Introduction to Algebraic Geometry

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December 3, 2018

Contents

Part I
Lecture Notes.

These lecture notes are from MATH 24400 Introduction to Algebraic Geometry taught in Autumn, 2018 by Benson Farb. The course loosely followed Fulton, *Algebraic Curves* and Arrondo, *Introduction to Projective Varieties*. Some material has been added or altered by the scribe's discretion.

Week 1. Affine Algebraic Sets.

1 Preliminaries. -Tuesday, 10.2.2018

1.1 Notation.

Unless specified otherwise:

1.) R is a commutative ring
2.) k is a field, often algebraically closed (or even just $\mathbb{C}$)

1.2 Algebraic Sets.

We start by looking at affine varieties.

Definition 1. The **affine n-space (over the field k)** is just the space $\mathbb{A}^n := \mathbb{A}^n_k := k^n$.

Of course, this has a vector space structure over k. However, we do not use this structure here (hence why we give it a new name), and in particular, there is nothing special about the origin.

Remark 2 (Polynomials Acting on Affine Spaces.). There is a natural action of polynomials over k on the affine space, namely if $f \in k[x_1, ..., x_n]$, then we have a map from $\mathbb{A}^n$ to k given by

$$a := (a_1, ..., a_n) \mapsto f(a) = f(a_1, ..., a_n) \tag{1}$$

This correspondence is very important since we define varieties in terms of polynomials.

Definition 3. The **zero set of a polynomial** $f \in k[x_1, ..., x_n]$ is the set

$$V(f) := \{a \in \mathbb{A}^n | f(a) = 0\}$$

and for a collection of polynomials $\{f_i\}$,

$$V(\{f_i\}) := \bigcap_i V(f_i) \tag{2}$$

A **set $X \subseteq \mathbb{A}^n$ is an affine algebraic set** if there exists some set $S \subseteq k[x_1, ..., x_n]$

$$X = V(S) \tag{3}$$

Remark 4. (As we will see later,) each equation cuts down at most one "degree of freedom" (= "dimension" of the algebraic set). So in particular, an algebraic set consisting of finitely many points must be given as the vanishing set of at least n equations. We say "at least" here because the equations may not always be independent (much like how linear dependence in linear equations give redundancy and how we have a notion of a rank of a system of equations).

Here some basic properties:

Proposition 5 (Basic Properties of the V Functor). Let $R := k[x_1, ..., x_n]$.

1.) V is inclusion reversing, i.e. if $I \supseteq J$ in R, then $V(I) \subseteq V(J)$.
2.) The zero set of a set is the zero set of the ideal generated by the set, i.e. for all $S \subseteq R$, $V(S) = V((S))$.

PROOF 6. *1.)* This follows from definitions. If $a \in V(I)$, then for all $f \in I$, $f(a) = 0$. But for any $g \in J \subseteq I$, we have $g(a) = 0$. Thus, $a \in V(J)$, so $V(I) \subseteq V(J)$. □

2.) One inclusion is trivial: since $(S) \supseteq S$, we must have $V(S) \supseteq V((S))$ by 1.).

For the other direction, observe that if $a \in V(S)$, then for all $g \in S$, $g(a) = 0$, so

$$\left(\sum_{i=1}^{n} f_i g_i\right)(a) = 0 \tag{4}$$

for any $f_i \in R, g_i \in S$. Thus, any element in (S) vanishes when evaluated at a, so we have the other inclusion.

□

Definition / Proposition 7. The **Zariski topology** is the *topology* on $\mathbb{A}^n$ generated by taking the algebraic sets to be the closed sets.

PROOF 8. We need to verify that

1.) Any intersection of algebraic sets is algebraic.
2.) Finite unions of algebraic sets is algebraic.

Observe that if $V(S_\alpha)$ is a family of algebraic sets, then

$$\bigcap_{\alpha \in A} V(S_\alpha) = V\left(\bigcup_{\alpha \in A} S_\alpha\right) \tag{5}$$

which proves the first claim.

Now if $V(S_1), V(S_2)$ are algebraic sets, then

$$V(S_1) \cup V(S_2) = V((S_1)(S_2)) \tag{6}$$

which gives us the claim.

□

Let's now look at some examples of algebraic sets.

Example 9 (Finite Sets.). $a = (a_1, ..., a_n) \in \mathbb{A}^n$ is an algebraic set since $\{a\} = V(\{x - a_1, ..., x - a_n\})$. But since finite unions of algebraic sets is an algebraic set, any finite set is an algebraic set.

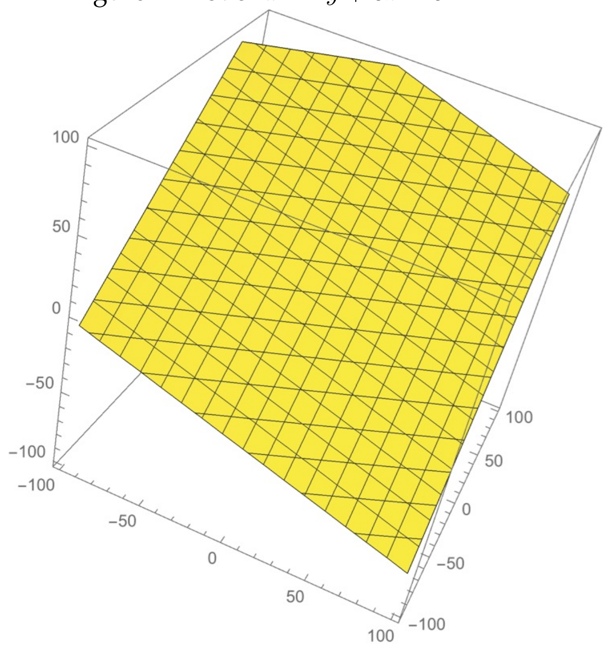
Example 10 (Linear Affine Subspaces). We have

$$\mathbb{A}^{n-2} \simeq (\{x_1, x_7\}) \subseteq \mathbb{A}^n \tag{7}$$

Another example is

$$V(x - 2y + 3z - 82) \subseteq \mathbb{A}^3 \tag{8}$$

Figure 1: Plot of $x - 2y + 3z - 82$ in $\mathbb{R}^3$



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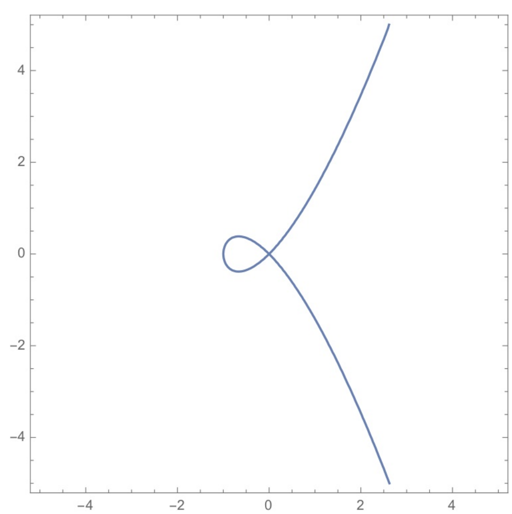
Example 11 (Conics.). A **conic** is an algebraic set of the form $V(p(x, y))$ where p is a degree 2 polynomial. We of course have the usual ellipse, parabola, hyperbola, and the circle. But there are also some degeneracies such as $p(x, y) = (x - y)(x + y)$ (two lines intersect at origin), $x^2 - 1$ (two parallel lines), x^2 ("double line.")

Example 12 (Nodal Cubic.). A **nodal cubic** is the algebraic set given by

$$V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2 \tag{9}$$

Figure 2: Nodal Cubic

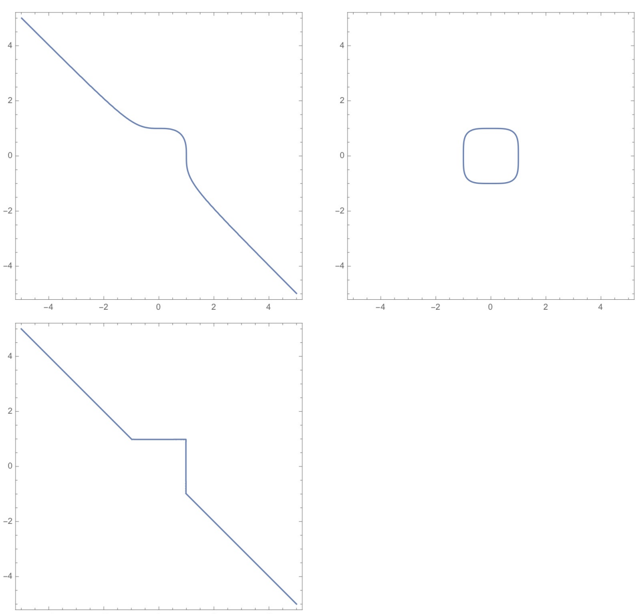


Example 13 (Fermat Curve.). The Fermat curve is given by

$$V(x^n + y^n - 1) \subseteq \mathbb{A}^2 \tag{10}$$

which we obtain by taking $x^n + y^n = z^n$ and dividing through by z^n. The fact that the above is empty over $\mathbb{Q}$ for $n \geq 2$ is equivalent to Fermat's Last Theorem.

Figure 3: The Fermat curves for $n = 3$ (top left), $n = 4$ (top right), and $n = 101$ (bottom left).



Example 14 (Algebraic Groups.).
Lie groups such as the n by n special linear group over k gives an algebraic set in $\mathbb{A}^{n^2}$ by as usual, seeing the matrix elements as just elements of k^{n^2}. More explicitly,

$$SL_n k = V(\det -1) \subseteq \mathbb{A}^{n^2} \tag{11}$$

Note that the determinant is a polynomial in the entries of the matrix.

Example 15 (Affine Hypersurface in $\mathbb{A}^n$.).

We note here that the algebraic sets in $\mathbb{A}^1$ are quite boring.

Example 16 (Algebriac sets in $\mathbb{A}^1$ are the finite sets.). This follows from the simple observation that

$$X = V(I) \subset V(f) \tag{12}$$

for any $f \in I$, and $V(f)$ by the Fundamental Theorem of Algebra.

Now we look at two nonexamples of algebraic sets.

Proposition 17. Any algebraic set $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is closed in the standard topology.

PROOF 18. This is an immediate consequence of the fact that polynomials are continuos with respect to the usual topology:

$$V((S)) = \bigcap_{F \in (S)} f^{-1}(0) \tag{13}$$

□

Example 19 (Open Ball in the Standard Topology.). As a consequence of the above proposition, the open ball in the standard topology is not an algebraic set.

Proposition 20. The interior of any algebraic set $X \subsetneq \mathbb{A}^n$ is empty.

PROOF 21. Any polynomial is determined by its values on an open set.

□

Example 22. The interior of a unit cube is nonempty, so the unit cube cannot be an algebraic set.

1.3 Hilbert Basis Theorem.

The proof presented in class is the standard one. (The following are taken from Reid, *Undergraduate Commutative Algebra*.)

Definition / Proposition 23. The **ring** A **is Noetherian** if it satisfies any of the following (*equivalent*) conditions:

1.) The set Σ of ideals of A has a.c.c.
2.) Every nonempty set S of ideals S has a maximal element (which is not necessarily a maximal ideal)
3.) Every ideal $I \subseteq A$ is finitely generated.

PROOF 24 (Noetherian Rings.). The first and second statement are equivalent just by definition of a.c.c. (so there is no ring theory there). However, the equivalent between the third statement and the other two requires ring theoretic considerations.

(1 $\implies$ 3: with Axiom of Choice) We build the sequence of ideals using the generators. Pick $f_1 \in I$. Inductively, choose $f_{k+1} \in I \setminus (f_1, ..., f_k)$. (Notice that we need to make countably many choices, and so we are using Axiom of Choice in this step.) Then by the a.c.c., the chain

$$(f_1) \subseteq (f_1, f_2) \subseteq ... \subseteq (f_1, ..., f_k) \subseteq ... \tag{14}$$

terminates at the Nth step which (by the construction) can only happen if $I = (f_1, ..., f_N)$. □

Proving 2 $\implies$ 3 will also give the claim.

(2 $\implies$ 3: without Axiom of Choice.) Let I be an ideal in A, and consider the set of finitely generated ideals contained in I. S is nonempty since it contains the trivial ideal. Thus, S has a maximal element J. We must have $J = I$, or else we can choose $f \in J \setminus I$ and construct $(J, f) \in S$ which violates the maximality of J. □

(3 $\implies$ 1) Take the increasing sequence of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ... \subseteq I_k \subseteq ... \tag{15}$$

Since union of increasing sequence of ideals are ideals, $\bigcup_{k=1}^{\infty} I_k$ is an ideal. But by hypothesis, this ideal must be finitely generated which can only happen if the sequence terminated at some point. Thus, a.c.c holds.

□

Proposition 25 (Hilbert Basis Theorem.). If R is a Noetherian ring, then the polynomial ring $R[x]$ is also a Noetherian ring and hence (by induction,) $R[x_1, ..., x_n]$ is Noetherian.

PROOF 26. By the third definition of Noetherian rings, we claim that an ideal $I \subseteq R[x]$ is finitely generated. The proof is fairly constructive. We consider the ideal of the leading coefficients of degree n polynomials in I, i.e.

$$J_n := \left\{ a \in R | \ \exists f \in I \text{ such that } f = ax^n + \sum_{k=0}^{n-1} b_k x^k \right\} \tag{16}$$

We can then relate this to the a.c.c. of the ring R which then gives us a finiteness condition. We then look at the generators of J_n (which also follows from Noetherianess of R), and thus the corresponding polynomials in $R[x]$. We then provide an inductive process which writes $f \in R[x]$ as a finite linear combination of these generator elements.

To use the a.c.c. in R, we must show that 1.) J_n is an ideal, and 2.) it is ascending. The first condition is immediate: if $a, b \in J_n$, then there are corresponding degree n polynomials $f, g \in I$ whose leading coefficients are a, b. But since I is an ideal, for any $\lambda_1, \lambda_2 \in R \subseteq R[x]$, we have $\lambda_1 f + \lambda_2 g \in I$, and this polynomial has degree n and has leading coefficient $\lambda_1 f + \lambda_2 g \in I$. Thus, $\lambda_1 a + \lambda_2 b \in J_n$, so J_n is an ideal.

For the second condition, if $a \in J_n$, then we have a corresponding degree n polynomial with leading coefficient R. But since I is an ideal, we must have $fX \in I$, and this is a degree $n+1$ polynomial with leading coefficient R. Therefore, we have $a \in J_{n+1}$, so $J_n \subseteq J_{n+1}$ thus we have an ascending chain of ideals

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq ... \subseteq J_n \subseteq ... \tag{17}$$

But from the ascending chain condition in R, this chain terminates, so for some n, $J_n = J_{n+1} = ...$

Now since R is Noetherian, the ideals $J_m, m \leq n$ are each finitely generated by elements $\{a_{m,i}\}_{i=1}^{r_m}$. These elements have corresponding polynomials (of degree m and leading coefficients $a_{m,i}$) in $R[x]$, call them $f_{m,i}$.

We now claim that $\{f_{m,i}\}_{\substack{1 \leq i \leq m \\ 1 \leq m \leq n}}$ is a generator of the ideal I which shows that $R[x]$ is Noetherian. Take $f \in I$. It suffices to construct a linear combination of f and $f_{m,i}$ whose degree is less than $\deg f$. Then we can do induction on the degree of f which must terminate in finitely many steps since $\deg f < \infty$ (this is crucial for obtaining a *finite* linear combination of the generators).

We do casework with the degree of f. If $\deg f =: m \leq n-1$, then the leading coefficient R belongs to J_m, so we have $b_i \in R$ with

$$a = \sum_{i=1}^{r_m} b_i a_{m,i} \tag{18}$$

and (this is the key trick) by taking the corresponding polynomials, we can cancel out the leading term of f to get a polynomial with degree less than m:

$$f - \sum_{i=1}^{r_m} b_i f_{m,i} \tag{19}$$

On the other hand, if the degree is big, i.e. $\deg f =: m \geq n$, then we can do the same trick but now use the fact that the chain of the ideals of leading coefficients terminate, i.e. $a \in J_m = J_n$. In other words, for any large degree, we can write the leading coefficient in terms of the generators in J_n. Thus, there are $b_i \in R$ such that

$$a = \sum_{i=1}^{r_n} b_i a_{n,i} \tag{20}$$

and accounting for the fact that $\deg f \geq n$, we can kill the leading term in f:

$$f - \sum_{i=1}^{r_m} b_i x^{n-m} f_{m,i} \tag{21}$$

In either case, we can reduce the degree of f. Consequently, we have a finite generator of the arbitrary ideal I and so, $R[x]$ is Noetherian.

□

2 Hilbert Nullstellensatz. -Thursday, 10.4.2018

Last time we discussed the Hilbert Basis Theorem which gives a finiteness condition inside $k[X_1, ..., X_n]$ and briefly mentioned the dictionary between commutative algebra and algebraic geometry. In this lecture, we will further develop this dictionary including one of the key results: Hilbert Nullstellensatz.

2.1 Ideals vs. Affine Algebraic Sets.

Last time, we discussed the properties of the $V(\cdot)$ functor which sends polynomial ideals to point sets in the affine space $\mathbb{A}_k^n$. We now give a means of going in the other direction.

Definition / Proposition 27. The **ideal of a set** $X \subseteq \mathbb{A}^n$ is the *ideal* given by

$$I(X) := \{f \in k[X_1, ..., X_n] | (\forall a \in X)(f(a) = 0)\} \tag{22}$$

PROOF 28. There is no idea here; we just unravel definitions. Take $f, g \in I$. Since a vanishes at all $a \in X$, for $p, q \in k[X_1, ..., X_n]$,

$$p(a)f(a) + q(a)g(a) = 0 \tag{23}$$

and so, $pf + qg \in I(X)$. Thus, $I(X)$ is an ideal.

□

Example 29. For $a := (a_1, ..., a_n) \in \mathbb{A}^n$, then we have

$$I(a) = (X_1 - a_1, ..., X_n - a_n) \tag{24}$$

We will see later that this is in fact a maximal ideal.

Let's prove this fact. For the inclusion $\supseteq$, if f is in the RHS, then by definition of an ideal

$$f = \sum_{i=1}^{n} g_i (X_i - a_i) \qquad g_i \in k[X_1, ..., X_n] \tag{25}$$

So, we get $f(a) = 0$ and thus $f \in I(a)$.

