as well prove the same result for 1-forms.

[26] i.e. $\det D\Phi(p) > 0$ for all $p \in U$

Lemma 71. Line integrals are invariant under pullbacks. i.e., for a smooth diffeomorphism $\Phi : U \to V$ of $\mathbb{R}^2$ and smooth curve $c : [a, b] \to U$, and 1-form ω

$$\int_c \Phi^*\omega = \int_{\Phi\circ c} \omega \tag{118}$$

PROOF 72. Again, the proof is straightforward from how we set up the machinery.

$$\int_{\Phi\circ c} \omega = \int_a^b \omega(\Phi\circ c(t))(\Phi\circ c)'(t)dt \tag{119}$$

$$= \int_a^b \omega(\Phi\circ c(t))(D\Phi\circ c)c'(t)dt \tag{120}$$

$$= \int_c \Phi^*\omega \tag{121}$$

$$\tag{122}$$

□

One more important result about the invariance of integrals under change of variables. We've only proved the Stokes theorem for a very small class of regions in the plane. We would now expand this class to a much broader category.

Lemma 73. For a 1-form ω on the open set $U \subset \mathbb{R}^2$ and if $\Omega \subset U$ is some region such that for a smooth orientation preserving diffeomorphism Φ, $\Phi(\Omega)$ is a triangle[27] on $\mathbb{R}^2$,

$$\int_{\partial(\Phi(\Omega))} \omega = \int_{\Phi(\Omega)} d\omega \tag{123}$$

i.e., the Stokes theorem is invariant under smooth orientation preserving diffeomorphism.

Now, we can use Stokes theorem not only to triangles, but figures which are smoothly diffeomorphic to triangles (with same orientations)!

PROOF 74. The proof is just straighten out the region Ω using the diffeomorphism and then apply the Stokes theorem downstairs, and then go back upstairs. Denote $\tilde{\Omega} := \Phi(\Omega)$. Then

$$\int_{\partial\Omega} \Phi^*\omega = \int_{\partial\tilde{\Omega}} \omega \qquad \text{(definition)} \tag{124}$$

$$= \int_{\tilde{\Omega}} d\omega \qquad \text{(Stokes on a plane)} \tag{125}$$

$$= \int_{\Omega} \Phi^* d\omega \qquad \text{(definition)} \tag{126}$$

$$= \int_{\Omega} d(\Phi^*\omega) \qquad \text{(pullback commutes with } d) \tag{127}$$

$$\tag{128}$$

which concludes the proof. Note that we still owe explanation since we never proved that pullbacks commute with exterior derivatives. We will proves this in pset 3.

□

Finally, there is a notion of *closed* forms.

[27] By this, we just mean a region on which Stokes theorem holds.

Definition 75. A **closed 1-form** on $U \subset \mathbb{R}^2$ satisfies $d\omega = 0$.

Lemma 76. An exact form is closed.

PROOF 77. We can just compute this. This is a direct consequence of Schwarz lemma, i.e. mixed partials are equal for C^2 functions. Thus, if $\omega = df$ for a smooth function f, then

$$d\omega = d\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\right) \tag{129}$$

$$= \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial x_j x_k} dx_j \wedge dx_k \tag{130}$$

$$= 0 \tag{131}$$

since $\frac{\partial^2 f}{\partial x_j x_k} dx_j \wedge dx_k = -\frac{\partial^2 f}{\partial x_k x_j} dx_k \wedge dx_j$.

□

A natural question is: does the converse hold? The answer in general is no. In fact, the whole point of DeRham cohomology is that closed forms are not exact; how they fail to be exact captures the topology of the space in which one works in.

Example 78. Once again, our favorite example, the angle form shows us why closed forms are not exact. It is exact since

$$d\omega_0 = d(\Psi^* d\theta) = \Psi^* d^2\theta = 0 \tag{132}$$

where Ψ is the coordinate transform from Cartesian to polar.

On the other hand, if there is indeed a smooth function f on the punctured plane such that $\omega_0 = df$, then

$$\int_\gamma \omega_0 = \int_{\Psi \circ \gamma} d\theta = 2\pi \tag{133}$$

where γ is a circle. As physicists would say, this shows that ω_0 **does not have a potential** or **is not conservative** (for instance, an electric field cannot look like the vector field given by an angle form).

However, a converse does hold *locally*, and that is the statement of the following.

Proposition 79 (Póincare Lemma.). Every closed form is locally exact. i.e., for a closed form ω, there is some neighborhood $U \subset \mathbb{R}^n$ and a smooth function $f : U \to \mathbb{R}$ such that $\omega = df$ on U.

Note that the above theorem (and the following proof) is for *1-forms*. This is the most intuitive case (not to mention, we can describe it using words from physics). We defer the more general proof to Spivak, *Calculus on Manifolds* ch.4 (which does things in a similar manner, but the construction of the potential function is more complicated.) We also use Stokes theorem which (strictly speaking), we've proven for triangles in $\mathbb{R}^2$. Since we use triangles in the following proof, we are allowed to do this; however, strictly speaking, we must restrict our attention to $\mathbb{R}^2$. (Alternatively, one could use the general Stokes theorem on manifolds (which is a bit over kill) which does not require the Poincaré lemma, so we do not get a tautology.)

PROOF 80. What we want to show is that locally, there is a potential function cooresponding to ω, in other words, in some neighborhood, $\omega(x)$ looks like a derivative of some function! Our task is to find what this function is. Naturally, our proof is constructive; we will specify what this potential function is. The essential point is that via the Stokes theorem, the vanishing of $d\omega$ is indeed what we need to get (local) path independence of the integral.

Let $B \subset \mathbb{R}^n$ be some ball (and hence simply connected[28]) in some domain of ω. Fix $x_0 \in B$ (we can take this wlog to be the origin, if we like.) Then define $f : B \to \mathbb{R}$ by

$$f(x) := \int_{x_0 \to x} \omega \tag{134}$$

where $x_0 \to x$ is *any* path from x_0 to x. This function is well defined because of the closedness of ω. We can show this by proving that the integral of ω over any closed loop vanishes. Let $\partial\Omega$ be any closed loop in B, and let $\Omega \subset B$[29] be the region it encloses. Then

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega = 0$$

by Stokes for 1-forms on $\mathbb{R}^n$. Therefore, the integral of ω is path independent, in other words, f is only a function of x and not the path $x_0 \to x$.
Since derivatives are unique, in order to show $\omega = df$, it suffices to show that

$$\lim_{\epsilon \to 0} f(x + \epsilon v) - f(x) = \omega(x)(v) \tag{135}$$

for some $x \in B$ and v such that $x + \epsilon v \in B$ for all $\epsilon \in (0, 1)$. Fix such points in B. Then

$$\begin{aligned} f(x + \epsilon v) - f(x) &= \int_{x \to x + \epsilon v} \omega \\ &= \int_0^1 \omega(x(1-t) + \epsilon t v)(x + \epsilon v) dt \\ &= \omega(x)(v) + o(\|\epsilon v\|) \end{aligned}$$

This last line is simply the mean value theorem for integral. Observe that

$$\lim_{\epsilon \to 0} \omega(x(1-t) + \epsilon t v)(x + \epsilon v) dt = \omega(x)(v) \tag{136}$$

The "remainder term" must vanish with $\|\epsilon v\|$, and so, by definition, we have $o(\|\epsilon v\|)$. Thus, $\omega = df$ in the neighborhood.

□

Consequences of Stokes Theorem: Homotopy Invariance and Exterior Derivative. -**Thursday, 4.12.2018**

We will provide two very important consequences of Stokes theorem, namely the homotopy invariance of the line integral and the motivation of exterior derivatives. In particular, we will not take the common approach of simply asserting the definition of the exterior derivative and deriving its properties but rather define it in such a way that Stokes theorem holds.

But first, we must define what a homotopy is.

[28] This is crucial, and this is where one needs the *localness* of the theorem. We need to kill the line integral of ω by killing the the integral of $d\omega$ over a disk, but of coure, *we need this disk to lie inside* B.
[29] Recall that we chose B *so that* any Ω enclosed in a closed loop is contained in B.

Definition 81. Let $U \subset \mathbb{R}^n$ be an open set, and let $c_0, c_1 : [0,1] \to U$ be (piecewise) C^1 curves[30] Additionally, let $p := c_0(0) = c_1(1), q := c_0(1) = c_1(1)$. Then the C^1 function $H : [0,1]^2 \to U$ is called a **C^1- homotopy** if it satisfies $H(0,s) = p, H(1,s) = q$ for all $0 \le s \le 1$ and $H(t,0) = c_0(t), H(t,1) = c_1(t)$. We say c_0, c_1 are **C^1-homotopic** if there exists a C^1-homotopy between them.

If instead, the curves are C_0 and H is C_0, then we call H simply a C^0-homotopy.

If c_0, c_1 are closed curves, and $H(t,s)$ is a closed curve for all s, then H is a (C^1 or C^0) **homotopy of closed curves**.

Proposition 82. Let $U \subset \mathbb{R}^2$ be open, and let c_0 and c_1 be C^1-homotopic curves in U. If ω is a closed 1-form on U, then

$$\int_{c_0} \omega = \int_{c_1} \omega \tag{137}$$

PROOF 83. This is immediate, once we pull back ω via H and then apply Stokes. If $\alpha := H^*\omega$, then

$$d\alpha = d(H^*\omega) = H^*(d\omega) = 0 \tag{138}$$

so, $d\alpha$ is a closed 1-form in $[0,1]^2$. Then

$$\begin{aligned}\int_{c_1} \omega - \int_{c_0} \omega &= \int_{\partial[0,1]^2} \alpha \qquad \text{(definition of } \alpha) \\ &= \int_{[0,1]^2} d\alpha \qquad \text{(Stokes)} \\ &= 0\end{aligned}$$

□

Before, we remarked that a line integral requires the curve to be C^1. Can we make sense of a line integral over a continuous curve? We cannot define such a thing if ω is not closed[31] In fact, this condition is indeed sufficient to define the integral.[32]

Let's go to the easier case when ω is exact on $U \subset \mathbb{R}^2$. If p, q are endpoints of our curve, then by fundamental theorem of calculus

$$\int_c \omega = f(q) - f(p) \tag{139}$$

We want to do something similar for when ω is *locally* exact. Since $c([0,1])$ is compact, cover it with finitely many open ball $D_j \subset U,\ 1 \le j \le N$ where $\omega = df_j$ on each D_j. We can then define

$$\int_c \omega := \sum_{j=1}^{n} f_j(t_j) - f_j(t_{j-1}) \tag{140}$$

for t_i partitions of the domain of f_j, i.e. $0 = t_0 < ... < t_N = 1$ and $c([t_j, t_{j+1}]) \subset D_i$ for $j \ge i$. More informally, we take the partitions of the unit interval fine enough that each segment of the curve lies in each neighborhood in which ω is exact.

Definitions of this form raises a few questions:

[30] Note that a line integral requires the curves to be C^1 because the derivative of the curve is in the definition.
[31] The key idea in the following discussion is using Poincare's lemma to get (local) primitives of ω and to patch these together in a manner similar to first fundamental theorem of calculus. We obviously cannot do this if we don't have closedness.
[32] The minimum requirement is rectifiability (see Rudin) of the curve.

- Does this agree with our old definition of a line integral, i.e. is $c \in C^1([0,1])$ case a special case of this definition?
- Do we still get the same value of the integral if we used a different partition?

Let's start with the first question. If $c \in C([0,1])$, then

$$\begin{aligned}\int_c \omega &= \int_0^1 \omega(c(t)) \cdot \dot{c}(t)dt \qquad \text{(definition)}\\ &= \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} \omega(c(t)) \cdot \dot{c}(t)dt \qquad \text{(partition the integral)}\\ &= \sum_{j=1}^{n} f_j(t_j) - f_j(t_{j-1}) \qquad \text{(FTC)}\end{aligned}$$

For the second, if we defined the integral in terms of two different partitions, we can simply take their common refinement. The integral defined with respect to a refinement must agree with the original integral by telescoping. Thus, the two integrals with respect to the two partitions agree with the common refinement, and thus, agree with each other.

Let's now prove the homotopy invariance of line integrals for this new definition.

Proposition 84. If c_0, c_1 are C_0 curves which are C_0 homotopic in $U \subset \mathbb{R}^2$ sharing endpoints, then for all closed 1-forms ω in U

$$\int_{c_0} \omega = \int_{c_1} \omega \tag{141}$$

PROOF 85. By our definition of $\int_c \omega$ for continuous curve c, our proof should start by producing an appropriate partition. After that, the primitives all cancel nicely.

Since H is a continuous function of a compact set, it is uniformly continuous, i.e. for all $\epsilon > 0$, there exists $\delta > 0$ such that $\left|(t,s) - (\tilde{t},\tilde{s})\right| < \delta$ implies $\left|H(t,s) - H(\tilde{t},\tilde{s})\right| < \epsilon$. Choose ϵ such that $\text{dist}(H([0,1]^2), \partial U) > \epsilon$. Thus, take a sect of rectangles R_j which partitions $[0,1]^2$ and such that their diameter is less than δ. This ensures that the image of R_j is contained in a neighborhood on whose image ω has a primitive on (as in the definition of $\int_c \omega$) *and is contained in* U. [33]

Now, since the integrals over the sides of R_j cancel for all except $\partial[0,1]^2$,

$$\begin{aligned}\int_{c_1} \omega - \int_{c_0} \omega = \int_{H(\partial[0,1]^2)} \omega \qquad &= \int_{\partial[0,1]^2} H^*\omega\\ = \sum_j \int_{\partial R_j} H^*\omega &\\ = \sum_j \int_{H(\partial R_j} \omega &\end{aligned}$$

[33] We need to do all of this point set topology since we must avoid U^c. Everything in this proof is only defined on U, and so, going outside of this region causes problems.

so, it suffices to show that $\int_{H(\partial R_j} \omega = 0$. However, if A, B, C, D are the images of the vertices of R_j under H, then we have a primitive f_j in a neighborhood of $H(R_j)$, so

$$\int_{H(\partial R_j} \omega = (f_j(A) - f_j(B)) + (f_j(B) - f_j(C)) + (f_j(C) - f_j(D)) + (f_j(D) - f_j(A)) = 0$$

just by the definition of $\int_{H(\partial R_j} \omega$. (Note that $H(\partial R_j$ is just a continuous curve on which a primitive is given everywhere.)

□

As another consequence of Stokes theorem, we will motivate the definition of exterior derivatives. In particular, we define exterior derivatives to be a unique expression that preserves Stokes theorem. The special role Stokes theorem plays in this is analogous to the relevance of the fundamental theorem of calculus.

Let $\omega = a dx_1 \wedge dx_2$ be a 2-form on $U \subset \mathbb{R}^2$. We want to find what $d\omega$ should be. For a cube $Q \subset U$, we want the Stokes theorem to hold, i.e.

$$\int_{\partial Q} \omega = \int_Q d\omega \tag{142}$$

We recall here that the Stokes theorem is a statement about *orientable* manifolds. Therefore, we need to define what we mean by an orientation.

First, define

$$Q := \prod_{j=1}^{3} [a_j, b_j] \subset U \tag{143}$$

to be a cube in the plane (where $b_1 - a_1 = b_2 - a_2 = b_3 - a_3$). We can then partition Q so that

$$\int_{\partial Q} \omega = \sum_j \int_{\partial Q_{j,\epsilon}} \omega \tag{144}$$

where $Q_{j,\epsilon}$ denotes a cube with side length ϵ. Therefore, it is sufficient to consider this small cube, and then pass to the limit $\epsilon \to 0$.

Observe that

$$\int_{\partial[-\epsilon,\epsilon]^3} a dx_1 \wedge dx_2 = \int_{\partial[-\epsilon,\epsilon]^2} a(x_1, x_2, \epsilon) - a(x_1, x_2, -\epsilon) dx_1 dx_2$$

since for faces with either x_1 or x_2 do not contribute to the integral since we are integrating over the 2-form $dx_1 \wedge dx_2$. Now, we can Taylor expand with respect to the third variable to get

$$\int_{\partial[-\epsilon,\epsilon]^2} \frac{\partial a}{\partial x_3}(x_1, x_2, 0) \cdot 2\epsilon + o(\epsilon^2) dx_1 dx_2$$

Now, since 2ϵ is the side length the square we are integrating over, so we can instead integrate over a *cube* with side length 2ϵ[34]

[34] Technically, we are using $\int_{[-\epsilon,\epsilon]} dx_3 = 2\epsilon$, and then using Fubini's theorem to turn the product of integrals into a volume integral.

So,

$$\int_{\partial[-\epsilon,\epsilon]^3} \frac{\partial a}{\partial x_3}(x_1, x_2, 0) + o(\epsilon^4) dx_1 dx_2$$

Now, additionally, by definition of derivatives, we have the equality

$$\frac{\partial a}{\partial x_3}(x_1, x_2, x_3) - \frac{\partial a}{\partial x_3}(x_1, x_2, 0) = \frac{\partial^2 a}{\partial x_3^2}(x_1, x_2, 0) x_3 + o(\epsilon) \tag{145}$$

where $x_3 < \epsilon$ (by the limit we are integrating over), so the integral now becomes

$$\int_{\partial[-\epsilon,\epsilon]^3} \frac{\partial a}{\partial x_3}(x_1, x_2, x_3) + o(\epsilon^4) dx_1 dx_2$$

where $\epsilon^3 \frac{\partial^2 a}{\partial x_3^2}(x_1, x_2, 0) x_3 = o(\epsilon^4)$. Finally, we can pull out $o(\epsilon^4)$ from the integral by observing that the integral over$[-\epsilon, \epsilon^3]$ is $\epsilon^3 = o(\epsilon^{-3})$, so

$$\int_{\partial[-\epsilon,\epsilon]^3} \frac{\partial a}{\partial x_3}(x_1, x_2, x_3) dx_1 dx_2 + o(\epsilon)$$

We thus have

$$\int_{\partial Q} a dx_1 \wedge dx_2 = \sum_j \int_{Q_{j,\epsilon}} \frac{\partial a}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 = o(\epsilon) \to \int_Q \frac{\partial a}{\partial x_j} dx_1 \wedge dx_2 \wedge dx_3 = \int_Q d\omega \tag{146}$$

as desired.

Another cleaner way of doing this is to just use fundamental theorem of calculus:

$$\int_{\partial[-\epsilon,\epsilon]^2} a(x_1, x_2, \epsilon) - a(x_1, x_2, -\epsilon) dx_1 dx_2 = \int_{[-\epsilon,\epsilon]^3} \frac{\partial a}{\partial x_3}(x_1, x_2, x_3) dx_1 dx_2 dx_3 \tag{147}$$

We can do the same calculation to get

$$\int_{\partial Q} b dx_1 \wedge dx_3 = -\int_Q \frac{\partial b}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_3 \tag{148}$$

We pick up a negative sign due to orientation.

All of this is consistent with the formal calculus of exterior derivatives:

$$d(a dx_1 \wedge dx_2) = da \wedge dx_1 \wedge dx_2 = \frac{\partial a}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 \tag{149}$$

and

$$d(b dx_1 \wedge dx_2) = db \wedge dx_1 \wedge dx_2 = -\frac{\partial b}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 \tag{150}$$

Definition 86. Let

$$\omega := \sum_I \omega_I dx_I \tag{151}$$

be a k-form on $\mathbb{R}^n$ (with $I = \{i_1 < ... < i_k\}$). Then we define the **exterior derivative of** ω by

$$d\omega := \sum_I d\omega_I \wedge dx_I \tag{152}$$

Proposition 87.
- $d : \Lambda_k(\mathbb{R}^n) \to \Lambda_{k+1}(\mathbb{R}^n)$ is a linear map.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ where ω, η are respectively k, l-forms.
- $d^2 = 0$

PROOF 88. The first is immediate from the linearity of d acting on smooth functions, which we already know.
The third is an immediate consequence of Schwarz lemma. For a k-form $\omega = \sum_I \omega_I dx_I$,

$$\begin{aligned} d^2\omega &= \sum_I d(d\omega_I \wedge dx_I) \\ &= \sum_I d^2\omega_I \wedge dx_I - d\omega_I \wedge d^2 x_I \end{aligned}$$

where the first term cancels because there are exactly two terms in its expansion containing any given partial, and they have the opposite signs because of the order of the differentials. The second term vanishes becase

$$d^2 x_I = dx_I \wedge dx_I = 0 \tag{153}$$

simply the definition of exterior derivatives.

□

Finally, we define differential forms on manifolds. We use the notion of tangent bundles introduced in pset 2.

Definition 89. Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$. The map $\alpha : \mathcal{M} \to \{\Lambda^k(T_p\mathcal{M})\}_{p\in\mathcal{M}}$ given by

$$\alpha : p \mapsto (\alpha_p : (T_p\mathcal{M})^k \to \mathbb{R}) \tag{154}$$

is a **differential form on the manifold** $\mathcal{M}$.

The space of alternating k-forms $\Lambda^k(T_p\mathcal{M})$ forms a real vector space whose basis elements are the dual basis of Euclidean space. This does not depend on chart or pullback by diffeomorphisms.

Week 4.

Stokes Theorem on a Manifold. - Tuesday, April, 17. 2018.

Our goal today is to state, understand, and prove the following theorem.

Proposition 90 (General Stokes Theorem). Let $\mathcal{M}$ be a smooth, n-dimensional, compact, orientable manifold with boundary. Let ω be an $n-1$ form on $\mathcal{M}$. Then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega \left(= \int_{\partial\mathcal{M}} i^*\omega \right) \tag{155}$$

(where $i : \partial\mathcal{M} \to \mathcal{M}$ is the natural embedding of the boundary into the manifold $\mathcal{M}$.[35])

There are many things in the above that we have not yet defined, namely:

- orientable manifold
- manifold with boundary
- differential form on a manifold
- integrate a differential form on a manifold

Let's define each of these concepts carefully, and once we do this, Stokes theorem will fall right out.

Definition 91. A manifold $\mathcal{M}$ is a **smooth n-dimensional manifold with boundary** if there exists a chart $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$[36] such that

- $\mathcal{M}$ is covered by the coordinate neighborhoods, i.e.

$$\mathcal{M} \subseteq \bigcup_{\alpha\in\mathcal{A}} U_\alpha \tag{156}$$

- Each coordinate space is open relative to the subspace topology, i.e. $\phi_\alpha : U_\alpha \to \tilde{U_\alpha}$ are homeomorphisms such that $\tilde{U_\alpha}$ is open in $\mathcal{H}^n$ in the subspace topology, i.e.

$$\tilde{U_\alpha} = V_\alpha \cap \mathcal{H}^n \tag{157}$$

- The change of coordinate maps are smooth, i.e. for all α, β, the maps

$$f_{\alpha\beta} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta) \tag{158}$$

  defined by $f_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$ and $f_{\beta\alpha}$ (defined analogously) are smooth.[37]

where $\mathcal{H}^n$ is the **half plane in** $\mathbb{R}^n$, i.e.

$$\mathcal{H}^n := \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 \leq 0\} \tag{159}$$

Additionally, we define the **boundary of the manifold** $\mathcal{M}$ to be

$$\partial\mathcal{M} := \left\{p \in \mathcal{M} : p \in U_\alpha, \phi_\alpha(p) \in \{0\} \times \mathbb{R}^{n-1}\right\} \tag{160}$$

[35] Inside the parenthesis is Do Carmo's notation which is strictly speaking the correct notation since ω lives on $\partial\mathcal{M}$. However, the abuse of notation has the advantage of being very symmetric, and indeed, this is how most people write the the theorem.

[36] The coordinate neighborhoods U_α are of course open in $\mathcal{M}$.

[37] Note that we do not ask that these change of coordinate maps are *diffeomorphisms*. Otherwise, the whole argument in the proof that $\partial\mathcal{M}$ is well-defined would not be necessary.

In other words, a manifold with boundary is one in which we can straighten out the boundary into the simplest possible boundary possible, i.e. the half plane $\mathcal{H}^n$.

The last part of the above definition leads to the following question: is the definition of boundary independent of the chart?

Proposition 92. The boundary $\partial\mathcal{M}$ is well-defined, i.e. the notion of boundary point does not depend on the choice of chart.

PROOF 93. The idea is to argue by contradiction, and produce a diffeomorphism out of the change of coordinate maps. This will guarantee the consistency of the coordinate spaces under different choices of charts.

Suppose the assertion is false. Then there is α, β such that for some $p \in \partial\mathcal{M}$, $\phi_\alpha(p) = (0, x')$ and $\phi_\beta(p) = (y_1, y')$ for $y_1 < 0$. Now, by definition of smooth manifold, $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ is a smooth map with the inverse $f_{\beta\alpha}$. But by chain rule, this gives

$$\begin{aligned} f_{\alpha\beta} \circ f_{\beta\alpha} &= \mathbf{Id} \\ (Df_{\alpha\beta}(y_1, y')) \circ Df_{\beta\alpha}(p) &= \mathbf{Id} \end{aligned}$$

Therefore, the map $Df_{\alpha\beta}(y_1, y')$ has an inverse, hence it is an invertible map. In particular, $\det Df_{\alpha\beta}(y_1, y') \neq 0$, and so, by inverse function theorem, there is some neighborhood W containing (y_1, y') such that $f_{\alpha\beta}$ is a C^1-diffeomorphism on $\mathbb{R}^n$. In particular $f_{\alpha\beta}$ is then an open map, and so, $f_{\alpha\beta}(W)$ is open. However, any neighborhood of $f_{\alpha\beta} = (0, x')$ contains a point with first coordinate positive. Therefore, we have a contradiction, and so, $\phi_\beta(p)$ cannot be an interior point of $\mathcal{H}^n$.

□

Since we are integrating over $\partial\mathcal{M}$ in Stokes theorem, we must verify that it has some nice properties.