For the other direction, if $f_1 := f \in I(a)$, then $f(a) = 0$. But now, if we take $f(X_1, a_2, ..., a_n)$, then we have a function in a single variable which vanishes at a_1 and thus by Fundamental Theorem of Algebra, we can write f as

$$f_1(X_1, a_2, ..., a_n) = (X_1 - a_1) f_2(X_1, a_1, ..., a_n) \tag{26}$$

Now for the ith variable, take

$$f_i(a_1, a_2, ..., a_{i-1}, X_i, a_{i+1}, ..., a_n) = (X_i - a_i) f_{i+1}(a_1, a_2, ..., a_{i-1}, X_i, a_{i+1}, ..., a_n) \tag{27}$$

Proceeding inductively,

$$f_1(X_1, X_2, ..., X_n) = \left(\prod_{i=1}^{n} (X_i - a_i) \right) f_{n+1}(X_1, X_2, ..., X_n) \in (X_1 - a_1, ..., X_n - a_n) \tag{28}$$

Remark 30 (Dictionary.). The I functor gives a map from algebraic subsets of $\mathbb{A}^n$ to the ideals in $k[X_1, ..., X_n]$ and for the other direction, we have the V functor.

Proposition 31 (Inclusion Reversing). Let $X \subset Y \subset \mathbb{A}^n$ be algebraic sets, and $J \subseteq K \subseteq k[X_1, ..., X_n]$ be ideals. Then I, V functors are inclusion reversing, i.e.

1.) $I(X) \supseteq I(Y)$
2.) $V(J) \supseteq V(K)$

PROOF 32. Once again, this is just by definition. If $f \in I(Y)$, then for all $a \in Y$, $f(a) = 0$. So, in particular, for all $b \in X$, $f(b) = 0$. Thus, $f \in I(X)$.

For V, if $a \in V(K)$, then for all $f \in K$, $f(a) = 0$. But in particular, this means for all $g \in J$, $g(a) = 0$. Thus, $a \in V(J)$.

□

Lets now look at other set theoretic properties of the V, I functors.

Proposition 33. For all algebraic sets $X \subseteq \mathbb{A}^n$,

$$V(I(X)) = X \tag{29}$$

So in particular, I is injective and V is surjective.

PROOF 34. One direction is immediate. We must have $V(I(X)) \supseteq X$ since if $a \in X$ and if $f \in I(X)$, then $f(a) = 0$. But this holds for any $f \in I(X)$, so a in in the vanishing set of $I(X)$.

For the other direction, if X is algebraic, then there exists a set S for which $X = V(S)$. Then since S is a set of functions on which annihilate X, so by definition, $S \subseteq I(X)$.

But since V is inclusion reversing, this gives

$$V(S) \supseteq V(I(X)) \tag{30}$$

□

Remark 35. Naively speaking, we also expect $I(V(J))$ will give back the ideal J. However, this is the blemish in our dictionary. (This is indeed the subject of the Nullstellensatz.)

Set theoretically, this would imply that V is injective and I is surjective. However we can look at examples to demonstrate that this cannot be the case.

In a polynomial ring (say over $\mathbb{R}$), (since we are working over integral domains) the zeros of X^2 are the zeros of X. So, $V(X^2) = V(X)$. Thus, V is not injective.

On the same note, we cannot have $I(Y) = (X^2)$ for any $Y \subseteq \mathbb{A}^n$ since (from the previous paragraph) if a point is annihilated by X^2 then it must also be annihilated by X, so we would get the ideal (X) instead of (X^2). Thus, I is not surjective.

We take a slight detour to introduce more commutative algebra definitions before stating the Nullstellensatz.

Proposition 36. $I(X)$ is a radical ideal [1]

PROOF 37. There is no idea here although the following trick is very useful as we will see throughout the rest of the lecture. If $f \in \text{rad}I$, then $f^m \in I$ for some m. So, for all $x \in X$, $(f(x))^m = 0$. But since k is an integral domain, this implies that $f(x) = 0$. Therefore $f \in I(X)$. $\square$

Proposition 38. If $f(x) = \prod_{i=1}^r (x - a_i)^{n_i}$ then

$$\text{rad}(f) = \left(\prod_{i=1}^r (x - a_i) \right) \tag{32}$$

PROOF 39. One inclusion is immediate. If f is in the RHS, then we can raise it to the power $\max_{1 \leq i \leq r} n_i$ to get a multiple of f.

On the other hand, if h in the LHS, then $h^n = pf$ for some $p \in k[X_1, ..., X_n]$. But now, since $h^n(a_1, b_2, ..., b_n) = 0$ for any $X_2, ..., X_n$, we have

$$h(a_1, b_2, ..., b_n) = 0 \tag{33}$$

since k is an integral domain. Thus, $h = (X_1 - a_1)h_2(X_1, X_2, ..., X_n)$ and inductively, we see that h has a factor of $\prod_{i=1}^r (x - a_i)$ hence we have the other inclusion. $\square$

The big result that answers our question about the composition $I(V(J))$ is the Hilbert Nullsellensatz.

Proposition 40 ((Strong) Nullstellensatz). Let k be an algebraically closed field. Then for all ideals $J \subseteq k[X_1, ..., X_n]$

$$I(V(J)) = \text{rad}J \tag{34}$$

Remark 41 (Dictionary.). This establishes a bijection between algebraic subsets of the affine space and the *radical* ideals of the polynomial ring.

[1] Recall that the **radical of an ideal** I is the *ideal* given by

$$\text{rad}I := \left\{ r \in R : (\exists d \geq 1)(r^d \in I) \right\} \tag{31}$$

and the fact that it is an ideal follows immediately from the binomial theorem. An **ideal** I **is a radical ideal** if it is equal to its radical: $I = \text{rad}I$.

We will also state the Weak Nullstellensatz which we use to prove the Strong Nullstellensatz.

Proposition 42 (Weak Nullstellensatz). Let k be an algebraically closed field. If J is a *proper* ideal in the polynomial ring $k[x_0, ..., x_n]$, then its zero set $V(J)$ is nonempty.

Remark 43 (Concrete Significance of the Nullstellensatz.). We can think of the Nullstellensatz as generalizations of the Fundamental Theorem of Algebra in the following sense. By definition of the vanishing set, having $V(J) \neq \emptyset$ is indeed saying that there is a simultaneous solution to the system of equations $f_1 = 0, ..., f_n = 0$ where f_i are the generators of J obtained by HBT.

In a similar manner, $I(V(J)) = \text{rad} J$ is just talking about what polynomials simultaneously vanish on the exact same points as a collection of polynomials J.

We thus look at the concrete version of the Nullstellensatz [2]

Concrete Weak Nullstellensatz. Let $P_1, ..., P_m \in k[x]$ be polynomials. over the algebraically closed field k. Then exactly one of the following statements hold:

1.) The system of equations $P_1(x) = ... = P_m(x) = 0$ has a solution $x \in \mathbb{A}^d$.
2.) There exists nontrivial $k[x]$-linear combinations of P_i such that $P_1Q_1 + ... + P_mQ_m = 1$.

The latter is the obvious obstruction to solving the system $P_1(x) = ... = P_m(x) = 0$. Indeed, if the latter did hold, then having a solution x would give an expression of the form $0 = 1$. In this sense, the Weak Nullstellensatz is a statement of the type "the only obstructions are the obvious obstructions."

The Strong Nullstellensatz is similar but now for a more general system:

Concrete Strong Nullstellensatz. Let $P_1, ..., P_m \in k[x]$ be polynomials. over the algebraically closed field k. Then exactly one of the following statements hold:

1.) The system of equations $P_1(x) = ... = P_m(x) = 0,\ R(x) \neq 0$ has a solution $x \in \mathbb{A}^d$.
2.) There exists polynomials $Q_1, ..., Q_m \in k[x]$ such that $P_1Q_1 + ... + P_mQ_m = R^r$ for some $r \geq 0$.

The above reduces to the weak case for $R = 1$. Thus, the Strong Nullstellensatz gives a condition for when a system $P_1(x) = ... = P_m(x) = 0$ has a solution with the additional condition that $R(x) \neq 0$.

2.2 Irreducible Varieties.

Definition 44. An **algebraic set** $X \subseteq \mathbb{A}^n$ **is irreducible** if X is written of two algebraic varieties, then one of them is equal to X and the other is empty. i.e. $\nexists\, X_1, X_2 \neq \emptyset, X$ such that

$$X = X_1 \cup X_2 \tag{35}$$

An **algebraic set is an algebraic variety** if it is also irreducible. [3]

Remark 45. This is obviously in analogy to the notion of an irreducible *element* of a ring.

[2] From Terry Tao's blog.
[3] This definition seems to be the standard terminology however some people (Professor Farb included) uses the two terms algebraic set and algebraic variety interchangeably.

Remark 46. Perhaps a naïve guess is that an irreducible component is the connected component. This is certainly not true as we can see from the example:

$$V(XY) = V(X) \cup V(Y) \subseteq \mathbb{A}^2 \tag{36}$$

which is the union of the X and Y axis.

Definition / Proposition 47 (Irreducible Decomposition.). The **irreducible decomposition of an algebraic set** $X \subseteq \mathbb{A}^n$ is the union

$$X = X_1 \cup ... \cup X_n \tag{37}$$

where each X_i is an irreducible algebraic set. If additionally $X_i \not\subseteq X_j$, then the decomposition is unique.

PROOF 48. We obtain the decomposition via an inductive process. If X is reducible, then take the decomposition $X_1 \cup X_1'$. Now if X_n' is reducible, then take the decomposition $X_n = X_n \cup X_n'$. From this, we get a decomposition. However, at this point, we do not know if the decomposition terminates. This follows from the Noetherian property of $k[X_1, ..., X_n]$. From the I operator, we have the chain

$$I(X_1') \subseteq I(X_2') \subseteq ... \tag{38}$$

But since the polynomial ring is Noetherian, there is some N for which $I(X_i') = I(X_{i+1}')$ for $i \geq N$. But applying the V functor to this, we get $X_i' = V(I(X_i')) = V(I(X_{i+1}')) = X_{i+1}'$ as desired. □

Uniqueness is also immediate from the definition. Suppose we have two irreducible decompositions

$$X_1 \cup ... \cup X_n = Y_1 \cup ... \cup Y_m \tag{39}$$

and X_n intersects one of the sets in the RHS, say Y_m.

□

Finally we have the following addition to the dictionary.

Proposition 49 (Dictionary: Irreducible algebraic sets correspond to prime ideals). A nonempty algebraic set $X \subseteq \mathbb{A}^n$ is irreducible iff $I(X)$ is prime.

PROOF 50. ($\Longleftarrow$.) We prove the contrapositive. If X is reducible, then $X = X_1 \cup X_2$ for $X_i \neq X$. So there exists $f_i \in I(X_i) \setminus I(X)$ and thus $f_1 f_2 \in I(X_1 \cup X_2)$.

But $f_i \notin I(X) = I(X_1 \cup X_2)$ so $I(X)$ is not a prime ideal. □

($\Longrightarrow$.) Contrapositive again. If $I(X)$ is prime, then there exists $f_1 f_2 \in I(X)$ such that $f_i \in I(X)$. Let $J_i := (I(X), f_i)$, then $V(J_i) \subsetneq X$ and

$$X = V(J_1) \cup V(J_2) \tag{40}$$

since for all $x \in X$, $f_1 f_2(x) = 0$, so $f_1 = 0$ or $f_2 = 0$.

□

Week 2. Projective Algebraic Sets.

3 Coordinate Ring and Definition of Projective Sets. -Tuesday, 10.8.2018

3.1 Announcements.

1.) Read introduction: Farb & Wolfson, *Resolvent Degree, Hilbert's 13th Problem and Geometry.* (Farb's Webpage)
   (a.) Example of using modern algebraic geometry to address questions from 17th century mathematics. Understanding roots of polynomials in multivariable.
2.) See new HW2; some problems are quite involved.
3.) Notion of irreducible variety depends on whether it is over $\mathbb{R}$ and $\mathbb{C}$. (See HW.)
   (a.) Think of conjugation as acting on complex algebraic variety (e.g. complex sphere).
   (b.) Think of complex curve intersected by plane.
4.) Algebraic geometry in $\mathbb{C}$ is natural.
   (a.) If we consider the complex torus, then we can slice it through by a plane by looking at the points fixed by the action of complex conjugation (i.e. the real points). The curve is then two circles. Therefore, restricting a connected complex algebraic set to the reals can give a nonconnected set. This is one indication that one should perhaps work in $\mathbb{C}$ rather than $\mathbb{R}$.
5.) Summary of the Commutative Algebra - Algebraic Geometry Dictionary. (k is algebraically closed.)
   (a.) Subvarieties of $\mathbb{A}^n_{\mathbb{C}}$ correspond to radical ideals in $\mathbb{C}[x_1, ..., x_n]$
   (b.) Irreducible algebraic sets correspond to prime ideals.
   (c.) Points in $\mathbb{A}^n$ correspond to maximal ideals in $\mathbb{C}[x_1, ..., x_n]$.
   (d.) Irreducible decomposition of varieties corresponds to any radical ideals $J \subseteq k[x_1, ..., x_n]$ is $J = J_1 \cap ... \cap J_n$

3.2 Coordinate Ring.

Let k be a field of characteristic 0. Let $Y \subseteq \mathbb{A}^n_k$ be an affine algebraic set. We build up to the notion of an (iso-)morphism on an algebraic set.

Definition 51. A **polynomial (function) on an affine algebraic set** Y is a map $F : Y \to k$ which is a restriction of some polynomial to the algebraic set, i.e. $F = P|_Y$ for some $P \in k[x_1, ..., x_n]$.

Remark 52. Note that the polynomial P in the definition is far from unique! In fact, for polynomials P, Q, $P|_Y = Q|_Y$ iff $(P - Q)|_Y = 0$ iff $P - Q \in I(Y)$.

E.g. $X = V(y - x^2) \subseteq \mathbb{A}^2$ and

$$P(x, t) = 3x - 2y^3$$
$$Q(x, t) = 3x - 2y^3 + y - x^2$$

define same polynomial on X since

$$P - Q = -y + x^2 = 0 \in k[x, y]/(y - x^2) \tag{41}$$

Definition 53. The **coordinate ring on the algebraic set** Y [4] is the quotient ring

$$k[Y] := k[x_1, ..., x_n]/I(Y) \tag{42}$$

Remark 54. We call this the *coordinate* ring because it is generated by the images of functions on the affine space. (HW 2 Problem 4:) Let k be any field and let $P \in k[x]$ be a polynomial of degree $d > 0$. Consider the natural quotient map $k[x] \to k[x]/(P)$, denoted by $w \to \overline{w}$. Prove that $\{\overline{1}, \overline{x}, ..., \overline{x}^{d-1}\}$ is a basis for $k[x]/(P)$ over k.

Remark 55. There are two important ways of viewing the coordinate ring $k[Y]$: as a function on Y and as an equivalence class of polynomials.

Proposition 56 (Properties of the Coordinate Ring). Let Y be an algebraic set.

1.) $k[Y]$ is a k-algebra [5].
2.) $k[\mathbb{A}^n] = k[x_1, ..., x_n]$
3.) When characteristic of k is positive, then $k[Y]$ is not equal to the set of polynomial maps from Y to k

Here is a proof for the first two statements:

PROOF 57. The first part is true in general for any quotient of a polynomial ring. The second is immediate because $I(\mathbb{A}^n) = 0$.

□

Here is an example of the third part of the proposition.

Example 58. Consider the case when $k = \mathbb{F}_p$ and $Y = \mathbb{A}^1_k$, in which case, $k[Y] = k[x]$.

Recall why polynomials are different from polynomial *maps*: if $F(x) := x^p - x \neq 0$ in $\mathbb{F}_p[x]$, then $F(a) = 0$ for every $a \in \mathbb{A}^1 = \mathbb{F}_p$ by Fermat's Little Theorem, so even though F and constant 0 are different polynomials, they define the same polynomial *map*. Therefore, $k[Y] = k[x]$ has elements mapping to the same polynomial map under the evaluation homomorphism, so indeed, $k[Y]$ cannot be the collection of all polynomial maps on Y.

Proposition 59. Y is an irreducible algebraic set iff $k[Y]$ is an integral domain.

PROOF 60. This is immediate from general theory. $k[Y] := k[x]/I(Y)$ is an integral domain iff $I(Y) \subseteq k[x]$ is prime, from basic ring theory. But the algebraic set Y is irreducible iff $I(Y)$ is prime, from the Dictionary.

□

Proposition 61. $k[\{y_1\}] \simeq k$ and $k[\{y_1, ..., y_n\}] \simeq k^n$, i.e. coordinate rings over algebraic sets consisting of n points is just k^n.

PROOF 62.

□

Here is the converse.

Exercise 63. $k[Y]$ finite dimensional as k-vector space, then Y is a finite set of points.

PROOF 64. □

Example 65 (HW2 S2). Let $I = (y^2 - x^2, y^2 + x^2) \subseteq \mathbb{C}[x, y]$. We would like to compute $\dim_C \mathbb{C}[x, y]/I$.

A priori, we know that the dimension is finite since if we let $I = I(Y)$, then

$$\begin{aligned} Y &= V(I(Y)) \\ &= V(y^2 - x^2, y^2 + x^2) \end{aligned} \tag{43}$$

But now, if $y^2 - x^2 = 0$, then $y = \pm x$ from which

$$y^2 + x^2 = 2x^2 = 0 \tag{44}$$

So, $x = 0$ and thus $y = 0$. So, $Y = \{(0, 0)\}$ is a finite set.

But we can show that $\mathbb{C}[x, y]/I$ is spanned by $\overline{1}, \overline{x}, \overline{y}, \overline{xy}$ so it is 4 dimensional.

Since we now have a notion of polynomials acting on Y, we will expand out dictionary to algebraic subsets of algebraic sets Y.

Dictionary. We now generalize our algebra/geometry dictionary for $Y \subseteq \mathbb{A}^n$.

1.) Subvarieties $W \subseteq Y$ corresponds to radical ideals *in the coordinate ring* $k[Y]$
2.) Irreducible varieties corresponds to prime ideals *in the coordinate ring*.
3.) Points corresponds to the maximal ideals *in the coordinate ring*.

Remark 66 (On General Math.). If we have functions with certain properties on a space, then those properties can be described for scalar valued maps as well.

Here is an example. Consider the smooth functions on manifolds. We can take a smooth function $F : M \to N$, then for any smooth function h on N, $h \circ F$ is a real valued smooth function on M.

3.3 Polynomial Maps (Morphisms) of Algebraic Sets.

We now make precise what we mean by isomorphisms of algebraic sets. This is analogous to continuous functions for topological spaces. Throughout, let $X \subseteq \mathbb{A}^m, Y \subseteq \mathbb{A}^n$.

Definition 67. A **map** $F : X \to Y$ **is a morphism (/ polynomial map)** if there exists component functions $F_1, ..., F_n \in k[x_1, ..., x_m]$ such that

1.) $F(x) = (F_1(x), ..., F_n(x))$
2.) For all $x \in X$, then $F(x) \in Y$.

Remark 68. Here is an easy condition to check. If $\pi_i : \mathbb{A}^n \to k$ is the ith projection map, then $F : X \to Y$ is a morphism iff $Y_j \circ F \in k[X]$.

[4] Perhaps a less confusing notation for this is to write $\Gamma[Y]$ (Fulton uses this notation). This way, there is no way for us to mistake $k[Y]$ for the polynomial ring with variable Y.
[5] k-algebras: Think matrix rings or polynomials.

Proposition 69. Morphisms are continuous with respect to Zariski topology, i.e. preimage of an algebraic set under a morphism is again an algebraic set.

PROOF 70. If $F : X \to Y$ is a morphism and $h_i \in k[Y]$, then we can just consider the pullbacks of the map h_i so that

$$F^{-1}(V(h_1, ..., h_r)) = V(h_1 \circ F, ..., h_r \circ F) \tag{45}$$

□

Definition / Proposition 71. A **morphism** $F : X \to Y$ **is an isomorphism of algebraic sets** if there exists a morphism $G : Y \to X$ such that $F \circ G = \mathbf{Id}_Y$ and $G \circ F = \mathbf{Id}_X$.

The two **algebraic sets** X, Y **are isomorphic** if they satisfy the following (*equivalent*) conditions:

1.) There exists an isomorphism $F : X \to Y$, i.e. a morphism with another morphism $G : Y \to X$ for which compositions are identities on the appropriate spaces.
2.) The coordinate rings $k[X], k[Y]$ are isomorphic as rings.

Note the similarity with the notion of *homeomorphisms* from general topology. (This will be conceptually important in a moment.)

Remark 72 (Open Problem: Jacobian Conjecture.). (HW2 Problem 5:) If $F : \mathbb{A}^n \to \mathbb{A}^n$ is a morphism, then its Jacobian is nonzero constant polynomial. This can be done without too much machinery. The converse of this is the **Jacobian conjecture** which is an open problem [6]

PROOF 73 (Isomorphism of Algebraic Sets).

□

Remark 74. Here is an important question: If a morphism is bijective, then is it an isomorphism? → NO.

Recall from topology: There are bijective continuous maps which are not homeomorphisms.

Note. Any morphism $F : X \to Y$ induces a map via "pullback wrt F"

$$F^* : k[Y] \to k[X] \tag{46}$$

$$h \mapsto h \circ F \tag{47}$$

(where $F : X \to Y, h : Y \to k$). Pullback is a homomorphism on k-algebras, i.e. a ring homomorphism such that

$$F^*(\lambda h) = \lambda F^*(h) \qquad \lambda \in k \tag{48}$$

Proposition 75. Pullback is **functorial (/natural)**, (i.e. $(F \circ G)^* = G^* \circ F^*$. In particular, if F is isomorphism, then the pullback is a isomorphism.

PROOF 76. This is immediate:

$$\begin{aligned}(F \circ G)^* h &= h \circ F \circ G \\ &= G^*(h \circ F) \\ &= (G^* \circ F^*) h\end{aligned} \tag{49}$$

[6] In fact, it is one of Stephen Smale's 1998 list of Mathematical Problems for the Next Century.

□

Now for some examples of morphisms.

Example 77. Take $\psi : \mathbb{A}^1 \to V(y - x^2)$ (i.e. maps to a paraboloid) defined by

$$t \mapsto (t, t^2) \tag{50}$$

This is an isomorphism of varieties. In fact, the inverse is just the projection $(x, t) \mapsto x$.

In fact, $\psi^* : \mathbb{C}[x, y]/(y - x^2) \to \mathbb{C}[t]$ is given by

$$\begin{aligned} x &\mapsto t \\ y &\mapsto t^2 \end{aligned}$$

is surjective with no kernel. ψ^* is an isomorphism of k-algebra.

Example 78 (Cuspidal Cubic). Consider $\psi : \mathbb{A}^1 \to V(y^2 - x^3) \subseteq \mathbb{A}^2$ given by

$$t \mapsto (t^2, t^3) \tag{51}$$

Inverse is $(x, t) \mapsto \frac{y}{x}$.

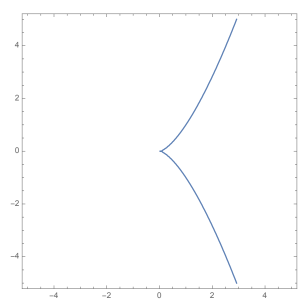
The pullback is

$$\psi^* : \mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[t] \tag{52}$$

$$\begin{aligned} x &\mapsto t^2 \\ y &\mapsto t^3 \end{aligned}$$

So, ψ^* is not surjective.

Figure 4: Cuspidal cubic. We obtain a trefoil knot when we intersect this curve with the unit 3-sphere in $\mathbb{C}^2$! See Wikipedia: Trefoil Knot



Proposition 79. Let V, W be affine varieties (perhaps in different affine spaces). If we have k-algebra homomorphism $\psi : k[W] \to k[V]$, then there exist unique morphism $F : V \to W$ such that $\psi = F^*$. Furthermore, ψ is a k-algebra isomorphism iff F is an isomorphism of varieties.

Thus there exists a bijection $F \mapsto F^*$ between morphisms $V \to W$ and k-algebra homomorphisms $k[W] \to k[V]$.

In other words, the coordinate rings are isomorphic iff the corresponding varieties are isomorphic.

PROOF 80.
Let

$$k[W] := k[x_1, ..., x_n]/I(W) \tag{53}$$

Let $\overline{x}_i$ be the image of $x_i \in k[x_1, ..., x_n]$ in $k[W]$. Let

$$F_i := \psi(\overline{x}_i) \in k[V] \tag{54}$$

Define

$$F : V \to \mathbb{A}^n \supseteq W \tag{55}$$

by

$$F(v) := (F_1(v), ..., F_n(v)) \tag{56}$$

In other words, we want coordinates of points in the domain to map to coordinates of the points in the range.

Check:

1.) F is morphism
2.) $F(V) \subseteq W$: must check
   (a.) $(F_1(v), ..., F_n(v)) \in W$ for all $v \in V$
3.) $\psi = F^*$: since $\{\overline{Y}_i\}$ generate $k[W]$ as a. k-algebra, enough to prove $F_i = F^*(\overline{Y}_i) = \psi(\overline{Y}_i)$

Images of generators are generators

□

3.4 Affine Change of Coordinates.

[7]

Definition / Proposition 81. A **polynomial map** $T : \mathbb{A}^n \to \mathbb{A}^n$ **is an affine change of coordinates on** $\mathbb{A}^n$ if for

$$T = (T_1, ..., T_n) \tag{57}$$

each T_i satisfies the following (*equivalent*) conditions:
1.) It is a polynomial of degree 1 and is a bijective map.
2.) $T = T'' \circ T'$ for a invertible linear map T' and a translation T''.

PROOF 82. We can address both directions at once. Immediate by definition of degree 1 polynomial that

$$T_i = \sum a_{i,j} x_j + a_{i,0} \tag{58}$$

which is a composition of a linear map and a translation. Translation is obviously bijective, and so, we just need the linear map to be bijective.

□

[7] See Fulton, 2.3 Coordinate Changes.

3.5 Projective Varieties.

Projective varieties are covered by affine varieties, so if we are working locally, we can pass to affine varieties.

One of the motivations for projective varieties:

Exercise 83. Over $\mathbb{R}$ or $\mathbb{C}$, affine varieties are never compact (in $\mathbb{A}^n$ for $n \geq 2$), and projective varieties are always compact wrt Euclidean topology. (HW3 S3 for $\mathbb{C}$ case.)

PROOF 84. (*Complex case.*) Take affine algebraic set W contained in the ball. $Y_i|_W \colon W \to \mathbb{C}$ would be bounded for all i. So, Y_i is constant by maximum principle.

$\square$

Just like in affine space, there is a background space.

Definition 85. The **projective n-space** is the set

$$\begin{aligned}\mathbb{P}^n := \mathbb{P}^n_k &:= \left\{\text{1-dim subspaces of } k^{n+1}\right\} \\ &= (\mathbb{A}^{n+1} \setminus \{0\})/\sim\end{aligned}$$

where $v \sim w$ iff $v = \lambda w$ for $\lambda \in k$.

The image of the quotient map $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ of $(a_0, ..., a_n)$ is denoted

$$[a_0 : ... : a_n] \tag{59}$$

called **homogeneous coordinates on** $\mathbb{P}^n$.

Remark 86 (How to Remember the Notation.). Note that the superscript (as is the case with the standard manifold notation $\mathcal{M}^n$) indicates its dimension as a smooth manifold (at least when it is over $\mathbb{R}$ or $\mathbb{C}$). The set of lines in $\mathbb{A}^2$ have exactly one degree of freedom (the angle), so it is $\mathbb{P}^1$. In $\mathbb{A}^3$, there are two degrees of freedom given by spherical coordinates, so it is $\mathbb{P}^2$. (Of course, we are working over $\mathbb{R}$ when we say all of this.)

The homogeneous coordinates show how we only care about the *ratio* of the coordinates (so this is a very good notation).

Example 87. For $n = 1$, they are just one dimensional subspaces in k^2.

$$\begin{aligned}\mathbb{P}^1 &= \{[a_0 : a_1] : a_0, a_1 \in k,\ a_0 a_1 \neq 0\} \\ &= \{[a_0 : a_1] : a_0 \neq 0\} \cup \{[0 : 1]\} \\ &= \{[1 : a_1] : a_1 \in k\} \cup \{[0 : 1]\}\end{aligned}$$

We call $\{[0 : 1]\}$ the **line at infinity** and

$$\{[1 : a_1] : a_1 \in k\} \simeq \mathbb{A}^1 \tag{60}$$

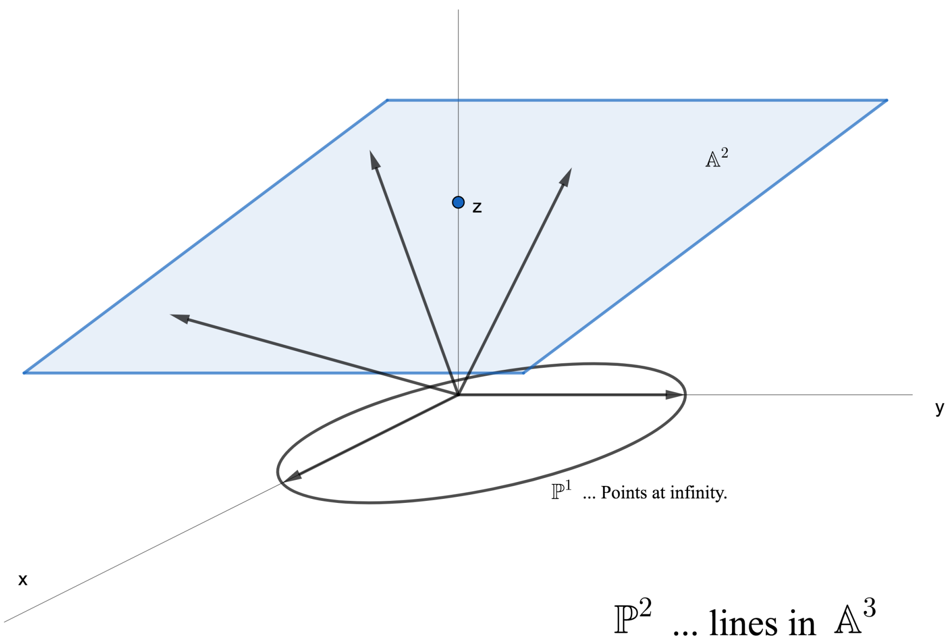
So, $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^0$. This is what we mean when we say projective space is covered by affine spaces.

Example 88. For $n = 2$, it is the lines in $\mathbb{A}^3$. Then

$$\begin{aligned}\mathbb{P}^2 &= \{[1 : a_1 : a_2] : a_1, a_2 \in k\} \cup \{[0 : a_1 : a_2] : a_1 \neq 0 \text{ or } a_2 \neq 0\}\\ &= \mathbb{A}^2 \cup \mathbb{P}^1\end{aligned}$$

We can think of $\mathbb{P}^1$ as the "$\mathbb{P}^1$-at infinity."

Figure 5: The projective 2-space can be thought of the union of the affine 2-space and the line at infinity which is a projective 1-space.



So for $n = 1$, we broke infinity by considering $x = 1$.

In general, for $\mathbb{P}^n$, for all $0 \leq i \leq n$, let

$$\begin{aligned}U_i &:= \{[a_0 : ... : a_n] \in \mathbb{P}^n : a_i \neq 0\}\\ &\simeq \{(a_0, ..., a_{i-1}, 1) \in \mathbb{A}^n\}\\ &\simeq \mathbb{A}^n\end{aligned}$$

where the isomorphism from U_i to $\mathbb{A}^n$ is given by

$$[a_0 : ... : a_n] \mapsto \left[\frac{a_0}{a_i} : ... : \frac{a_n}{a_i}\right] \tag{61}$$

For any i, we have $\mathbb{P}^n = U_i \cup \mathbb{P}^{n-1}$, and by recursion,

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup ... \cup \mathbb{A}^1 \cup \mathbb{A}^0 \tag{62}$$

is the **standard affine cover of** $\mathbb{P}^n$. Once again, a projective space is covered by affine spaces with ascending dimensions.

Example 89. From before

$$\mathbb{P}^n = \mathbb{A}^2 \cup \{[0 : 1]\} \tag{63}$$

So if we have a parabola $V(y - x^2) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$. We can take the line they are asymptotes to to get ellipse. So, hyperbola, parabola and ellipse are just breaking the symmetry in different ways!

But polynomials are not well defined on projective space, so we need to be careful.

4 Lecture 4. -Thursday, 10.11.2018

4.1 Announcements.

1.) Arrondo Problem 1.8 - Just do $d = 2$ case.

4.2 Projective Varieties.

Last time we discussed projective spaces as the quotient

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\to \mathbb{P}^n \\ v &\mapsto [\lambda v] \qquad \lambda \in k^* \end{aligned}$$

So in other words, the projective n-space takes all lines in the affine $n + 1$ space, only caring about the *directions*. We now develop projective varieties.

Consider the polynomial map induced by $f \in k[x_1, ..., x_n]$ which is

$$\begin{aligned} \mathbb{A}^{n+1} &\to k \\ (a_0, ..., a_n) &\to f(a_0, ..., a_n) \end{aligned}$$

It is not usually true that

$$f(\lambda a_0, ..., \lambda a_n) = f(a_0, ..., a_n) \qquad \forall \lambda \in k^* \tag{64}$$

So, f does not give a map from the projective space to the field. Thus, we need to restrict a smaller class of polynomials.

Definition 90. $f \in k[x_1, ..., x_n]$ **is homogeneous of degree** $d \geq 0$ written $f \in k[x_1, ..., x_n]_{(d)}$ if every monomial term in f has degree d, i.e. each term is of the form

$$x^I = x_0^{d_0} x_1^{d_1} ....x_n^{d_n} \tag{65}$$

and

$$\deg x^I = \sum_{i=0}^{n} d_i \tag{66}$$

Nonexample.

$$p(x, y) = x^3 + 2xy \tag{67}$$

is not homogeneous because the first term is degree 3 and second term is degree 2.

Proposition 91. f is homogeneous of degree d iff for all $\lambda \in k^*$,

$$f(\lambda a_0, ..., \lambda a_n) = \lambda^d f(a_0, ..., a_n) \tag{68}$$

So, while $f \in k[x_0, ..., x_n]_{(d)}$ still doesn't give a well-defined function from the projective space to the field, we still have a well-defined zero set

$$V(f) = \{[a_0 : ... : a_n] \in \mathbb{P}^n | f(a_0, ..., a_n) = 0\} \tag{69}$$

Remark. Some of the stuff in projective is very similar to affine but others are very different!

Definition 92. A **projective variety** is a set $X \subseteq \mathbb{P}^n$ for some n such that

$$X = V(S) := \bigcap_{p \in S} V(p) \tag{70}$$

for some set of homogeneous polynomials S. i.e., it is just the vanishing sets of homogeneous polynomials, not necessarily of the same degree.

Remark 93. We can once again use HBT to make this finitely generated.

Example 94. $V((0)) = \mathbb{P}^n$

Example 95.

$$J = (\{x_0, ..., x_n\}) \tag{71}$$

which is the ideal of polynomials with constant term 0 (so it is a proper ideal).

BUT $V(J) = \emptyset$ since projective space does not contain the origin!

Example 96. Finite sets.

Example 97 (Hypersurfaces). A **hypersurface of** $d \geq 1$ is the projective set

$$X = V(f) \qquad f \in k[x_0, ..., x_n] \tag{72}$$

and for $d = 1$,

$$\mathbb{A}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n \tag{73}$$

and subspace V

$$W \to \mathbb{P}(W) = \mathbb{P}^k \qquad k = \dim W - 1 \tag{74}$$

We call these hyperplanes because V of a linear polynomial.

Example 98 (Zariski Topology). Now take the set of projective varieties in $\mathbb{P}^n$ is closed under arbitrary intersection [8] and finite unions. So, this gives the **Zariski topology in projective space**.

Example 99 (Grassmannians). The **Grassmanian** is

$$G(k, n) := \left\{ V < \mathbb{C}^{n+1} | \dim V = k \right\} \tag{75}$$

i.e. linear subspaces of dimension k. (There is also a notion of affine Grassmanians.)

Example 100 (Segre embedding). The **Segre embedding** gives an embedding of

$$\mathbb{P}^n \times \mathbb{P}^n \tag{76}$$

which makes it into a projective variety. Or, if X, Y are projective variety, then their product is a projective variety.

Remark. Restriction of Zariski from affine to projective agrees with the Zariski on projective.

4.3 Commutative Algebra of Homogeneous Polynomials.

Let's now look at the commutative algebra of homogeneous polynomials

Proposition 101. Any polynomial $f \in k[x_1, ..., x_n]$ has a unique decomposition

$$p = \sum_{i=0}^{d} p^{(i)} \qquad p^{(i)} \in k[x_1, ..., x_n]_{(i)} \tag{77}$$

where $d = \deg p$.

Example 102. If we have

$$p(x, y, z) = 18z^3 - 6(i+2)xy^2 + 3xy + 2z^2 + z + 1 \tag{78}$$

then

[8] Intersection of bunch of vanishing sets is just the vanishing set of all those polynomials. Note that this is also finitely generated by HBT.

$$p^{(3)} = 18z^3 - 6(i+2)xy^2$$
$$p^{(2)} = 3xy + 2z^2$$
$$p^{(1)} = z$$
$$p^{(0)} = 1$$

Definition / Proposition 103. The **graded k-algebra**

$$k[x_0, ..., x_n] = \oplus_{d \geq 0} k[x_0, ..., x_n]_{(d)} \tag{79}$$

is a graded ring.

Definition / Proposition 104. An **ideal** $I \subseteq k[x_0, ..., x_n]$ **is a homogeneous ideal** if it satisfies the following (*equivalent*) conditions

1.) I is generated by homogeneous polynomials (not necessarily of the same degree)
2.) For all $p \in I$, $p^{(d)} \in I$ for all $d \geq 0$. i.e., all the homogeneous ideals of an element in I belongs to the ideal.

PROOF 105 (Homogeneous Ideal). The idea is to think finitely and constructively, i.e. take the finite generators of ideals, and build the desired object from them. The argument is straightforward.