Proposition 94. $\partial\mathcal{M}$ is an $n-1$-dimensional smooth manifold.

PROOF 95. This proof is trivial in that there is only one way to proceed: construct charts from the charts already given to us by the manifold.

Let $\{(\phi_\alpha, U_\alpha)\}$ be the atlas on $\mathcal{M}$. We claim that $\{(\phi'_\alpha, U'_\alpha)\}$ defined by

$$U'_\alpha := U_\alpha \cap \partial\mathcal{M}, \ \phi'_\alpha := \phi_\alpha|_{\partial\mathcal{M}} \tag{161}$$

is an atlas on $\partial\mathcal{M}$.

Firstly, ϕ'_α is a smooth chart since it is a restriction of the smooth chart ϕ_α. Also, for any α, β such that $p \in \partial\mathcal{M}$,

$$\phi'_\beta \circ (\phi'_\alpha)^{-1} : (0, x') \to (0, y') \tag{162}$$

by the lemma (which applies to $\phi'_\alpha, \phi'_\beta$ because they are restrictions). Therefore, the pairs we exhibited are indeed an atlas on the boundary.

□

Definition 96. The smooth manifold $\mathcal{M}$ is **orientable** if there exists an atlas such that

$$\det D(\phi_\beta \circ \phi_\alpha^{-1}) > 0 \tag{163}$$

where defined, i.e. on $\phi_\alpha(U_\alpha \cap U_\beta) \neq \emptyset$.

In Stokes theorem, it suffices for us to require that only the manifold is orientable because of the following.

Proposition 97. [38] If the smooth manifold $\mathcal{M}$ is orientable, then the boundary $\partial\mathcal{M}$ is an orientable manifold.

PROOF 98. The idea is to look at the submatrix of the total derivative, and find what the signs are for the first row.

Let $(0, x')$ be a point in the coordinate space of some point on the boundary of $\mathcal{M}$. Then

$$D(\phi_\beta \circ \phi_\alpha^{-1})(0, x') = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & A \\ B & D(\phi'_\beta \circ (\phi'_\alpha)^{-1}(0, x')) \end{pmatrix} \tag{164}$$

By assumption, the determinant of the above matrix is positive, and our claim is that the $(n-1) \times (n-1)$ submatrix $D(\phi'_\beta \circ (\phi'_\alpha)^{-1}(0, x'))$ has positive determinant as well.

Now, the row vector $A \in \mathbb{R}^{n-1}$ is all zeros since $y_1(x)$ has no $x_2, ..., x_n$ dependence; in particular, we know from the well-definedness of $\partial\mathcal{M}$ that $y_1 = 0$ iff $x_1 = 0$. Therefore, ∇y_1 at $(0, x')$ only depends on x_1.

Consequently, we have

$$\det D(\phi_\beta \circ \phi_\alpha^{-1})(0, x') = \frac{\partial y_1}{\partial x_1} \det D(\phi'_\beta \circ (\phi'_\alpha)^{-1}(0, x')) > 0 \tag{165}$$

We also have $\frac{\partial y_1}{\partial x_1} > 0$; the corresponding difference quotient

$$\frac{0 - \epsilon_y}{0 - \epsilon_x} > 0 \tag{166}$$

since $(\epsilon_y, y_2, ..., y_n), (\epsilon_x, x_1, ..., x_n) \in \mathcal{H}^n$ hence $\epsilon_x, \epsilon_y < 0$.

Therefore,

$$\det D(\phi'_\beta \circ (\phi'_\alpha)^{-1}(0, x')) > 0 \tag{167}$$

as desired.

□

We can now address the differential forms on manifolds and how one integrates them. Differential forms should always be dealt with in coordinates by pulling them back to the coordinate space:

Definition 99. η is a **k-form on the smooth manifold** $\mathcal{M}$ (with or without boundary) if for all $p \in \mathcal{M}$, $\eta(p)$ is an alternating k-form on $T_p\mathcal{M}$, and in charts, $p \mapsto \eta(p)$ is smooth, i.e. for a chart ϕ_α,

$$(\phi_\alpha^{-1})^*\eta(x) = \sum_I a_I(x) dx_I \; I = \{i_1 < ... < i_k\} \tag{168}$$

where $a_I(x)$ is smooth function.

Note that again, this last part of the definition is independent of the choice of charts; if we take a different chart ϕ_β, we would just get a composition of smooth maps, i.e.

$$(\phi_\beta^{-1})^*\eta(x)v = (\phi_\beta^{-1} \circ \phi_\alpha \circ \phi_\alpha^{-1})^*\eta(x)v = (\phi_\beta^{-1} \circ \phi_\alpha)^*(\phi_\alpha^{-1})^*\eta(x)v \tag{169}$$

and so, $(\phi_\beta^{-1})^*\eta(x)$ is smooth.
We now want to define what the integral of a k-form is. We note that the following definition requires us to integrate k-forms on k-dimensional submanifolds. We are sill allowed to integrate k-forms in n-manifolds

[38] See also Do Carmo, *Differential Forms and its Applications* p.62.

as long as we are integrating over k-dimensional *sub*manifolds.

We need to break down the definition of the integral depending on the topology of $\mathcal{M}$.

Definition 100 (Compact support case.). Let η be a k-form with compact support on a k-manifold. Let $\text{supp}\eta \subset U_\alpha$ where U_α is some coordinate neighborhood, and

$$\eta = a dx_1 \wedge ... \wedge dx_k \tag{170}$$

Then we define the **the integral of the k-form η over $\mathcal{M}^k$** to be

$$\int_{\mathcal{M}} \eta := \int_{\mathbb{R}^k} a(x_1, ..., x_k) dx_1 ... dx_k \tag{171}$$

Note that this integral must exist because the k-form has compact support.

Proposition 101. $\int_{\mathcal{M}} \eta$ is well defined; i.e. the value of the integral does not depend on the choice of charts.

PROOF 102. Let $(\phi_\alpha, U_\alpha), (\phi_\beta, U_\beta)$ be charts on $\mathcal{M}$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Let $(\phi_\alpha^{-1})^* \eta = a dx_1 \wedge ... \wedge dx_n$ and $(\phi_\beta^{-1})^* \eta = b dy_1 \wedge ... \wedge dy_n$. Then

$$f_{\beta\alpha}^*(b dy_1 \wedge ... \wedge dy_n) = a dx_1 \wedge ... \wedge dx_n \tag{172}$$

for the change of coordinate map $f_{\beta\alpha} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$. But now,

$$\begin{aligned}
f_{\beta\alpha}^*(b dy_1 \wedge ... \wedge dy_n) v &= b(f_{\beta\alpha}(x))(dy_1 \wedge ... \wedge dy_n) \cdot D f_{\beta\alpha}(x) v \\
&= b(f_{\beta\alpha}(x))(dy_1 \wedge ... \wedge dy_n) \cdot [D f_{\beta\alpha}(x) v_1 ... D f_{\beta\alpha}(x) v_n] \\
&= b(f_{\beta\alpha}(x)) \det[D f_{\beta\alpha}(x) v_1 ... D f_{\beta\alpha}(x) v_n] \\
&= b(f_{\beta\alpha}(x)) \det D f_{\beta\alpha}(x) \det[v_1 ... v_n] \\
&= b(f_{\beta\alpha}(x)) \det D f_{\beta\alpha}(x) (dx_1 \wedge ... \wedge dx_n) v
\end{aligned}$$

Therefore,

$$a(x) = b(f_{\beta\alpha}(x)) \det D f_{\beta\alpha}(x) \tag{173}$$

or using the notation $y = f_{\beta\alpha}(x)$,

$$a(x) = b(y) \det D y(x) \tag{174}$$

But now, by the change of variables formula for integration on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} a(x) dx = \int_{\mathbb{R}^n} b(y) \det D y(x) dy \tag{175}$$

so indeed, the integrals given by the two charts are equal. $\square$

We will now provide a definition for $\int_{\mathcal{M}} \eta$ when $\mathcal{M}$ is a compact manifold (as in the statement of Stokes theorem). This case reduces to the case where η has compact support via partitions of unity.

Definition 103 (Compact manifold case.). Let $\mathcal{M}$ be a smooth compact k-manifold, and let $\{(\phi_\alpha, U_\alpha)\}_{\alpha=1}^N$. Let η be a k-form on $\mathcal{M}$. For the partitions of unity $\theta_\alpha \in C^\infty(\mathcal{M})$ with compact support and $\text{supp}\theta_\alpha \subset U_\alpha$, the **integral of the k-form η on $\mathcal{M}$** is

$$\int_{\mathcal{M}} \eta := \sum_{\alpha=1}^N \int_{\mathcal{M}} \theta_\alpha \eta \tag{176}$$

Note that the finiteness of the atlas is automatic from the compactness. We also don't need to worry about the convergence of the sum because of this finiteness. The integrals in the summand on the RHS reduces to our definition of integrals of differential forms with compact support.

Proposition 104. The $\int_{\mathcal{M}} \eta$ is well defined for the compact manifold $\mathcal{M}$, i.e. the value of the integral is independent of the choice of the partition of unity.

PROOF 105. The idea is the same as when we checked that Riemann sums are well-defined, i.e. we take refinements.

Let θ_α and ψ_β be distinct partitions of unity on $\mathcal{M}$. Then

$$\begin{aligned}\int_{\mathcal{M}} \eta &= \sum_\alpha \int_{\mathcal{M}} \theta_\alpha \eta \\ &= \sum_\beta \psi_\beta \sum_\alpha \int_{\mathcal{M}} \theta_\alpha \eta \\ &= \sum_{\alpha,\beta} \int_{\mathcal{M}} \psi_\beta \theta_\alpha \eta \\ &= \sum_\beta \int_{\mathcal{M}} \psi_\beta \eta\end{aligned}$$

□

We now understand the statement of Stokes theorem, and we are ready to prove it. Indeed, the proof is not too hard, once we understand what the theorem says.

PROOF 106. Let $\{\theta_\alpha\}_{\alpha=1}^N$ be a partition of unity on $\mathcal{M}$, and let $i : \partial\mathcal{M} \to \mathcal{M}$ be the natural embedding $x' \mapsto (0, x')$. Then it suffices to prove the theorem for compactly supported charts because

$$\begin{aligned}\int_{\partial\mathcal{M}} i^*\omega &= \sum_{\alpha=1}^N \int_{\partial\mathcal{M}} i^*(\theta_\alpha \omega) \\ &= \sum_{\alpha=1}^N \int_{\mathcal{M}} d(\theta_\alpha \omega) \qquad \text{(Stokes for compact supp. charts)} \\ &= \int_{\mathcal{M}} d\left(\sum_{\alpha=1}^N \theta_\alpha \omega\right) \\ &= \int_{\mathcal{M}} d\omega\end{aligned}$$

Let

$$\omega = \sum_{j=1}^n a_j a dx_1 \wedge \dots \wedge d\hat{x}_j \wedge \dots \wedge dx_n \tag{177}$$

be a an n-form on $\mathcal{M}$ with compact support. Here, we use Spivak's notation, i.e.

$$dx_1 \wedge \dots \wedge d\hat{x}_j \wedge \dots \wedge dx_n := dx_1 \wedge \dots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \dots \wedge dx_n \tag{178}$$

Then

$$d\omega = \sum_{j=1}^{n} (-1)^{j+1} \frac{\partial a_j}{\partial dx_j} dx_1 \wedge ... \wedge dx_n \tag{179}$$

where we get $(-1)^j$ because $dx_1 \wedge ... \wedge dx_n$ is alternating. Therefore,

$$\int_{\mathcal{M}} d\omega = \sum_{j=1}^{n} (-1)^{j+1} \int_{\mathcal{H}^n} \frac{\partial a_j}{\partial x_j}(x) dx \tag{180}$$

where dx is the volume measure in $\mathbb{R}^n$. The integral in the summand converges because the partial derivative has compact support. We now claim that only the $j = 1$ term contributes to the integral, and everything else vanishes.

For $j \neq 1$,

$$\begin{aligned}
\int_{\mathcal{H}^n} \frac{\partial a_j}{\partial dx_j}(x) dx &= \int_{\mathcal{H}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial a_j}{\partial dx_j}(x) dx_j \right) dx_1 ... \widehat{dx_j} ... dx_n \\
&= \int_{\mathcal{H}^{n-1}} (a_j) \bigg|_{-\infty}^{\infty} dx_1 ... \widehat{dx_j} ... dx_n = 0
\end{aligned}$$

where in the last step, we used the fact that a_j has compact support, and therefore, vanishes at $\pm\infty$. To get to the second line, we just use fundamental theorem of calculus. Thus, $j \neq 1$ contributes nothing to the integral.

Now, for $j = 1$, there are two cases:

- $U_\alpha \cap \partial\mathcal{M} = \emptyset$
- $U_\alpha \cap \partial\mathcal{M} \neq \emptyset$

For the first case, we can do the same computation as above to get

$$\int_{\mathcal{H}^n} \frac{\partial a_j}{\partial dx_j}(x) dx = \int_{\mathbb{R}^{n-1}} (a_1) \bigg|_{-\infty}^{0} dx_2 ... dx_n \tag{181}$$

this vanishes because the support of a_1 is contained in U_α which is disjoint from $\partial\mathcal{M} \ni 0$. Thus, this term also contributes 0.

Thus, if anything contributes to the integral, it is the second case. Here, we see that

$$\int_{\mathcal{H}^n} \frac{\partial a_j}{\partial dx_j}(x) dx = \int_{\mathbb{R}^{n-1}} a_1(0, x_2, ..., x_n) dx_2 ... dx_n \tag{182}$$

which is nonzero since $0 \in \text{supp} a_1$. Therefore,

$$\int_{\mathcal{M}} d\omega = \int_{\mathbb{R}^{n-1}} a_1(0, x_2, ..., x_n) dx_2 ... dx_n \tag{183}$$

We just need to show that $\int_{\partial\mathcal{M}} i^*\omega$ is equal to the RHS in local coordinates. Since $i : x' := (x_2, ..., x_n) \mapsto (0, x')$, we have $i^* dx_1 = 0$. Therefore, all terms vanish but $j = 1$ when pulled back by i:

$$i^*\omega = a_1 dx_2 ... dx_n \tag{184}$$

We recall that we defined the positive orientation on the boundary to be such that $dx_1 dx_2 ... dx_n$ is a positive orientation. Therefore, this is a positive orientation. Since $\partial\mathcal{M} = \{0\} \times \mathbb{R}^{n-1}$, we get

$$\int_{\partial\mathcal{M}} i^*\omega = \int_{\mathbb{R}^{n-1}} a_1(0, x') dx_2 ... dx_n \tag{185}$$

which is what we got earlier. This establishes the desired equality.

□

It is worth noting that the only big idea in the proof of Stokes theorem is the fundamental theorem of calculus. Other than that, we just kept track of notation.

Post-Midterm.
Week 4.

Surface Theory: First and Second Fundamental Forms - Thursday April, 19. 2018, Thursday April, 26. 2018.[39]

We now switch to surface theory. We are going to study surface theory with a *metric* which is a means of measuring length upstairs on a manifold. Smooth manifolds do not have this; we instead worked in coordinates to do all of our calculus.

We all know what a surface is. Let's make this more precise using the language of manifolds.

Definition 107. Let $x : U \to \mathbb{R}^3$ be an embedding. A **surface** is a 2-dimensional submanifold $\mathcal{M} = x(U) \subset \mathbb{R}^3$.

Note that a surface has an *ambient space* $\mathbb{R}^3$. This was also the case with any compact manifold via the Whitney embedding theorem.

We also have a **Euclidean structure**, i.e. a usual inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ on the tangent space $T_p\mathcal{M}$ for $p \in \mathcal{M}$ (denoted $\langle \cdot, \cdot \rangle$ when it's clear from context that we are talking about tangent spaces).

The goal of this lecture is make precise and provide a way of computing lengths, areas, and curvatures on a surface. In order to do this, we must start by talking about curves.

The curve $\gamma(t)$ on the surface $\mathcal{M}$ is just $\gamma(t) = x(\eta(t))$ where $\eta(t) = (v(t), u(t))$ is a curve downstairs, i.e. in U.

Definition 108. The **length of a curve** γ **on the surface** $\mathcal{M}$ is the number

$$L(\gamma) := \int_0^1 \|\gamma'(t)\| \, dt \tag{186}$$

where $\|v\| := \langle v, v \rangle$ is the Euclidean norm on $\mathbb{R}^3$.

We will now motivate the first fundamental form via chain rule on γ. Firstly,

$$\gamma'(t) = x_u(\eta)\dot{u} + x_v\dot{v} \tag{187}$$

where of course, we are suppressing the arguments of u, v, η and x_u, x_v denotes partial derivatives. We thus have

$$\|\gamma'(t)\|^2 = E(\eta)\dot{u}^2 + 2F(\eta)\dot{u}\dot{v} + G(\eta)\dot{v}^2 \tag{188}$$

where E, F, G are smooth functions on U which we define in the following.

Definition 109. For $p = x((u, v)(0))$, the **first fundamental form** $I(p)(u, v)$ is the quadratic form in $\dot{u}(0), \dot{v}(0)$ given by

$$I(p)(u, v) := \|\gamma'\|^2 = \left\langle \begin{pmatrix} E & F \\ F & G \end{pmatrix} ((u, v)(0)) \begin{pmatrix} \dot{u}(0) \\ \dot{v}(0) \end{pmatrix}, \begin{pmatrix} \dot{u}(0) \\ \dot{v}(0) \end{pmatrix} \right\rangle \tag{189}$$

where $E := x_u^2, \ F := x_u \cdot x_v, \ G := x_v^2$.

[39] Much of the content of April, 26th was a review of the April 19th, so we just make the two lectures into a single section.

We also call the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}(p)$ the **first fundemental form**[40] and denote it by $I(p)$ when it is clear from the context whether we are referring to the matrix or the quadratic form.

The first fundamental form is an inner product acting on the tangent vectors written in coordinates (i.e. downstairs). Linear algebra says that such an inner product can be written in terms of the usual inner product and a matrix.

Lemma 110. The matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}(p)$ is positive definite, and so, the first fundamental form is a positive definite inner product.

PROOF 111. Firstly, we know that $I(p) \geq 0$ because we defined it to be $\|\gamma'\|^2$ which is nonnegative. Furthermore, it is 0 iff $(u, v) = 0$ because

$$\|\eta'(t)\| = \|Dx(\eta)\eta'\| = 0 \tag{190}$$

iff $\eta' = 0$ since Dx has full rank which follows from x being an embedding. $\square$

We will see later that this observation is crucial for us when computing an explicit formula for the Gaussian curvature.

Alternatively, we can look at the first fundamental form as the pullback of the inner product on $\mathbb{R}^3$ via the embedding. More explicitly, for a point $(u_0, v_0) \in \mathbb{R}^2$ and vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$,

$$\begin{aligned} \langle v, w \rangle_{I(u_0,v_0)} &:= \mathbf{x}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} (u_0, v_0)(\mathbf{v}, \mathbf{w}) \\ &:= \langle D\mathbf{x}(u_0, v_0)\mathbf{v}, D\mathbf{x}(u_0, v_0)\mathbf{w} \rangle_{\mathbb{R}^3} \\ &:= \left\langle (D\mathbf{x}(u_0, v_0))^t \, D\mathbf{x}(u_0, v_0)\mathbf{v}, \mathbf{w} \right\rangle_{\mathbb{R}^2} \end{aligned}$$

and so, we define

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}(u_0, v_0) := (D\mathbf{x}(u_0, v_0))^t \, D\mathbf{x}(u_0, v_0) \tag{191}$$

We want to introduce here the **(Einstein) summation convention** in which the first fundamental form can be denoted as

$$ds^2 = g_{i,j} dx^i dx^j \tag{192}$$

where $g_{i,j} := \partial_i x \cdot \partial_j x$.

Now that we can measure length, we want to know how to measure area. More precisely, if we had a compact triangle K in the coordinate space, then we want to find the area of $x(K)$ upstairs. We will find that the sensible definition for this is

$$\int_K |x_u \wedge x_v| \, dudv \tag{193}$$

[40] We would like to remark here that Shifrin defines (the matrix representation of) the first fundamental form on p.39 *upstairs* in the basis $\mathbf{x}_u, \mathbf{x}_v$, but this gives the same matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$. Why? We recall that $\mathbf{x}_u, \mathbf{x}_v$ are simply coordinate vector fields, i.e. if ∂_1, ∂_2 are the standard basis downstairs in $\mathbb{R}^2$, then $\mathbf{x}_u = Dx\partial_1, \mathbf{x}_v = Dx\partial_2$. So, if we had a vector $v = a\partial_1 + b\partial_2$ downstairs, then the corresponding vector upstairs is $Dxv = a\mathbf{x}_u + b\mathbf{x}_v$. In other words, a basis representation of a vector in $\mathbf{x}_u, \mathbf{x}_v$ basis is the same invariant under the pullback (with respect to the chart). This is what is meant when we say that "the coordinate vector fields are equal to $(1, 0, .., 0), (0, 1, 0, ..., 0)$."

The motivation for this is as follows. Suppose that we split the figure K downstairs into "straightened" parallelograms of side length $\Delta u, \Delta v$. Then the linearization for this (hence the dimensions for the parallelogram upstairs) is $x_u \Delta u, x_v \Delta v$. This then gives us the wedge product of the two which we recall is the volume element spanned by the two vectors. Note that this is the same as the magnitude of the cross product.

Since this is defined in terms of the partials of x, can we somehow relate this to the first fundamental form of $\mathcal{M}$? If we think more geometrically, $|x_u \wedge x_v|\, dudv$ gives the area of the parallelogram spanned by both x_u, x_v. Let α be the angle between the two vectors. Then we can write the area of the parallelogram as

$$\begin{aligned}
|x_u \wedge x_v|\, dudv &= \|x_u\|\,\|x_v\| \sin\alpha \\
&= \|x_u\|\,\|x_v\| \sqrt{1-\cos^2\alpha} \\
&= \|x_u\|\,\|x_v\| \sqrt{1-\left(\frac{x_u}{x_v}\right)^2} \\
&= \sqrt{x_u^2 x_v^2 - (x_u x_v)^2} \\
&= \sqrt{EG - F^2} \\
&= \sqrt{\det I(u,v)}
\end{aligned}$$

Definition 112. For a compact region $K \subset \mathbb{R}^2$, the **area of** $x(K) \subset \mathcal{M}$ is the number

$$\int_K \sqrt{\det I(u,v)} dudv \tag{194}$$

We are now ready to talk about curvature of a surface. We first define the notion of curvature for curves which is what we used to define the notion for surfaces.

The curvature of a curve c is the speed at which the tangent vector turns. One can imagine a bug on the surface of tha curve moving along with a rod in the direction of the curve, looking at how fast the rod turns. However, we do not want this measurement to be determined by how fast the bug is moving; we need to have a standardized clock for this. The standard tool for this is the arc length parametrization of c, which requires that $\|\dot{c}(t) = 1\|$ for all t.

Definition 113. A curve c is **regular** if $\dot{c}(t) \neq 0$ for all t.

Definition 114. [41] Let c be a curve. c is **parametrized by arclength** if $s(t) = t$ for all $T \in [0,1]$ where

$$s(t) := \int_0^t \|\dot{c}(t)\|\, du \quad 0 \le t \le 1 \tag{195}$$

Definition 115. For an arclength parametrized, regular curve, then the **curvature of** c is the function (of t) given by

$$\mathscr{K}(t) := \|\ddot{c}(t)\| \tag{196}$$

[41] From Shifrin, *Differential Geometry of Curves and Surfaces.*

Example 116. For instance, consider the circle

$$c(t) = r(\cos t, \sin t) \tag{197}$$

for which we have $\|\dot{c}(t)\| = r$. We can then let $\tilde{c}(s) := c\left(\frac{s}{r}\right)$, and indeed,

$$\frac{d}{ds}\tilde{c}(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right) \tag{198}$$

whose norm is 1. The curvature is then $\mathscr{H}(t) = \|\ddot{c}(t)\| = \frac{1}{r}$. This agrees with the intuition; a small coin looks a lot more curved than the great circle on earth because the radius is small.

We give an alternative formula for the curvature. Since c is arclength parametrized, $\dot{c}\cdot\dot{c} = 1$, and so, we can differentiate via product rule to get $\ddot{c}\cdot\dot{c} = 0$. But now, since the normal vector is given by $n = -\frac{\ddot{c}}{\|\ddot{c}\|}$, we get

$$\mathscr{H}(s) = \ddot{c}\cdot n \tag{199}$$

Indeed, in the previous example, we have $n = (\cos\frac{s}{r}), \sin\frac{s}{r}$, and so this formula holds.

This motivates the second fundamental form which is related to the curvature of a surface. We define this in analogoue to the above formula.

Definition 117. The map $S(p) : T_p\mathcal{M} \to T_p\mathcal{M}$ given by

$$S(p)\mathbf{v} := -D\mathbf{N}(p)\mathbf{v} \tag{200}$$

is the **shape operator**, where $\mathbf{N}$ is the **normal vector of the surface** $\mathcal{M}$ which is a function $\mathbf{N} : \mathcal{M} \to \mathbb{R}^3$ given by

$$\mathbf{N} := \frac{x_u \times x_v}{\|x_u \times x_v\|} \tag{201}$$

Definition 118. Let $p := \mathbf{x}(u_0v_0)$. The **second fundamental form** $\mathbb{I}(p)$ is a quadratic form in the coordinates of $v, w \in T_p\mathcal{M}$ defined by the formula

$$\mathbb{I}(p)(\mathbf{V}, \mathbf{W}) := -\langle D\mathbf{N}(p)\mathbf{V}, \mathbf{W}\rangle_{\mathbb{R}^3} = \langle S_pD\mathbf{x}(u_0, v_0)\mathbf{v}, D\mathbf{x}(u_0, v_0)\mathbf{w}\rangle_{\mathbb{R}^3} \tag{202}$$

where $\mathbf{V} := D\mathbf{x}(u_0, v_0)\mathbf{v}, \mathbf{W} := D\mathbf{x}(u_0, v_0)\mathbf{w} \in \mathbb{R}^3$ for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$. We then denote the matrix

$$\mathbb{I}(p) := (D\mathbf{x}(u_0, v_0))^t\, S_pD\mathbf{x}(u_0, v_0) \tag{203}$$

and call it the **second fundamental form**.