(2 $\implies$ 1.) This direction is trivial if we use HBT. Let

$$I = (p_1, ..., p_r) \tag{80}$$

some (not necessarily homogeneous) $p_i \in k[x_0, ..., x_n]$. (HBT is not necessarily, but makes proof easier.)

Now by assumption, each $p_j^{(d)} \in I$ so

$$I = \left(\left\{p_j^{(d)}\right\}\right) \tag{81}$$

$\square$

(1 $\implies$ 2.) Let

$$I = (p_1, ..., p_r) \tag{82}$$

with each p_i homogeneous. Let $p \in I$ be given

$$p = \sum a_i p_i \qquad a_i \in k[x_0, ..., x_n] \tag{83}$$

Then

$$p^{(d)} = \sum a_i^{(d - \deg p_i)} p_i \in I \tag{84}$$

where $a_i^{(d-\deg p_i)} \in k[x_0, ..., x_n]_{(d-\deg p_i)}$.

$\square$

Here is an immediate facts:

Proposition 106. If I, J are homogeneous ideals, then $I + J, I \cap J, \text{rad} I$ are homogeneous.

4.4 The Dictionary for Projective Space.

Just as we had our Dictionary for affine spaces, we also have a Dictionary for projective spaces. We will start by looking at the maps $I(\cdot)$ and $V(\cdot)$.

Definition 107. Let $X \subseteq \mathbb{P}^n$ be a projective variety. The **homogeneous ideal of** X is the set

$$I(X) := \{p \in k[x_0, ..., x_n] | (\forall z \in X)(p(z) = 0)\} \tag{85}$$

Proposition 108. Let X be projective set.

1.) $I(X)$ is a homogeneous ideal.
2.) By HBT, $I(X)$ can be finitely generated by homogeneous elements

Dictionary. So, we have maps from the set of homogeneous ideals in $k[x_0, ..., x_n]$ to projective variety in $\mathbb{P}^n$ ($V(\cdot)$) and vice versa $I(\cdot)$.

We now want to look at the properties of the map $V(\cdot)$. In the affine case, the Nullstellensatz says

START HERE!!! 10.18.2018

$$J \subsetneq k[x_0, ..., x_n] \implies V(J) \neq \emptyset \tag{86}$$

BUT THIS IS NOT TRUE HERE!! The obstruction here is the **irrelevant ideal**

$$J := (x_0, ..., x_n) \tag{87}$$

Arrondo gives a nice self-contained proof of projective Nullstellensatz, but here we will deduce from affine.

Proposition 109 (Projective Nullstellensatz). Assume k is algebraically closed. Let $J \subseteq k[x_0, ..., x_n]$ be a homogeneous ideal. Then

1.) $V(J) = \emptyset$ iff there exists some N such that J contains all homogeneous polynomials of degree $\geq N$, i.e. rad$J \supseteq (x_0, ..., x_n)$
  (a.) Takes care of irrelevant ideal.
  (b.) We can do dumb things such as squaring
2.) If $V(J) \neq \emptyset$ then $I(V(J)) = \text{rad}J$
  (a.) Are there other polynomials that vanish? This says only powers!

Remark 110 (Concrete Projective Nullstellensatz). It is probably worthwhile here to present the concrete version of the above theorem, just as we did for the affine case.

Suppose we are trying to solve a system

$$P_1(x) = ... = P_m(x) = 0, \; R(x) \neq 0 \tag{88}$$

where now $P_i, R \in k[x]$ are *homogeneous* ideals. Then the projective nullstellensatz says that the two possible obstructions are

Here is a corollary:

Proposition 111 (Projective Dictionary). There exists a bijection between projective varieties in $\mathbb{P}^n$ and homogeneous radical ideals in $k[x_0, ..., x_n]$ given by the I and V maps.

Note that the irrelevant ideal is not radical.

PROOF 112 (Projective Nullstellensatz). For all projective varieties $Z = V(J) \subseteq \mathbb{P}^n,\ J \subseteq k[x_0, ..., x_n]$, there exists associated to it an **affine cone over** Z

$$Z_{aff} := V_{aff}(J) = V(J) \subseteq \mathbb{A}^{n+1} \tag{89}$$

(as opposed to $Z = V_{proj}(J);$).

If $J \neq k[x_0, ..., x_n]$, then

$$Z_{aff} = \pi^{-1}(Z) \cup \left\{0 \in \mathbb{A}^{n+1}\right\} \tag{90}$$

where π is the projective map giving the projective space.

Note that $(a_0, ..., a_n) \in Z_{aff}$ iff $(\lambda a_0, ..., \lambda a_n)$ for all $\lambda \in k^*$. Also, $f(\lambda a_0, ..., \lambda a_n) = 0$ for all $\lambda \in k^*$ iff $f^{(d)}(\lambda a_0, ..., \lambda a_n) = 0$ for all d (i.e. vanish iff all homogeneous components vanish).

Now,

1.) $V(J) = \emptyset$ iff $V_{aff}(J) \subseteq \{0\}$ iff rad$J \supseteq (x_0, ..., x_n)$ by affine Nullstellensatz
2.) If $V(J) \neq \emptyset$ then $f \in I(V(J))$ iff $I(V_{aff}(J)) =$ rad$J$, again by affine Nullstellensatz.

□

4.5 Projective Closure.

Definition 113. Let $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be given by the usual map

$$\{[x_0, ..., x_n, 1] : x_i \in k\} \tag{91}$$

(i.e. the plane of the lines.) The **projective closure of** X **in** $\mathbb{P}^n$ is the smallest projective variety in $\mathbb{P}^n$ containing X. So, in particular, it is just the closure with respect to the projective Zariski topology. (When $k = \mathbb{C}$, it is the usual topological closure of X.)

Existence is immediate since projective sets are just ideals (by the Projective Dictionary), so we can just take intersections.

Example 114 (Key Example: Conics.). Let $X = V(y - x^2)$ and $Y = (xy - 1)$.

The closure adds the line at infinity (as mentioned at the end last time)! We add another variable to make the polynomial homogeneous. So

$$\begin{aligned} \overline{X} &= \left\{[x : y : z] : zy - x^2 = 0\right\} \\ &= V_{proj}(zy - x^2) \end{aligned}$$

so we just added $[0 : 1 : 0]$.

Now

$$\mathbb{P}^2 = U_x \cup U_y \cup U_z \tag{92}$$

where

$$U_z := \{[x:y:z] | z \neq 0\} = \{[x:y:1]\} \tag{93}$$

and

$$\mathbb{P}^1_z = \{[x:y:0] | xy \neq 0\} \tag{94}$$

then

$$\mathbb{P}^2 = U_z \cup \mathbb{P}^1_z \tag{95}$$

Then

$$\overline{X} \cap U_z = \left\{[x:y:1] | 1 \cdot y - x^2 = 0\right\} \tag{96}$$

$$\overline{X} \cap \mathbb{P}^1_z = \{[x:y:0] | 0 \cdot y - x = 0\} = \{[0:1:0]\} \tag{97}$$

Now let's look at Y. Then

$$\overline{Y} = V_{proj}(xy - z^2) \simeq V_{proj}(zy - x^2) \tag{98}$$

as projective varieties. So why $X \not\simeq Y$ but $\overline{X} \simeq \overline{Y}$?

Remark. Map is isomorphism iff the map is a polynomial if restricted to affine chart.

Let's now make more explicit the procedure we did above.

Definition 115. The **homogenization of a polynomial** $p \in k[x_1, ..., x_n]$ is $\overline{p} \in k[x_0, ..., x_n]_{(\deg p)}$ given by

$$\overline{p} = x_0^d + x_0^{d-1} p^{(1)} + ... + x_0 p^{(d-1)} + p^{(d)} \tag{99}$$

when

$$p = \sum_{d=0}^{\deg} p p^{(d)} \tag{100}$$

Remark 116. Note that

$$\overline{p}|_{x_0 = 1 \text{ hyperplane in } \mathbb{A}^{n+1}} = p \tag{101}$$

i.e.

$$\overline{p}(1, x_1, ..., x_n) = p(x_1, ... x_n) \tag{102}$$

So, if $Y = V_{aff}(P) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ (for collection of polynomials P) is $\overline{Y} = V_{proj}(\overline{P})$? $\rightarrow$ NO (See HW3 Problem 6)

Proposition 117. Let $Y \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ be an affine variety. Let

$$\overline{I} = \{p : p \in I(Y)\} \tag{103}$$

(This is the **homogenization of** I). Then $\overline{I} = I(\overline{Y})$.

Remark. Segre and Veronese are quite hard. At least try the lowest case. Just try to do it on your own.

Veronese: just give the inverse map of the isomorphism.

Remark 118 (General Advice.). Talk to fellow students and TA. People who are struggling tend to be the people who don't talk to people. It will spiral. Do not get behind.

4.6 Discussion Section. - Friday, 10.12.2018

1.) General note:
  (a.) Try to use theorems from class for HW
    i.) e.g. If use transcendentality of e for HW1 problem 6, then it is trivial.
2.) **Linear subspace of** $\mathbb{P}^n$: Vanishing set of a homogeneous polynomial of degree 1
  (a.) Geometry of $\mathbb{P}^2$ is the geometry of $\mathbb{A}^2$ plus infinity. Homogenization of polynomial passes from affine to projective.
  (b.) See Fulton and Shafarevich (volume 1) on geometric intuition of projective spaces.
3.) Arrondo Ex 1.4
  (a.) Use something like
$$V(I) \cup V(J) = V(I \cap J) \tag{104}$$
  and for coprime, $I \cap J = IJ$.
  (b.) As a general principle, this will get nasty.
  (c.) "Interpret geometrically": idk.
    i.) Projective space: think as patches of affine spaces; pass to projective space via homogenization
    ii.) Google: Desargues' theorem and Pappus theorem, inversion (Möbius) transformations
  (d.) Similar question: Find ideal for $(1, 3, -2)$?
4.) Problem 1
  (a.) General way of thinking:
    i.) Can you do this for the 0 ideal? $\to$ nilradical
    ii.) Hint: From general situation, you can reduce to this special case. (Take quotient by the ideal.)
    iii.) Localization. (Formally inverting elements in the ring.)
5.) Problem 6
  (a.) Generally, just assume algebraically closed field unles specified otherwise.
6.) Problem 3 $\to$ recall what we discussed in discussion section last time
7.) Problem 5
  (a.) We have not done tangent spaces. (For $\mathbb{C}$, it is quite clear.)
8.) Problem 4: easy.
9.) Problem 2: very similar to last pset. adapt solution from Zariski topology from pset 1.

Motivation: Veronese Embeddings.
Here is some motivation: thinking geometrically can be better than thinking just algebraically. We start with something similar to last time. Take the ideal

$$I = (wz - xy, wz - x^2, xz - y^2) \subseteq k[w, x, y, z] \tag{105}$$

WTS: $V(I)$ is irreducible.

By the dictionary, we just need to show I is prime. The general strategy for this is to either show the quotient wrt I is an integral domain or go by definition.

This is nontrivial since there is no nice factorization. So take the substitution

$$\overline{w} = s^3, \ \overline{z} = t^3 \tag{106}$$

We then get

$$s^3 t^3 = \overline{xy} \tag{107}$$

$$s^3 \overline{y} = \overline{x}^2 \tag{108}$$

$$\overline{x} t^3 = \overline{y}^2 \tag{109}$$

where $\bar{\cdot} = \cdot + I$.

From the first ans second equations, $\overline{x} = s^2t$ and from the last two, $\overline{y} = st^2$.

So,

$$k[x, y, z, w] = k[s^2t, st^2, t^3, s^3] \tag{110}$$

This is the Veronese embedding!

Veronese Embedding.
Take the map from $\mathbb{P}^1$ to $\mathbb{P}^3$ given by

$$[s, t] \mapsto [s^3, s^2t, st^2, t^3] \tag{111}$$

and $\mathbb{P}^1$ to $\mathbb{P}^2$ given by

$$[u, v] \mapsto [u^2, uv, v^2] \tag{112}$$

Idea is to go higher dimensional prokective space in order to consider hypersurfaces. Let's consider the latter. But the hyperplane in $\mathbb{P}^2$ is of the form $\alpha x + \beta y + \gamma z$.

Passing the polynomial

$$f(u, v) = 2v^3 - 3uv + v^2 \tag{113}$$

to $\mathbb{P}^2$ gives

$$g(x, y, z) = 2x - 3y + z \tag{114}$$

So, studying f becomes just studying the subset of g!

Going back to our earlier motivation problem,

$$V(I) \simeq [s^3 : s^2t : st^2 : t^3] \simeq \mathbb{P}^1 \tag{115}$$

Segre Embedding.
Show that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective set. (Note: in previous pset, showing this for *affine* was easy.)

Let's consider this for the special case $n = m = 1$. We map from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ [9] given by

$$([x : y], [u : v]) \mapsto [xu : xv : yu : yv] \tag{116}$$

From this, we see that the target space is just

$$V(\alpha_0\alpha_3 - \alpha_1\alpha_2) \tag{117}$$

where $[\alpha_0, ..., \alpha_3]$.

Why is this the correct approach? We work the other way around. Take

$$\alpha_0\alpha_3 = \alpha_1\alpha_2 \tag{118}$$

General remark: Different embeddings for different contexts:

[9] Note that Veronese embedding maps into a much bigger space then this.

1.) Topology: $\mathbb{C}^{n+1} \to \mathbb{P}^n$
2.) Alg Geo: $\mathbb{A}^n \to \mathbb{P}^n$

Hint for exercise: $m \times m$ has rank 1 iff and cofactor is 0 which is equivalent to $\alpha_0\alpha_3 - \alpha_1\alpha_2 = 0$.

Veronese: hyperplanes are much easier to think about

Week 3. Projective Equivalence, 5 Points on a Conic.

5 Lecture 5. -Tuesday, 10.16.2018

5.1 Announcements.

1.) See HW3

5.2 The Veronese Map.

Recall (from Arrondo Exercise 1.8 from HW2) the monomials in $k[x_1, ..., x_n]$ are just expressions of the form

$$x_I = x_1^{m_1} ... x_n^{m_n} \qquad \sum_i m_i = d,\ I = (m_1, ..., m_n) \tag{119}$$

Then $\{x_I\}$ is a basis for the k-vector space $V_{d,n} := k[x_1, ..., x_n]_{(d)}$.

The dimension is just

$$\dim V_{d,n} = \binom{d+n}{d} \tag{120}$$

and

$$\mathbb{P}(V_{d,n}) \simeq \mathbb{P}^{\binom{d+n}{n}-1} \tag{121}$$

i.e. the parameter space of degree d hypersurface in $\mathbb{P}^{n-1}$ (e.g. conics in projective 2-space which we will see at the end of the lecture) [10] .

Definition 119. For $d \geq 1, n \geq 1$, the **Veronese map**

$$\nu_d : \mathbb{P}^n \to \mathbb{P}(V_{d,n}) \tag{122}$$

defined by

$$\nu_d([x_0 : ... : x_n]) = [x_0^d : x_0^{d-1}x_1 : ... : x^I : ... : x_n^d] \tag{123}$$

(where the RHS is ordered *lexicographically*).

We proved the following in HW2.

Proposition 120. $\nu_d(\mathbb{P}^n) \subseteq \mathbb{P}(V_{d,n})$ is a projective variety.

Here are some interesting properties of ν_d.

1.) ν_d gives an interesting embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^m$. e.g. $\nu_3 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by

$$\nu_3([x : y]) = [x^3 : x^2y : xy^2 : y^3] \tag{124}$$

2.) Translating problems about degree d hypersurfaces to linear ones, e.g.

$$X := V(x^2 - 3yz) \subseteq \mathbb{P}^2_{[x:y:z]} \tag{125}$$

[10] Do not get confused here. We are just considering the space $V_{d,n}$ of homogeneous polynomials in n-variables of degree d and considering the space to be just coordinates given by the basis $\{x_I\}$. The source of (potential) confusion is that homogeneous polynomials *themselves* also act on projective spaces. (That was the whole point of our discussion up until this point.) Here, we are just looking at it as just a "moduli space." (I use quotations because we have not discussed the formal definition of this term.)

and

$$\nu_2(X) = [x^2 : xy : xz : y^2 : yz : z^2] \subseteq \mathbb{P}^5 \tag{126}$$

If we call the six coordinates respectively s_i, then

$$\nu_2(X) = V(s_1 - 3s_5) \cap \nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5 \tag{127}$$

so, it is just a copy of $\mathbb{P}^2$ hitting it with a hypersurface (i.e. a linear thing).

Here is an application.

Proposition 121. Let $X = V(S) \subseteq \mathbb{P}^n$ be a projective variety. Let $f \in k[x_0, ..., x_n]_{(d)}$. Then $X \setminus (V(f) \cap X)$ is an affine variety. In particular, $\mathbb{P}^n \setminus V(f)$ is an affine variety.

The following comes up all the time.

Example 122. Consider $\mathbb{A}^n_k$, i.e. monic n degree polynomials in $k[x]$. Let $\mathrm{Poly}_n k$ be the monic, degree n square free polynomials in $k[x]$. But now

$$\mathrm{Poly}_n k = \mathbb{A}^n_k \setminus V(\Delta_n) \tag{128}$$

where Δ_d is the discriminant.

For instance,

$$\begin{aligned}\mathrm{Poly}_2 &= \left\{x^2 + bx + c : b^2 - 4c \neq 0\right\} \\ &= \left\{(b, c)) : b^2 - 4c \neq 0\right\}\end{aligned}$$

i.e. a complement of some parabola in the plane!

Another example: for $d = 3$, we have

$$\Delta_3(b, c, d) = b^2c^2 - 4c^3 - 4b^3c - 27d^2 + 18bcd \tag{129}$$

Now, $\mathrm{Poly}_n\mathbb{C}$ is isomorphic to the unordered configurations of n distinct points in $\mathbb{C}$. The isomorphism is given by P mapping to roots of P. The inverse map is the **Vieté map** given by

$$(\lambda_1, ..., \lambda_n) \mapsto \left(\sum_i \lambda_i, \sum \lambda_i\lambda_j, ..., \prod \lambda_i\right) \tag{130}$$

Here is the proof for the theorem.

PROOF 123. (*$d = 1$ case.*) WTS: X is a union of algebraic sets defined by linear polynomials. (e.g. $\mathbb{P}^2 \setminus (\{x_1 = 0\} \cup \{x_2 = 0\})$).

Indeed, we can see inductively that

$$\mathbb{P}^n \setminus H_1 = \mathbb{A}^n \tag{131}$$

(which is a copy of affine space for which $x_0 \neq 0$), and

$$\mathbb{P}^n \setminus X = \mathbb{A}^n \setminus \left(\bigcup_{i \geq 2} (H_i \cap \mathbb{A}^n)\right) \tag{132}$$

If $X = V(g_1, ..., g_n)$ and

$$\widehat{g}_i = g_i(1, x_1, ..., x_n) \tag{133}$$

Then

$$X \cap \{x_1 \neq 0\} = V(\{\widehat{g}_i\}) \tag{134}$$

(*General Case: Key Case.*) Applying v_d,

$$\nu_d(X) \subseteq \mathbb{P}(N_{d,n}) \tag{135}$$

and

$$\nu_d(V(f)) = V(\text{linear}) \subseteq \mathbb{P}(V_{d,n}) \tag{136}$$

Now applying the linear case,

$$\nu_d(X \setminus V(f)) \simeq \nu_d(X) \setminus V(\text{linear}) \tag{137}$$

□

Here is one more application.

Remark 124 (On General Math.). If we want to know about one object, look at *all* objects!

Example 125. Recall that a **conic in** $\mathbb{P}^n$ is just

$$C = V(P(x, y, z)) \tag{138}$$

where $P \in k[x, y, z]$ is *homogeneous* and has degree 2. Note that we can take $Q(x, y) := P(x, y, 1)$, then

$$V(Q(x, y)) = V(P(x, y, z)) \cap \{z \neq 0\} \tag{139}$$

is a conic in $\mathbb{A}^2$.

For instance,

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{A}^2_{\mathbb{R}} \cup \mathbb{P}^1_{\mathbb{R}} \tag{140}$$

where $\mathbb{P}^1_{\mathbb{R}}$ is the "circle at infinity."

Now lines L in $\mathbb{P}^2$ correspond to algebraic set of the form $V(ux + vy + wz),\ u, v, w \in k$ and lines passing through $(x, y, 0)$ correspond to $L = V(ux + vy)$ and point at infinity is just $u/v \in \mathbb{S}^1$.

Now $L^1 := V(u^1, v^1) \subseteq \mathbb{A}^2$, then L^1, L intersect at infinity iff $u'/v' = u/v$. As a corrolary, any two lines in $\mathbb{P}^2$ meet at a point! (So, there are no parallel lines in projective geometry.)

Note that the only idea above was to decompose the projective 2-space into an affine 2-space and a projective 1-space. It then suffices to show that lines either intersect at infinity (i.e. in $\mathbb{P}^1$) or they intersect in the affine 2-space $\mathbb{A}^2$.

Example 126 (Hyperbolas in $\mathbb{P}^2$ Are Ellipses.). Consider the hyperbola

$$H := V\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - z^2\right) \tag{141}$$

So in the affine plane for which $z \neq 0$ (this is called the chart U_z), we just get the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If we take $H \cap \mathbb{P}^1_{z=0} = \{[a : \pm b : 0]\}$. (The hyperbola is then approaching the asymptote.)

But in the affine chart $U_x \subseteq \mathbb{P}^2$, we have $H \cap U_x$ in which the coordinates are $u = y/x, v = z/x$ and

$$H : \frac{1}{a^2} - \frac{u^2}{b^2} = v^2 \tag{142}$$

and so,

$$H \cap U_x = V\left(\frac{u^2}{(b/a)^2} + \frac{v^2}{(1/a)^2} - 1\right) \tag{143}$$

is an ellipse in $U_x \simeq \mathbb{A}^2$. The whole point is that if we take a ellipse in projective 2-space, whether we get a hyperbola or an ellipse is just a matter of choice of affine charts.

Example 127 (Degenerate Conics). (This is where schemes become useful.) We also have the following conics:

$$V(x^2 - 1) = \mathbb{A}^2 \qquad \text{(two parallel lines)} \tag{144}$$

and

$$V(x^2 - y^2) \subseteq \mathbb{P}^2 \qquad \text{(two intersecting lines)} \tag{145}$$

and

$$V(x^2) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2 \qquad \text{(double line)} \tag{146}$$

5.3 Projective Equivalence

Back to general theory! Consider

$$GL_3k := \{A \in M_3(k) : \det A \neq 0\} = GL(k^3) \tag{147}$$

An element in this set acts on subspaces in k^3 which then induces an action of GL_3k on $\mathbb{P}^2$.

For instance, GL_2k acts on

$$\mathbb{P}^1 = \{[x : 1] | x \in k\} \cup \{[1 : 0]\} \tag{148}$$

and the action gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix} & x \neq -d/c \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & x = -d/c \end{cases} \tag{149}$$

Similarly, GL_nk gives automorphisms of $\mathbb{P}^n$ which induces PGL_nk by modding out the scalars $\{\lambda\mathbf{Id}_n : \lambda \in k^*\}$. It turns out $PGL_nk \simeq \text{Aut}(\mathbb{P}^n)$.

Note: diagonal matrices which has different scalars in the entries do not fix diagonal lines will (so it is not included in this set).

Definition 128. **Curves $C_1, C_2 \subseteq \mathbb{P}^2$ are projectively equivalent** if there exists $A \in PGL_2k$ for which $A(C_1) = C_2$.

Proposition 129. Any conic $C \subseteq \mathbb{P}^2_{\mathbb{C}}$ is projective equivalent to precisely one of

1.) $V(x^2 + y^2 + z^2)$
2.) $V(x^2 + y^2)$
3.) $V(x^2)$

The proof is on the HW3. In $\mathbb{A}^2_{\mathbb{C}}$, there are 6 types (HW3S2).

What about cubic curves in $\mathbb{P}^2$?

5.4 5 Points in $\mathbb{P}^2$ Determine a Conic.

Proposition 130. Given 5 distinct points in $p_1, ..., p_5 \in \mathbb{P}^2$. Then

1.) There are conics $C \subseteq \mathbb{P}^2$ such that $\{p_1, ..., p_5\} \subseteq C$
2.) The conic C is unique unless 4 of the p_i are collinear.
3.) C is nondegenerate if 3 points are collinear.

PROOF 131. We only prove the existence part. The uniqueness is left to HW3 Problem 3.

(As we see in a moment) the set of all conics in $\mathbb{P}^2$ is just $\mathbb{P}(V_{2,3})$ (i.e. the projective space of degree 2 polynomials in 3 variables) which is isomorphic to $\mathbb{P}^5$ (from the general isomorphism $\mathbb{P}(V_{d,n}) \simeq \mathbb{P}^{\binom{d+n}{n}-1}$ discussed at the beginning of lecture). The first isomorphism is PGL_3k invariant!

Fix a point $p_1 = [x : y : z] \in \mathbb{P}^2$. The conic $C = V(f)$ passes through p_1 iff

$$ax^2 + bxy + ... + eyz + fz^2 = 0 \tag{150}$$

Fixing x, y, z, we have a *linear equation* in $a, b, c..., f$

$$\begin{pmatrix} a \\ \vdots \\ \vdots \\ f \end{pmatrix} \cdot \begin{pmatrix} x^2 \\ \vdots \\ \vdots \\ z^2 \end{pmatrix} = 0 \tag{151}$$

Here let

$$p_i = [x_i : y_i : z_i], \quad v_i := (x_i^2, ..., z_i^2) \qquad i = 1, ..., 5 \tag{152}$$

So, the space of all conics H_i satisfies the isomorphism

$$H_i := \left\{(a, b, ..., f) \in \mathbb{P}^5 : (a, b, ..., f) \cdot v_i = 0\right\} \simeq \mathbb{P}^4 \subseteq \mathbb{P}^5 \qquad i = 1, ..., 5 \tag{153}$$

since it is just a hypersurface defined by the one (linear) equation (so, it is just one dimension lower than the entire space).

Let $\pi : \mathbb{A}^6 \setminus \{0\} \to \mathbb{P}^5$ be the projection map and

$$W_i := v_i^{\perp} = (x_i^2 : x_i y_i : ... : z_i^2)^{\perp} \subseteq \mathbb{A}^6 \setminus \{0\} \tag{154}$$

Here, taking the orthogonal complement makes sense since we are just exploiting the vector space structure of $\mathbb{A}^n$. The space W_i is just the copy of H_i in the affine space, and indeed, we have that $H_i = \pi(W_i)$, i.e. H_i is just the projection of W_i onto the projective 5-space. In words, W_i are just the possible coefficients for the conic *with redundancy* since obviously scaling the coefficients still gives the same conic. In order to eliminate this redundancy, we mod out by scalars to get the space H_i which we defined earlier.

But p_i are distinct iff v_i are distinct iff H_i are distinct (all of which follow immediately from the definitions).

Now from our definitions,

$$\begin{aligned} H_1 \cap ... \cap H_5 &= \pi\left(\bigcap_{i=1}^{5} W_i\right) \\ &\supseteq \pi(\text{line}) \\ &\supseteq \text{points in } C \end{aligned}$$

where we get the second inclusion since it is an intersection of 5 hyperplanes W_i in affine 6-space. We have the last inclusion because by definition, a point which is in all of $H_1, ..., H_5$ is a valid set of coefficients for a conic. This establishes the existence.

□

Remark 132. "Generic hypersurface": throw away bunch of Zariski closed sets from the set of all hypersurfaces

6 Lecture 6. -Thursday, 10.18.2018

Where we're headed.

1.) Quasiprojective varieties
2.) Primary ideals: prime powers
3.) Main invariant of this class: Hilbert polynomial

6.1 Irreducible Components of Projective Varieties

A lot of this follows Arrondo.

Definition 133. A **quasiprojective variety** Y is the intersection of a projective variety $X \subseteq \mathbb{P}^n$ with a Zariski open set in $\mathbb{P}^n$, i.e.

$$Y = X \setminus Z \tag{155}$$

with $Z \subseteq X$ is a projective variety.

Zariski closed sets are just subvarieties.

Example 134. Projective varieties.

Example 135. Affine varieties.

Example 136. $\mathbb{P}^2 \setminus$ (curve)

Example 137. Many moduli (/parameter) spaces, for instance

$$\begin{aligned}\mathrm{poly}_n\mathbb{C} &= \{\text{square free, monic polynomials}\} \\ &= \mathbb{C}^n \setminus \{\Delta_n = 0\} \\ &= \mathrm{conf}_n\mathbb{C} = \left(\mathbb{C}^n \setminus \bigcup_{i\neq j} \{z_i = z_j\}\right)/\mathbb{S}_n\end{aligned}$$

The following is only useful in AG.

Definition 138. A **topological space** X **is irreducible** if

$$X = Z_1 \cup Z_2; Z_1, Z_2 \text{ closed} \implies X = Z_1 \text{ or } X = Z_2 \tag{156}$$

Exercise 139. If $X \subseteq Z_1 \cup Z_2$, Z_i closed, then X is contained in one of them.

Remark 140. Arrondo reserves quadiprojective for irreducible varities.

Remark 141 (On General Math.). Irreducibility: we do it so that if we prove stuff for only irreducible, then we get the property for everything via unique decomposition of varieties.

E.g. 3-manifolds are prime: any 3 manifold has unique prime decomposition.

Proposition 142. A projective variety is irreducible iff $I(X)$ is prime.

PROOF 143. Same as in affine, via Dictionary between irreducible projective variety and prime homogeneous proper ideals.

□

6.2 Primary Ideals.

Definition 144. An **ideal** I **in a commutative ring** R **is primary** if every zero divisor in R/I is nilpotent [11]

Exercise 145. The above is equivalent to Arrondo's definition: if $FG \in I$, $F \notin I$, then $G^n \in I$ for some $n \geq 1$.

Example 146. If I is a maximal ideal, then I is prime, so I is primary.

Example 147 (Key Example.). The primary ideals in $\mathbb{Z}$ are exactly prime powers (p^d) and the trivial one (0).

[11] Nilpotent: in matrix ring, just think of stuff above the diagonal.

Proposition 148. If J is primary, then its radical is prime.

PROOF 149. Homework: If $P := \text{rad}J$, then J is P**-primary**. □

Remark 150. Converse is false. For $\mathbb{Z}$, the converse is true (via classification of the ideals.)

Consider $R = k[x, y]$, $I = (xy, y^2)$. So, rad$I = (xy, y) = (y)$ which is prime. But I is not primary because $xy \in I$ and $y \notin I$ but $x^n \notin I$.

Remark 151 (Nice thing about commutative rings.). Helps reason algebraic varieities in terms of just rings, but stuff may not be true....

Remark 152. Make sure you know how to justify details, e.g. for why (y) is prime!

Definition 153. A **commutative ring** S **is graded** if

$$S = \oplus_d S_d \tag{157}$$

as abelian groups (or as vector spaces if S), and such that

$$S_i \cdot S_j \subseteq S_{i+j} \tag{158}$$

Example 154 (Key Example.). Polynomials. Then S_d are the degree d homogeneous polynomials:

$$S = k[x_0, ..., x_n] \tag{159}$$

then

$$S_d := k[x_0, ..., x_n]_{(d)} \tag{160}$$

Example 155. If I is a homogeneous irreducible ideal , then

$$S = k[x_0, ..., x_n]/I \tag{161}$$

then

$$S_d := k[x_0, ..., x_n]_{(d)}/I_{(d)} \tag{162}$$

Notation. From this point on, S will always be a graded Noetherian commutative ring (e.g. polynomial ring), and R is a commutative Noetherian ring.

Definition 156. A **homogeneous ideal** I **is irreducible** if

$$I = I_1 \cap I_2 \implies I = I_1 \text{ or } I = I_2 \tag{163}$$

We work up to the primary decomposition.

Proposition 157. Any homogeneous ideal I in R, is a finite intersection of irreducible ideals.

PROOF 158. If I is irreducible, then nothing to prove. Otherwise, take $I = I_1 \cap I_2$ and $I \neq I_j$, and is both irreducible, then we're done. If not, $I = (I_3 \cap I_4) \cap I_2$ and this terminates because Noetherian.

□

Proposition 159. In R, irreducible ideals I are primary.

PROOF 160. Consider the **annihilator**

$$\text{Ann}(J) = \{x \in R | xJ = 0\} \tag{164}$$

By passing to a ring $S := R/I$ (not graded), it's enough to prove that if $(0) \subseteq S$ is irreducible then it is primary. (Just consider the quotients and nilpotent elements!)

Suppose $xy = 0$, $y \neq 0$ for $x, y \in S$. We want $x^d = 0$ for some $d \geq 0$. Consider

$$\text{Ann}(x) \subseteq \text{Ann}(x^2) \subseteq \text{Ann}(x^3) \subseteq ... \tag{165}$$

BUT S is Noetherian! So, there exists n for which $\text{Ann}(x^n) = \text{Ann}(x^{n+1})$.

Now we want to show that $(x^n) \cap (y) = 0$. $a \in (y)$ implies $ax = 0$. So, if $a \in (x^n)$, then $a = bx^n$. So $ax = bx^{n+1} = 0$. So,

$$b \in \text{Ann}(x^{n+1}) = \text{Ann}(x^n) \tag{166}$$

so, $bx^n = 0$.

But (0) is irreducible by assumption, so since $(y) \neq 0$ and $(x^n) = 0$. Thus, $x^n = 0$ which concludes the proof. □

The previous two propositions imply the following.

Definition / Proposition 161. Every ideal I in a Noetherian ring R *has* a **primary decomposition**

$$I = I_1 \cap I_2 \cap .... \cap I_r \tag{167}$$

where each I_j is primary.

We have *uniqueness* if

1.) can take $\text{rad} I_k \neq \text{rad} I_l$
2.) can take $I_j \not\subset \bigcap_j I_j$
3.) ...

Example 162 (Key Example). For $R = \mathbb{Z}$, this is just the decomposition into prime powers.

Take $(72) \subseteq \mathbb{Z}$. Then

$$(72) = (2^3) \cdot (3^2) \tag{168}$$

where the radical of the RHS are respectively $(2), (3)$.

Example 163. Let

$$I = (x^2, xy) \subseteq k[x, y] \tag{169}$$

Then

$$I = P_1 \cap P_2^2 \tag{170}$$

for $P_1 = (x), P_2 = (x, y)$. They are both prime, and P_2^2 is primary.

Example 164. Consider

$$(x^3 - xy^3) \subseteq k[x, y] \tag{171}$$

$$(x^3 - xy^3) = (x) \cap (x^2 - y^3) \tag{172}$$

and the RHS are both prime. Then

$$V(x^3 - xy^3) = V(x) \cup V(x^2 - y^3) \tag{173}$$

Proposition 165. Any projective variety $X \subseteq \mathbb{P}^n$ decomposes

$$X = Z_1 \cup ... \cup Z_r \tag{174}$$

where each $Z - i$ is irreducible projective variety (which are the **irreducible components of** X). They are unique if $i \neq j$ then $Z_i \not\subseteq Z_j$.

Compare with affine proof:

PROOF 166. Let

$$I(X) = I_1 \cap ... \cap I_r \tag{175}$$

then

$$\text{rad} I(X) = \text{rad} I_1 \cap ... \cap \text{rad} I_r \tag{176}$$

and let $Z_i = V(\text{rad} I_i)$. Continue as in affine case.

□

6.3 Hilbert Polynomials.

Hilbert polynomials is probably the deepest invariant from Comm. Alg. (There are other cool stuff from topology.)

Here is the setup. Let k be a field, and let $R = k[x_0, ..., x_n]$. $I \subseteq R$ is a homogeneous ideal and $S := R/I$.

Definition 167. The **Hilbert function of a commutative ring** $S = \oplus_{d \geq 0} S_d$ is

$$h_S : \mathbb{N} \to \mathbb{N} \tag{177}$$

$$h_S(d) = \dim_k(S_d) \tag{178}$$

Notation. In Arrondo,

$$h_I := h_{k[x_0,...,x_n]/I} \tag{179}$$

and for projective variety $X \subseteq \mathbb{P}^n$,

$$h_X := h_{I(X)} \tag{180}$$

Example 168. Suppose X contains just a point, e.g. $X = \{[1:0:...:0]\} \subseteq \mathbb{P}^n$, then

$$I = (x_1, ..., x_n) \tag{181}$$

so,

$$\begin{aligned} h_X(d) &= h_{k[x_0,...,x_n]/(x_1,...,x_n)}(d) \\ &= \dim_k \left(k[x_0, ..., x_n]_{(d)}/(x_1, ..., x_n)_{(d)}\right) \\ &= \dim_k \left(k[x_0]_{(d)}\right) \\ &= 1 \ \forall d \\ &\geq 0 \end{aligned} \tag{182}$$

where

$$(x_1, ..., x_n)_{(d)} = (x_1, ..., x_n) \cap k[x_0, ..., x_n]_{(d)} \tag{183}$$

and the quotient $k[x_0, ..., x_n]_{(d)}/(x_1, ..., x_n)_{(d)}$ is a quotient of *vector spaces*, not quotients of *rings*.

Exercise 169. After linear change of variables on the projective space, Hilbert polynomial does not change.

Example 170. Take $X = \mathbb{P}^n$, $I(X) = (0)$ and

$$h_{\mathbb{P}^n}(d) = \dim_k(k[x_0, ..., x_n]_{(d)}/(0)) = \binom{n+d}{d} \tag{184}$$

Notice that clearly by definition, for all $X \subseteq \mathbb{P}^n$,

$$h_X(d) \leq h_{\mathbb{P}^n}(d) = \binom{n+d}{d} \tag{185}$$

(which is a trivial linear algebra fact). So, we have an upper bound on the Hilbert polynomial.