Note that since x is an embedding, x_u, x_v are linearly independent, so $x_u \times x_v \neq 0$. We can also make sense of $DN(p)v$ since this is simply the partial derivative of $\mathbf{N}$ in the direction v:

$$D\mathbf{N}(p)v = \left.\frac{d}{dt}\right|_{t=0}\mathbf{N}(\gamma(t)) \tag{204}$$

where $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. In particular, if $v = (1, 0) = \frac{\partial}{\partial u}$[42], then $D\mathbf{N}(p)u = \mathbf{N}_u$

We want to extend our notion of curvature on curves to a notion on a surface. If we have the surface $\mathcal{M}$, we can take an arbitrary plane π going through $N(p)$. Then, $\pi \cap \mathcal{M}$ is some curve γ_π going through p for which one can find the curvature $\mathscr{H}_\pi(s)$. Our claim is then that

$$\mathbb{I}(p)(\dot{\gamma_\pi}(0), \dot{\gamma_\pi}(0)) = \mathscr{H}_\pi(0) \tag{205}$$

[42] Recall that we use the standard notation $\frac{\partial}{\partial u} := (1, 0),\ \frac{\partial}{\partial v} := (0, 1)$.

i.e., the second fundamental form acting on the tangent vectors of the intersection curves γ_π gives the curvature of the curves. This gives us a better sense of what the second fundamental form means.

We will define the principal curvatures of $\mathcal{M}$ to be the eigenvalues of $\mathbb{I}(p)$. In order for these eigenvalues to exist, we will need to apply the spectral theorem. Hence, we will need to show that this quadratic form is indeed symmetric.

Proposition 119. $\mathbb{I}(p)$ is a symmetric quadratic form on $T_p\mathcal{M}$.

PROOF 120. Since $\mathbb{I}(p)$ is bilinear, it suffices to check this for the basis vectors for $T_p\mathcal{M}$. Since the tangent space has dimensions x_u, x_v, and since x_u, x_v are linearly independent (since Dx has full rank by definition of an embedding), $\{x_u, x_v\}$ is an embedding on $T_p\mathcal{M}$. Now, unraveling definitions, letting $p = x(u_0, v_0)$

$$\begin{aligned}\mathbb{I}(p)(x_u(u_0,v_0), x_v(u_0,v_0)) &= \langle -D\mathbf{N}(p)x_u, x_v\rangle \\ &= \left\langle -\frac{\partial}{\partial u}\mathbf{N}(x)(u_0,v_0), x_v\right\rangle \\ &= -\langle \mathbf{N}(x), x_{uv}\rangle\end{aligned}$$

In the last step, we used that

$$\langle \mathbf{N}(p), x_u\rangle = 0 \tag{206}$$

which by product rule gives

$$\langle \partial_u \mathbf{N}, x_v\rangle + \langle \mathbf{N}, x_{vu}\rangle = 0 \tag{207}$$

Now, we can do the exact same manipulation to get

$$\mathbb{I}(p)(x_v, x_u) = -\langle \mathbf{N}(x), x_{vu}\rangle \tag{208}$$

which is equal to the above by Schwarz lemma.

□

Now, for a symmetric quadratic form, there is a cooresponding symmetric matrix. Denote this by

$$\begin{pmatrix} L & M \\ M & N\end{pmatrix}(p) \tag{209}$$

whose entries are functions of the partials of the normal vector **N**.

Definition 121. The **principal curvature of $\mathcal{M}$ at p** is the maximum and minimum values of the second fundamental form of $\mathcal{M}$ at p, i.e.

$$k_1 := \max_{v\in T_p\mathcal{M}, \|v\|=1} \mathbb{I}(p)(v,v),\ k_2 := \min_{v\in T_p\mathcal{M}, \|v\|=1} \mathbb{I}(p)(v,v) \tag{210}$$

Recall that *the min/max of a quadratic form is just the eigenvalue of the corresponding matrix*, $\begin{pmatrix} L & M \\ M & N\end{pmatrix}$. We also give the product of the principal curvatures a name.

Definition 122. The **Gaussian curvature K of a surface $\mathcal{M}$ at the point p** is the product of the principal curvatures at p, i.e.

$$K := K(p) := k_1(p)k_2(p) \tag{211}$$

Can we find an explicit formula for this using the first and second fundamental forms? We recall from the proof of the spectral theorem that we can compute eigenvalues using Lagrange multipliers.

More specifically, we are trying to optimize $\mathbb{I}(p)$ with the constraint that the length of the curve is 1, i.e. $I(p)\dot{(c)} = 1$. Therefore, we have

$$\nabla \mathbb{I}(p)(v,v) = \lambda \nabla I(p) v \tag{212}$$

for a nonzero vector v. We can turn this coordinate free expression in to an expression in coordinates using the basis representation in $\{x_u, x_v\}$ on the tangent plane $T_p\mathcal{M}$. Denote $v := \begin{pmatrix} \xi \\ \eta \end{pmatrix} \neq 0$. We then recall that

$$\begin{aligned} \nabla I(p)v &= \nabla(E^2\xi^2 + 2F\xi\eta + G^2\eta^2) \\ &= (2E^2\xi + 2F\eta, 2F\xi + 2G\eta) \\ &= 2\begin{pmatrix} E & F \\ F & G \end{pmatrix} v \end{aligned}$$

and by the exact same computation,

$$\nabla \mathbb{I}(p)v = 2\begin{pmatrix} L & M \\ M & N \end{pmatrix} v \tag{213}$$

Therefore, our original equation becomes

$$(B - \lambda A)v = 0 \tag{214}$$

where $A := \begin{pmatrix} E & F \\ F & G \end{pmatrix}, B := \begin{pmatrix} L & M \\ M & N \end{pmatrix}$.
Since $v \neq 0$, $B - \lambda A$ must have nontrivial kernel, and so,

$$\det(B - \lambda A) = 0 \tag{215}$$

But since A is positive definite, we can take its square root, i.e. there exists $A^{1/2} \neq 0$ such that $A = A^{1/2} \cdot A^{1/2}$. Therefore, we can write

$$\det A^{1/2} \det(A^{-1/2}BA^{-1/2} - \lambda \mathbf{Id}) \det A^{1/2} = 0 \tag{216}$$

so,

$$\det(A^{-1/2}BA^{-1/2} - \lambda \mathbf{Id}) = 0 \tag{217}$$

Therefore, λ is an eigenvalue of $A^{-1/2}BA^{-1/2}$.[43] Therefore, the principal curvature is the product of the maximum and minimum values of λ, and so, it is just the determinant of $A^{-1/2}BA^{-1/2}$:

$$K = \det(A^{-1/2}BA^{-1/2}) = \frac{\det B}{\det A} = \frac{LN - M^2}{EG - F^2} \tag{218}$$

This is an extraordinary fact because we defined K extrinsically in terms of (ultimately) the parametrization of $\mathcal{M}$. However, the above (and a little more work) shows that K is in fact intrinsic[44] in $\mathcal{M}$.[45]

[43] The general principle is that Lagrange multipliers always have meaning; in this case, it is the eigenvalue of some matrix.
[44] Note the difference between *intrinsic* and *geometric*. The former means the quantity depends only on the length measurement of a surface (i.e. the first fundamental form) and does not make reference to the ambient space (i.e. use the normal vector). The latter just means coordinate invariant.
[45] See Shifrin, *Differential Geometry on Curves and Surfaces* Theorem 3.1.

A corollary to this is that a sphere and a plane are not locally isometric[46] since the former has $K = 1$ (everywhere) whereas a plane has $K = 0$. A plane and and a cylinder is locally isometric since (the cylinder has a $k_2 = 0$, and therefore) they have $K = 0$.

Proposition 123 (Theorema Egregium). The Gaussian curvature is an intrinsic property, i.e. K can be written entirely from first fundamental form.

PROOF 124.

□

Shape Operator. -Monday, 6.4.2018

We would like to discuss a very important identity from Shifrin (p.53) that we never discussed in class:

$$S_p = \mathrm{I}_p^{-1}\mathrm{II}_p \tag{219}$$

where the above are linear maps on $T_p\mathcal{M}$. As we shall see, a natural interpretation comes up once we look at the quadratic forms corresponding to these maps.

We remarked before that the vector/matrix representation on $T_p\mathcal{M}$ are the same as the representation downstairs in $\mathbb{R}^2$[47]. So, even though we defined in class that $\mathrm{I}_p, \mathrm{II}_p$ are matrices/quadratic forms downstairs, we can naturally associate a map in $T_p\mathcal{M}$.

Once we understand the meaning of $\mathrm{I}_p, \mathrm{II}_p$, the above identity is transparent. Using Shifrin's definition (consistent with ours from class),

$$\mathrm{II}_p(U, V) := S_p(U) \cdot V \tag{220}$$

where $U, V \in T_p\mathcal{M}$ and the $\cdot$ is the inner product on $\mathcal{T}_p\mathcal{M} \simeq \mathbb{R}^2$. (Pulling this back downstairs gives us the definition from class.) On the other hand we defined I_p as the pullback of the ambient space ($\mathbb{R}^3$) inner product, i.e.

$$\langle U, V\rangle_{\mathbb{R}^3} = \langle \mathrm{I}_p U, V\rangle_{T_p\mathcal{M}} \tag{221}$$

Can we give a similar interpretation to I_p^{-1}?

If $U, V \in T_p\mathcal{M}$, then it seems natural that

$$\left\langle \mathrm{I}_p^{-1} U, V\right\rangle_{\mathbb{R}^3} = \langle U, V\rangle_{T_p\mathcal{M}} \tag{222}$$

i.e., I_p^{-1} is the pullback of the tangent space metric to the ambient space metric. But this is indeed the case; if we let $U = \mathrm{I}_p\tilde{U}$ (since I_p is invertible, $\tilde{U}$ is still an arbitrary vector in $T_p\mathcal{M}$), then the above becomes

$$\left\langle \mathrm{I}_p^{-1} U, V\right\rangle_{\mathbb{R}^3} = \left\langle \tilde{U}, V\right\rangle_{\mathbb{R}^3} = \left\langle \mathrm{I}_p\tilde{U}, V\right\rangle_{\mathbb{R}^3} \tag{223}$$

which we know is true.

Going back to our original identity,

$$\begin{aligned}\langle S_p U, V\rangle_{T_p\mathcal{M}} &= \langle \mathrm{II}_p U, V\rangle_{T_p\mathcal{M}} \\ &= \left\langle \mathrm{I}_p^{-1}\mathrm{II}_p U, V\right\rangle_{\mathbb{R}^3}\end{aligned}$$

[46] Two surfaces are **isometric** if their first fundamental forms agree. See MathOverflow
[47] See footnote to Surface Theory: First and Second Fundamental Forms - Thursday April, 19. 2018, Thursday April, 26. 2018

Notice that this is precisely what we wanted; when we compute the inner product of $U = a\mathbf{x}_u + b\mathbf{x}_v, V = c\mathbf{x}_u + d\mathbf{x}_v$ in the basis $\mathbf{x}_u, \mathbf{x}_v$, we are really using the metric in $\mathbb{R}^3$ rather than the tangent space metric:

$$U \cdot V = ac + bd \tag{224}$$

Therefore, when we compute the matrix for S_p for the basis $\mathbf{x}_u, \mathbf{x}_v$, we need to work in the inner product we use to compute the quadratic form, i.e. the inner product of the ambient space $\mathbb{R}^3$.

Week 5. Midterm & Review of surface theory. (See 4.19.2018 notes.)

Week 6.

Surface Theory: Christoffel Symbols and Covariant Derivatives - Tuesday May, 1. 2018.

We continue with out discussion of surface theory. We will start by proving the Theorem Egregium.

Definition 125. The Christoffel symbols $\Gamma^{\circ}_{\circ\circ} : U \subset \mathbb{R}^2 \to \mathbb{R}$ are functions such that

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + \ell\mathbf{N} \\
\mathbf{x}_{uv} &= \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v + m\mathbf{N} \\
\mathbf{x}_{vv} &= \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v + n\mathbf{N}
\end{aligned}$$

We can find such numbers at each point on the surface since $\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}$ form a basis in $\mathbb{R}^3$.

To reiterate, for the Chrsitoffel symbol Γ^a_{bc} denotes the coefficient of $\mathbf{x}_a$ for the basis representation of the second derivative $\mathbf{x}_{bc}$; thus, superscript corresponds to the basis vectors (the first derivative) and the subscript corresponds to what is written as a linear combination (the second derivative).

Proposition 126. The Christoffel symbols are intrinsic.

PROOF 127. Since $\mathbf{N}$ are orthogonal to $\mathbf{x}_u, \mathbf{x}_v$,

$$\begin{aligned}
\frac{1}{2}E_u &= \mathbf{x}_{uu} \cdot \mathbf{x}_u = \Gamma^u_{uu}E + \Gamma^v_{uu}F \\
\frac{1}{2}E_v &= \mathbf{x}_{uv} \cdot \mathbf{x}_u = \Gamma^u_{uv}E + \Gamma^v_{uv}F \\
F_v - \frac{1}{2}G_u &= \mathbf{x}_{vv} \cdot \mathbf{x}_u = \Gamma^u_{vv}E + \Gamma^v_{vv}F
\end{aligned}$$

and

$$\begin{aligned}
F_u - \frac{1}{2}E_v &= \mathbf{x}_{uu} \cdot \mathbf{x}_v = \Gamma^u_{uu}F + \Gamma^v_{uu}G \\
\frac{1}{2}G_u &= \mathbf{x}_{uv} \cdot \mathbf{x}_v = \Gamma^u_{uv}F + \Gamma^v_{uv}G \\
\frac{1}{2}G_v &= \mathbf{x}_{vv} \cdot \mathbf{x}_v = \Gamma^u_{vv}F + \Gamma^v_{vv}G
\end{aligned}$$

So, we can solve for each Christoffel symbol in terms of E, F, G and so, they are intrinsic.

□

Proposition 128. The Gaussian curvature is intrinsic, i.e. K can be computed entirely of the first fundamental form.

PROOF 129. Since we have the formula

$$K = \frac{\ell n - m^2}{EG - F^2} \tag{225}$$

It suffices to show that $\ell n - m^2$ is intrinsic. Thus,

$$\begin{aligned}
\ell n - m^2 &= (\ell \mathbf{N} \cdot n\mathbf{N}) - (m\mathbf{N}) \cdot (m\mathbf{N}) \\
&= (\ell \mathbf{N} \cdot n\mathbf{N}) - (m\mathbf{N}) \cdot (m\mathbf{N}) \\
&= (\mathbf{x}_{uu} - (\Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v)) \cdot (\mathbf{x}_{vv} - (\Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v)) - (\mathbf{x}_{uv} - (\Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v))^2 \\
&= \mathbf{x}_{uu} \cdot \mathbf{x}_{vv} - \mathbf{x}_{uv} \cdot \mathbf{x}_{uv} + (\text{intrinsic})
\end{aligned}$$

We saw in the proof of the lemma that products of second partial derivatives and first partial derivatives of $\mathbf{x}$ are intrinsic. We just need to show that the remaining part is intrinsic. For this, we use something similar to what we did in the lemma.

We can rewrite the above as

$$\begin{aligned}
\mathbf{x}_{uu} \cdot \mathbf{x}_{vv} - \mathbf{x}_{uv} \cdot \mathbf{x}_{uv} &= (\mathbf{x}_u \cdot \mathbf{x}_{vv})_u - (\mathbf{x}_u \cdot \mathbf{x}_{uvv}) - (\mathbf{x}_u \cdot \mathbf{x}_{uv})_v + (\mathbf{x}_u \cdot \mathbf{x}_{uvv}) \\
&= (\mathbf{x}_u \cdot \mathbf{x}_{vv})_u - (\mathbf{x}_u \cdot \mathbf{x}_{uv})_v \\
&= (F_v - \frac{1}{2}G_u)_u - \left(\frac{1}{2}E_v\right)_v
\end{aligned}$$

which is intrinsic.

□

Definition 130. Let $\mathcal{M} \subset \mathbb{R}^3$ be a surface, and let $\mathbf{V}$ be a vector field in $\mathbb{R}^3$ along $\gamma \subset \mathcal{M}$ (i.e. $\mathbf{V}(t) \in T_{\gamma(t)}\mathcal{M}$). The **vector field V is parallel** if (the acceleration) $\dot{V}(t) = 0$.

More generally, $\dot{V}(t)$ must be normal[48] to the surface, i.e.

$$\pi_t(\dot{V}(t)) = 0 \tag{226}$$

where π_t projects onto $T_{\gamma(t)}\mathcal{M}$

Definition 131. Given a curve $c(t) := \mathbf{x}(\gamma(t))$ on $\mathcal{M}$, and a vector field $\mathbf{V}(t)$ along c, the **covariant derivatives of V along** c is

$$\frac{D\mathbf{V}}{dt} = \pi_{T_{c(t)}\mathcal{M}}(\dot{V}(t)) \tag{227}$$

$\mathbf{V}$ is parallel along c if $\frac{D\mathbf{V}}{dt} = 0$. The curve c is a **geodesic** if c' is parallel along c.

In other words, the covariant derivative is just the projection of the acceleration along the tangent plane. Parallel simply means that the vector field is not allowed to change along the tangent space of the curve. Geodesic just means the velocity of the curve is parallel along the curve.[49]

Proposition 132. If c is a geodesic, then $|\dot{c}(t)|$ is a constant.

[48] The idea is that the particle cannot experience any tangential force, and so, the acceleration projected onto the tangent space must vanish.

[49] Note that a geodesic is *locally* the shortest path between two points. Consider for instance, two points on the great circle on a sphere. Then an ant can travel from one point to the other via the shorter route on the great circle, or the longer route.

PROOF 133.

$$\begin{aligned}\frac{d}{dt}\langle \dot{c}, \dot{c}\rangle &= 2\langle \dot{c}, \ddot{c}\rangle \\ &= 2\left\langle \dot{c}, \frac{D\dot{c}}{dt}\right\rangle = 0\end{aligned}$$

In the last step, we used the fact that $\langle \dot{c}, \rangle$ kills the normal component of c, and so, what remains is the the component living in the tangent space, i.e. $\frac{D\dot{c}}{dt}$. □

Proposition 134. If $\mathbf{V}, \mathbf{W}$ are vector fields along any curve c, then

$$\frac{d}{dt}\langle \mathbf{V}, \mathbf{W}\rangle = \left\langle \mathbf{V}, \frac{D\mathbf{W}}{dt}\right\rangle + \left\langle \frac{D\mathbf{V}}{dt}, \mathbf{W}\right\rangle \tag{228}$$

In particular, if $\mathbf{V}, \mathbf{W}$ are parallel along c, then $\langle \mathbf{V}, \mathbf{W}\rangle$ is constant.

PROOF 135. The idea is the same as in the previous proposition, we just project onto the tangent plane.

$$\begin{aligned}\frac{d}{dt}\langle \mathbf{V}, \mathbf{W}\rangle &= \left\langle \dot{\mathbf{V}}, \mathbf{W}\right\rangle + \left\langle \mathbf{V}, \dot{\mathbf{W}}\right\rangle \\ &= \left\langle \mathbf{V}, \frac{D\mathbf{W}}{dt}\right\rangle + \left\langle \frac{D\mathbf{V}}{dt}, \mathbf{W}\right\rangle\end{aligned}$$

where we get the last step by the same reason as in the previous proposition. □

Now we want to ask: how do we compute covariant derivatives, vector fields, and geodesics in coordinates (i.e. downstairs)? Also, are these quantities intrinsic?

Let $\mathbf{v}(t) := (v_1(t), v_2(t))$ be a vector field in $\mathbb{R}^2$, $c(t) := \mathbf{x}(\gamma(t))$ be a curve for $\gamma(t) := (\gamma_1(t), \gamma_2(t))$. Then the vector field $\mathbf{V}$ on the manifold along c is just the pushup of the vector field $\mathbf{v}$ downstairs:

$$\begin{aligned}\mathbf{V}(t) &= D\mathbf{x}(\gamma_1, \gamma_2)(v_1, v_2) \\ &= \mathbf{x}_u(\gamma_1, \gamma_2)v_1 + \mathbf{x}_v(\gamma_1, \gamma_2)v_2\end{aligned}$$

which is the basis representation of $\mathbf{V}$ in terms of the basis vectors $\mathbf{x}_u, \mathbf{x}_v$. Now the derivative is

$$\dot{V}(t) = (\mathbf{x}_{uu}(\gamma_1, \gamma_2)\dot{\gamma_1} + \mathbf{x}_{uv}(\gamma_1, \gamma_2)\dot{\gamma_2})v_1 + (\mathbf{x}_{vu}(\gamma_1, \gamma_2)\dot{\gamma_1} + \mathbf{x}_{vv}(\gamma_1, \gamma_2)\dot{\gamma_2})v_2 + \mathbf{x}_u(\gamma_1, \gamma_2)\dot{v_1} + \mathbf{x}_v(\gamma_1, \gamma_2)\dot{v_2}$$

by chain rule. We can now compute the covariant derivative of $\mathbf{V}$. We would like to write this in the basis $\mathbf{x}_u, \mathbf{x}_v$. We can expand each second derivative of $\mathbf{x}$ via Christoffel symbols and the normal vector terms vanish since the covariant derivative is a projection onto the tangent plane (and thus, normal vectors do not contribute anything). Thus,

$$\begin{aligned}\frac{D\mathbf{V}}{dt}(t) =&(\Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v)\dot{\gamma_1}v_1 + \\ &(\Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v)\dot{\gamma_2}v_1 + \\ &(\Gamma^u_{vu}\mathbf{x}_u + \Gamma^v_{vu}\mathbf{x}_v)\dot{\gamma_1}v_2 + \\ &(\Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v)\dot{\gamma_2}v_2 \\ &+ \mathbf{x}_u\dot{v_1} + \mathbf{x}_v\dot{v_2}\end{aligned} \tag{229}$$

where we suppresed the arguments of the Christoffel symbols for convenience of notation (they are all evaluated at γ). Since the covariant vector lives in $T_{c(t)}\mathcal{M}$, we can write it as a linear combination $\xi_1\mathbf{x}_u+\xi_2\mathbf{x}_v$ for the basis $\mathbf{x}_u, \mathbf{x}_v$. Collection terms, we see that

$$\begin{aligned}\xi_1 &= \dot{v_1} + \Gamma^u_{uu}\dot{\gamma_1}v_1 + \Gamma^u_{uv}\dot{\gamma_2}v_1 + \Gamma^u_{vu}\dot{\gamma_1}v_2 + \Gamma^u_{vv}\dot{\gamma_2}v_2 \\ \xi_2 &= \dot{v_2} + \Gamma^v_{uu}\dot{\gamma_1}v_1 + \Gamma^v_{uv}\dot{\gamma_2}v_1 + \Gamma^v_{vu}\dot{\gamma_1}v_2 + \Gamma^v_{vv}\dot{\gamma_2}v_2\end{aligned}$$

And so, we have a means of computing the covariant derivative in coordinates. We also note here that covariant derivatives are intrinsic since Christoffel symbols are intrinsic.

We would like to take a brief digression on Riemanian geometry to give this discussion more context. Riemannian manifolds do not have an ambient space, and so we must work with the metric, i.e. work downstairs.

Let $\mathcal{M}^n$ be a smooth n-dimensional manifold with **metric** $\langle\cdot,\cdot\rangle(x) : T_x\mathcal{M}\times T_x\mathcal{M}\to\mathbb{R}, x\in\mathcal{M}$ defined by

$$\langle \mathbf{V}, \mathbf{W}\rangle(x) := \sum_{i,j} g_{i,j}v_iw_j \tag{230}$$

where $(g_{i,j})^n_{i,j=1} := (g_{i,j}(x))^n_{i,j=1}$ is a positive definite matrix of smooth functions, and $\mathbf{V}, \mathbf{W}$ are vector fields on $\mathcal{M}$. For $n = 2$,

$$(g_{i,j})^2_{i,j=1}(p) = I_p = \begin{pmatrix} E & F \\ F & G\end{pmatrix} \tag{231}$$

(This is what we mean when we say surface theory is a subset of Riemannian geometry since the first fundamental form is a metric.)

How does the vector field transform under change of parametrization? Let $\mathbf{x}, \mathbf{y}$ be two different parametrizations of $\mathcal{M}$ with overlapping coordinate neighborhoods.[50] Let $X := \sum_{j=1}^n X_j\frac{\partial}{\partial x_j}, \tilde{X} := \sum_{j=1}^n \tilde{X}_j\frac{\partial}{\partial x_j}$ be some vector fields in $\mathcal{M}$, and let $Y := \sum_{j=1}^n Y_j\frac{\partial}{\partial y_j}, \tilde{Y} := \sum_{j=1}^n \tilde{Y}_j\frac{\partial}{\partial y_j}$ be the pullbacks of these vector fields under the $x(y) := \mathbf{x}^{-1}\circ\mathbf{y}(y)$.[51]
We can then write

$$\begin{aligned}X &= \sum_{j=1}^n X_j\frac{\partial}{\partial x_j} \\ &= \sum_{k=1}^n\sum_{j=1}^n X_j\frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial y_k}\end{aligned}$$

[50] Note the distinction in notation. $\mathbf{x}, \mathbf{y}$ denote the charts whereas x, y denote the parameters living in the coordinate space downstairs (i.e. $\mathbf{x}, \mathbf{y}$ eats x, y).

[51] Since this is the first time in the lecture to use this notation (and it will keep coming up), let's define this more carefully. A **coordinate vector field** ∂_i is the pushup of the standard basis vectors in the coordinate space downstairs, with respect to the parametrization $\mathbf{x}$. In formulas, this is simply $\partial_i(q) := \mathbf{x}_* x_i = D\mathbf{x}(p)x_i$ where $q := \mathbf{x}(p)$ and x_i is the ith standard basis vector downstairs. Note that since a coordinate vector field is a *vector field*, it lives *upstairs*. Notice that its action on smooth function f (as a derivation) is just partial differentiation in that direction, i.e.

$$\partial_i(f)(p) = df(\partial_i)(p) = Df(p)(\partial_i(p)) = \frac{\partial f}{\partial x_i} \tag{232}$$

where $\frac{\partial f}{\partial x_i}$ is really a vector field upstairs (which is really $\frac{\partial(f\circ\mathbf{x})}{\partial x_i}$).

which in turn shows that

$$Y_k = \sum_{j=1}^{n} X_j \frac{\partial y_k}{\partial x_j} \tag{233}$$

and so,

$$Y(y) = \frac{\partial y}{\partial x}(x) X(x) \tag{234}$$

where $\frac{\partial y}{\partial x}(x)$ is the total derivative of the diffeomorphism $y := y(x)$ between the two coordinate spaces. Likewise,

$$\tilde{Y}(y) = \frac{\partial y}{\partial x}(x) \tilde{X}(x) \tag{235}$$

We would like to have the metric be invariant under parametrization, i.e.

$$\left\langle Y, \tilde{Y} \right\rangle (y) = \left\langle X, \tilde{X} \right\rangle (x) \tag{236}$$

The left hand side is then

$$\begin{aligned} \left\langle Y, \tilde{Y} \right\rangle (y) &= \sum_{k,l=1}^{n} h_{k,l}(y) Y_k(y) \tilde{Y}_l(y) \\ &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} h_{k,l}(y) X_j(x) \frac{\partial y_k}{\partial x_j}(x) \tilde{X}_i(x) \frac{\partial y_l}{\partial x_i}(x) \end{aligned}$$

On the other hand,

$$\left\langle X, \tilde{X} \right\rangle (x) = \sum_{i,j=1}^{n} g_{i,j}(x) X_i \tilde{X}_j$$

So,

$$g_{i,j}(x) = h_{k,l}(y) \frac{\partial y_k}{\partial x_j}(x) \frac{\partial y_l}{\partial x_i}(x) \tag{237}$$

[52]

We can now go back to our discussion of covariant derivatives, and state what we found in the language of metrics. For a vector field $X(t) = (X_1, X_2) := (X \circ \gamma)(t)$ in $\mathbb{R}^2$ along the curve $\gamma(t) = (\gamma_1, \gamma_2)$,

$$\left(\frac{DX}{dt} \right)_i = \dot{X}_i + \Gamma^i_{k,l}(\gamma(t)) X_k(t) \dot{\gamma}^\ell(t) \tag{238}$$

[52] This is actually not terribly abstract; this is just the change of coordinates for quadratic forms. If $X, \tilde{X}$ are vectors, $B(\cdot, \cdot)$ is a quadratic form corresponding to the matrix B, and $Y = AX,\ \tilde{Y} = A\tilde{X}$, then

$$\begin{aligned} B(Y, \tilde{Y}) &= (BY, \tilde{Y}) \\ &= (BAX, A\tilde{X}) \\ &= (A^t BAX, \tilde{X}) \end{aligned}$$

and here, we are saying that A is the total derivative of the change of coordinates map $y(x)$ and B is the metric for the parametrization in y and $A^t BA$ is the metric in the variable x.

Existence and Uniqueness of ODEs. - Thursday May, 3. 2018.

Last lecture, we defined what is meant for a vector field to be parallel along a curve. This is defined in terms of ODEs. In this lecture, we will discuss the existence uniqueness theorem for ODEs.

Recall from last time that a vector field V is parallel along a curve γ if $\frac{D\mathbf{V}}{dt} = 0$, or

$$\dot{V}^i + \sum_{k,l=1}^{2} \Gamma^i_{j,k}(\gamma(t)) v^j \dot{\gamma}^k(t) = 0 \tag{239}$$

In particular, if $\Gamma^{\circ}_{\circ,\circ} = 0$ for all indices, then parallel transport is given by $\dot{V}^i = 0$, i.e. $\mathbf{V}$ is constant. Some examples of this is the flat plane and the cylinder[53]. Note that we can do a parallel transport around both of these without changing their directions.

We now develop an important tool that we will need later. The question is, given a value for $\mathbf{V}(0)$, does there exist a unique solution to equation 239? The answer is in the affirmative, and this is what we will prove now.

Proposition 136 (Existence-Uniqueness for ODEs in $\mathbb{R}^n$). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipshitz, i.e. there exists some $L \in \mathbb{R}_{>0}$ such that

$$\|F(x) - F(y)\| \leq L \|x - y\| \qquad \forall x, y \in \mathbb{R}^n \tag{241}$$

Then the ODE

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{242}$$

has a unique, global (i.e. valid for all t) solution, i.e. there exists $x \in C^1(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(0) = x_0, \ \dot{x}(t) = F(x(t)) \tag{243}$$

We note here that the F in equation 239 is

$$F = -\sum_{j,k=1}^{2} \begin{pmatrix} \Gamma^1_{j,k}(\gamma(t)) v^j \dot{\gamma}_k \\ \Gamma^2_{j,k}(\gamma(t)) v^j \dot{\gamma}_k \end{pmatrix} \tag{244}$$

which has time dependence, not just $\mathbf{V}$. We call such a system a nonautonomous system. In such cases, the trick to make the system autonomous is to take

$$F(u, \mathbf{V}) \tag{245}$$

where u is a new variable such that $\frac{du}{dt} = 1$.

PROOF 137. There are two main ideas to the proof: Picard iteration and time-stepping. We also make very heavy use of the ML-lemma.

[53] As an aside, note that these two surfaces are locally isometric, i.e. there exists a change of coordinates map that preserves the first fundamental form. A cylinder is give by

$$\mathbf{x}(u, v) = (R\cos(u/R), R\sin(u/R), v) \tag{240}$$

whose first fundamental form is the identity matrix. The first fundamental form for the plane is also the identity matrix. Therefore, in fact, we can just take the local isometry to be the trivial map.

Let $Y := C^0([0,T],\mathbb{R}^n)$[54] for some $T > 0$. By fundamental theorem of calculus, we can define $A : Y \to Y$ such that

$$Ax(t) := x_0 + \int_0^t F(x(s))ds \tag{246}$$

Then the original system is just $Ax = x$, so we need to find the fixed point of A in Y. Now, take the norm

$$d(x,\tilde{x}) := \max_{t\in[0,T]} \|x(t) - \tilde{x}(t)\| \tag{247}$$

on Y. Then

$$\begin{aligned} d(Ax, A\tilde{x}) &= \max_{t\in[0,T]} \left\| \int_0^t F(x(s)) - F(\tilde{x}(s))ds \right\| \\ &= \max_{t\in[0,T]} \int_0^t \|F(x(s)) - F(\tilde{x}(s))\| \, ds \\ &\leq \max_{t\in[0,T]} \int_0^t L \|x - y\| \, ds \\ &\leq LT \|x - y\| \end{aligned}$$

So, A is a contraction if we choose $T < 1$. By contraction mapping principle, there exists a unique fixed point of A inside Y. This gives existence and uniqueness on the interval $[0,T]$.

To get a global solution, we use a techinique called "time-stepping." Let x_1 be a solution for $\left[0, \frac{4}{3L}\right]$. Then there exists some solution x_2 on $[\frac{1}{2L}, \frac{1}{L}]$ such that $x_1\left(\frac{1}{2L}\right) = x_2\left(\frac{1}{2L}\right)$. But by uniqueness of solution on intervals of length T, the piecewise defined solution

$$x = \begin{cases} x_1 & t \in \left[0, \frac{4}{3L}\right]] \\ x_2 & t \in [\frac{1}{2L}, \frac{1}{L}] \end{cases} \tag{248}$$

must be the solution on $[0, \frac{1}{L}]$. We can continues this inductively to construct a unique solution for all time. Thus, there exists a unique, global solution.

□

Here is a variant of the above theorem.

Proposition 138. Let (Y, d) be complete metric space, and let $A : Y \to Y$ be continuous. Additionally, let there be some $N \geq 1$ such that A^N is a contraction. Then A has a unique fixed point.

PROOF 139. By the contraction mapping theorem, A^N has a unique fixed point y_0. So,

$$A^N y_0 = y_0 \tag{249}$$

But now,

$$AA^N y_0 = A^N(Ay_0) = Ay_0 \tag{250}$$

so, Ay_0 is also a fixed point of A. But by uniqueness of fixed points, $Ay_0 = y_0$.

□

Here is an alternative proof to the existence uniqueness theorem using the above.

[54] Note that we only require C^0 because A takes C^0 to C^1, so the fixed point must be in C^1 anyways. We might as well choose the weaker condition.

PROOF 140. We observe that

$$\begin{aligned}
\|Ax - A\tilde{x}\|_T &\leq LT\,\|x-\tilde{x}\|_T \qquad \text{for any } T \\
\left\|A^2x - A^2\tilde{x}\right\|_T &\leq \int_0^T L\,\|Ax(s) - A\tilde{x}(s)\|\,ds \\
&\leq \int_0^T L^2 s\,\|Ax(s) - A\tilde{x}(s)\|\,ds \\
&= \frac{1}{2}L^2T^2\,\|x-\tilde{x}\|_T
\end{aligned}$$

where in the third line, instead of using the Lipshitz bound twice, we just used

$$\begin{aligned}
\|Ax(s) - A\tilde{x}(s)\| &= \left\|\int_0^s x(v) - \tilde{x}(v)dv\right\| \\
&\leq \int_0^s \|x(v) - \tilde{x}(v)\|\,dv \\
&\leq s\,\|x(v) - \tilde{x}(v)\|_T
\end{aligned}$$

By induction, we then get the bound

$$\|A^n x - A^n\tilde{x}\|_T \leq \frac{1}{n!}(LT)^n\,\|x-\tilde{x}\|_T \tag{251}$$

Note that since $\frac{1}{n!}(LT)^n$ is the nth term of the Taylor expansion of e^{LT}, it must be less than 1 for some large n. Therefore, for some large n, A^n is a contraction mapping, and so, by the previous proposition, A has a unique fixed point on $[0,T]$.

We can also prove uniqueness directly. If x, y are two different fixed points, then

$$\begin{aligned}
\|x-y\| &\leq \int_0^t \|Ax(s) - Ay(s)\|\,ds \\
&\leq Lt\,\|x(s) - y(s)\|_T
\end{aligned}$$

Therefore, if $\|x(t) - y(t)\| > 0$ for some t, then,

$\square$

Example 141. The following example shows that Lipshitz is necessary in order to get a *global* solution. For the system

$$\begin{cases} \dot{x}(t) = x^2(t) \\ x(0) = 1 \end{cases} \tag{252}$$

the function $F(x) = x^2$ is not globally Lipshitz since

$$|F(x) - F(\tilde{x})| = |x-\tilde{x}|\,|x+\tilde{x}| \tag{253}$$

Since the ODE is separable, we find the solution

$$x(t) = \frac{1}{1-t} \tag{254}$$

which blows up after finite time. The proof fails for this since T blows up (since the "Lipshitz constant $|x-\tilde{x}|$" is unbounded.)

What if the function F has a restricted domain, and the leaf must flow in a confined open region in $\mathbb{R}^n$? We do still have a solution in this case.

Proposition 142. Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. Let $x_0 \in U$, then the ODE

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0 \end{cases} \tag{255}$$

has a local solution, i.e. there exists $0 < T \leq \infty$ and a smooth function x on $[0, T]$ which satisfies the ODE.

PROOF 143. Let $r > 0$ such that $B(x_0, 2r) \subseteq U$. Then

$$\Phi(x) := \begin{cases} 1 & \|x - x_0\| \leq r \\ 0 & \|x - x_0\| \geq 2r \end{cases} \tag{256}$$

Then take $\tilde{F} := \Phi(x)F(x)$ which is a function defined on all of $\mathbb{R}^n$. But since F is smooth on the compact ball $B(x_0, r)$, $\tilde{F}$ is Lipshitz on the ball, and since $\tilde{F}$ vanishes elsewhere, $\tilde{F}$ is globally Lipshitz on $\mathbb{R}^n$.

Now, we can do the exact same argument as before[55], namely we can write

$$Ax(t) := x_0 + \int_0^t \tilde{F}(x(s))ds, \ t > 0 \tag{257}$$

and we can show that this is a contraction map.

We also need to find the bound for t, since we know we might not have a global solution. By this, we want some T such that $\|x(t) - x_0\| \leq r$ for all $t \in [0, T]$. But observe that from our integral equation above,

$$\begin{aligned} \|x - x_0\| &= \left\| \int_0^t \tilde{F}(x(s))ds \right\| \\ &\leq \int_0^t M ds = Mt \end{aligned}$$

where $M := \max_{y \in \mathbb{R}^n} \tilde{F}(y)$[56]. So we just take $T \leq \frac{r}{M}$.

□

[55] Notice we obviously cannot blindly apply existence uniqueness here since it is very possible that the solution is not global; the leaf could just wander out of our little region, and not come back. Or alternatively, we can just apply the existence uniqueness *as an ODE on* $\mathbb{R}^n$ (rather than in U), and then observe how much time it takes for the particle to flow out of U.

[56] Note that we used the compact support of M here.

Week 7.

Method of Moving Frames. - Tuesday May, 8. 2018.

The purpose of this lecture is to develop the machinery of moving frames. In particular, we would like to develop the machinery on a general n-dimensional which we will specialize to surfaces in the next lecture. This discussion will build the framework for the proof of Gauss-Bonnet as in the later two chapeters of DoCarmo, *Differential Forms and Applications*.

We start by proving proposition 2 on page 49 of DoCarmo. Unlike DoCarmo, we will *derive* the result, rather than stating it and verifying its validity.

Proposition 144. Let ω be a 1-form on an n-dimensional manifold $\mathcal{M}$, and let X, Y be smooth vector fields. Then

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \tag{258}$$

where $[\cdot, \cdot]$ denotes Lie brackets, i.e.

$$[X, Y] := X(Y^i) - Y(X^i) \tag{259}$$

Remark 145. Note that we did not have to specify where the 1-form ω lives; since differential forms commute with respect to pullbacks, the above statement is invariant under pullbacks. Equivalently, the above statement is coordinate free, i.e. does not make reference to the parametrization.

As we remarked before, we will derive the formula. The hardest part of this is probably to remember how to compute using vector fields. Thus, we should develop a few basic lemmas.

Proposition 146. For a vector field X, Y on $\mathcal{M}^n$, smooth maps $f, g : \mathcal{M} \to \mathbb{R}$, and 1-form ω on $\mathcal{M}$,

$$df(X) = X(f) \tag{260}$$

and (**product rule**)

$$X(fg) = X(f)g + fX(g) \tag{261}$$

In other words, the first proposition says that *a vector field X (as a derivation) acts on a function as a partial derivative*, i.e.

$$X(f)(x) = df(x)(X(x)) = \frac{d}{dt}\bigg|_{t=0} f(x + tX(x)) \tag{262}$$

PROOF 147. Both identities follow trivially from definition, but let's go through them carefully. For the first, first note that $df(X) : \mathcal{M}^n \to \mathbb{R}$ since df is a 1-form, and $X(p) \in T_p\mathcal{M}$ for $p \in \mathcal{M}$. So,

$$\begin{aligned} df(X) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i(X) \\ &= \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} \\ &=: X(f) \end{aligned}$$

For the second identity, just

$$\begin{aligned} X(fg) &= \sum_{i=1}^{n} X_i \partial_i (fg) \\ &= \sum_{i=1}^{n} X_i (f\partial_i g + g\partial_i f) \\ &= fX(g) + gX(f) \end{aligned}$$

□

PROOF 148 (Proof: Deriving the DoCarmo proposition.). We will use the summation convention. Take

$$\omega = \omega_i dx^i, X = X^j \partial_j, Y = Y^k \partial_k \tag{263}$$

Then

$$\begin{aligned} d\omega(X,Y) &= (d\omega_i \wedge dx^i)(X,Y) \\ &= d\omega_i(X)dx^i(Y) - d\omega_i(Y)dx^i(X) \\ &= d\omega_i(X)Y^i - d\omega_i(Y)X^i \\ &= X(\omega_i)Y^i - Y(\omega_i)X^i \\ &= X(\omega_i Y^i) - \omega_i X(Y^i) - Y(\omega_i X^i) + \omega_i Y(X^i) \\ &= X(\omega_i Y^i) - Y(\omega_i X^i) - \omega_i(X(Y^i) - Y(X^i)) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega_i(X(Y^i) - Y(X^i)) \end{aligned}$$

Notice that

$$[X,Y] = (X(Y^j) - Y(X^j))\partial_j$$

and so,

$$\begin{aligned} \omega([X,Y]) &= (X(Y^j) - Y(X^j))\omega(\partial_j) \\ &= (X(Y^j) - Y(X^j))\omega_j \end{aligned}$$

and so,

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \tag{264}$$

□

Definition 149. Let $U \subset \mathbb{R}^n$ be an open set. The collection $\{\mathbf{e}_1(x), ..., \mathbf{e}_n(x)\}$ of smooth vector fields on U be a **moving frame on** U if $\{\mathbf{e}_1(x), ..., \mathbf{e}_n(x)\}$ is an orthonormal basis of $\mathbb{R}^n$ for each $x \in U$.

The collection of 1-forms (on U) $\{\omega_1(x), ..., \omega_n(x)\}$ is said to be the **dual frame of** $\{\mathbf{e}_1(x), ..., \mathbf{e}_n(x)\}$ if $\omega_j(\mathbf{e}_k) = \delta_{j,k}$.[57]

[57] We suppress the argument x for the sake of convenience.

Notice that the moving frames take points $x \in U \subset \mathbb{R}^n$ *downstairs* whereas the dual frames (since they eat the moving frames) eat vectors upstairs.

We will now derive the first and second structure equations. They are the differential equations satisfies by the dual forms.

Definition 150. For a moving frame $\{\mathbf{e}_1(x), ..., \mathbf{e}_n(x)\}$, the **connection forms** $\omega_{j,k}$ are 1-forms on $U \subset \mathbb{R}^n$ such that

$$D\mathbf{e}_j = \sum_{k=1}^{n} \omega_{j,k}\mathbf{e}_k \tag{265}$$

where $\mathbf{e}_k$ is a basis element in the moving frame, and D is the total derivative.

DoCarmo denotes the total derivative by d. We need to note that this is different from an exterior derivative, and we must be careful of this distinction.

Another point we must remember is that $\omega_{j,k}$ are 1-forms, rather than smooth functions. This is not the best notation, but we must learn to use this since this is the convention.

Notice that $\mathbf{e}_j : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, so $D\mathbf{e}_j$ is a total derivative (an $n \times n$ matrix), and thus $(D\mathbf{e}_j)(x)\mathbf{v}$ is simply the directional derivative of $\mathbf{e}_j$ in the direction $\mathbf{v}$ at x. Also notice that we are expanding with respect to the frame rather than the standard basis. We will continue to do this for the following derivations.

To make more sense of what the above definition means, take $x \in U$ and $\mathbf{v} \in \mathbb{R}^n$. Then

$$(D\mathbf{e}_j)(x)\mathbf{v} = \sum_{j=1}^{n}(\omega_{jk}(x)\mathbf{v})\mathbf{e}_k(x) \tag{266}$$

which is just the directional derivative of e_j at x in the direction of $\mathbf{v}$. So, we are merely writing the vector $(De_j)(x)\mathbf{v}$ in the basis $\mathbf{e}_i(x)$.

Proposition 151. The connection forms are antisymmetric in their indices, i.e.

$$\omega_{jk} = -\omega_{kj} \tag{267}$$

PROOF 152. By definition, $(\mathbf{e}_i \cdot \mathbf{e}_j) : U \subset \mathbb{R}^n \to \mathbb{R}$ satisfies

$$(\mathbf{e}_i \cdot \mathbf{e}_j)(x) = \delta_{ij} \tag{268}$$

for each $x \in U$. Therefore,

$$D(\mathbf{e}_i \cdot \mathbf{e}_j)(x)\mathbf{v} = 0 \tag{269}$$

and so, by product rule,

$$
\begin{aligned}
D\mathbf{e}_i(x)\mathbf{v}\cdot\mathbf{e}_j(x) + D\mathbf{e}_j(x)\mathbf{v}\cdot\mathbf{e}_i(x) &= 0 && (270)\\
\left(\sum_{k=1}^{n}(\omega_{ik}(x)\mathbf{v})\mathbf{e}_k(x)\right)\cdot\mathbf{e}_j(x) + \left(\sum_{l=1}^{n}(\omega_{jl}(x)\mathbf{v})\mathbf{e}_l(x)\right)\cdot\mathbf{e}_i(x) &= 0 && (271)\\
\left(\sum_{k=1}^{n}(\omega_{ik}(x)\mathbf{v})\delta_{k,j}\right) + \left(\sum_{l=1}^{n}(\omega_{jl}(x)\mathbf{v})\delta_{l,i}\right) &= 0 && (272)\\
(\omega_{ij}(x) + \omega_{ji}(x))\mathbf{v} &= 0 && (273)\\
& && (274)
\end{aligned}
$$

and so,

$$
\omega_{i,j} = -\omega_{j,i} \tag{275}
$$

as desired.

□

We would like to find a means of computing with these dual forms and connection forms. In fact, this is what Cartan's structure equations are good for.

Proposition 153 (First Structure Equation.). For the dual form ω_i and connection form $\omega_{i,l}$ on $U \subset \mathbb{R}^n$,

$$
d\omega_i = \sum_{l=1}^{n} \omega_{i,l} \wedge \omega_l \tag{276}
$$

for $1 \le i \le n$.

PROOF 154. Since we are dealing with the 2-form $d\omega$ eating two vector fields, we must use the proposition from DoCarmo.

First by definition,

$$
\omega_i(\mathbf{e}_j) = \delta_{i,j} \tag{277}
$$

and so, differentiating with respect to $\mathbf{e}_k, \mathbf{e}_j$ (in the sense of vector fields),

$$
\mathbf{e}_k(\omega_i(\mathbf{e}_j)) = 0,\ \mathbf{e}_j(\omega_i(\mathbf{e}_k)) = 0 \tag{278}
$$

Therefore, from equation 258 in the proposition of DoCarmo,

$$
\begin{aligned}
d\omega_i(\mathbf{e}_j, \mathbf{e}_k) &= \mathbf{e}_j(\omega_i(\mathbf{e}_k)) + \mathbf{e}_k(\omega_i(\mathbf{e}_j)) - \omega_i([\mathbf{e}_j, \mathbf{e}_k])\\
&= -\omega_i([\mathbf{e}_j, \mathbf{e}_k])
\end{aligned}
$$

Note that this is not zero since $\mathbf{e}_j, \mathbf{e}_k$ are not coordinate vector fields, but rather moving frames.[58]

[58] As an aside, we mention here that Lie brackets of vector fields vanish iff the vector fields commute.

Let ϕ_t, ψ_s are paths from the origin O. Then the vector fields giving ϕ_t, ψ_s "commute" if taking ϕ_t and then ψ_s results in the same place as taking the other order. When we say a vector fields "gives" a path, we mean the path is the solution to the ODE given by the vector field.

Notice that the vanishing Lie brackets of coordinate vector fields becomes more intuitive in this point of view. It does not matter what order of path we take if we are walking on a grid; we end up at the same point even if we change the order of the path.

Now,

$$\begin{aligned}
[\mathbf{e}_j, \mathbf{e}_k](x) &= \mathbf{e}_j(\mathbf{e}_k)(x) - \mathbf{e}_k(\mathbf{e}_j)(x) \\
&= (D\mathbf{e}_k)(x)\mathbf{e}_j - (D\mathbf{e}_j)(x)\mathbf{e}_k \\
&= \sum_{l=1}^{n} \omega_{kl}(x)(\mathbf{e}_j)\mathbf{e}_l - \sum_{l=1}^{n} \omega_{jl}(x)(\mathbf{e}_k)\mathbf{e}_l \\
&= \sum_{l=1}^{n} \omega_{kl}(x)(\mathbf{e}_j)\mathbf{e}_l - \sum_{l=1}^{n} \omega_{jl}(x)(\mathbf{e}_k)\mathbf{e}_l
\end{aligned}$$

Note that the second step follows from the action of a vector field on a function, i.e. as a directional derivative. In other words, $\mathbf{e}_j(\mathbf{e}_k)(x)$ is the directional derivative of the vector function $\mathbf{e}_k$ at x in the direction of $\mathbf{e}_j(x)$, or $(D\mathbf{e}_k)(x)\mathbf{e}_j(x)$.