Example 171. Let $p, q, r \in \mathbb{P}^2$ be distinct. Let X consist of these points. Then

$$h_X(1) = \begin{cases} 2 & p, q, r \text{ collinear} \\ 3 & \text{o/w} \end{cases} \tag{186}$$

(Think why geometrically by passing from projective to affine. It's just linear algebra)

Here's the proof. By definition

$$h_X(1) = \dim_k \left(k[x, y, z]/I(X)_{(1)}\right) \tag{187}$$

But $f \in I(X)_1$ iff f is linear and vanishes at all three points. But if $V(f)$ is a line, then this is true iff p, q, r is on the line. So,

$$I(X)_{(1)} = \begin{cases} 0 & p, q, r \text{ not collinear} \\ f & \text{if collinear on } V(f) \end{cases} \tag{188}$$

6.4 Discussion Section. - 10.18.2018

Books.

1.) Hartshorne, ch.1
  (a.) See for Hilbert polynomial
2.) Shafarevich
3.) Reid
4.) Gathmann, *Plane Algebraic Curves*
5.) Milne, *Algebraic Geometry*
6.) Commutative Algebra:
  (a.) Atiyah-MacDonald
  (b.) Eisenbud
  (c.) Milne

Comments on HW 2.

1.) Problem 1: Take $P \subseteq R \setminus S$... note that maximal ideal of a family is not maximal ideal of the ring
2.) Noetherian Rings:
  (a.) Ideals are finitely generated
  (b.) $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ ... terminates
  (c.) Every nonempty family of ideals has a maximal element (*not* ideal)
3.) Arrondo 1.4
  (a.) Note that $\alpha x + \beta y + \gamma z$ does *not* give a line

Homework 3.

1.) S3: use quotient topology
2.) 6 c.)
  (a.) consider the charts $\mathbb{A}_{x_i \neq 0}$ for $i = 1, 2, 3$.
  (b.) see if we have a decomposition in $k[x, y, z, w]/(wz - xy, wy - x^2)$
  (c.) Zariski closure: look at $\mathbb{A}_{w \neq 0}$ and consider $V(y - x^2, z - x^2)$; obtain irreducibility
3.) Arrondo 1.17 (Plücker embedding)
  (a.) Grassmannian in projective space
4.) 5. Projective Geometry
  (a.) Work in $\mathbb{A}$, i.e. points as lines, and lines as planes
  (b.) Practice Problem: $p_1, p_2, p_3 \in \mathbb{P}^1$; wts: there exists a unique projective transformation ($PGL_1\mathbb{C}$) sending (p_1, p_2, p_3) to $((1:0), (0:1), (1:1))$. Idea: just work in $\mathbb{A}^2$
  (c.) Practice Problem: $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$ no plane containing them all ; wts: there exists a unique projective transformation ($PGL_2\mathbb{C}$) sending (p_1, p_2, p_3, p_4) to (Q_1, Q_2, Q_3, Q_4).
5.) Arrondo 0.8
  (a.) I homogeneous iff generated by homogeneos elements:
  (b.) If I is homogeneous then the quotient S/I has a natural structure of graded ring.
    i.) Take graded ring $S = \oplus_d S_d$.
    ii.)
  (c.)

Some More Clarifications on HW2.

1.) 3 b. Similar Problem: Find decomposition of

$$X := V(x^2 - y^2, x^3 + xy^2 - y^3 - x^2y - x + y) \tag{189}$$

  (a.) If we let f, g be the respective expressions, then we can factor

$$f = (x - y)(x + y), \; g = (x - y)(x^2 + y^2 - 1) \tag{190}$$

Therefore,

$$\begin{aligned} X &= V(x-y) \cup V(x+y, x^2+y^2-1) \\ &= V(x-y) \cup V(x+y, 2x^2-1) \\ &= V(x-y) \cup \left\{ \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}} \right) \right\} \end{aligned} \tag{191}$$

Week 4. Hilbert Polynomials.

7 Lecture 7. -Tuesday, 10.23.2018

7.1 Announcements.

1.) First Mditerm
  (a.) Next Thursday, Nov. 1
  (b.) In class closed books; just bring pencil
  (c.) Problems:
    i.) Everything up until next Tuesday
    ii.) Use commonsense
    iii.) Not everything from books
    iv.) 5 questions; one is short answers (T/F)
    v.) If it takes more than half a page, you're doing something wrong
    vi.) Computational; very basic stuff: e.g. vanishing set, primary decompositions
    vii.) Know: definitions, examples... not necessarily all the proofs; just know how they work
  (d.) Farb: won't give points if illegible; Faidon is nicer... emphasis on communication; generous grading tho

7.2 Hilbert Functions and Graded Modules.

We will do some proofs that are different from Arrondo.

Let R be a graded ring. (We can of course think $R = k[x_0, ..., x_n]$.)

Definition 172. Over a graded ring R, an R**-module is graded** if

$$M = \oplus_{i \in \mathbb{Z}} M_i \tag{192}$$

as abelian group and $R_i \cdot M_j \subseteq M_{i+j}$.

Remark 173 (How to Remember This Definition.). On top of the usual structure of a graded ring (which we can just think of as just as a generalized polynomial ring), we have a vector space (/module) over it which respects the grading of the ring. So in particular, we certainly need a *graded module* to be over a *graded ring*.

Three big examples:

Example 174. $M = R$

Example 175. $M =$ any homogeneous ideal I in R; $I_n = I \cap R_n$.

Example 176. If M is a graded R-module, and N= graded submodule. Then the quotient [12] M/N is a graded R-module with

$$\begin{aligned} (M/N)_n &= (M_n + N)/N \\ &= \{m + N : m \in M_n\} \end{aligned} \tag{193}$$

[12] Think the key example: R/I for ideal I.

Example 177. If $J := (x^2 - y^2)$

$$(k[x,y]/J)_2 = \langle x^2 + J, xy + J \rangle \tag{194}$$

Definition 178. The **Hilbert function of a finitely generated graded** $k[x_0, ..., x_n]$**-module** M is

$$h_M(d) = \dim_k(M_d) \tag{195}$$

where the dimension is in the sense of vector spaces.

Remark 179. If $X \subseteq \mathbb{P}^n$ is a projective variety, then

$$\begin{aligned} h_X(d) &= \dim_k \left(k[x_0, ..., x_n]_{(d)}/I(X)_d\right) \\ &= \dim_k \left(k[x_0, ..., x_n]_{(d)}\right) - \dim_k I(X)_d \end{aligned}$$

We can take this as a definition. (Arrondo does.)

Example 180. Let $X \subseteq \mathbb{P}^n$ be a nonempty curve i.e. $X = V(F)$ and $F \in k[x_0, x_2, x_2]_{(d)}$ irreducible. As a k-vector space,

$$I(X) = (F) = \{pF : p \in k[x_0, x_2, x_2]\} \tag{196}$$

So $I(X)_d$ is a $k[x_0, x_2, x_2]$-module (with respect to usual multiplication) and

$$\begin{aligned} I(X)_m &= \{pF : p \in k[x_0, x_2, x_2]_{m-d}\} \qquad m \geq d \\ &\simeq k[x_0, x_2, x_2]_{m-d} \end{aligned}$$

Thus, for $m \geq d$,

$$h_X(m) = \begin{cases} \binom{m+2}{2} & m < d \\ \binom{m+2}{2} - \binom{(m-d)+2}{2} = dm - \frac{d(d-3)}{2} & m \geq d \end{cases} \tag{197}$$

This is a linear polynomial in m for $m \geq d$. This is a general fact for. (Hilbert-Szergy(?) theorem...see Arrondo)

Remark 181. Grading of a coset: in R/I with x^{40} then x^{50} is grading 10. (Work out this example for the definition of graded R-module.)

The last example leads to the following.

Proposition 182 (Hilbert). Let $R = k[x_0, ..., x_n]$, and let M be a graded R-module (usually $M = R/I(X)$ for some $X \subseteq \mathbb{P}^n$). Then there exists $D \geq 0,\ P_I \in \mathbb{Q}[T], \deg P_I \leq n$ (the **Hilbert polynomial of** I) for which

$$h_{R/I}(d) = P_I(d) \qquad d \geq D \tag{198}$$

Remark 183. Arrondo sometimes calls this h_I to mean $h_{R/I}$.

Remark 184. Consider a concrete example, say $(x^7, x^2y^5, x^{10}yz^5)$. Multiply elements with polynomials, and evantually get a relation!

We give a different proof of Arrondo.

PROOF 185. *1.)* Consider an *exact sequence of vector spaces*

$$0 \to A_m \xrightarrow{\psi_m} A_{m-1} \to \ldots \xrightarrow{\psi_1} A_0 \xrightarrow{\psi_0 = 0} 0 \tag{199}$$

where A_i are vector spaces, $\psi_i : A_i \to A_{i-1}$ linear maps, and (recall:) **exact** if

$$\ker\psi_r = \mathrm{im}\psi_{r+1} \tag{200}$$

so in particular, ψ_1 is onto and ψ_m is injective.

Lemma. For all exact sequences, the alternating sums

$$\sum_{i=1}^{m} (-1)^i \dim_k A_i = 0 \tag{201}$$

Proof of Lemma in the Special Case. This has all the ideas for general case. Consider the short exact sequence

$$0 \to A \xrightarrow{\psi_2} B \xrightarrow{\psi_1} C \to 0 \tag{202}$$

Then as vector spaces, we have $B/A \simeq C$. So,

$$\dim C = \dim B - \dim A \tag{203}$$

$\square$

2.) Let $F : \mathbb{Z} \to \mathbb{Z}$, and suppose there exists $D \geq 0$ for which there exists $p \in \mathbb{Q}[T]$ with $\deg p \leq n-1$ such that

$$F(d) - F(d-1) = P(d) \qquad d \geq D \tag{204}$$

Then there exsits $Q \in \mathbb{Q}[T], \deg Q \leq n$ such that $F(d) = Q(d), d \geq D$. (Proof: Exercise, since it is elementary.)

3.) Idea: Induct on n.

For $n = 0$: Since M is finitely generated k-module (i.e. M is just a finite dimensional k-vector space),

$$M = \oplus_{d \in \mathbb{Z}} M_d \tag{205}$$

so $M_d = 0$ for d sufficiently large.

For induction step: The idea is to "mod out by a variable."Assume true for $k[x_0, \ldots, x_{n-1}]$, $n \geq 1$. Consider the map $\psi : M \to M$ given by

$$\psi : m \mapsto x_n \cdot m \tag{206}$$

We want this map to preserve gradings. Let $M(-1)$ be the graded R-module $M(-1)_d := M_{d-1}$. So,

$$\psi : M(-1) \to M \tag{207}$$

$$m \mapsto x_n \cdot m \tag{208}$$

which preserves the grading.

Aside. Consider

$$M = k[x, y, z]/(x^2x - xyz) \tag{209}$$

then

$$\psi : M(-1)_3 \to M_3 \tag{210}$$

$$m \mapsto zm \tag{211}$$

then ker$\psi \to x^2 - xy$.

Let K = kerψ is a graded R-module. Then there exists an exact sequence of graded k-vector spaces with degree preserving maps

$$0 \to K(-1) \to M(-1) \xrightarrow{\psi} M \to M/x_n \cdot M \to 0 \tag{212}$$

is exact. (Exercise.)

But now, by part 1., for all d,

$$h_M(d) - h_M(d-1) = h_{M/x_nM} - h_K(d-1) \tag{213}$$

and $M/x_nM, K$ are finitely generated as $k[x_0, ..., x_{n-1}]$-modules.

So by induction, the RHS agrees with the polynomial of degree $\leq n-1$ for all $d \geq D$ for some fixed D since it is just a difference of polynomials. Now we are done by part 2.

□

There will be geometric intuition for this later. Here is a corollary (of part 1 in the above proof).

Proposition 186. If $I, J \subseteq k[x_0, ..., x_n]$ homogeneous ideals, then $h_{R/(I+J)} = h_{R/I} + h_{R/J} - h_{R/(I\cap J)}$.

PROOF 187. There exists an exact sequence of graded vector spaces (in fact R-modules)

$$0 \to R/(I \cap J) \to R/I \times R/J \to R/(I+J) \to 0 \tag{214}$$

and in the above

$$\overline{f} \to (\overline{f}, \overline{f}) \tag{215}$$

in the first map and

$$(\overline{g}, \overline{h}) \to \overline{g} - \overline{h} \tag{216}$$

in the second. Now apply part 1.

□

Remark 188 (Moral of the Story). If we know what exact sequence, then we can prove it!

Remark 189 (Geometric intuition). If we have a variety X in projective space and intersect with hyperplane(?), we somehow want the dimension to be $d-1$ dimensions.

In terms of polynomials, intersection is where *both* polynomials vanish. This is what we are doing here.

Here is the second corollary.

Proposition 190. If $X, Y \subseteq \mathbb{P}^n$ disjoint, then there exist a D such that

$$h_{X \cup Y}(d) = h_X(d) + h_Y(d) \qquad d \geq D \tag{217}$$

i.e., $P_{X \cup Y} = P_X + P_Y$. In particular, if $X = \{p_1, ..., p_r\} \subseteq \mathbb{P}^n$ then $h_X(d) = r$ for all $d \geq r - 1$.

PROOF 191. If X, Y are disjoint, then

$$I(X \cup Y) = I(X) \cap I(Y) \tag{218}$$

Now apply the previous corollary. □

7.3 Dimension of Variety

Definition 192. Let $X \subseteq \mathbb{P}^n$ be a projective variety. The **dimension of** X is

$$\dim X := \deg P_X \tag{219}$$

where P_X is the Hilbert polynomial of X.

If $\dim X = 1$, X is a **curve**.

If $\dim X = 2$, X is a. **surface**.

If $\dim X = 3$, X is a **3-fold**.

See HW4 for equivalent definition of dimension.

Remark 193 (Dimension of Manifolds). On any Zariski open point on variety, it will be a manifold. (Avoid singularities.)

Example 194. The dimension of a finite set of points is 0 by the second corollary.

Example 195. $\dim \mathbb{P}^n = n$ since

$$h_{\mathbb{P}^n}(m) = \binom{m+n}{m} \sim m^n \tag{220}$$

Example 196. $\dim(V(F)) = n - 1$ if $F \in k[x_0, ..., x_n]_{(d)}, d \geq 1$.

Example 197.

$$V(xy, xz) = V(x) \cup V(y, z) \subseteq \mathbb{P}^3 \tag{221}$$

Exercise 198 (Arrondo. Prop. 5.7 (iv)). If $X = X_1 \cup ... \cup X_r$, X_i irreducible, then

$$\dim X = \max_i(\dim X_i) \tag{222}$$

We will state the following geometrically (as opposed to ideals as Arrondo does):

Proposition 199. Let $X = V(J) \subseteq \mathbb{P}^n$ and $Y = V(F), F \in k[x_0, ..., x_n]_d, d \geq 1$ be a hypersurface in $\mathbb{P}^n$. Assume Y does not contained any irreducible component of X. (Or equivalently, F is not contained in any prime ideal in the primary decomposition of J.)

Then $\dim(X \cap Y) = \dim X - 1$, i.e. if we intersect a variety with a hypersurface, the dimension goes down by 1 unless it is contained in the hypersurface.

1 Lecture 8. -Thursday, 10.25.2018

1.1 Theory of Dimensions and Varieties

Proposition 1. Let $I \subseteq k[x_0, ..., x_n]$ be a homogeneous ideal. Let $F \in k[x_0, ..., x_n]_{(d)}$ such that F is not in any prime associated to the prime decomposition of I. Then

$$P_{I+(F)}(\ell) = P_I(\ell) - P_I(\ell - d) \qquad \forall \ell \geq d \tag{1}$$

Key: doesn't depend on F, only of $\deg F$.

Equivalently, for all projective varieties $X \subseteq \mathbb{P}^n$, for all hypersurface

$$Y := V(F) \qquad \deg F \geq 1 \tag{2}$$

If $Y \not\supset$ any irreducible component of X, then

$$P_{X \cap Y}(\ell) = P_X(\ell) - P_X(\ell - d) \qquad \forall \ell \geq d \tag{3}$$

Example 2. If we had $V(xy)$ and $V(x)$, we have the same dimension. We want to rule this out.

Again: find the correct exact sequence and apply argument about dimensions.

PROOF 3. Let $R := k[x_0, ..., x_n]$ and

$$(R/I)_{(-e)}(d) := (R/I)_{e-d} \tag{4}$$

(so we allow for negative grading). Let

$$\psi : (R/I)(-d)_e \to R/I \tag{5}$$

$$\overline{f} \mapsto \overline{Ff} \tag{6}$$

which is a grading preserving linear map of graded vector spaces.

The key step is to show that

$$0 \to (R/I)(-d) \xrightarrow{\psi} (R/I) \to R/(I + (F)) \to 0 \tag{7}$$

is exact. (Arrondo lemma 3.12) This uses the hypothesis: $(F) \subset I$. We need hypothesis in order to avoid multiplying by just 0.

From here, we just apply the same argument then we are done.

□

Here is a corollary.

Proposition 4 (Theorem 1). Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $Y := V(F)$ be any hypersruface not containing any irreducible component of X. Then

$$\dim(X \cap Y) = \dim X - 1 \tag{8}$$

Once again we need the hypothesis to avoid $V(x), V(xy)$ situation.

PROOF 5. Just expand the polynomial and apply lemma.

$$\begin{aligned}
P_{X\cap Y}(\ell) &= P_X(\ell) - P_Y(\ell - d) \\
&= (a_0\ell^m + a_1\ell^{m-1} + ...) - (a_0(\ell-d)^m + a_1(\ell-d)^{m-1}) \\
&= a_0\ell^m - \big(a_0\ell^m - ma_0\ell^{m-1}d + a_1\ell^{m-1} + \text{lower order}\big) + a_1\ell^{m-1} \\
&= (a_1 + a_0dm - a_1)\ell^{m-1} + \text{lower order}
\end{aligned}$$

where $d := \deg F$. Let $m := \deg X$. Therefore, $\mathbb{P}_{X\cap Y} = m - 1$.

□

Example 6. The case where $X = \mathbb{P}^n$ is noteworthy.

$$\dim(\text{hypersurface } Y \subseteq \mathbb{P}^n) = n - 1 \tag{9}$$

and the converse is also true. If $X \subseteq \mathbb{P}^n$ with $\dim X = n - 1$, then $X = V(F)$ for some F.

Remark 7. Number of equations to dimension of the variety is not immediately obvious because of independence of equation issue.

Here are some corollaries.

Proposition 8 (Corollary 1). If $X \subseteq Y$ and Y is irreducible and they have the same dimension, then they are the same variety

Remark 9. It's like not having 2 dimensional submanifold, just 1.