Therefore,

$$\begin{aligned}
\omega_i([\mathbf{e}_j, \mathbf{e}_k]) &= \omega_i\left(\sum_{l=1}^{n}(\omega_{kl}(x)(\mathbf{e}_j))\mathbf{e}_l - \sum_{l=1}^{n} \omega_{jl}(x)(\mathbf{e}_k)\mathbf{e}_l\right) \\
&= \omega_{ki}(x)(\mathbf{e}_j) - \omega_{ji}(x)(\mathbf{e}_k)
\end{aligned}$$

Thus,

$$d\omega_i(\mathbf{e}_j, \mathbf{e}_k) = -\omega_{ki}(x)\mathbf{e}_j + \omega_{ji}(x)\mathbf{e}_k \tag{279}$$

We now want to eliminate $\mathbf{e}_j, \mathbf{e}_k$ from the equation. Observe that

$$\begin{aligned}
-\omega_{ki}(x)(\mathbf{e}_j) + \omega_{ji}(x)(\mathbf{e}_k) &= \omega_{ik}(x)(\mathbf{e}_j) - \omega_{ij}(x)(\mathbf{e}_k) \\
&= \sum_{l=1}^{n} \omega_{il}(x)(\mathbf{e}_j)\omega_k(\mathbf{e}_j) - \omega_{il}(x)(\mathbf{e}_k)\omega_l(\mathbf{e}_j) \\
&= \left(\sum_{l=1}^{n} \omega_{il} \wedge \omega_k\right)(\mathbf{e}_j, \mathbf{e}_k)
\end{aligned}$$

Therefore,

$$d\omega_i = \sum_{l=1}^{n} \omega_{il} \wedge \omega_k \tag{280}$$

□

Proposition 155 (Second structure equation.). Let $\omega_{j,k}$ be a connection form. Then

$$d\omega_{j,k} = \sum_{l=1}^{n} \omega_{j,l} \wedge \omega_{l,k} \tag{281}$$

PROOF 156. By how we defined $\omega_{i,j}$ the only approach we could possibly take is taking the second derivative of $\mathbf{e}_j$.

For the standard vector field ∂_r, ∂_s,

$$\begin{aligned} D^2\mathbf{e}_j(\partial_r, \partial_s) &= \partial_s \left(D\mathbf{e}_j(\partial_r) \right) \\ &= \partial_s \left(\sum_{k=1}^{n} \omega_{j,k}(\partial_r)\mathbf{e}_k \right) \end{aligned}$$

Note that here again we used the fact that vector fields act on functions as directional derivative, in this case the vector field ∂_s (corresponding to directional derivative in the x_s direction)
But since second derivatives are symmetric, this is also equal to

$$\partial_r \left(\sum_{k=1}^{n} \omega_{j,k}(\partial_s)\mathbf{e}_k \right) \tag{282}$$

Therefore by product rule,

$$\begin{aligned} 0 = &\sum_{k=1}^{n} \left(\partial_s \left(\omega_{j,k}(\partial_r) \right) - \partial_r \left(\omega_{j,k}(\partial_s) \right) \right) \mathbf{e}_k \\ &+ \sum_{k=1}^{n} \omega_{j,k}(\partial_r)\partial_s\mathbf{e}_k - \omega_{j,k}(\partial_s)\partial_r\mathbf{e}_k \end{aligned}$$

By the definition of exterior derivative, the first two terms gives

$$d\omega_{j,k}(\partial_s, \partial_r) = \partial_s \left(\omega_{j,k}(\partial_r) \right) - \partial_r \left(\omega_{j,k}(\partial_s) \right) \tag{283}$$

Now the last two terms can be written as

$$\sum_{k=1}^{n} \omega_{j,k}(\partial_r)D\mathbf{e}_k(\mathbf{e}_s) - \omega_{j,k}(\partial_s)D\mathbf{e}_k(\mathbf{e}_r)$$

since $\partial_s\mathbf{e}_k, \partial_r\mathbf{e}_k$ are directional derivatives. We can then relabel these for convenience:

$$\sum_{l=1}^{n} \omega_{j,l}(\partial_r)D\mathbf{e}_l(\mathbf{e}_s) - \omega_{j,l}(\partial_s)D\mathbf{e}_l(\mathbf{e}_r) \tag{284}$$

So, by definition of connection forms,

$$\sum_{l=1}^{n} \omega_{j,l}(\partial_r) \sum_{k=1}^{n} \omega_{l,k}(\partial_s)\mathbf{e}_k - \omega_{j,l}(\partial_s) \sum_{k=1}^{n} \omega_{l,k}(\partial_r)\mathbf{e}_k \tag{285}$$

Thus, we have

$$\sum_{k=1}^{n}\sum_{l=1}^{n} (\omega_{j,l}(\partial_r)\omega_{l,k}(\partial_s) - \omega_{j,l}(\partial_s)\omega_{l,k}(\partial_r))\mathbf{e}_k \tag{286}$$

which is just

$$-\sum_{k=1}^{n}\sum_{l=1}^{n} (\omega_{j,l} \wedge \omega_{l,k})(\partial_s, \partial_r)\mathbf{e}_k \tag{287}$$

So putting together the summands of $\sum_k$, with what we had for $d\omega_{j,k}$,

$$0 = d\omega_{i,j}(\partial_s, \partial_r) - \sum_{l=1}^{n} (\omega_{j,l} \wedge \omega_{l,k})(\partial_s, \partial_r) \tag{288}$$

which gives

$$d\omega_{i,j}(\partial_s, \partial_r) = \sum_{l=1}^{n} (\omega_{j,l} \wedge \omega_{l,k})(\partial_s, \partial_r) \tag{289}$$

as desired.

□

We now stop for a moment and prove the two Cartan lemmas. These are linear algebra facts (as opposed to facts about forms).

Proposition 157 (First Cartan Lemma.). Let V be a finite dimensional real vector space ($\dim V = n$), and let $\omega_1, ..., \omega_n \in V^*$ be linearly independent. Suppose that $\theta_1, ..., \theta_r \in V^*$ such that

$$\sum_{i=1}^{n} \theta_i \wedge \omega_i = 0 \tag{290}$$

Then there exists unique $a_{i,j}$

$$\theta_i = \sum_{j=1}^{n} a_{i,j}\omega_j \tag{291}$$

where $a_{i,j} = a_{j,i}$ and $1 \leq i \leq r$.

PROOF 158. Expand $\omega_1, ..., \omega_r$ to a basis

$$\omega_1, ..., \omega_r, ..., \omega_n \tag{292}$$

in V^*. Then there exists unique $a_{i,j}, b_{i,k}$ such that

$$\theta_i = \sum_{j=1}^{r} a_{i,j}\omega_j + \sum_{k=r+1}^{n} b_{i,k}\omega_k \tag{293}$$

Then by hypothesis,

$$\begin{aligned}
0 &= \sum_{i=1}^{n} \theta_i \wedge \omega_i \\
&= \sum_{i,j=1}^{r} a_{i,j}\omega_j \wedge \omega_i + \sum_{k=r+1}^{n} b_{i,k}\omega_k \wedge \omega_i \\
&= \sum_{1 \leq i < j \leq r} (a_{i,j} - a_{j,i})\omega_j \wedge \omega_i + \sum_{k=r+1}^{n} b_{i,k}\omega_k \wedge \omega_i
\end{aligned}$$

But now, since $\omega_j \wedge \omega_i$ are linearly independent, $a_{i,j} - a_{j,i} = b_{i,k} = 0$, as desired.

□

Proposition 159 (Second Cartan lemma.). Suppose the 1-forms $\omega_1, ..., \omega_n$ on $U \subset \mathbb{R}^n$ are linearly independent. Suppose that for a different collection of 1-forms $\{\omega_{i,j}\}_{i,j=1}^n$,

$$d\omega_i = \sum_{k=1}^n \omega_{i,k} \wedge \omega_k \tag{294}$$

and $\omega_{i,k} = -\omega_{k,i}$ for each i. Then the $\omega_{i,j}$ are unique.

Note that if we apply this to our moving frames, this just says that the connection forms are unique.

PROOF 160. Suppose the assertion is false. Then there exists a second collection $\{\tilde{\omega}_{i,j}\}_{i,j=1}^n$. Then

$$\sum_{k=1}^n (\omega_{i,k} - \tilde{\omega}_{i,k}) \wedge \omega_k = 0 \tag{295}$$

by hypothesis. By the first Cartan lemma, taking $\theta_{i,k} := \omega_{i,k} - \tilde{\omega}_{i,k}$, there exists unique coefficients $B^l_{i,k}$ such that

$$\theta_k = \sum_{l=1}^n B^l_{i,k} \omega_l \tag{296}$$

where $B^l_{i,k} = B^i_{l,k} = B^k_{i,l}$. Furthermore, by hypothesis, $B^l_{i,k} = -B^l_{k,i}$.

Our claim is that $B^l_{i,k} = 0$. But we can obtain this by permuting the indices:

$$\begin{aligned} B^k_{i,j} &= B^j_{i,k} = -B^j_{k,i} \\ &= -B^i_{k,j} = B^i_{j,k} \\ &= B^k_{j,i} = -B^k_{i,j} \end{aligned}$$

So, $B^k_{i,j} = 0$.

$\square$

Notice that this new machinery of moving frames is irreplaceable with what with had before. For surfaces, if we took $\mathbf{x}_u, \mathbf{x}_v$ to be orthonormal, then we would have $I_p = \mathbf{Id}$, and so, we would be on the plane, up to isometry.

Theorema Egregium Revisited. - Thursday May, 10. 2018.

In the last lecture, we developed the machinery of moving frames, proved the structure equations, and showed the Cartan lemmas. We will now use all of these machinery to provide an alternative proof to Gauss' Theorema Egregium.

For this lecture, we will restrict our attention to $n = 3$, i.e. a surface embedded in $\mathbb{R}^3$. Also, note that we will constantly abuse notation and denote a function, form, etc. by the same symbol as its pullback; this is common notation, and in some sense, it is natural because we are associating to the object upstairs to its counterpart downstairs.

Here is a brief overview of this lecture. We start from the connection and dual forms we discussed last time, and find a means of connecting this to the second fundamental form/shape operator and hence to Gaussian curvature. We then show that this expression is in fact invariant under isometry, and so the Gaussian curvatures are the same.

Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathcal{M} \subset \mathbb{R}^3$ be the parametrization. Recall that $\{\mathbf{e}_i\}_{i=1,2,3}$ is an moving orthonormal frame. We choose this to be such that $(\mathbf{e}_1, \mathbf{e}_2)(x)$ is a tangent field on $\mathcal{M}$ (and consequently, $\mathbf{e}_3$ is the normal to $\mathbf{e}_3$).

Let's motivate the computation. Recall that, the shape operator gives the directional derivative of the normal vector on which the operator is acting upon. So, by definition of the shape operator and the connection forms,

$$
\begin{aligned}
S(p)\mathbf{v} &= -D\mathbf{N}(p)\mathbf{v} \\
&= -D\mathbf{e}_3(p)\mathbf{v} \\
&= -\sum_{j=1}^{2}(\omega_{3j}(p)\mathbf{v})\mathbf{e}_j
\end{aligned}
$$

[59]

If we can find an expression for $\omega_{3j}(p)$ in terms of the dual forms, then that would make it easier to do linear algebra. Fortunately, we recall that the first Cartan lemma would give a conclusion of such form. Hence, we will do some computations so to satisfy the hypothesis of the lemma.

This is actually not terrible hard to get; first observe that

$$
\begin{aligned}
\mathbf{x}^*(d\omega_3) &= d(\mathbf{x}^*\omega_3) \\
&= (\mathbf{x}^*\omega_{31}) \wedge (\mathbf{x}^*\omega_1) + (\mathbf{x}^*\omega_{32}) \wedge (\mathbf{x}^*\omega_2)
\end{aligned}
$$

[60]

Since $x^*\omega_1$ and $x^*\omega_2$ are linearly independent functionals at each point, if we can show that $x^*(d\omega_3)$ is 0, then the hypothesis for Cartan's first lemma will be satisfied.

However, this is immediate from the definition of ω_3; recall that ω_3 is the dual to the normal vector $\mathbf{e}_3$. Therefore,

$$
\mathbf{x}^*\omega_3(x_0)\mathbf{v} = \omega_3(\mathbf{x}(x_0))\,(D\mathbf{x}(x_0)\mathbf{v}) = 0
$$

since $D\mathbf{x}(x_0)\mathbf{v} \in T_{x_0}\mathcal{M}$.

Therefore, by Cartan's first lemma, there exists smooth functions h_{ij}, $i, j = 1, 2$ on U and $h_{ij} = h_{ji}$ so that

$$
\begin{aligned}
\omega_{13} &= h_{11}\omega_1 + h_{12}\omega_2 \\
\omega_{23} &= h_{21}\omega_1 + h_{22}\omega_2
\end{aligned}
$$

If we now go back to our expression for the shape operator, and write $\mathbf{v} = (\xi_1, \xi_2)$ in the moving frame basis $\mathbf{e}_1, \mathbf{e}_2$, then

[59] Note here that $\mathbf{N}$ does not change in the normal direction $\mathbf{e}_3$. If it did, then the length of $\mathbf{N}$ would change, and so, it would no longer be a normal vector.

[60] We will start dropping the x^* when we write pullbacks (i.e. $x^*\omega_1$ will just be written as ω_1) since it is common practice. This does not cause a huge problem, because we usually work downstains (i.e., in coordinates) anyways.

$$
\begin{aligned}
S(p)\mathbf{v} &= -\sum_{j=1}^{2}(\omega_{3j}(p)\mathbf{v})\mathbf{e}_j \\
&= ((h_{11}\omega_1 + h_{12}\omega_2)\mathbf{e}_1 + (h_{21}\omega_1 + h_{22}\omega_2)\mathbf{e}_2)\,\mathbf{v} \\
&= (h_{11}\xi_1 + h_{12}\xi_2)\mathbf{e}_1 + (h_{21}\xi_1 + h_{22}\xi_2)\mathbf{e}_2 \\
&= \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
\end{aligned}
$$

and if we recall that

$$
\mathrm{II}_p(\mathbf{v},\mathbf{v}) = \left\langle \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\mathbf{v}, \mathbf{v} \right\rangle \tag{297}
$$

then the matrix (h_{ij}) is the matrix for the second fundamental form.

Now, since the Gaussian curvature is just the determinant of this matrix, we have

$$
\begin{aligned}
d\omega_{12} &= \omega_{13} \wedge \omega_{32} \\
&= -\omega_{13} \wedge \omega_{23} \\
&= -(h_{11}\omega_1 + h_{12}\omega_2) \wedge (h_{21}\omega_1 + h_{22}\omega_2) \\
&= -(h_{11}h_{22} - h_{12}^2)\omega_1 \wedge \omega_2 \\
&= -K\omega_1 \wedge \omega_2
\end{aligned}
$$

We are now ready to prove the Theorema Egregium..

Proposition 161. Let $\mathcal{M}, \tilde{\mathcal{M}}$ be locally isometric surfaces, i.e. for each $x_0 \in U$ and $y_0 := \phi(x_0)$, there is a local isometry $\phi : U \subset \mathbb{R}^2 \to V \subset \mathbb{R}^2$ (in other words $\mathrm{I}_{x_0}(v_1, v_2) = \mathrm{I}_{y_0}(D\phi(x_0)v_1, D\phi(x_0)v_2)$). Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathcal{M} \subset \mathbb{R}^3$ and $\mathbf{y} : V \subset \mathbb{R}^2 \to \tilde{\mathcal{M}} \subset \mathbb{R}^3$ be the parametrizations. Then

$$
K = \tilde{K} \circ \phi \tag{298}
$$

where $K(x), \tilde{K}(y)$ are the Gaussian curvaures of $\mathcal{M}, \tilde{\mathcal{M}}$.

PROOF 162. First, we can choose a moving frame $\{\mathbf{e}_i\}$ on $\mathbf{x}(U)$ which are tangent fields on defined a neighborhood of $\mathbf{x}(U)$. For this, we can take the canonical basis on the tangent spaces $\mathbf{x}_u, \mathbf{x}_v$ and then by Gram-Schmidt orthonormolization,

$$
\mathbf{e}_1 := \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|},\ \mathbf{e}_2 := \alpha\mathbf{x}_u + \beta\mathbf{x}_v,\ \mathbf{e}_3 := \mathbf{e}_1 \times \mathbf{e}_2 \tag{299}
$$

where $\alpha, \beta \in \mathbb{R}$.[61]
Now let $\{\mathbf{f}_i\}$ be the basis living downstairs in U that is orthonormal with respect to the first fundamental form; in other words, $\mathbf{e}_i$ is a pushforward of $\mathbf{f}_i$:

$$
\mathbf{e}_i(p) = d\mathbf{x}(x_0)\mathbf{f}_i(x_0) \tag{300}
$$

[61] Technical Point: We need to extend the above frames to some neigborhood of $\mathbf{x}(U)$ in the ambient space $\mathbb{R}^3$. This is because we need to take derivatives in the $\mathbf{e}_3$ direction and differentials of ω_3 which both require a *neighborhood* in the ambient space. One way to extend a vector field on $\mathcal{M}$ (and in particular a frame on $\mathcal{M}$) to a neighborhood is by taking the vector field $(X_1(x,y), X_2(x,y), 0)$ on the straightened manifold, and then locally extend beyond the straightened manifold onto the ambient space using the same formula: $X(x,y,z) = X(x,y,0)$. We can then unstraighten this, and we will have our extension.

Now define $\tilde{\mathbf{f}}_i$ to be the pushforward of $\mathbf{f}_i$ with respect to the isometry ϕ, i.e.

$$\tilde{\mathbf{f}}_i(y_0) := D\phi(x_0)\mathbf{f}_i(x_0) \tag{301}$$

Notice that since ϕ is an isometry, $\tilde{f}_i$ are orthonormal with respect to the first fundamental form on $\tilde{\mathcal{M}}$:

$$\mathrm{I}_{y_0}(\tilde{\mathbf{f}}_i(y_0), \tilde{\mathbf{f}}_i(y_0)) = \mathrm{I}_{y_0}(D\phi(x_0)\mathbf{f}_i(x_0), D\phi(x_0)\mathbf{f}_i(x_0)) = \mathrm{I}_{x_0}(\mathbf{f}_i(x_0), \mathbf{f}_i(x_0)) \tag{302}$$

From there, we can define the moving frames on $\tilde{\mathcal{M}}$ to be the pushforwards of $\tilde{\mathbf{f}}_i$:

$$\tilde{\mathbf{e}}_i := D\mathbf{y}(y_0)\tilde{\mathbf{f}}_i \tag{303}$$

We then claim that the dual frame on $\mathcal{M}$ is just the pullback of the dual frame on $\tilde{\mathcal{M}}$ with respect to the isometry[62]:

$$\omega_i = \phi^*\tilde{\omega}_i \tag{304}$$

This follows from unraveling definitions. Observe that

$$\begin{aligned}
\phi^*\tilde{\omega}_i(x_0)(\mathbf{f}_j) &= \tilde{\omega}_i(\phi(x_0))(D\phi(x_0)\mathbf{f}_j(x_0)) \\
&= \tilde{\omega}_i(y_0)(\tilde{\mathbf{f}}_j(y_0)) \\
&= \delta_{ij} = \omega_i(x_0)(\mathbf{f}_j)
\end{aligned}$$

Of course, we also want the pullbacks of connection forms to also be connection forms:

$$\omega_{12} = \phi^*\tilde{\omega}_{12} \tag{305}$$

One the one hand, if we take the dual frame on $\mathcal{M}$ and pull it back with respect to the parametrization, by the first structure equation,

$$\begin{aligned}
x^*(d\omega_1) &= x^*(\omega_{12} \wedge \omega_2 + \omega_{13} \wedge \omega_3) \\
&= x^*\omega_{12} \wedge x^*\omega_2
\end{aligned}$$

where we used $\mathbf{x}^*(\omega_3) = 0$ again. We right the above (in our usual abuse of notation) as

$$d\omega_1 = \omega_{12} \wedge \omega_2 \tag{306}$$

On the other hand, pulling back the dual frames on $\tilde{\mathcal{M}}$ with respect to the parametrization and then by the local isometry,

$$\begin{aligned}
\phi^*(d\tilde{\omega}_1) &= \phi^*(\tilde{\omega}_{12} \wedge \tilde{\omega}_2 + \tilde{\omega}_{13} \wedge \tilde{\omega}_3) \\
&= \phi^*(\tilde{\omega}_{12} \wedge \tilde{\omega}_2) \\
&= \phi^*\tilde{\omega}_{12} \wedge \phi^*\tilde{\omega}_2 \\
&= \phi^*\tilde{\omega}_{12} \wedge \phi^*\tilde{\omega}_2 \\
&= \phi^*\tilde{\omega}_{12} \wedge \phi^*\tilde{\omega}_2
\end{aligned}$$

But now, from what we showed before, $d\omega_1 = \phi^*(d\tilde{\omega}_1)$. So we get

$$\omega_{12} \wedge \omega_2 = \phi^*\tilde{\omega}_{12} \wedge \phi^*\tilde{\omega}_2 \tag{307}$$

[62] Note that this does not make sense unless these "dual frames" are pullbacks of dual frame upstairs pulled back downstairs via the parametrizations.

But since connection forms are unique (second Cartan lemma), indeed we have $\omega_{12} = \phi^*\tilde{\omega}_{12}$.

Now that we have developed the machinery, the rest is straightforward. Take our relation we had for K, and pulling it back with respect to the isometry, we get

$$\begin{aligned} d\tilde{\omega}_{12} &= -\tilde{K}\tilde{\omega}_1 \wedge \tilde{\omega}_2 \\ \phi^* d\tilde{\omega}_{12} &= -\phi^*(\tilde{K}\tilde{\omega}_1 \wedge \tilde{\omega}_2 \\ d\phi^*\tilde{\omega}_{12} &= -(\tilde{K}\circ\phi)\phi^*\tilde{\omega}_1 \wedge \phi^*\tilde{\omega}_2 \\ d\omega_{12} &= -(\tilde{K}\circ\phi)\omega_1 \wedge \omega_2 \end{aligned}$$

Therefore,

$$(K - \tilde{K}\circ\phi)\omega_1 \wedge \omega_2 = 0 \tag{308}$$

But since the area form $\omega_1 \wedge \omega_2 \neq 0$, we thus have $K = \tilde{K}\circ\phi$, as desired.

□

Example 163 (Surface of revolution.). Let's now try using this method of moving frames to compute the Gaussian curvature on the surface of revolution. The parametrization is

$$\mathbf{x}(s,v) = (r(s)\cos v, g(s), r(s)\sin v) \tag{309}$$

where $r(s)$ is the radius and s is the arclength along the generator. Since $\mathbf{x}(s,v_0)$ is an arclength parametrization for fixed v_0,

$$\begin{aligned} L &= \int_0^L \|\mathbf{x}(s,v_0)\|\, ds \\ 1 &= \|\mathbf{x}(s,v_0)\| = \sqrt{\dot{r}^2(s) + \dot{g}^2(s)} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{x}_s &= (\dot{r}(s)\cos v, \dot{g}(s), \dot{r}(s)\sin v) \\ \mathbf{x}_v &= (-r(s)\sin v, 0, r(s)\cos v) \end{aligned}$$

and so $|\mathbf{x}_s| = 1, |\mathbf{x}_v| = r(s), \mathbf{x}_s\cdot\mathbf{x}_v = 0$. Thus, we take

$$\mathbf{e}_1 := \mathbf{x}_s,\ \mathbf{e}_2 := \frac{\mathbf{x}_v}{r(s)} \tag{310}$$

Thus,

$$\mathbf{f}_1 = \frac{\partial}{\partial s},\ \mathbf{f}_2 = \frac{1}{r(s)}\frac{\partial}{\partial v} \tag{311}$$

and

$$\omega_1 = ds,\ \omega_2 = rdv,\ \omega_1 \wedge \omega_2 = rds \wedge dv \tag{312}$$

And so,

$$d\omega_1 = 0,\ d\omega_2 = \dot{r}ds \wedge dv \tag{313}$$

Now taking $a, b \in \mathbb{R}$ such that

$$\omega_{12} = a\omega_1 + b\omega_2 \tag{314}$$

On the other hand, since $d\omega_1 = 0$, by the first structure equation,

$$\omega_{12} \wedge \omega_2 = 0 \tag{315}$$

So, $a = 0$, and so $\omega_{12} = b\omega_2$. It then follows that

$$\begin{aligned} d\omega_2 &= -\omega_{12} \wedge \omega_1 \\ &= -b\omega_2 \wedge \omega_1 \end{aligned}$$

Comparing coefficients from what we had before, we thus have $b = \frac{\dot{r}}{r}$. But now,

$$\omega_{12} = \frac{\dot{r}}{r}\omega_2 = \dot{r}dv \tag{316}$$

And so finally,

$$d\omega_{12} = \ddot{r}ds \wedge dv$$

which is equal to

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 = -Krds \wedge dv \tag{317}$$

So, we must have $K = -\frac{\ddot{r}}{r}$.

In summary, the idea is to compute $d\omega_{12}$ in two different ways. One is to write ω_{12} as a linear combination of ω_1, ω_2. The other is to compute what $\omega_1 \wedge \omega_2$ is. To get to this branching point, we must compute all of the basic quantities, e.g. the moving frame, the frame downstairs, and the dual frames.

Week 8.

Gauge Invariance of Gaussian Curvature and Geodesic Curvature. - Tuesday May, 15. 2018.

In the last two lectures, we have discussed the Gaussian curvature from the point of view of moving frames. In particular, we proved the Theorema Egregium which states that the Gaussian curvature is invariant under isometry. However, we have not shown that it is independent of the choice of the frames. Since frames (by definition) form an orthonormal basis, and so, two such frames will vary by a (smoothly varying) rotation matrix. In this lecture, we will prove this *gauge invariance* of the Gaussian curvature and discuss how we can compute the angle between two different moving frames. This will ultimately lead to the notion of a geodesic curvature.[63]

Throughout this discussion, let $\mathcal{M}$ be a 2-dimensional Riemannian manifold (not necessarily with ambient space) with moving frame orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ on $\Omega \subseteq \mathcal{M}$ and metric

$$\langle v, w\rangle (x_0) := \left\langle \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle \tag{318}$$

Additionally, we require that there is a dual frame ω_1, ω_2 and the (unique) connection form ω_{12} satisfying the first structure equation and antisymmetry property (as given in problem 3 in pset 7). Also, let $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$ be another moving frame with the same orientation and let the corresponding dual frame be $\underline{\omega}_1, \underline{\omega}_2$.

Definition 164. The Gaussian curvature $K : \mathcal{M} \to \mathbb{R}$ is the function (uniquely[64]) given by

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \tag{319}$$

From problem set 7 problem 1, $\omega_1 \wedge \omega_2$ is the area form. So, the differential of the connection form is just the area form weighted by the Gaussian curvature.

Notice that the above definition (seems to) depends on our choice of ω_1, ω_2.[65] The following says that we do not have to make this specification.

Proposition 165 (Gauge Invariance of Gaussian Curvature.). Let the two moving frames, dual frames, connection forms, Riemannian metric, and the manifold $\mathcal{M}$ be as above. Let $K, \underline{K}$ be the Gaussian curvature given by ω_i and $\underline{\omega}_i$ respectively, i.e.

$$\begin{aligned} d\omega_{12} &= -K\omega_1 \wedge \omega_2 \\ d\underline{\omega}_{12} &= -\underline{K}\, \underline{\omega}_1 \wedge \underline{\omega}_2 \end{aligned}$$

Then $K = \underline{K}$.

We will prove this by examining how the dual frames and connection forms transform under the change of frames. Notice that since we chose the underlined frames to be of the same orientation as the original frames, so there exists some $f, g \in C^\infty(\Omega, \mathbb{R})$ such that

[63] As usual, we will use results from problem set. The relevant content for this lecture is problem set 7.

[64] Notice that the uniqueness is given by the fact that the 2-forms on the two dimensional space $T_p\mathcal{M}$ is a one dimensional vector space (spanned by $\omega_1 \wedge \omega_2$.)

[65] Notice that we also specified the underlined frames to be the *same orientation* as the original frames. Gaussian curvature is invariant under change of orientation, so indeed, Gaussian curvature is invariant under any choice of moving frames.

$$\underline{\mathbf{e}}_1 = f\mathbf{e}_1 + g\mathbf{e}_2$$
$$\underline{\mathbf{e}}_2 = -g\mathbf{e}_1 + f\mathbf{e}_2$$

and $|\underline{\mathbf{e}}_1| = |\underline{\mathbf{e}}_2| = f^2 + g^2 = 1$. Call this linear transformation R. This linear transformation can be written as the matrix (*with respect to the basis* $\mathbf{e}_1, \mathbf{e}_2$)

$$\begin{pmatrix} f & g \\ -g & f \end{pmatrix} \tag{320}$$

which is an orthogonal matrix on $T_p\mathcal{M}$ (as a two dimensional vector space) *with respect to the inner product (i.e. the metric)*. Additionally, it preserves orientation, so it is a special orthogonal matrix (i.e. a rotation). We can extract f, g from $\mathbf{e}_i, \underline{\mathbf{e}}_i$ by projections:

$$f = \langle \underline{\mathbf{e}}_1, \mathbf{e}_1 \rangle_p\,,\ g = \langle \underline{\mathbf{e}}_1, \mathbf{e}_2 \rangle_p \tag{321}$$

We now have the necessary data to state the proposition.

Proposition 166 (Transformation of Dual Frames and Connection Forms.). Let the two moving frames, dual frames, connection forms, functions f, g, Riemannian metric, and the manifold $\mathcal{M}$ be as above. Then

1. The dual frame transforms by the same matrix[66], i.e.

$$\underline{\omega}_1 = f\omega_1 + g\omega_2$$
$$\underline{\omega}_2 = -g\omega_1 + f\omega_2$$

   or equivalently,

$$\omega_1 = f\underline{\omega}_1 - g\underline{\omega}_2$$
$$\omega_2 = g\underline{\omega}_1 + f\underline{\omega}_2$$

2. $\underline{\omega}_{12} = \omega_{12} + (f, g)^*\omega_0$ where ω_0 is the angle form.

PROOF 167 (Proof of first statement.). We would have to *derive* the expression either from general principle (see footnote), inutition that rotating the orthonormal basis will also rotate the projection maps, or by backtracking from the following computation.

We just need to verify that the given formula is indeed a dual frame to $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$. Observe that

$$\begin{aligned}(f\omega_1 + g\omega_2)(\underline{\mathbf{e}}_1) &= (f\omega_1 + g\omega_2)(f\mathbf{e}_1 + g\mathbf{e}_2) \\ &= f^2 + g^2 = 1\end{aligned}$$

by definition of f, g. In a similar manner,

[66] In general, the dual frame transforms by the inverse transpose of the transformation on the frame, which for orthogonal matrices, gives back the original matrix.

$$\begin{aligned}(f\omega_1 + g\omega_2)(\underline{\mathbf{e}}_2) &= (f\omega_1 + g\omega_2)(-g\mathbf{e}_1 + f\mathbf{e}_2) \\ &= -gf + fg = 0\end{aligned}$$

and so, $f\omega_1 + g\omega_2$ must be the unique functional with the above property which is dual frame ω_1. By the exact same computation, we can verify the second identity.

The equivalent identity follows immediate from taking the inverse of the rotation matrix. □

PROOF 168 (Proof of Second Statement.). This is also a straightforward computation modulo some crucial observations. We will compute $d\underline{\omega}_1$ and isolate the 2-form with $\underline{\omega}_2$ and via the first structure equation deduce ω_{12}.

Observe from above that

$$\begin{aligned}d\underline{\omega}_1 &= df \wedge \omega_1 + dg \wedge \omega_2 + f d\omega_1 + g d\omega_2 \\ &= df \wedge \omega_1 + dg \wedge \omega_2 + f(\omega_{12} \wedge \omega_2) - g(\omega_{12} \wedge \omega_1) \\ &= (df - g\omega_{12}) \wedge \omega_1 + (dg + f\omega_{12}) \wedge \omega_2 \\ &= (df - g\omega_{12}) \wedge (f\underline{\omega}_1 - g\underline{\omega}_2) + (dg + f\omega_{12}) \wedge (g\underline{\omega}_1 + f\underline{\omega}_2) \\ &= (fdf - fg\omega_{12} + gdg + gf\omega_{12}) \wedge \underline{\omega}_1 + (-gdf + g^2\omega_{12} + fdg + f^2\omega_{12})\underline{\omega}_2 \\ &= \frac{1}{2}\left(d(f^2 + g^2)\right) \wedge \underline{\omega}_1 + (-gdf + fdg + \omega_{12})\underline{\omega}_2 \\ &= \left(\frac{-gdf + fdg}{f^2 + g^2} + \omega_{12}\right)\underline{\omega}_2 \\ &= ((f, g)^*\omega_0) + \omega_{12})\underline{\omega}_2\end{aligned}$$

which gives

$$\underline{\omega}_{12} = (f, g)^*\omega_0 + \omega_{12} \tag{322}$$

as desired. □

We are now ready to prove the gauge invariance.

PROOF 169 (Proof of gauge invariance of Gaussian curvature.). From the above, since ω_0 is a closed form [67]

$$\begin{aligned}d\underline{\omega}_{12} &= d((f, g)^*\omega_0) + d\omega_{12} \\ &= (f, g)^* d\omega_0 + d\omega_{12} \\ &= d\omega_{12} \\ &= -K\omega_1 \wedge \omega_2\end{aligned}$$

[67] We can show closedness of angle forms by direct computation. Observe that

$$\begin{aligned}d(f, g)^*\omega_0 &= df \wedge dg - dg \wedge df \\ &= 2df \wedge dg \\ &= 0\end{aligned}$$

where in the last step, we used the fact that f, g are *dependent* on each other, i.e. g can be written as a function of f since $f^2 + g^2 = 1$ by hypothesis.

On the other hand,

$$d\underline{\omega}_{12} = -\underline{K}\ \underline{\omega}_1 \wedge \underline{\omega}_2 \tag{323}$$

and since

$$\omega_1 \wedge \omega_2 = \underline{\omega}_1 \wedge \underline{\omega}_2 \tag{324}$$

[68]

we have $K = \underline{K}$ as desired.

□

Since we know go the angle form as a result of the above calculation, it is natural to ask: can we compute this angle? In particular, on some disk D on $\mathcal{M}$[69], By Poincaré's lemma, there is a smooth function $\phi : D(0,r) \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$(f,g)^*\omega_0 = d\phi \tag{327}$$

We would like to establish a coorespondence between ϕ and rotation given by f, g.

Proposition 170. Let $c(t)$ be a curve on Ω and $t \in [0, t_0]$, and

$$\phi(t) := \int_{c:0\to t} (f,g)^*\omega_0 + \phi_0 \tag{328}$$

where the integral is the line integral from $c(0)$ to $c(t)$ along c, and ϕ_0 is such that[70]

$$\begin{pmatrix} f & g \\ -g & f \end{pmatrix}(c(0)) = \begin{pmatrix} \cos\phi_0 & \sin\phi_0 \\ -\sin\phi_0 & \cos\phi_0 \end{pmatrix} \tag{329}$$

Then

$$f(c(t)) = \cos(\phi(t))$$
$$g(c(t)) = \sin(\phi(t))$$

for all $t \in [0, t_0]$.

PROOF 171. The main idea here is that both the vectors (f,g) and $(\cos\phi, \sin\phi)$ lie in $\mathbb{S}^1$. Therefore, if their inner product is 1 for all t, then they are equal.[71]

[68] We can show that area forms are invariant under choice of frames by a simple calculation. Since

$$\underline{\omega}_1 = f\omega_1 + g\omega_2,\ \underline{\omega}_2 = -g\omega_1 + f\omega_2 \tag{325}$$

then

$$\underline{\omega}_1 \wedge \underline{\omega}_2 = (f^2+g^2)\omega_1 \wedge \omega_2 = \omega_1 \wedge \omega_2 \tag{326}$$

Intuitively, this just means that the area of a square is the same even if we rotate the square.

[69] A disk on a manifold without ambient space is just a homeomorphic image of a disk in $\mathbb{R}^2$, i.e. $D = h(D(0,r)) \subset \Omega$ where $h : \mathbb{R}^2 \to \mathcal{M}$ is a (local) homeomorphism (or a chart), and $D(0,r) \subset \mathbb{R}^2$ is a disk.

[70] Notice that we can always write the matrix in this form since $\begin{pmatrix} f & g \\ -g & f \end{pmatrix}(c(0))$ is just an special orthogonal matrix, and so it can be written as a matrix of trigonometric functions.

[71] More quantitatively, if $\rho \equiv 1$, then

$$\begin{aligned}(f(c(t)) - \cos\phi)^2 + (g(c(t)) - \sin\phi)^2 &= f^2 - 2f\cos\phi + \cos^2\phi + g^2 - 2g\sin\phi + \sin^2\phi \\ &= 0\end{aligned}$$

and so, $f(c(t)) = \cos\phi(t), g(c(t)) = \sin\phi(t)$.

Thus, define

$$\rho(t) := a\cos\phi + b\sin\phi \tag{330}$$

where we denote $a := f(c(t)), b := g(c(t))$. We claim that for all $t \in [0, t_0]$, $\rho(t) = 1$. By hypothesis, $\rho(0) = 1$

However once we realize this, we just need to compute. Observe that

$$\dot{\rho}(t) = \dot{a}\cos\phi + \dot{b}\sin\phi - a\dot{\phi}\sin\phi + b\dot{\phi}\cos\phi$$

where by definition of ϕ and by first fundamental theorem of calculus,

$$\begin{aligned}
\dot{\phi} &= (f,g)^*\omega_0(\dot{c}) \\
&= f(c(t))dg(c(t))(\dot{c}(t)) - g(c(t))df(c(t))(\dot{c}(t)) \\
&= a\dot{b} - b\dot{a}
\end{aligned}$$

and so,

$$\begin{aligned}
\dot{\rho}(t) &= (\dot{a} + b\dot{\phi})\cos\phi + (\dot{b} - a\dot{\phi})\sin\phi \\
&= (\dot{a} + b(a\dot{b} - b\dot{a}))\cos\phi + (\dot{b} - a(a\dot{b} - b\dot{a}))\sin\phi \\
&= (\dot{a} + ab\dot{b} - b^2\dot{a})\cos\phi + (\dot{b} - a^2\dot{b} - a\dot{a}b))\sin\phi \\
&= ((1-b^2)\dot{a} + ab\dot{b})\cos\phi + ((1-a^2)\dot{b} - ba\dot{a}))\sin\phi \\
&= (a^2\dot{a} + ab\dot{b})\cos\phi + (b^2\dot{b} - ba\dot{a}))\sin\phi \\
&= \frac{a}{2}\frac{d}{dt}(a^2+b^2)\cos\phi + \frac{b}{2}\frac{d}{dt}(a^2+b^2)\sin\phi = 0
\end{aligned}$$

Thus, $\dot{\rho}(t) = 0$, and so, ρ is constant.

□

Now that we have a formula for the angle ϕ at time s, it is natural to ask how ϕ changes with time. Take a curve c with no self intersections and speed 1 (i.e. $|\dot{c}| = 1$). Take the frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ near $c(s)$ such that $\dot{c}(s) = \mathbf{e}_1(c(s))$, i.e. the frame is aligned to the velocity vector of the curve. Now take a vector field V along c which is parallel:

$$\nabla_{\dot{c}} V(s) = 0 \tag{331}$$

and WLOG, let $|V| = 1$.Associate to this a second frame $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$ so that $V(s) = \underline{\mathbf{e}}_1$. Using our earlier notation, we can then write

$$V(s) = \cos\phi(s)\mathbf{e}_1(c(s)) + \sin\phi(s)\mathbf{e}_2(c(s)) \tag{332}$$

But from the general formula for covariant derivatives in terms of frames[72], we have

$$\nabla_{\dot{c}} V(s) = (-\dot{\phi}\sin\phi - \omega_{21}(\dot{c})\sin\phi)\mathbf{e}_1 + (-\dot{\phi}\cos\phi + \omega_{12}(\dot{c})\cos\phi)\mathbf{e}_2 \tag{333}$$

But by hypothesis, this vanishes. Therefore,

$$\dot{\phi}(s) = \omega_{12}(\dot{c}(s)) \tag{334}$$

We give this quantity a name.

[72] See pset 7 problem 2.

Definition 172. Let c, ϕ, ω_{12} be as above. A **geodesic curvature of a curve** c **at time** s is the number

$$k_g(s) := \dot{\phi}(s) = \omega_{12}(\dot{c}(s)) \tag{335}$$

Intuitively, the geodesic curvature is how much the derivative of a curve deviates from being parallel. In particular, a geodesic has a vanishing geodesic curvature everywhere because by definition, its tangent is parallel along the entire geodesic.

Example 173. Let's compute the geodesic curvature of a circle of radius R. Suppose $V(0) = (1, 0)$ (at the bottom of the circle). If α is the central angle, then $\phi(\alpha) = \alpha$, and so,

$$\frac{d\phi}{d\alpha} = 1 \tag{336}$$

But we are interested in the *time* derivative of ϕ. By chain rule,

$$\frac{d\phi}{ds} = \frac{d\phi}{d\alpha}\frac{d\alpha}{ds} = \frac{1}{R} \tag{337}$$

which is what we got for the usual curvature of a circle. Notice that in the limit $R \to \infty$, we get 0 which is the geodesic curvature of a striaght line on the plane (which is a geodesic).

Examples of Geodesic Curvature and Intuition of Gauss-Bonnet. - Thursday May, 17. 2018.

In the last lecture, we explored how the dual frame and connection forms depended on the gauge change, and we were then led to the notion of the rotation of the frame with respect to the vector field. The means of measuring this change was the geodesic curvature. We ended by computing the geodesic curvature of the circle. We come back to this example, and combining this with the idea of parallel transport, we motivate the Gauss-Bonnet theorem.

For completeness, we state the Gauss-Bonnet theorem here (which we will prove in the next lecture).

Proposition 174 (Gauss-Bonnet for Sufaces with Boundary.). [73]
Let $\mathcal{M}$ be an oriented, compact, two dimensional differentiable manifold with boundary $\partial\mathcal{M}$. Let X be a differentiable vector field on $\mathcal{M}$ such that it is transversal to $\partial\mathcal{M}$ (i.e. X is nowhere tangent to $\partial\mathcal{M}$). Assume that the singularities $p_1, ..., p_k$ of X are isolated, do not belong to $\partial\mathcal{M}$ and denote their indices by $I_1, ..., I_k$.

Then, for any Riemannian metric on $\mathcal{M}$,

$$\int_{\mathcal{M}} K\sigma + \int_{\partial\mathcal{M}} k_g ds = 2\pi \sum_i I_i \tag{338}$$

where k_g is the geodesic curvature of $\partial\mathcal{M}$ and ds is the arc element of $\partial\mathcal{M}$.

Remark 175. Note that the setting of this theorem is the same as Stokes (the condition imposed on the manifold is the same as the conditions for which we proved Stokes). As we will see, the two are related. The 2π in fact comes from how $(f, g)^*\omega_0$ from the last lecture fails to be exact on a region (at least this is the case with a disk on the manifold, as we will see).

[73] From DoCarmo p.104

We will now resume with the examples of geodesic curvature.

Example 176 (Circle.). Take $\mathcal{M} := \mathbb{S}^1$ with the usual Euclidean metric, i.e.

$$ds^2 = dx^2 + dy^2 \tag{339}$$

Consider the starting point $(R, 0)$, and for a vector field V, we have $V(0) = (0, 1)$. Consider the frame (corresponding to the velocity vector of the circle), and let $\phi(s)$ be the angle between this frame vector and the vector field, where we take s to be the arclength along the circle.

From Euclidean geometry,

$$\phi(s) = \frac{s}{R} \tag{340}$$

and so,

$$k_g := \dot{\phi}(s) = \frac{1}{R} \tag{341}$$

Integrating this quantity around the circle gives

$$\int_{\mathbb{S}^1} k_g(s) ds = \frac{1}{R} \cdot 2\pi R = 2\pi \tag{342}$$

Example 177 (Conformally changing the metric.). Let's now consider the previous example but with a different metric. Conformal simply means to preserve the angles, so let's keep the metric orthogonal, but scale the metric. Take $\lambda > 0$, and define the metric

$$ds^2 = \lambda^2(dx^2 + dy^2) \tag{343}$$

which then of course is equivalent to $(g_{i,j}) = \lambda^2\mathbf{Id}$. Since we scaled the metric, the arclength will also be scaled[74]. So,

$$s = \lambda R\phi \tag{344}$$

[75] Therefore,

$$k_g := \dot{\phi} = \frac{1}{\lambda R} \tag{345}$$

Thus, the geodesic curvature depends on the metric.

However, if we compute its integral,

$$\int_{\partial D} k_g ds = \frac{1}{\lambda R} \cdot 2\pi R\lambda = 2\pi \tag{346}$$

which is the same as before. (This is the invariance under metric that Gauss-Bonnet is talking about.)

Example 178 (Part of a circle.). We now artificially cut a piece of a circle off so that its boundary consists of a large arc and a line segment. Call the region inside Ω, and let the metric be the standard metric on $\mathbb{R}^2$. The arc has geodesic curvature $k_g = \frac{1}{R}$ and the segment has $k_g = 0$.

[74] An intuition for this is to take a meter stick which is 2 meters long rather than 1 meter. Under this meter stick, all lengths will be twice as long as before.

[75] Notice that the radius R is simply a *number*. It is given by the coordinate of the point on the circle $(R, 0)$. Notice that positions given from coordinates are different from distances given by the metric.

Now, if we call the central angle formed by the two edges of the segment φ, then again from Euclidean geometry,

$$\int_{\partial\Omega} k_g = \frac{1}{R} \cdot R(2\pi - \varphi) = 2\pi - \varphi \tag{347}$$

Where did the φ go? Of course, the natural culprit is the artificial slice we put in the circle. If we extend the tangent of the arc, we notice that this makes the angle $\frac{\varphi}{2}$ with the line segment. In total, the two sharp corners give φ, and so,

$$\int_{\partial\Omega} k_g + 2 \cdot \frac{\varphi}{2} = 2\pi \tag{348}$$

and once again, we get 2π. This makes sense because if we consider the parallel transport of a vector V initially tangent to the circle, then ϕ measures how much the frame tangent to the circle deviates from V at arc length s. Thus,

$$\begin{aligned}\int_{\partial\Omega} k_g ds &= \int_{\partial\Omega} \dot{\phi}(s) \\ &= \phi(\text{length of } \partial\Omega) - \phi(0) \\ &= 2\pi\end{aligned}$$

where we used the Fundamental Theorem of Calculus in the second step. Notice that in order to do this, we require $\partial\Omega$ to be smooth. Therefore, we must approximate $\partial\Omega$ with smooth curves, do the line integral along these curves, and then pass to the limit.

Another way to see this is the following. The contribution of the two sharp corners are given by

$$\int_{\text{sharp}} \dot{\phi} = -\phi(s_1+) + \phi(s_0-) \tag{349}$$

where the top and bottom corners are denoted s_1, s_0 respectively, and $\phi(s_1+), \phi(s_0-)$ are respectively the changes in angle measured from the arc side respectively.

Example 179 (Sphere.). Now let's consider the sphere $\mathcal{M} := \mathbb{S}^2$. Fix an altitude $\theta = \alpha$, and take the circle c of radius $\sin\alpha$. Here, recall that the parallel transport is nontrivial.

Take the metric induced by spherical coordinates, i.e.

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{350}$$

Then[76] we get the frame

$$\mathbf{e}_1 = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi},\ \mathbf{e}_2 = -\frac{\partial}{\partial\theta} \tag{351}$$

Notice that $\dot{c} = \mathbf{e}_1$. The corresponding dual frame are

$$\omega_1 = \sin\theta d\phi,\ \omega_2 = -d\theta \tag{352}$$

and so,

[76] We can derive this result using the notion of *scale factors*. See Boas, *Mathematical Methods in the Physical Sciences.* 2nd Edition. Section 10.8. Scale factors are very useful; they give a general method to compute gradients, divergence, Laplacians, etc. in a much cleaner way than change of variables.

Alternatively, just note that the coordinate vector fields under spherical coordinates (in positive orientation) are $d\theta, \sin\theta d\phi$ (since by defintion, these are the pushforwards of the basis vectors in the θ, ϕ space under the spherical coordinate diffeomorphism).

$$d\omega_1 = \cos\theta d\theta \wedge d\phi = \omega_{12} \wedge \omega_2$$
$$d\omega_2 = 0 = -\omega_{12} \wedge \omega_1$$

and so, $\omega_{12} = \cos\theta d\phi$. We are now ready to compute the geodesic curvature of c. Recall from last lecture that $c^*\omega_{12} = k_g$, so

$$\begin{aligned} k_g &= c^*\omega_{12} \\ &= \omega_{12}(\mathbf{e}_1) \\ &= \cos\alpha d\phi \left(\frac{1}{\sin\alpha}\frac{\partial}{\partial\phi}\right) \\ &= \cot\alpha \end{aligned}$$

Notice that the geodesic curvature is the same around the circle. This is no surprise because the rotation with respect to the north-south axis is an isometry (or more loosely, the sphere is symmetric).

Now, once again, we integrate:

$$\int_c k_g = \cot\alpha \cdot 2\pi\sin\alpha = 2\pi\cos\alpha \tag{353}$$

We now have a factor of $\cos\alpha$. Where did the other $1-\cos\alpha$ go? If we integrate on the cap D bounded by c, we have

$$\int_D d\sigma = \int_D \omega_1 \wedge \omega_2 = \int_0^{2\pi}\int_0^{\alpha} \sin\theta d\theta \wedge d\phi = 2\pi(1-\cos\alpha) \tag{354}$$

And so,

$$\int_c k_g + \int_D K d\sigma = 2\pi \tag{355}$$

where K is the Gaussian curvature. The above computation is a preview of the Gauss-Bonnet theorem. It also invokes the question: why do we keep getting back 2π?

We will now start to understand the meaning of Gauss-Bonnet. Take the two dimensional Riemannian manifold $\mathcal{M}^2$. Let D be a disc in $\mathcal{M}$ such that its closure is contained in some coordinate neighborhood. This (via Gram-Schmidt on the coordinate vector fields) gives us an orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ defined on a neighborhood of D.

Now take a different orthonormal frame $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2\}$ on ∂D. Then (via the same topological considerations as the Theorema Egregium, i.e.) we can straighten D to be the usual disk on the plane $\mathbb{R}^2$, i.e.

$$\overline{D} = \Phi(D(0,1)) \tag{356}$$

for a straightening diffeomorphism Φ and $D(0,1) \subset \mathbb{R}^2$. We can also pull the frames back downstairs. We can then take an *annulus* around the $\partial D(0,1)$, and extend the frames on this annulus (by using the same formula as on $\partial D(0,1)$).

The INCORRECT (but naïve) thing to do now is Stokes:

$$\begin{aligned}\int_{\partial D}\dot{\phi} &= \int_{\partial D}\underline{\omega}_{12}\\ &= \int_D d\underline{\omega}_{12}\\ &= \int_D -K\omega_1\wedge\omega_2\end{aligned}$$

and from here conclude that

$$\int_{\partial D}\dot{\phi} + \int_D K\omega_1\wedge\omega_2 = 0 \tag{357}$$

Why is this calculation so BAD? Where is the ERROR? The problem here is that we ignored the fact that $\underline{\omega}_{12}$ is defined on an annulus containing ∂D.[77] Another way of looking at this is pretenging that $\tau := (f,g)^*\omega_0$ from the last lecture is exact on all of D.

So what can we do to fix this? We can do a gauge change. Observe that

$$\begin{aligned}\int_{\partial D} k_g &= \int_{\partial D}\underline{\omega}_{12}\\ &= \int_{\partial D}\omega_{12}+\tau\\ &= \int_D \omega_{12} + \int_{\partial D}\tau\\ &= \int_D -Kd\sigma + \int_{\partial D}\tau\end{aligned}$$

and so,

$$\int_{\partial D} k_g + \int_D Kd\sigma = \int_{\partial D}\tau \tag{358}$$

But why do we get

We out aside the discussion of Gauss-Bonnet to examine another (very important) example.