PROOF 10. If $X \subsetneq Y$, then $I(Y) \subsetneq I(X)$ thus there exists an $F \in I(X) - I(Y)$ homogeneous. So, $X \subseteq Y \cap V(F)$ implies $\dim X = \dim Y - 1$. This contradicts the hypotheses.

□

So, the only way two things of the same dimension to contain each other is if it is an irreducible component.

Proposition 11 (Corollary 2). Suppose $X \subseteq \mathbb{P}^n$ projective variety, $r := \dim X$. Then for all homogeneous polynomials $F_1, ..., F_s$ for $s \leq r + 1$, then

$$\dim(X \cap V(F_1, ..., F_s)) \geq r - s \tag{10}$$

(Note: $\dim Z < 0$ just means $Z = \emptyset$)

So every hypersurface you intersect by drops *at most* one dimension. At most because (again) dependence of the equations.

Remark 12 (For the purpose of the course.). Just know all the theorem for the irreducible case since it only adds technicalities.

Remark 13 (Grading). Can correct score. We can discuss stuff. (Personal reason.) Will actually listen.

PROOF 14. Reduce to the case X irreducible, and by induction done by theorem 1.

□

Remark 15. Rational maps and birational maps: every variety isomorphic to smooth hypersurface

Another corollary.

Proposition 16 (Corollary 3). $X \subseteq \mathbb{P}^n,\ n \geq 1$ is a hypersurface iff $\dim X = n-1$.

PROOF 17. We just did the forward case. For the reverse direction, if X irreducible, then $I(X)$ is prime. Then $\dim X = n-1$ implies there is a homogeneous polynomial F with $\deg F > 0$ such that $F \in I(X)$. By taking an irreducible factor of F; can assume F is irreducibel, so (F) is prime so $V(F)$ is irredcible.

Now $I(X) \supseteq (F)$ then $V(F) \subseteq V(I(X)) = X$ and X irreducible, so $\dim V(F) = n-1 = \dim X$ so $V(F) = X$. □

Remark 18 (Implicit Fact.). If $F = F_1^{r_1}...F_m^{r_m}$ is irreducible decomposition, then the irreducible decompositio is

$$V(F) = V(F_1^{r_1}) \cup ... \cup V(F_m^{r_m}) \tag{11}$$

Proposition 19 (Arrondo Lemma 3.16; rephrased). Let $X \subseteq \mathbb{P}^n$ *nonempty* Then

$$\dim X = \min\left\{n-d : \text{any linear } \mathbb{P}^d \subseteq \mathbb{P}^n \text{ intersects } X\right\} \tag{12}$$

This can be one of the definitions of dimension.

Remark 20. m-primary components; $(x_0,...,x_n)$ vanishing locus is empty, so we don't want this. (This is $X \neq \emptyset$.)

Remark 21 (Intuition for Dimension). Recall from linear algebra: Let W be an n-dimensional vector space over k, and let $U, V < W$ be subspaces. Then the codimensions add:

$$\text{codim}(U \cap V) \leq \text{codim}U + \text{codim}V \tag{13}$$

And generically, they are equal. Equiavalently, if the dimension does not add up to dimension of whole space, i.e. if they exceed, then they are going to intersect.

Exercise. Theorem is true for X some linear $\mathbb{P}^r \subseteq \mathbb{P}^n$ for some r.

PROOF 22 (Special Case.). Let k algebraically closed (i.e. $\mathbb{C}$). Let $X = V(F) \subseteq \mathbb{P}^n$ hypersurface. This meets any line $L \subseteq \mathbb{P}^n$ but not any point ($V(F)$ does not meet every arbitrary point), so $\dim X = n-1$.

Take $V(F) \cap L$ for which

$$V(A_2,...,A_n),\ A_i \in k[x_0,...,x_n] \tag{14}$$

(for instance $L = V(x_2,...,x_n)$). So

$$V(F) \cap L = V(F|_L) \tag{15}$$

But $F|_L$ is a homogeneous polynomial in two variable. Now restricting to each affine space U_i, it is a polynomial in one variable [1] and for at least some i, $F|_{L\cap U_i}$ is a nonconstant one variable polynomial, so by FTA, $F|_{L\cap U_i}$ has a root.

[1] We can get a constant polynomial, e.g. x_0^3. Consider $F(x_0, x_1) = x_0^2 x_1 - 3x_1$

$\square$

Remark 23. Given $X = V(F)$ and a line $L := AL_0$ for $A \in GL_n k$. WTS: $X \cap L \neq \emptyset$.

$$\begin{aligned} A^{-1}X \cap L_0 &\neq \emptyset \\ X \cap AL_0 = X \cap L &\neq \emptyset \end{aligned}$$

2 Problem Session. Friday, 10.26.2018

2.1 On HW3.

1.) Problem 2.
  (a.) Approach 1. Complete the square.
    i.)
  (b.) Approach 2. Geometric -
    i.) Translate to origin $\to$ eliminates ax, by
    ii.) Rotate
    iii.)
2.) Problem 3.
  (a.) 4 points collinear
    i.) Nonuniqueness is immediate because we can take any line for the fifth point
    ii.) Veronese: $(x : y : z) \mapsto (x^2 : \ldots)$ then dimension less than 3
  (b.) Pf 2:
    i.) Look at $\lambda f + \mu g$
3.) S2: $V(x^2) = V(x)$, so irreducible.
4.) Problem 6:
  (a.) Pass to affine chart

2.2 HW4

1.) Arrondo 3.19
  (a.) Check: $x^2 + y^2(y-1)^2$
2.) 2.
  (a.) Hilbert functions: nonstandard;
  (b.) Dimension of a ring
    i.) Let X be a topological space. Supremum of all irreducible closed sets is a dimension.
    ii.) Krull dimension: take chains of prime ideals. Dimension is the supremum of all such chains.
    iii.) Take $X = V(I)$ variety: dimension of X as a topological space is same as dimesion of coordinate ring.
  (c.) 3 a.) just give a chain.
  (d.) Hilbert polynomials:
    i.) Filterations of modules and vector spaces;
    ii.) $\mathbb{A}^{n-1} \subseteq \mathbb{A}^n$; nonzero then $n-1$ and zero then n
    iii.) Theorem: k-algebra A; dimension of A if transcendence degree; adding a non-algebraic element raises degree 1
    iv.)
3.) Hilbert Polynomial of Veronese variety
  (a.) Sample computations. $(xy - z^2)$
    i.) Check for $d = 0, 1, 2$ and $d \geq 3$ and do combinatorics
  (b.) $(x^2 + y^2 - 1)$
    i.)

Week 5. Midterm.

8 Lecture 9. -Tuesday, 10.30.2018

Midterm.

1.) True or false
2.) 5 or 6 questions, maybe a bonus question for fun
3.) Stick to basics, e.g. Hilbert polynomial, dimension, Hilbert Basis Theorem
4.) Can't do anything too involved; there has to be a two line answer...
5.) No quasiprojective stuff; none of the obscure stuff

8.1 Review of Hilbert Polynomial.

1.) Recall that if $I, J \subseteq k[x_0, ..., x_n]$ are homogeneous ideals, then

$$h_{I+J} = h_I + h_J - h_{I\cap J} \tag{223}$$

hence

$$P_{X\cup Y} = P_X + P_Y \qquad X \cap Y = \emptyset \tag{224}$$

So, we can use this for a collection of finitely many points. So, the Hilbert function only depends on the number of points, but the Hilbert polynomial will depend on the *configuration* of points.

Remark 200 (Primary Decomposition). Recall that prime ideals correspond to irreducible varieties in *affine* space whereas prime *not containing the irrelevant ideal* (i.e. not whole ring and irrelevant ideal) correspond to irreducible in projective space.

2.) Let $J \subseteq k[x_0, ..., x_n]$ is a homogeneous ideal. Let $F \in k[x_0, ..., x_n]_{(d)}, d \geq 1$. Suppose $F \notin J'$ for any prime ideal J' associated to J. Then

$$P_{J+(F)}(\ell) = P_I(\ell) - P_I(\ell - d) \qquad \forall \ell \geq d \tag{225}$$

(This is what to use any time we intersect and induct.)
Equivalently, if $X = V(F) \subseteq \mathbb{P}^n$ is nonempty, then it is a hypersurface in $Y = V(F) \subseteq \mathbb{P}^n$.

If $F \in J$ then $V(F) \supseteq X$ or more generally, if $F \in J'$.

If Y strictly contains any irreducible component of X, then

$$P_{X\cap Y}(\ell) = P_X(\ell) - P_X(\ell - d) \tag{226}$$

where the RHS does not depend on F but only on $\deg F$. In particular,

$$\dim(X \cap Y) = \dim X - 1 \tag{227}$$

as long as $d \geq 1$.

Remark 201 (Primary Decomposition). The primary decomposition of $I(X)$ is $I_1 \cap ... \cap I_r$ iff

$$X = X_1 \cup ... \cup X_r \tag{228}$$

where $X_i = V(\text{rad}(I_i)), 1 \leq i \leq r$ is irreducible.

The only thing that could go wrong is just $I(V(J)) = \text{rad}J$. i.e. just because you are in the vanishing set of an ideal does not mean you are in the ideal.

A powerful method in AG is to intersect with hypersurface and *induct*!

3.)

Proposition 202 (Arrondo Lemma 3.16). Let $X \subseteq \mathbb{P}^n$ nonempty projective variety. Then

$$\dim X = \max\{d : \forall W \subseteq \mathbb{P}^n \text{ linear subspace with codim}W = d, X \cap W \neq \emptyset\} \tag{229}$$

PROOF 203. Let $W \subseteq \mathbb{P}^n$ be a linear subspace with codimension at least $\dim X$ (i.e. the space is large). We claim that W is disjoint from X.

Write

$$W = V(A_1, ..., A_r) \qquad A_i \in k[x_0, ..., x_n]_{(i)} \tag{230}$$

and A_i are linearly independent. This is true for $d = 0$ by just linear algebra.

Then we have the SES

$$R/I(X) \to R/I(X) \to R/(I + (A_1)) \to 0 \tag{231}$$

where $R/(I + (A_1)) = V(X \cap V(A_1))$. So,

$$P_{X \cap V(A_1)}(\ell) = P_{I(X)+(A_1)}(\ell) = P_{I(X)}(\ell) - P_{(A_1)}(\ell - 1) \tag{232}$$

where this is the polynomial of degree $\dim X - 1$.

So, $X \cap V(A_i) \neq \emptyset$ for $d \geq 1$. Then we can iterate:

$$\begin{aligned} P_{X \cap W}(\ell) &= P_{X \cap (V(A_1) \cap ... \cap V(A_r))}(\ell) \\ &= P_{I(X)+I(A_1)+...+I(A_{r-1})}(\ell) - P_{I(X)+I(A_1)+...+I(A_{r-1})}(\ell - 1) \end{aligned}$$

So the degree is ≥ 0. We are using linear independence in the application of 2) (just by linear algebra).

And we can keep going to get

$$\begin{aligned} \dim(X \cap V(A_1)) &\geq \dim X - 1 \\ : \dim(X \cap V(A_1) \cap ... \cap V(A_{\dim X})) &\geq \dim X - \dim X = 0 \end{aligned}$$

and this,

$$X \cap V(A_1) \cap ... \cap V(A_r) \neq \emptyset \tag{233}$$

Now we just need to show that there exists a linear subspace $W \subseteq \mathbb{P}^n$ with $\text{codim}W = \dim X + 1$ such that it is disjoint from X.

Since X is nonempty, its ideal $I(X)$ does npt contain the irrelevant ideal (or any point of it). Geometrically, this just says that the complement of the variety does not contain every single hyperplane, i.e. the variety is empty.

So, there exists $A \in k[x_0, ..., x_n]_{(1)}$ which is not in $I(X)$. Therefore,

$$P_{I(X+(A_1)}(\ell) = P_{I(X)}(\ell) - P_{I(X)}(\ell - 1) \tag{234}$$

has degree $\dim X - 1$.

We repeat this (i.e. keep intersecting with $\dim X + 1$ hyperplanes).

□

Exercise 204. Hilbert polynomial has degree 0, then there are points. If the dimension is negative one, then it is an empty set.

Proposition 205. If $X = X_1 \cup ... \cup X_r$ is the irreducible decomposition of X, then

$$\dim X = \max_i \{\dim X_i\} \tag{235}$$

PROOF 206. ($\geq$.) This is immediate from just $X_i \subseteq X$ from which we get $\dim X_i \leq \dim X$.

($\leq$.) This is induction on r. $r = 1$ iff X is irreducible, so done. So take $r \geq 2$, and we must find some X_i with $\dim X_i = \dim X$.

But

$$\begin{aligned} I(X) &= I(X_1 \cup ... \cup X_r) \\ &= I(X_1 \cup ... \cup X_{r-1}) \cap I(X_r) \end{aligned}$$

So,

$$P_X = P_{X_1 \cup ... \cup X_{r-1}} + P_{X_r} - P_{(X_1 \cup ... \cup X_{r-1}) \cap X_r}$$

where $X_1 \cup ... \cup X_{r-1}) \not\supseteq X_r$, so the dimension goes down.

But we have

$$\deg P_X = d,\ \deg P_{X_1 \cup ... \cup X_{r-1}} \leq d,\ \deg P_{(X_1 \cup ... \cup X_{r-1}) \cap X_r} < d \tag{236}$$

(so just from numbers) either $\deg P_{X_1 \cup ... \cup X_{r-1}} = d$ or $\deg P_{X_r} = d$

□

See proof of projective Nullstellensatz in Arrondo.

8.2 Dimension via Chains (Noetherian Dimensions.)

Proposition 207.

$$\dim X = \max \{d : \exists X_0 \subsetneq ... \subsetneq X_d = X\} \tag{237}$$

and each X_i is a irreducible projective variety.

PROOF 208 (Proof for X irreducible). For $r \leq \dim X$, follows from $\dim X_{i+1} \geq \dim X_i + 1$ since X_{i+1} is irreducible and $X_i \subsetneq X_{i+1}$.

So $r \geq \dim X$, we must find some chain with $d = \dim X$. We induct on $\dim X$.

For $\dim X = 0$, X irreducible then X is just a point, so this is immediate.

For $\dim X \geq 1$, take $F \notin I(X)$. This implies

$$\dim(X \cap V(F)) = \dim X - 1 \tag{238}$$

So by induction, there exists a chain

$$X_0 \subsetneq ... \subsetneq X_{d-1} \tag{239}$$

where X_{d-1} can be a component of $X \cap V(F) \subsetneq X$. But this is a chain of length $d+1$.

□

Remark 209. Arrondo proves that the dimension of the quasiprojective set $X \setminus Z$, X irreducible, then

$$\dim(X \setminus Z) = \dim X \tag{240}$$

The left hand side is defined in terms of chains.

8.3 Review Sessions. - Faidon. Tuesday, 10.30.2018.

1.) k algebraically closed field, characteristic 0. We want to establish language between polynomial ring over k in n variables and the space $\mathbb{A}_k^n$.

The first of this is the vanishing set

$$V(f) = \{x \in \mathbb{A}_k^n | f(x) = 0\} \tag{241}$$

for $f \in k[x_1, ..., x_n]$.

Some elementary properties of the V operator:

1.) $V(S) = V((S))$, i.e. the vanishing set of a set is the same as the vanishing set of the ideal generated by the set.
2.) V operator is inclusion reversing: $I \supseteq J$ then $V(I) \subseteq V(J)$
3.) $V\left(\bigcup_\alpha I_\alpha\right) = \bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$
4.) $V(I \cdot J) = V(I) \cup V(J) = V(I \cap J)$ (where we used the fact that $I + J = R$ (coprime), then $IJ = I + J$)

And similarly, for the I operator:

1.) It is union reversing, i.e. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$
2.) Inclusion reversing: $X_1 \subseteq X_2$ then $I(X_1) \supseteq I(X_2)$
3.) $I((a_1, ..., a_n)) = (x_1 - a_1, ..., x_n - a_n)$
4.) $I(\emptyset)$ is the whole polynomial ring
5.) $I(\mathbb{A}^n)$ is the empty set.
6.) $I(X)$ is a radical ideal.

Some major theorems we discussed:

Hilbert Basis Theorem. R Noetherian, then $R[x]$ Noetherian. In particular, the polynomial ring over k is Noetherian, so vanishing sets of ideals can be written as vanishing sets of finitely many polynomials.

Nullstellensatz. If I is an ideal in the polynomial ring, then $I(V(I)) = \text{rad}I$.

Dictionary.
1.) Algebraically closed sets in $\mathbb{A}^n$ correspond to radical ideals in the polynomial ring.
2.) Irreducible algebraic sets (i.e. it cannot be written as union of closed sets one not contained in the other and vice versa) correspond to prime ideals
3.) Points correspond to maximal ideals $(x_1 - a_1, ..., x_n - a_n)$.

All three of the above came from the Nullstellensatz. We black boxed that any maximal ideal in a polynomial ring is of the form $(x_1 - a_1, ..., x_n - a_n)$.

Standard approach uses Zariski lemma.
Irreducible Decomposition. An algebraic set can be written as a union of its irreducible components

$$V(I) = V(p_1) \cup ... \cup V(p_r) \tag{242}$$

Recall from HW1 S3 that: if $f, g \in k[x, y]$ with $\gcd(f, g) = 1$, then $V(f, g) = V(f) \cap V(g)$ is a finite number of points.

Coordinate Rings and Morphisms. If I is radical, the we have the coordinate ring

$$\Gamma(I) := k[x_1, ..., x_n]/I \tag{243}$$

which are the ring of functions on $V(I)$.

And a morphism is a map $\varphi : V \to W$ such that

$$\varphi(a_1, ..., a_n) = (f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n)) \tag{244}$$

where $f_i \in k[x_1, ..., x_n]$.

Recall that morphisms on varieties are the same as morphisms on coordinate rings (via the notion of pullback). In abstract notations,

$$\text{Hom}_{Var}(V, W) \simeq \text{Hom}_{rings}(\Gamma(W), \Gamma(V)) \tag{245}$$

Primary Decompositions. Recall from HW3 Problem 1 that

$$\sqrt{I} = \bigcap_{P \supseteq I, \text{prime}} P \tag{246}$$

We can avoid using primary ideals in this way since (recall that) $\sqrt{I}$ prime iff I primary, and so I is just the intersections of primary ideals.

We never proved the proof of the primary decomposition theorem. See Atiyah-MacDonald for any commutative algebra questions.

Projective Spaces. Recall that we defined

$$\mathbb{P}^n_k; = \mathbb{A}^{n+1} \setminus \{0\} /\sim \tag{247}$$

where

$$(a_0, ..., a_n) \sim (b_0, ..., b_n) \tag{248}$$

iff there exists $\lambda \in k^*$ such that $a_i = \lambda b_i$.