Example 180 (Hyperbolic Plane.). The **hyperbolic plane** $\mathbb{H}^2$ is a two dimensional Riemannian manifold defined to be the open upper half plane in $\mathbb{R}^2$ given the metric

$$ds^2 = \frac{dx^2+dy^2}{y^2} \tag{359}$$

A frame orthonormal in this metric is

$$\mathbf{e}_1 = y\frac{\partial}{\partial x},\ \mathbf{e}_1 = y\frac{\partial}{\partial y} \tag{360}$$

and so the dual frame is

$$\omega_1 = \frac{1}{y}dx,\ \omega_2 = \frac{1}{y}dy \tag{361}$$

[77] This is analogous to pretending a meromorphic function does not have a pole in a region, and then invoking Cauchy's theorem to kill the integral. If we could do such a thing, the theory of contour integration and calculus of residues would not exist.

Therefore,

$$d\omega_1 = \frac{1}{y^2} dx \wedge dy,\ d\omega_2 = 0 \tag{362}$$

Now from the first structure equation,

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 \\ d\omega_2 &= -\omega_{21} \wedge \omega_1 \end{aligned}$$

and so,

$$\omega_{12} = \frac{1}{y} dx = \omega_1 \tag{363}$$

Consequently,

$$d\omega_{12} = d\omega_1 = \omega_1 \wedge \omega_2 \tag{364}$$

So, $K = -1$.

What are the geodesics of the hyperbolic plane? Let

$$c(s) := (x(s), y(s)) \tag{365}$$

We want to use the expression from problem set 7 problem 2 for the covariant derivative, so take

$$\dot{c}(s) = a\mathbf{e}_1(s) + b\mathbf{e}_2(s) \tag{366}$$

where $a, b : \mathcal{M} \subset \mathbb{H}^2 \to \mathbb{R}$ such that $|\dot{c}|^2 = a^2 + b^2 = 1$ by (wlog) choosing arclength parametrization. (We will abuse notation in the following and write $\dot{a} := \frac{d}{dt}(a \circ c)$ and likewise for b.) If c is a geodesic, then $\nabla_{\dot{c}}\dot{c} = 0$ which can be rewritten as

$$\begin{aligned} \dot{a} + \omega_{21}(\dot{c})b &= 0 \\ \dot{b} + \omega_{12}(\dot{c})a &= 0 \end{aligned}$$

which is a system of linear ODEs giving the components of $\dot{c}$. Since $\omega_1 = \omega_{12}$ for the hyperbolic plane, this becomes much simpler:

$$\begin{aligned} \dot{a} - ab &= 0 \\ \dot{b} + a^2 &= 0 \end{aligned}$$

which is a system of linear ODEs. However, notice that a, b are two dependent functions since $a^2 + b^2 = 1$. We can instead take

$$a = \cos\phi,\ b = \sin\phi \tag{367}$$

Here, we implicitly assume that $a(0) = 1, b(0) = 0$. (This is equivalent to saying $\dot{c}(0) = (1, 0)$.) We can then write

$$
\begin{aligned}
-\dot{\phi}\sin\phi - \sin\phi\cos\phi &= 0 \\
\dot{\phi}\cos\phi + \cos^2\phi &= 0
\end{aligned}
$$

or

$$\dot{\phi} = -\cos\phi \tag{368}$$

which is the pendulum equation. This does not how a closed solution.

Instead of giving up, we make a change of variables in hopes of getting something cleaner. Recall that we defined

$$\dot{c} = (\dot{x}, \dot{y}) = a\mathbf{e}_1 + b\mathbf{e}_2 \tag{369}$$

where $(\dot{x}, \dot{y})$ is in the standard basis. But since

$$\mathbf{e}_1 = y\frac{\partial}{\partial x_1}, \mathbf{e}_2 = y\frac{\partial}{\partial x_2} \tag{370}$$

we have

$$\dot{x} = ay, \dot{y} = by \tag{371}$$

The second equation *looks* separable, except that the derivative on the left is with respect to s, and b is a function of $\phi(s)$. We now use the fact that $\frac{d\phi}{ds} = -\cos\phi$[78] By chain rule,

$$
\begin{aligned}
\frac{dy}{d\phi}\frac{d\phi}{ds} &= \sin\phi\, y \\
\frac{dy}{d\phi} &= -\tan\phi\, y
\end{aligned}
$$

which is now certainly a separable ODE. Thus,

$$
\begin{aligned}
\frac{dy}{d\phi} &= -\tan\phi\, y \\
\int\frac{dy}{y} &= \int\frac{-\sin\phi}{\cos\phi}d\phi \\
\log y &= \log\cos\phi + C \\
y(\phi) &= K\cos\phi
\end{aligned}
$$

Now the first ODE gives

$$
\begin{aligned}
\frac{dx}{d\phi}\frac{d\phi}{ds} &= \cos\phi\, y \\
\frac{dx}{d\phi} &= -y \\
\frac{dx}{d\phi} &= -K\cos\phi \\
x(\phi) &= -K\sin\phi + C
\end{aligned}
$$

[78] Of course we will use this information (or the ODE $\nabla_{\dot{c}}\dot{c}$) somewhere; this is what specifies that c is a geodesic.

Now we imposed the initial conditions

$$a(0) = \left(\frac{\frac{dx}{d\phi} \cdot \frac{d\phi}{ds}}{y} \right)(0) = \left(\frac{K\cos^2\phi}{K\cos\phi} \right)(0) = \cos\phi(0) = 1 \tag{372}$$

Thus, $\phi(0) = 2\pi n$ for $n \in \mathbb{Z}$. The other initial condition

$$b(0) = \left(\frac{\frac{dy}{d\phi} \frac{d\phi}{ds}}{y} \right)(0) = \left(\frac{K\sin\phi\cos\phi}{K\cos\phi} \right)(0) = \sin\phi(0) = 0 \tag{373}$$

which gives us $\phi(0) = \pi m$ for $m \in \mathbb{Z}$. Thus, our solution is

$$\begin{cases} y(\phi) & = K\cos\phi \\ x(\phi) & = -K\sin\phi + C \\ \phi(0) & = 0 \end{cases} \tag{374}$$

where K, C are constants determined by what point the geodesic goes through.

Week 9.

Gauss-Bonnet Theorem. - Tuesday May, 22. 2018.

The goal of this lecture is to prove the (local and global) Gauss-Bonnet theorem (which is not as complicated as one may first think). We will start with some examples; this lecture is relatively pictorial compared to the other lectures.

We will recall the following from the lecture on May 15th (on gauge invariance of Gaussian curvature) since it is central to some of our arguments. First, recall that we defined the geodesic curvature k_g to be the deviation of the tangents of a curve from being completely parallel along the curve. We measured this through the function

$$\phi := \int_{c:0\to t} (\underline{\omega}_{12} - \omega_{12}) + \phi_0 \tag{375}$$

i.e., as the total gauge shift of the connection forms arising from the two different moving frames (one is aligned along the tangent of the curve, the other is a parallel vector field along the curve).

Here is the special case of the Gauss-Bonnet we proved last time.

Proposition 181 (Gauss-Bonnet for a disk on a surface.). Let $\mathcal{M}^2 \subset \mathbb{R}^3$ be a smooth surface, and let $D \subset \mathcal{M}$ be a disk such that the boundary ∂D is piecewise C^1. Additionally, let α_i be the angles of the corners. Then

$$\int_D K\sigma + \int_{\partial D} k_g ds + \sum_i \alpha_i = 2\pi \tag{376}$$

Remark 182. Recall from last lecture that $\sum_i \alpha_i$ is the contribution to the geodesic curvature term $\int_{\partial D} k_g ds$ due to the sharp corners; last time, we approximated sharp corners with smooth ones, integrated over them, and deduced that the sum of the exterior angles are indeed the integral of the geodesic curvature over the approximating curves. (In fact, in the following discussion, we ommit the exterior angle α_i, but we can easily recover them through this idea.)

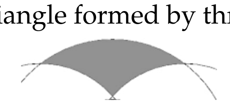
Remark 183. Notice that the left hand side of the above equation is purely *geometric*. The Gaussian curvature arises from the metric, and the geodesic curvature comes from the gauge change in the moving frame, which is also a geometric thing (and via the previous remark, α_i is also geometric). On the contrary, the 2π is a *topological* invariant since it does not depend on the choice of metric. We will see later that this is the nature of Gauss-Bonnet: it connects the geometric and the topological aspects of the manifold.

Remark 184. As an application of the previous remark, the above theorem gives us a necessary condition for a Riemannian manifold $\mathcal{N}$ to be homeomorphic to a disk, namely the expression on the left hand side evaluated for $\mathcal{N}$ must be 2π.

Example 185 (Hyperbolic Plane.). The Gauss-Bonnet gives some of the most interesting properties of the hyperbolic plane.

Consider the hyperbolic triangle T formed by three geodesic semicircles.

Figure 1: Hyperbolic triangle formed by three geodesic semicircles.



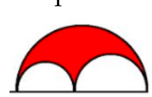
Call the inner angles $\alpha_1, \alpha_2, \alpha_3$ (from the left one). From Gauss-Bonnet, we then have

$$\text{Area}(T) + \sum_i (\pi - \alpha_i) = 2\pi \tag{377}$$

So $\text{Area}(T) = \pi - \sum_i \alpha_i$. Notice that as a conseque¡nce, $0 < \text{Area}(T) < \pi$ in the hyperbolic plane (and $0 < \sum_i \alpha_i < \pi$). This is astonishing since in Euclidean geometry, we can take a triangle as large as we want.

An example of an **ideal triangle**[79] is a hyperbolic formed by three geodesic circles[80]:

Figure 2: An example of an ideal triangle.



The only property of this object that we care about right now is that it has maximal area, i.e. $\text{Area}(T) = \pi$. This is one of the peculiarities of hyperbolic geometry; this object has no analogue in classical geometry.

We will now build more machinery for global Gauss-Bonnet.

Definition 186. Let X be a smooth vector field on $\mathcal{M}^2$. A point $p \in \mathcal{M}^2$ is a **singularity of** X if $X(p) = 0$. A singularity is **isolated** if it is isolated in the sense of point set topology.

In other words, a singularity is just a zero of the vector field.

Definition 187. (In addition to the previous definition,) let U be an open set on $\mathcal{M}$ and take a moving frame such that $\underline{\mathbf{e}}_1 := \frac{X}{|X|}$ on $U \setminus \{p\}$, and take the associated connection form[81] $\underline{\omega}_{12}$. Take *any* different frame $\{\mathbf{e}_1, \mathbf{e}_2\}$ from which we get the gauge change $\tau := \underline{\omega}_{12} - \omega_{12}$. The **index of the vector field** X **at** $p \in \mathcal{M}$ is the integer

$$I(X, p) := \frac{1}{2\pi} \int_c \tau \tag{378}$$

where c is a simple closed curve in $U \setminus \{p\}$ around p.

Remark 188. The above definition obviously depends on the choice of curve C and frame $\{\mathbf{e}_1, \mathbf{e}_2\}$. It is a theorem that in fact the index does not depend on these two things. We will show this later.

We also need to show that the index is an integer. For the this, the intuitive reason is because τ is the gauge change between the two moving frames, and so it can only make a multiple of 2π amount of rotations. Or, τ is the pullback of the angle form with respect to the rotation (f, g), so the index is just the *winding number* of this rotation curve $\gamma(t) = (f(C(t), g(C(t))))$ where $C(t)$ is the curve on which we are computing the index.

Now we will show that the index is indeed independent of the choice of curve.

Proposition 189 (Index does not depend on the choice of curves.). With the same hypothesis as in the previous definition,

$$\frac{1}{2\pi} \int_{c_1} \tau = \frac{1}{2\pi} \int_{c_2} \tau \tag{379}$$

where c_1, c_2 are a simple closed curves in $U \setminus \{p\}$ around p.

[79] See wikipedia.

[80] These three geodesics are parallel to each other, and they become infinitely close to each other near the "points of infinity."

[81] Recall that a connection form is uniquely determined, given a dual frame (see problem set 7, problem 3). Also, the second vector in the frame is determined uniquely by the orientation of the manifold.

PROOF 190. There is only one idea in the proof: Stokes on annulus. The second case will reduce to the first.

Case 1.: Curves are disjoint. Suppose that $c_1 \cap c_2 = \emptyset$. Let the annulus bound by c_1, c_2 be A, and so by Stokes,

$$\int_{c_2} \tau - \int_{c_1} \tau = \int_A d\tau = 0 \tag{380}$$

since τ is a pullback of a closed form, it is also closed.

Case 2. Curves are not disjoint. Now if $c_1 \cap c_2 \neq \emptyset$, take a third simple closed curve c_3 in $U \setminus \{p\}$ around p such that c_1, c_2 lies in the interior of the curve. We can then use the first case to establish that

$$\int_{c_2} \tau - \int_{c_3} \tau = \int_A d\tau = 0 \tag{381}$$

and

$$\int_{c_1} \tau - \int_{c_3} \tau = \int_A d\tau = 0 \tag{382}$$

which then implies the conclusion.

□

Here is an alternative characterization of indices.

Proposition 191. Take a manifold $\mathcal{M}$ and open set U so that we can define indices of vector fields. Then the limit $\lim_{r\to 0} \int_{c_r} \underline{\omega}_{12}$ exists and

$$I(X,p) = \frac{1}{2\pi} \lim_{r\to 0} \int_{c_r} \underline{\omega}_{12} \tag{383}$$

where c_r is a curve of radius r in $U \setminus \{p\}$ around $p \in U$.

PROOF 192. Of course, there are two parts to this proof: existence of the limit and the equality. Let

$$\overline{I} := \frac{1}{2\pi} \lim_{r\to 0} \int_{c_r} \underline{\omega}_{12} \tag{384}$$

We want to show that $\overline{I} = I$.

For the former, observe that by Stokes,

$$\int_{c_{r_1}} \underline{\omega}_{12} - \int_{c_{r_2}} \underline{\omega}_{12} = \int_{A_{r_1,r_2}} d\underline{\omega}_{12} \tag{385}$$

where c_{r_1}, c_{r_2} are curves around the singularity $p \in U$ of radius r_1, r_2, and A_{r_1,r_2} is the annulus they bound. But now, if f is the chart of $\mathcal{M}^2$, and for the 2-form $f^* d\underline{\omega}_{12} = a dx_1 \wedge dx_2$ on $\mathbb{R}^2$, and if $|a| \leq M$ on $f^{-1}(U \setminus \{p\})$,then

$$\left| \int_{A_{r_1,r_2}} d\underline{\omega}_{12} \right| \leq \text{Area}(A_{r_1,r_2}) \cdot M \tag{386}$$

Thus, as $r_1, r_2 \to 0$, the integral goes to 0, and hence $\int_{c_{r_i}} \underline{\omega}_{12}$ forms a Cauchy sequence. Thus, the limit exists.

Now, to get equality, observe that

$$\begin{aligned}
\int_{c_{r_1}} \underline{\omega}_{12} - \lim_{r_2 \to 0} \int_{c_{r_2}} \underline{\omega}_{12} &= \int_{c_{r_1}} \underline{\omega}_{12} - 2\pi \overline{I} \\
&= \int_{\partial D_{r_1}} \underline{\omega}_{12} - 2\pi \overline{I} \\
&= \int_{D_{r_1}} d\underline{\omega}_{12} - 2\pi \overline{I} \\
&= \int_{D_{r_1}} -K \underline{\omega}_1 \wedge \underline{\omega}_2 - 2\pi \overline{I} \\
&= \int_{D_{r_1}} -K d\sigma - 2\pi \overline{I}
\end{aligned}$$

where D_{r_1} is of course the disk bounded by the curve c_1. Here we used Stokes, definition of Gaussian curvature, and problem 1 of pset 7 to get that $\underline{\omega}_1 \wedge \underline{\omega}_2$ is the area form.

On the other hand, by gauge invariance of Gaussian curvature,

$$\begin{aligned}
\int_{D_{r_1}} -K d\sigma &= \int_{D_{r_1}} d\omega_{12} \\
&= \int_{c_{r_1}} \omega_{12} \\
&= \int_{c_{r_1}} \underline{\omega}_{12} - \int_{c_{r_1}} \tau \\
&= \int_{c_{r_1}} \underline{\omega}_{12} - 2\pi I
\end{aligned}$$

Therefore, we get

$$\int_{c_{r_1}} \underline{\omega}_{12} - 2\pi I = \int_{c_{r_1}} \underline{\omega}_{12} - 2\pi \overline{I} \tag{387}$$

from which we get the conclusion.

□

Here is a very important corollary of what we just proved.

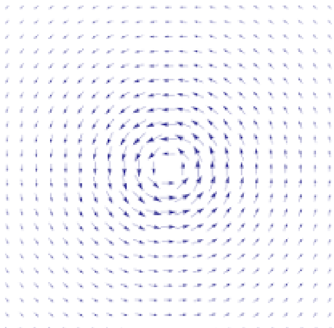
Proposition 193 (The index does not depend on the choice of the second frame.). If $\tilde{\tau}$ is given by a different choice of frame $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ in definition 187, then

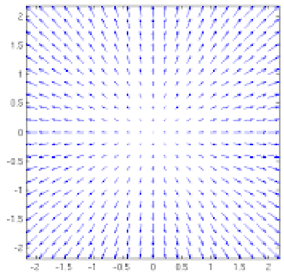
$$\int_c \tau = \int_c \tilde{\tau} \tag{388}$$

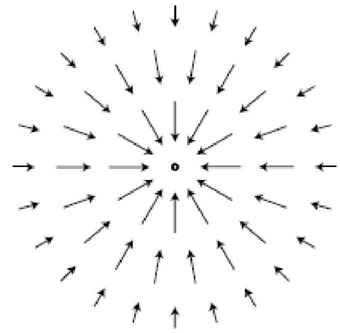
PROOF 194. The above two integrals both satisfy the definition of an index, so they must be equal to the limit given in the previous proposition 191. But now, this limit only depends on $\underline{\omega}_{12}$, so indeed, the two integrals must be equal.

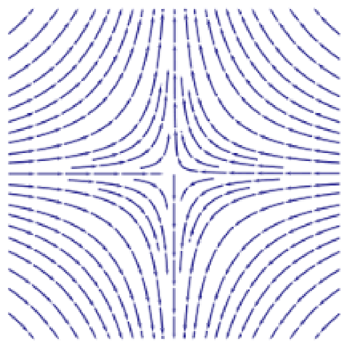
□

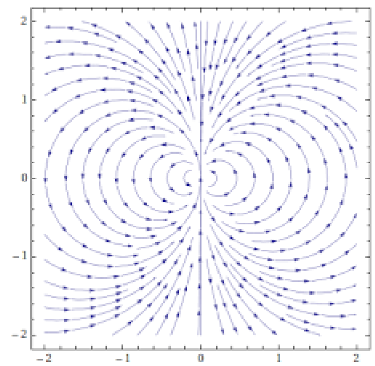
Example 195. Here are some examples of indices of vector fields. We can in fact find them by inspection, rather than computing an actual integral.


Swirl


Source


Sink

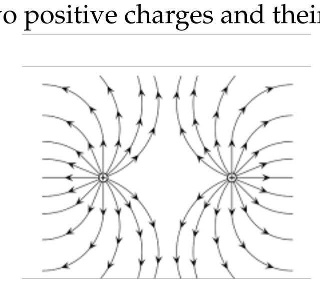

Saddle Point


Dipole

We can just take the unit circle around the origin to be our curve, and see how much the gauge changes under one rotation, i.e. how much the vector field rotates. Here, we are taking counterclockwise to be the positive direction. Thus, by inspection, the first three vector fields have index 1 at the origin, the saddle point has index -1, and the dipole has index 2.

Another (intuitive and informal) way to think about this is via electric fields.[82] Take the source example. We can imagine that this is a limiting case where we take two positive point charges and move them close to each other. Notice that there is a singularity between the two point charges. This has index -1 and the two positive charges has +1, so in total the index is +1 for the entire system.

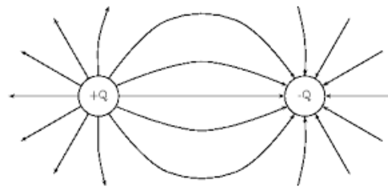
Figure 3: Two positive charges and their electric field



The dipole can then be thought of as a limiting case of one positive and one negative charge. Here the total index is 0.

[82] Obviously, this does not quite apply to the swirl vector field since it is nonconservative, and electric fields must be conservative (or equivalently, the 1-form corresponding to the vector field is nonexact).

Figure 4: One positive and one negative charge and their electric field.



This notion of "limiting case" can be formalized as a continuous deformation. This whole construction works because the index is an invariant under continuous deformations.

Proposition 196 (Gauss-Bonnet Theorem without boundary.). Let $\mathcal{M}^2$ be a smooth compact oriented manifold, and let X be a vector field with isolated singularities $\{p_i\}$. Then

$$\int_{\mathcal{M}} K d\sigma = 2\pi \sum_i I(X, p_i) \tag{389}$$

PROOF 197. Take balls $B_i(p_i, r_i)$, and take $\underline{\mathbf{e}}_1 := \frac{X}{|X|}$. On $\mathcal{M} \setminus \bigcup_i B_i$, there is a connection form $\underline{\omega}_{12}$ induced by this frame. Then

$$\int_{\mathcal{M} \setminus \bigcup_i B_i} -K d\sigma = -\sum_i \int_{\partial B_i} \underline{\omega}_{12} \tag{390}$$

by Stokes. (The negative sign on the LHS accounts for the orientation since $\mathcal{M} \setminus \bigcup_i B_i$ is outside the curves ∂B_i.) But now from our second characterization of indices, we can just pass to the limit $r_i \to 0$ to get

$$\int_{\mathcal{M}} K d\sigma = \lim_{r_i \to 0} \int_{\mathcal{M} \setminus \bigcup_i B_i} K d\sigma = \lim_{r_i \to 0} \sum_i \int_{\partial B_i} \underline{\omega}_{12} = 2\pi \sum_i I(X, p_i)$$

as desired.

□

We give a name to the topological invariant appearing on the RHS.

Definition 198. An **Euler characteristic of a (two dimensional) manifold** M^2 is the number

$$\chi(\mathcal{M}) := \sum_i I(X, p_i) = \frac{1}{2\pi} \int_{\mathcal{M}} K d\sigma \tag{391}$$

Remark 199. One implication of Gauss-Bonnet theorem is that the Euler characteristic does not depend on the choice of vector field, since the vector field appears on only one side of the equation. Or conversely, we are allowed to choose a specific vector field when we compute Euler characteristics, as we will see in the proof of the next proposition.

We saw in previous examples that we can find the Euler characteristic of a manifold by simply looking at it.

Here is an alternative characterization of Euler characteristics.

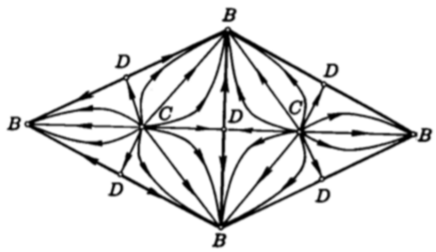
Proposition 200. Take a triangulation of $\mathcal{M}$.Then

$$\chi(\mathcal{M}) = V - E + F \tag{392}$$

where V, E, F are the total number of vertices, edges, and faces in the triangulation.

PROOF 201. The proof is by picture; just look at a pair of triangles, and choose a specific vector field.

Figure 5: The points labeled B are the sink, C the source, and D the saddle points.



□

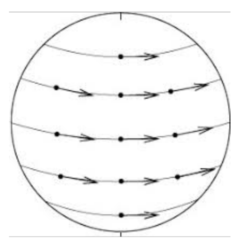
Remark 202. Of course, one would need to show that $V - E + F$ does not depend on the choice of triangulation. However, we can do this easily using refinements.

Example 203. Let's compute the Euler characteristic of $\mathbb{S}^2$. We can choose the vector field to be the wind blowing from west to east. Then the singularities are at the poles at which we see that $I = 2$. Thus,

$$\int_{\mathbb{S}^2} d\sigma = 4\pi \tag{393}$$

which we know is correct!

Figure 6:



Example 204. Let's compute the Euler characteristic for a torus. We know that the torus can be embedded into $\mathbb{R}^4$ (recall the flat torus from chapter 5 problem 1 of DoCarmo) at which $K = 0$. We thus expect the Euler characteristic to vanish. Indeed, if the wind is blowing from one tip of the donut A to the opposite point B, then the index at A, B are both +1. But the inner side C, D of the donut at A, B have index -1. So, in total we get 0 as expected.

We now prove Gauss-Bonnet for surfaces with boundary.

Definition 205. Let $\mathcal{M}^2$ be a smooth surface with boundary. A **vector field** X **is transverse to the boundary** $\partial\mathcal{M}$ if X does not have a singularity on $\partial\mathcal{M}$ and X is not tangent to $\partial\mathcal{M}$ at any point.

Proposition 206 (Gauss-Bonnet for surfaces with boundary.). Let $\mathcal{M}^2$ be a smooth, compact, orientable manifold with C^1 boundary. Let X be a vector field which is transverse to $\partial\mathcal{M}$. Then

$$\int_{\partial\mathcal{M}} k_g + \int_{\mathcal{M}} K\sigma = 2\pi \sum_i I(X, p_i) \tag{394}$$

PROOF 207. Again the proof is very short; there is probably one or two ideas here. For each singularity p_i, let $B_i := B(p_i, r_i)$ be an open ball. Take $\mathbf{e}_1$ tangent to $\partial\mathcal{M}$. Now since $-K d\sigma = -K\underline{\omega}_1 \wedge \underline{\omega}_2 = \underline{\omega}_{12}$ (by definition of K), by Stokes,

$$\int_{M\setminus\bigcup_i B_i} -K d\sigma = \int_{\partial\mathcal{M}} \underline{\omega}_{12} - \sum_i \int_{\partial B_i} \underline{\omega}_{12}$$

and passing to the limit $r_i \to 0$

$$\begin{aligned}\int_M -K d\sigma &= \int_{\partial\mathcal{M}} (\omega_{12} + \tau) - 2\pi \sum_i I(X, p_i) \\ &= \int_{\partial\mathcal{M}} k_g - 2\pi \sum_i I(X, p_i)\end{aligned}$$

where we used the limit characterization of index. Here, we used

$$\int_{\partial\mathcal{M}} \tau ds = 0 \tag{395}$$

which we get from the transversality of X. If this integral is nonzero, then in particular, $\mathbf{e}_1$ must make at least one full rotation, by definition of k_g as $\dot{\phi}$. However, by intermediate value theorem on ϕ, a full rotation means $\phi = 0$ at some point. Then $\mathbf{e}_1$ would be parallel to $\partial\mathcal{M}$ which violates transversality.

We thus have

$$\int_M K d\sigma + \int_{\partial\mathcal{M}} k_g = 2\pi \sum_i I(X, p_i) \tag{396}$$

□

Morse Theory. - Thursday 5.24.2018

[83]

In the last lecture, we proved the Gauss-Bonnet theorem. A key idea in Gauss-Bonnet was the notion of an Euler characteristic of a surface. In this lecture, we develop Morse theory which gives us a different perspective on Euler characteristics.

We start with some key definitions.

Definition 208. A smooth map $f \in C^\infty(\mathcal{M})$, has a **nondegenerate critical point at** $p \in \mathcal{M}$ if $df(p) = 0$ and (if we write f as a function in local coordinates, say $\tilde{f}(x) := f(\phi(x))$ such that $\phi(0) = p$, then)

$$\det D^2 f(0) \neq 0 \tag{397}$$

i.e. the Hessian matrix is invertible.

Proposition 209. The notion of a nondegenerate critical point does not depend on the choice of coordinates.

[83] A part of the proof for Morse Theorem is from Tuesday 5.29.2018

PROOF 210. The only idea here is the change of variables. Wlog, take 0 to be the nondegenerate critical point of f. We can just work downstairs because we can straighten the manifold. We claim that if the smooth function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has a nondegenerate singularity at the origin, then for any local diffeomorphism ψ on $\mathbb{R}^n$ near the origin such that $\psi(0) = 0$[84], $d\psi(0) = 0$ and $D^2(f \circ \psi)(0)$ is invertible.

Let $\psi(y) = x$ and $v = (v_1, ..., v_n) \in \mathbb{R}^n$. Then the linearization is

$$\begin{aligned} d(f \circ \psi)(y)(v) &= df(x)(D\psi(y)(v)) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(x) \frac{\partial \psi^k}{\partial y^i}(y) \right) v_i \end{aligned}$$

So, if $y = 0$, then $x = 0$, so by hypothesis, $\frac{\partial f}{\partial x^k}(0) = 0$. Consequently, $d(f \circ \psi)(0) = 0$ as desired.

For the Hessian, we also do a change of coordinates. Observe that

$$\frac{\partial^2}{\partial y^i \partial x^j}(f \circ \psi)(y) = \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial x^k \partial x^l}(x) \frac{\partial \psi^k}{\partial y^i}(y) \frac{\partial \psi^l}{\partial y^j}(y) \tag{398}$$

This allows us to see that the entire Hessian matrix is given by just

$$D^2(f \circ \psi)(x) = (D\psi)^t(y) D^f(\psi(x))(D\psi)(y) \tag{399}$$

where $D\psi = \left(\frac{\partial \psi^k}{\partial y^i}\right)_{k,i=1}^{n}$ is an invertible matrix. Therefore, in for $x = 0$, since $D^f(\psi(x))$ is invertible, the matrix $D^2(f \circ \psi)(0)$ must also be invertible.

□

Now we can introduce the following definition.

Definition 211. Let $\mathcal{M}$ be a compact manifold. We say that $f \in C^\infty(\mathcal{M})$ is a **Morse function** if f has only nondegenerate critical points.

Proposition 212. The nondegenerate singularities $p \in \mathcal{M}$ of a Morse function f is isolated.

PROOF 213. Write $\nabla f(x, y) = X_1(x, y)\partial_1 + X_2(x, y)\partial_2$ in local coordinates $g : U \subset \mathbb{R}^2 \to \mathcal{M}$ where $g(0, 0) = p$. Then we can use the inverse function theorem on df since the Hessian of f (i.e., the derivative of df) is invertible. We thus obtain a neighborhood $V \subseteq U$ on which df is bijective. Now, $\nabla f(x, y) = 0$ (in the sense of vector fields) iff $df(x, y) = 0$ (in the sense of linear maps) iff $x = y = 0$ by bijectivity. In other words, there are no singular pointss other than p in $g(V)$.

□

Proposition 214. Morse functions only have finitely many critical points.

PROOF 215. Suppose f is a Morse function, and it had infinitely many critical points. Since we defined Morse functions to be functions from compact manifolds, there is a subsequence $q_1, q_2, ...$ of the critical points such that they have a limit point, say q. But now, since f' is continuous and vanishes on each of q_i, so q must also be a critical point. Then q is not an isolated critical point. This is a contradiction from the previous proposition.

□

[84] We need this part of the hypothesis because we need to show that the preimage of 0 under ψ is a nondegenerate critical point of $f \circ \psi$.

Before we prove the Morse theorem, we recall that Planet-Moon lemma which gives a nicer way to state the proof of Morse theorem.

Proposition 216 (Morse Theorem.). Let $\mathcal{M}$ be a compact orientable 2-dimensional manifold. If $f : \mathcal{M} \to \mathbb{R}$ is a Morse function, then

$$M - s + m = \chi(\mathcal{M}) \tag{400}$$

where M, s, m are respectively the number of maximum, saddle, and minimum points with respect to the Morse function.

Remark 217. Intuitively (and indeed, this is one of the main ideas of the proof), we can invision a liquid trickling down from the local maxima (as liquid would behave in gravity), and keep trickling until it reaches either the saddle point of the minima, at which they would stop moving. The liquid would continue moving at the saddle point if the position is perturbed slightly. The point is that we can associate a natural vector field to the Morse function (i.e. gradient) to the Euler characteristic.

Remark 218. In continuation to the previous remark, vector fields such as the dipole is not allowed because of the Morse function condition. More specifically,

PROOF 219. [85]
As in the remark, the main idea is to compute the indices for ∇f (we still need to show this reduction rigorously). In order to do this, we will look at the second order approximation of f (the first order vanishes by hypothesis). We then compute the index for this vector field.
Put any metric $\langle , \rangle_p$ on $\mathcal{M}$. We will establish that ∇f contains all the information we really need.[86]
Then we have

$$df(p)v = \langle \nabla f(p), v \rangle_p \qquad v \in T_p\mathcal{M} \tag{401}$$

Therefore, in particular, $\nabla f(p) = 0$ iff $df(p) = 0$ iff p is a critical point of f.[87] Now by proposition 214, f has finitely many critical points.

Now let's abuse notation and near critical point $p \in \mathcal{M}$, write $f(x)$ (instead of $\phi^* f(x)$ where ϕ is the local coordinate and x is a point downstairs in the coordinate space).

For $v = (v^1, ..., v^k) \in \mathbb{R}^2$, we can write

$$\begin{aligned}
\langle \nabla f(x), v \rangle_p &= \sum_{j,k=1}^{2} g_{j,k}(x)(\nabla f)^j(x)v^k \\
&= df(x)(v) \\
&= \sum_{k=1}^{2} \frac{\partial f}{\partial x^k} v^k
\end{aligned}$$

[85] Some parts of this proof is from the lecture on Tuesday, 5.29.2018. We include this here for cohesiveness.
[86] Note the distinction between $df(p)(v)$ and $\langle \nabla f(p), v \rangle_p$. The former is a *linear transformation* on $T_p\mathcal{M}$ (or equivalently a 1-form on $\mathcal{M}$) whereas $\nabla f(p)$ is a *vector field*. Therefore, even though $df(p)(v) = \langle \nabla f(p), v \rangle_p$, $df(p)$ and $\nabla f(p)$ are fundamentally different objects. In fact, we need to *define* $\nabla f(p)$ in terms of the Riemannian metric so that the equality holds. In general linear algebraic terms, ∇f is the vector induced by df in the natural correspondence $v \mapsto \langle v, \cdot \rangle$ given the metric $\langle \cdot, \cdot \rangle$.
[87] This is the key fact behind the association stated in remark 217.

So in particular for the basis vectors $(1,0),(0,1)$ we get

$$\frac{\partial f}{\partial x_k}(x)=\sum_{j=1}^{2} g_{j,k}(x)(\nabla f)^j(x) \tag{402}$$

and so, taking the inverse of the metric[88], and Taylor expanding with respect to x,

$$\begin{aligned}(\nabla f)^l &= \sum_{k=1}^{2} g^{k,l}(x)\frac{\partial f}{\partial x^k}(x)\\ &= \sum_{k=1}^{2}\left(g^{k,l}(0)+O(|x|)\right)\left(\frac{\partial}{\partial x^k\partial x^j}(0)x^j+O(|x|^2)\right)\\ &= \sum_{k=1}^{2} g^{k,l}(0)\frac{\partial}{\partial x^k\partial x^j}(0)x^j+O(|x|^2)\end{aligned}$$

and so, we can approximate the components of the vector field ∇f using the second derivatives of f. In full vector field notation,

$$\nabla f(x)=g^{-1}(0)D^2f(0)x+O(|x|^2) \tag{403}$$

where g denotes the vector cooresponding to the metric, and $O(|x|^2)$ is now a vector function.

Let $Y(x):=g^{-1}(0)D^2f(0)x$. Notice $Y=0$ iff $x=0$ (and hence Y has a unique singularity at the origin) because f is a Morse function and g must be invertible.

We now move on to the core of the proof.

Claim 1. $I(\nabla f,0)=I(Y,0)$ for small x, say $|x|<\epsilon$.

By definition,

$$\begin{aligned}I(\nabla f,0)&=\int_{C_r}\tau\\ I(Y,0)&=\int_{C_r}\tilde{\tau}\end{aligned}$$

Recall the definition of $\tau,\tilde{\tau}$: they are rate of change of angles between the given vector fields ($\nabla f, Y$) and a parallel vector field.

This should become immediate as soon as we write down the expressions for the normal vectors; if the expression for the two vector fields coincide, then the index must also coincide. Observe that

$$\frac{Y}{|Y|}=\frac{B\frac{x}{|x|}}{\left|B\frac{x}{|x|}\right|} \tag{404}$$

where $B:=g^{-1}(0)D^2f(0)$. On the other hand,

[88] Recall that the metric is simply the pullback (with respect to the parametrization) of the inner product on the tangent space, so by the "inverse of the metric," we really mean the pullshforward of the inner product in the coordinate space up to the tangent space. Also, this is why we need to make a distinction between the subscript and superscript; subscript denotes the usual metric, and superscript denotes inverse.

$$\begin{aligned}\frac{\nabla f}{|\nabla f|} &= \frac{Bx + O(|x|^2)}{\left|Bx + O(|x|^2)\right|} \\ &= \frac{B\frac{x}{|x|} + O(|x|)}{\left|B\frac{x}{|x|} + O(|x|)\right|}\end{aligned}$$

But now, we only wanted a local statement, i.e. for $|x| \leq \epsilon$, and so, indeed, $\frac{\nabla f}{|\nabla f|} = \frac{Y}{|Y|}$ for $|x| \to 0$. □
Here is an equivalent proof of claim 1 in terms of winding numbers.

Alternate Proof of Claim 1.
Let

$$Y(x) := g^{-1}(0)D^2 f(0)x,\ Z(x) := O(|x|^2) \tag{405}$$

Now $\nabla f = Y + Z$, and

$$\int_{C_r} \tau = 2\pi n(Y \circ C_r; 0) \tag{406}$$

by definition of winding number (and likewise for $\tilde{\tau}$). Intuitively, C_r is a curve with Y moving around on it, and the arrow tip of Y traces out a different curve as Y moves around. We are interested in the winding number of this new curve.

Thus, our claim can be restated as

$$n(Y \circ C_r; 0) = n(Y \circ C_r + Z \circ C_r; 0) \tag{407}$$

which is exactly what we can prove using planet-moon lemma. In order to show this, the only thing we need to check is that the straight-line homotopy

$$F(t,s) := Y \circ C_r(t) + sZ \circ C_r(t) \tag{408}$$

for $s \in [0,1]$ is indeed a homotopy of the punctured plane $\mathbb{R}^2_*$, i.e. $F(s,t) \neq 0$ for all s, t.

But now, since $Z = O(|x|^2)$, for small enough r (which we will determine in a moment), by triangle inequality, we have

$$|Y \circ C_r(t) + sZ \circ C_r(t)| \geq |Y \circ C_r(t)| - |Z \circ C_r(t)| > 0 \tag{409}$$

So, what remains to do is to compute the bound on r. In order to get the above bound, we need r small enough such that

$$|Z \circ C_r(t)| \leq Kr^2 \leq |Y| = |BC_r| \tag{410}$$

where $K > 0$ is a fixed constant (obtained from $O(|x|^2)$), $B := g^{-1}(0)D^2 f(0)$, and $C_r = C_r(t)$ is the position vector of the curve. Can we then find a lower bound for $|Y| = |BC_r|$ so that we get an upper bound to Kr^2?

Notice that on the one hand

$$\|B\|\,|x| \geq |Bx| \tag{411}$$

where the norm is in the sense of operator norm. We can turn this bound on this own head; namely, if we write $x = B^{-1}Bx$, then

$$\begin{aligned}
\left\|B^{-1}\right\| |Bx| &\geq |x| \\
|Bx| &\geq \left\|B^{-1}\right\|^{-1} |x|
\end{aligned}$$

and so, we can take

$$Kr^2 < \left\|B^{-1}\right\|^{-1} r \tag{412}$$

or,

$$r < \frac{\left\|B^{-1}\right\|^{-1}}{K} \tag{413}$$

This proves the first claim. □

Claim 2.

$$I(Bx;0) = \begin{cases} 1 & \det D^2 f(0) > 0 \\ -1 & \det D^2 f(0) < 0 \end{cases} \tag{414}$$

This requires some trickery. We first notice that

$$B = g^{-1}(0) D^2 f(0) \tag{415}$$

loses the symmetry of the Hessian $D^2 f(0)$ because it lacks the factor $g(0)$ on the right. We will show that this does not matter.

If we instead had $B = g^{-1}(0)D^2 f(0)g(0)$, then

$$B = g^{-1}(0)D^2 f(0)g(0) \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{416}$$

up to change of basis, via spectral theorem. Now, under this assumption, the vector field ∇f looks like a source if $\lambda_1, \lambda_2 > 0$, a sink if $\lambda_1, \lambda_2 < 0$, and saddle if $\lambda_1 \lambda_2 < 0$. But the first two case are equivalent to $\det D^2 f(0) > 0$, and the last is equivalent to $\det D^2 f(0) < 0$. So, indeed, if $B = g^{-1}(0)D^2 f(0)g(0)$, then we have the claim.

Now we must return to the original case, $B = g^{-1}(0)D^2 f(0)$. In fact, this case reduces to the above easy case. Recall that an invertible function deforms a circle into an ellipse. But now,the index for an ellipse is the same as the index for a circle, so we can just reduce to the previous case. This gives us our claim. □

Here are elementary examples of the Morse theorem.

Example 220 (Sphere.). Consider the sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$. Consider the Morse function $f : \mathbb{S}^2 \subset \mathbb{R}^3 \to \mathbb{R}$ given by

$$f : (x_1, x_2, x_3) \mapsto x_3 \tag{417}$$

What are the critical points of f, i.e. points at which the gradient of f vanishes? Intuitively, we see immediately that the north and south pole are the critical points.

If we give the poles the weight of $+1$, then their sum is 2 which is indeed the Euler characteristic of $\mathbb{S}^2$.

Example 221 (Topological Sphere.). Now imagine a distorted sphere, one which has two maximas, a saddle point between the two, and then right below it a minimum (where the extrema are with respect to the projection onto the x_3 coordinate again). The rule is then to assign the weights of $+1$ to the extrema and then -1 to the saddle point. This gives the Euler characteristic $\chi(\mathcal{M}) = 2$. In general, if they have n bumps on top and m bumps on the bottom, we get

$$\chi(\mathcal{M}) = n - (n-1) - (m-1) + m = 2 = \chi(\mathbb{S}^2) \tag{418}$$

Example 222 (Topological Sphere with g holes.). Now consider the topological sphere from the the previous example with n bumps on top and m bumps on the bottom but now has g holes. Let's concentrate on the holes. Each highest/lowest points of the holes are saddle points, and so they each contribute -1 to the Euler characteristic. In all, for g holes, we get

$$\chi(\mathcal{M}) = n - (n-1) - (m-1) + m - 2g = 2 - 2g = \chi(g - \text{torus}) \tag{419}$$

The above examples should suffice to illustrate that the quantity $M - s + m$ is indeed a topological invariant, and that in particular, they are equal to the Euler characteristic.

Week 10.

Cohomology. - Tuesday 5.29.2018

89

In the last lecture, we discussed the basic ideas in Morse theory. Here, we would like to address the question:

What are the indices of vector fields and the statement of Morse theorem in higher dimension?

We start with some basic facts and definitions. Due to time constraints, we did not prove any of these results.[90]

Definition 223. Let f be a Morse function on a compact orientable n-dimensional manifold $\mathcal{M}$. The **index of the point** p is the largest dimension of a subspace $V \subseteq T_p\mathcal{M}$ such that $D^2 f(0)$ is negative definite, i.e. for all $\xi \in V$,

$$\left\langle D^2 f(0)\xi, \xi \right\rangle < 0 \tag{420}$$

Notice that in order to specify the matrix $D^2 f(0)$, we need to choose a coordinate system. We certainly need this to be invariant under the choice of coordinates:

Proposition 224 (Sylvester's Law of Inertia.). [91] Let $A \in M_n(\mathbb{R})$ be such that $\det A \neq 0$ and $A = A^t$. Then the number of negative eigenvalues of A are the same as the number of negative eigenvalues of BAB^t where $B \in M_n(\mathbb{R})$ is an invertible matrix.

The Morse theorem (which we will state without proof[92]) generalizes to the following:

Proposition 225. If f is a Morse function on a compact manifold $\mathcal{M}^n$ then

$$\sum_{df(p)=0} (-1)^{I(f,p)} = \chi(\mathcal{M}) \tag{421}$$

Now we will talk about cohomology. We already know some of the basic ideas in low dimensions. In 2-dimensions, we had the gauge change τ and the winding number (both purely 2-dimesional objects). We computed these quantities on a curve, which is a one dimensional object. To go to 3-dimensions, we need to use something 2-dimensional.

Let's continue to work with a curve so to generalize to higher dimensions. Consider $\mathbb{S}^1$ and a vector field X on $\mathbb{R}^2$ with an isolated singularity at 0. Fix $r > 0$, and define the map $f : \mathbb{S}^1 \subset \mathbb{R}^2 \to \mathbb{S}^1 \subset \mathbb{R}^2$ given by

$$f : x \mapsto \frac{X}{|X|}(rx) \tag{422}$$

Proposition 226. α is an exact 1-form on $\mathbb{S}^1$ iff

$$\int_{\mathbb{S}^1} \alpha = 0 \tag{423}$$

[89] See notes for Thursday 5.26.2018 for proof of Morse Theorem proof from this lecture.
[90] A good reference for the following is Madsen and Tornehave, *From calculus to cohomology : de Rham cohomology and characteristic classes.*.
[91] See a brief introduction in Wikipedia and Stackexchange. Find a proof in Wellesley.
[92] See Madsen and Tornehave page 122, Theorem 12.16.

PROOF 227. The key idea is just that $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and so, we can associate periodic functions on $\mathbb{R}$ to functions on $\mathbb{S}^1$.

From exactness, take the primitive $f \in C^\infty(\mathbb{S}^1)$. But since $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, this is equivalent to having a smooth function on the real line with period 1, i.e. $f \in C^\infty$ and $f(x) = f(x+1),\ x \in \mathbb{R}$.

Now since the space of alternating forms on $\mathbb{S}^1$ is a one dimensional vector space, there exists a smooth function $a \in C^\infty(\mathbb{S}^1)$ (and from the above argument, this is equivalent to $a \in C^\infty$ and $a(x) = a(x+1),\ x \in \mathbb{R}$), such that

$$\alpha(x) = a(x)dx \tag{424}$$

So, in particular,

$$\int_0^1 a(y)dy = 0 \tag{425}$$

Now, since f is the primitive, this gives

$$f(x) = \int_0^x a(y)dy \tag{426}$$

And so, putting the above together, α is exact iff there exists $f(x) = \int_0^x a(y)dy$ such that $f(0) = f(1)$ iff $\int_0^1 a(y)dy = \int_{\mathbb{S}^1} \alpha = 0$.

□

Brief Note on Gauss-Bonnet.- Friday 6.1.2018.

[93]

We provide a brief note on the Gauss-Bonnet theorem. Let's first recall the statement.

Proposition 228 (Gauss-Bonnet with Boundary.). Let $\mathcal{M}$ be a 2 dimensional compact orientable Riemannian manifold. Then for any vector field X on $\mathcal{M}$ with isolated singularities,

$$\int_{\mathcal{M}} K d\sigma = 2\pi \sum_{X(p_i)=0} I(X; p_i) \tag{427}$$

Let's stare at the statement for a moment and think of why we need each hypothesis. We need two dimensionality because that is the only version of the statement we know. We need compactness because we need to integrate over the entire manifold. We need orientability we need to use the volume form $d\sigma$. Finally, Riemannian metrics are necessary in order to have Gaussian curvature.

PROOF 229. Take $X \neq 0$. Since (by compactness of $\mathcal{M}$) X has finitely many singularities, it is sufficient to consider a single singularity p_1. Take a small[94] neighborhood[95] D_1 around p_1. Then $X \neq 0$ on $\mathcal{M} \setminus D_1$, so we can define

$$\mathbf{e}_1 := \frac{X}{|X|} \tag{428}$$

Since $\mathcal{M}$ is orientable, we can choose $\mathbf{e}_2$ so that $\mathbf{e}_1, \mathbf{e}_2$ is a frame defined everywhere outside D_1. Now since $d\sigma = \omega_1 \wedge \omega_2$ and

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \tag{429}$$

[93] We had a review session on the Friday before the final exam.
[94] By small, we mean contained in coordinate neighborhood.
[95] Recall that a neighborhood on a manifold is simply a diffeomorphic image of a disc in the coordinate space.

we have

$$\begin{aligned}\int_{\mathcal{M}\setminus D_1} K d\sigma &= -\int_{\mathcal{M}\setminus D_1} d\omega_{12}\\ &= -\int_{\partial(\mathcal{M}\setminus D_1)} \omega_{12}\\ &= -\int_{\partial D_1} \omega_{12}\end{aligned}$$

Notice that we are allowed to use Stokes since the frame $\mathbf{e}_1, \mathbf{e}_2$ and hence the connection form is defined everywhere on $\mathcal{M} \setminus D_1$. Notice that the curve ∂D_1 must be taken clockwise to account for the fact that we are integrating outside D_1.

But now, we ultimately want the area form back via Stokes inside the disk. We cannot do this of course because the connection form is not well defined inside the disc. Therefore, instead we take a gauge change

$$\omega_{12} = \underline{\omega}_{12} + \tau \tag{430}$$

where $\tau = d\phi$[96] and $\underline{\omega}_{12}$ is a connection form corresponding to a constant vector field inside the disk. This is obviously well-defined, and so, the connection form is also well-defined. And so, we can safely use Stokes on this new connection form.

Thus we get

$$\begin{aligned}\int_{-\partial D_1} \omega_{12} &= \int_{-\partial D_1} \underline{\omega}_{12} + \int_{-\partial D_1} \tau\\ &= \int_{D_1} d\underline{\omega}_{12} + \int_{\partial D_1} d\phi\\ &= \int_{D_1} -K\underline{\omega}_1 \wedge \underline{\omega}_2 + \int_{\partial D_1} d\phi\\ &= -\int_{D_1} K d\sigma + 2\pi I(X, p_1)\end{aligned}$$

[97]Notice that we used the negative sign from before to reverse the orientation of the curve to get $-\partial D_1$ which is in the positive orientation. We also used the fact that the area form is gauge invariant.

We thus have the equality

$$\int_{\mathcal{M}\setminus D_1} K d\sigma = -\int_{D_1} K d\sigma + 2\pi I(X, p_1) \tag{431}$$

or moving $-\int_{D_1} K d\sigma$ to the LHS,

$$\int_{\mathcal{M}} K d\sigma = 2\pi I(X, p_1) \tag{432}$$

[96] Note that this gauge change is only defined locally. The angle difference ϕ between the two frames is well-defined up to integer multiples of 2π which gets annihilated by d. Alternatively, ϕ is well-defined on the slit plane.

[97] In the proof we provided previously (for both Gauss-Bonnet with and without boundary), we used the limit characterization of indices of vector fields, and then passed to the limit. We do not need to pass to the limit here because we can simply add the area form the LHS of the equation.

as desired.

Alternatively, we could have taken the limit as $D_1 \to p_1$. Then

$$\int_{\mathcal{M}\setminus D_1} K d\sigma \to \int_{\mathcal{M}} K d\sigma \tag{433}$$

since it is just a continuous function on a compact set.

□

Remark 230. We note here that the essence of the above proof is in integrating

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \tag{434}$$

which is a *differential form* (in the sense physicists use it in electromagnetism) of Gauss-Bonnet.