We then had projective coordinates $[a_0 : ... : a_n]$, and the notion of an affine chart which gave a covering

$$\mathbb{P}^n = \bigcup_{i=0}^{n} \mathbb{A}^n_{a_i \neq 0} \tag{249}$$

where we used

$$\left\{ \left(\frac{a_0}{a_i}, .., 1, ..., \frac{a_n}{a_i} \right) \right\} \simeq \mathbb{A}^n_k \subseteq \mathbb{P}^n \qquad a_i \neq 0 \tag{250}$$

and

$$\{(a_0 : ... : 0 : ... : a_n)\} \simeq \mathbb{P}^{n-1} \subseteq \mathbb{P}^n \qquad a_i = 0 \tag{251}$$

which is called the hyperplane at infinity and we can write $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$.

Projective Dictionary. We also have the dictionary between homogeneous ideals in the polynomial ring and the algebraic sets in P^n_k given by the maps V, I.

Projective Nullstellensatz. For the projective Nullstellensatz, we had to additionally worry about the irrelevant ideal $(x_0, ..., x_n)$.

Here are the two most important morphisms:

Segre Embedding. The Segre mebedding s the map $\varphi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ given by the map

$$(x_0 : ..., x_n)(y_0 : ... : y_m) \mapsto (x_0 y_0 : ... : x_n y_n) \tag{252}$$

Veronese Embedding. $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$ and

$$[x_0 : ... : x_n] \mapsto [x_0^d : x_0^{d-1} x_1 : ... : x_n^d] \tag{253}$$

Some points on exercises:

1.) Don't hesitate to use geometric transformations.
   (a.) HW1 Problem 7(?): empty, whole set, or finitely many lines
      i.) Move it to the setting $x_1 = ...x_n = 0$. In this setting, your polynomial is just in x_0.
   (b.) Similarly for $PGL(n+1, k) = GL(n+1, k)/k^*$
2.) Projective Closures:
   (a.) (Hartshorne Exer 2.9, p.12): Take the set $Y \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ and we want to see $\overline{Y} \subseteq \mathbb{P}^n$. If we have $I(Y) = J$, then $I_h(\overline{Y}) = J^h$ where h indicates homogenization.
3.) Projective Geometry:
   (a.) The key observation is $\mathbb{A}^{n+1} \to \mathbb{P}^n$, then we have the correspondence

line points
planes lines

Generally, when finding generators with nice properties, we need Groebner bases. (So we don't need to do this...)

Homogenization of a Polynomial. If you have $g(y_1, ..., y_n) \in \mathbb{A}^n = \mathbb{A}^n_{x_0 \neq 0}$ then we can identify this with $g(x_1/x_0, ..., x_n/x_0)$ is degree r, then the homogenization is $x_0^r g(x_1/x_0, ..., x_n/x_0)$.

Look at the easier proofs (HW 1, HBT, geometric transformations) Homogenization is dirty, so probably not this. Neither is primary decomposition.

Know the Segre embedding for $n = 1$ because it gives twisted cubic, and it is a low dimensional thing.

Graded rings: not much to say... Just know the definitions.

Dimensions: For computations, use Hilbert polynomial. For intuition, use the chain definition from HW4 Problem 3.

Ways to show that stuff are irreducible: if I radical ideal, then $V(I)$ irreducible iff I prime iff $k[x_1, ...x_n]/I$ is integral domain. Another useful fact is, if X is irreducible, then every open set is dense inside it, i.e.

$$X \neq Y_1 \cup Y_2 \implies \emptyset \neq Y_1^c \cap Y_2^c \tag{254}$$

So, at the end of the day, irreducibility comes down to showing the ideal is prime.

9 Midterm.

Week 6. Degree of a Projective Set; Bézout's Theorem.

10 Lecture 10. -Tuesday, 11.6.2018

Announcements.

1.) See new psets.
2.) Danii Rudenko is teaching on Thursday.

10.1 Degree of a Variety.

The degree is very important. Hilbert polynomial allows us to do intersection theory very easily.

We already saw $X = \mathbb{P}^n$ and $d = \dim X$. Then the **hypersurface lemma** (...degree n-1 intersect any hypersurface; we have been using this implicitly all this time) Now taking a generic hypersurface $V(H_1)$ (i.e. take space of hypersurfaces which is projecive space, so take Zariski open set and also take it so that it does not contain any irreducible component of X) so that

$$\dim(X \cap V(H_1)) = d - 1 \tag{255}$$

and $H_i \in k[x_0, ..., x_n]_{(1)}$.

Now

$$P_{X \cap V(H_1)}(r) = P_{I(X)+(H_1)} = P_X(r) - P_X(r-1) \tag{256}$$

where the RHS is degree $d - 1$.

For general hypersurface $V(H_1), ..., V(H_d)$ for $H_i \in k[x_0, ..., x_n]_{(d)}$ then

$$X \cap V(H_1, ..., H_d) \tag{257}$$

is 0-dimensional, so it is finite. Also,

$$P_{X \cap V(H_1,...,H_d)} = a(d!) \tag{258}$$

is a constant polynomial, and $P_X(\ell)$ has leading term $a\ell^d$.

Remark 210. I want to change intersection into union, but union of ideal is not an ideal, so that smallest ideal containing it is the sum.

Remark 211. When taking intersection, we leave the world of irreducible. (Consider intersection of two curves which intersect at two points.)

Definition 212. The **degree of a subvariety of** $X \subseteq P^n$ is

$$\deg X = a \cdot ((\dim X)!) \tag{259}$$

where $P_X(t)$ has leading term $at^{\dim X}$.

Remark 213. If we had a hypersurface defined by polynomial of degree d, then the degree of the subvariety should be degree d.

Example 214 (Projective Space). We have

$$P_{\mathbb{P}^n}(r) = \binom{n+r}{n} = \frac{1}{n!}r^n + \ldots \tag{260}$$

so $\deg \mathbb{P}^n = \frac{1}{n!}(\dim \mathbb{P}^n)! = 1$.

Example 215. If $X = V(F)$ for $F \in k[x_0, ..., x_n]_{(d)},\ d \geq 1$, then $\deg X = \deg F$. This is an easy case of the hypersurface formula.

Example 216 (Finite Collection of Points).

$$\deg(\{p_1, ..., p_m\}) \geq m \tag{261}$$

We see this from

$$P_{\{p_1,...,p_m\}} = m \tag{262}$$

and the dimension of collection of points is 0, so $\deg(\{p_1, ..., p_m\}) = m \cdot 0! = m$.

THe inequality we get from the case $V(x^2) = V(x)$.

Recall that dimension is intrinsic; i.e. does not matter what projective space we are embedded in. Keeping this in mind, here are some remarks:

Remark 217. The degree is **projective invariant**, i.e. if

$$\phi \in \text{Aut}(\mathbb{P}^n) = PGL_{n+1}(k) \tag{263}$$

then $\deg \phi(x) = \deg X$.

This follows from the fact that $P_X, P_{\phi(X)}$ have the same leading coefficients since ϕ is just a linear change of coordinates.

Remark 218. Unlike dimension, the degree is *not* isomorphism invariant. For example, for $\mathbb{P}^1 \subseteq \mathbb{P}^1$

$$\deg \mathbb{P}^1 = 1 \tag{264}$$

but if we take $\nu_d(\mathbb{P}^1) \subseteq \mathbb{P}^d$, then

$$\deg \nu_d(\mathbb{P}^1) = d \tag{265}$$

Remark 219 (Arrondo Lemma 5.13). If $X, Y \subseteq \mathbb{P}^n$ projective variety with same dimension and has no common component, then

$$\deg(X \cup Y) = \deg X + \deg Y \tag{266}$$

e.g. For any conic $C \subseteq \mathbb{P}^n$, for $C = V(F)$ with $F \in k[x_0, ..., x_n]_{(2)}$ then $\deg C = 2$, i.e. always even. For instance $F(x, y, z) = xy$.

Remark 220. Recall that there are only two ways of doing hilbert polynomial: exact sequences and hypersurface lemma (which we proved using exact sequences).

10.2 Basic Intersection Theory

We only consider finite intersection case. Our goal is Bezout's theorem.

Remark 221 (The General Case). It's hard to do the general case even after a grad level diff top class... It's a second year level topic... As noted before, perturbation is very hard in AG. If we had a curve intersecting with a line, and if we kept moving the line, then we would have 2,1, and 0 intersections. This is indeed an invariant...

Question: When $X, Y \subseteq \mathbb{P}^n$ has $X \cap Y$ finite, what is expected for $|X \cap Y|$? For example the number of common solutions of $x^8 + y^8 + z^8 = 0$ and $2x^2y^2z^2 - 38x^3y^3 + 11z^6 = 0$?

Example 222. Let $X = V(F)$ be a degree d hypersurface so that $F \in k[x_0, ..., x_n]_{(d)}$. Also let $L := V(H)$ be a generic hyperplane (i.e. L and X do not contain one another) in $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$. Then

$$|X \cap L| = |V(F|_L)| \tag{267}$$

and $F|_L$ is a homogeneous polynomial is 2 variables, so wlog we can restrict to a chart so we get a polynomial in 1 variable. But by FTA, there exists $\deg F$ roots not necessarily distinct.

If we now have

$$F(x) = (z - a_1)^{m_1} ... (z - a_r)^{m_r} \tag{268}$$

where $\deg F = m_1 + ... + m_r$ and there are r distinct roots. Then $|V(F|_L)| \leq \deg F$.

Example 223. In $\mathbb{P}^2$, if we take

$$X = V(zy - x^2),\ L_1 = V(y - x),\ L_2 = V(y) \tag{269}$$

then $|X \cap L_1| = 2$ and $|X \cap L_2| = 1$.

We also have

$$\deg(X \cap L_1) = 2 = \deg(X \cap L_2) \tag{270}$$

and

$$\begin{aligned} P_{X \cap L_1}(r) &= P_{I(X)+I(L_1)}(r) \\ &= P_{I(X)}(r) - P_{I(X)}(r-1) \\ &= (2r + c) - (2(r-1) + c) \\ &= c + 2 - c = 2 \end{aligned}$$

So $\deg(X \cap L_1) = 2 \cdot 0! = 2$.

By the exact same thing as above (i.e. the above calculation holds for any line), $\deg(X \cap L_2) = 2$. Note that if we did not use the hypersurface lemma, then it becomes a bit weird mainly because $(x^2) \neq (x)$ but the vanishing locus are the same for the two ideals.

Now we give an upper bound on the number of solutions.

Proposition 224 (Weak Bezout Theorem). Let $X \subseteq \mathbb{P}^n$ with dimension at least 1. Let $F \in k[x_0, ..., x_n]$ be homogeneous with degree degree at least 1. Assume also that no irreducible component of X is contanined in $V(F)$. Then

$$\deg(X \cap V(F)) \leq (\deg X) \cdot \deg F \tag{271}$$

Remark 225. The $n = 2$ is a good example. The hypersurface is just a curve.

PROOF 226. Let m be the dimension of X. Then

$$P_X(t) = \frac{\deg X}{m!} t^m + O(t^{m-1}) \tag{272}$$

Then

$$P_{X \cap V(F)}(t) = P_X(t) - P_X(t - \deg(F)) \tag{273}$$

where first equality is by hypersurface lemma. Now plug and chug.

□

Remark 227. So far there are *two* degrees associated to a hypersurface: the degree as a subvariety and the degree of the polynomial defining it. We will prove that in fact the two notions are equal.

Here are some corollaries.

Proposition 228. $\deg(V(F)) = \deg F$

PROOF 229. Apply weak Bezout with $X = \mathbb{P}^n$.

□

Proposition 230. Take $X \subseteq \mathbb{P}^n$ a curve and $V(F) \subseteq \mathbb{P}^n$ a hypersurface. Then

1.) $|X \cap V(F)| \leq \deg X \cdot \deg F$
2.) For all curves $X, Y \subseteq \mathbb{P}^2$ with no common component $|X \cap Y| \leq \deg X \cdot \deg Y$.

Next time:

$$\deg(X \cap V(F)) = \sum_{P \in X \cap V(F)} I_P(X, V(F)) \tag{274}$$

which is the general Bezout. So, the LHS depends on the choice of the hypersurface. The terms on the RHS does depend, but the sum does not!!!

11 Lecture 11: Bézout's Theorem. -Thursday, 11.8.2018

Today, Danii Rudenko is giving the lecture.

11.1 Intersection Multiplicities.

Imagine we have a curve intersecting with another curve. The number of intersection points is just the product of the degrees! Note that if we have an intersection point, it can be very complicated (in the sense of topology)! We can understand this precisely via commutative algebra.

The definitions we give here is quite simple, but it captures all of this complicated structure.

Recall that the primary decomposition is just writing an ideal as an intersection of primary ideals:

$$I = I_1 \cap \ldots \cap I_n \tag{275}$$

where $\mathrm{rad} I_k$ is prime, and in particular

$$\mathrm{rad} I = \bigcap_k \mathrm{rad} I_k \tag{276}$$

We also assume $\mathrm{rad} I_k$ are distinct (since intersection of radicals is a radical). We also have for all k,

$$I_k \supsetneq \bigcap_{i \neq k} I_i \tag{277}$$

Under these assumptions, the set of $p_k := \mathrm{rad} I_k$ is unique.

For example, we can write

$$(x^2, xy) = (x) \cap (x, y) = (x) \cap (x^2, y) \tag{278}$$

where (x) is just a line, and the other two ideals give a point. So in this sense, it is not unique.

Definition 231. Let $X, Y \subseteq \mathbb{P}^n$ be projective sets. Let p be a point which is an irreducible component of $X \cap Y$ where $|X \cap Y| < \infty$.

Then the **intersection multiplicity of** X, Y **at** p is

$$I_p(X, Y) := h_J$$

(which is a constant polynomial, and) where J is the $I(p)$-primary component of $I(X) + I(Y)$, i.e. $\mathrm{rad} J = I(p)$.

Remark 232. The picture is as follows: it is just one of the irreducible components of the intersection of two curves.

J is the primary component of the intersection. We then look at the Hilbert polynomial of this ideal, then it is a constant polynomial.

This definition is very deep because a priori, it is *very* simple. With more machinery, one can come up with *a* definition, but compared to those, this definition is very, very simple...

Example 233 (Two Lines). Consider the axis in $\mathbb{P}^2$, i.e. $L_1 = V(x), L_2 = V(y)$. Then

$$(x) + (y) = (x, y) = I((0)) \tag{279}$$

where (x, y) is prime, and $I_{(0,0)}(L_1, L_2) = 1$. Indeed,

$$k[x, y, z]/(x, y) = k[z] \tag{280}$$

which of course is integral domain.

The power of the following example is that it illustrates how there are invariants under perturbation.

Example 234. Consider line and parabola:

$$W = V(yz - x^2),\ L_1 := V(y - x),\ L_2 := V(y) \tag{281}$$

Then

$$I(W) + I(L) = (yz - x^2, y - x) = (x, y) \cap (x - y, y - z, z - x) \tag{282}$$

The rightmost decomposition corresponds to the two intersections, the origin and the point where all three coordinates are equal. We just did

$$(yz - x^2, y - x) = (yz - yx, y - x) = (y, y - x) \cap (z - x, y - x) \tag{283}$$

Consequently,

$$I_{(0:0:1)}(W, L_1) = I_{(1:1:1)}(W, L_1) = \{(1 : 1 : 1)\} \tag{284}$$

What about the other line? We have

$$(yz - x^2, y) = (x^2, y) \tag{285}$$

and the RHS is primary.

Now,

$$k[x, y, z]/(x^2, y) = k[z] + xk[z] \tag{286}$$

and looking at the grading, we see that $h_{(x^2,y)} = 2$.

11.2 Bézout's Theorem.

The statement is nothing fancy. Rather, it just says that definition of multiplicity is correct.

Remark 235 (On General Mathematics.). It's common in mathematics that having the right definitions makes the theorem trivial...

We can think of this as a generalization of the Fundamental Theorem of Algebra.

Proposition 236 (Bézout's Theorem). Take $X \subseteq \mathbb{P}^n$ with $r = \dim X, d = \deg X$. Let $X_1, ..., X_r$ be hypersurfaces of degree $d_1, ..., d_r$ with finite intersection $|X \cap X_1 \cap ... \cap X_r| < \infty$. Then

$$\sum_{p \in X \cap X_1 \cap ... \cap X_r} I_p(X, X_1 \cap ... \cap X_r) = dd_1 ... d_r \tag{287}$$

Remark 237. Main example: intersection with hyperplanes.

Remark 238 (Generalization of FTA.). It is just a statement of how many solutions a system of equations have.

Remark 239 (Technical Points.). Intersection is finite: this is delicate since we can perturb and have infinitely many intersections. Instead, we can have a stronger formulation where we phrase in terms of inclusions of components.

There is also a notion of intersection index.

Remark 240 (Recall: Geometric Idea of Degree of Algebraic Set). Geometrically, we take the algebraic set intersect with hyperplane, if there is infinitely intersection, take another hyperplane. If we keep going, we will eventually have finitely many intersections. This is the degree.

PROOF 241 (Bézout: Linear Case). We prove the linear case, i.e. $X_i := V(H_i)$ are hyperplanes. Note that we need to take the hyperplanes generic enough...

Assume $|X \cap X_1 \cap ... \cap X_r| < \infty$. Then (from Arrondo Theorem 5.15)

$$\sum I_p(X, X_1 \cap ... \cap X_r) = \deg X \tag{288}$$

Let's start by understanding what the degree is. We have

$$\deg X = P_{X \cap X_1 \cap ... \cap X_r}(t) \tag{289}$$

which is constant.

The primary decomposition

$$I(X) + I(X_1 \cap ... \cap X_r) = I_1 \cap ... \cap I_s \tag{290}$$

What is the radical? We should have either rad$I_j = I(W_j)$ for W_j irreducible component of the intersection or rad$I_j = \mathfrak{M}$ for the irrelevant ideal $\mathfrak{M}$.

Take $I_s = \mathfrak{M}$ and for the others, rad$I_k = I(W_k)$. Now $s-1$ is the number of points in the intersection. Then we have

$$P_{I_1 \cap ... \cap I_{s-1} \cap I_s} = P_{I_1 \cap ... \cap I_{s-1}} \tag{291}$$

i.e., we can throw out the irrelevant ideal.

But now for all i, j, radI_i + rad$I_j = I(W_i, W_j) = I(\emptyset)$ since the respectively correspond to two distinct points W_i, W_j. So, $I_i + I_j = \mathfrak{M}$.

So,

$$P_{I_1 \cap ... \cap I_{s-1}} = \sum_{k=1}^{s-1} P_{I_k} = \sum_{k=1}^{s-1} I_{W_k}(X, X_1 \cap ... \cap X_r) \tag{292}$$

since $P_{J_1+J_2} = P_{J_1} + P_{J_2} - P_{J_1 \cap J_2}$.

$\square$

11.3 Bézout Application 1: Pascal's Theorem

Proposition 242 (Pascal's Theorem: Mystical Hexagon.). If we took 6 points on a conic, say $p_1, ..., p_6$, and if we took the pairwise lines between the first and last three points, then the intersection is collinear.

Remark 243 (Principle of Irrelevance of Algebraic Inequalities). [13]

Remark 244. The condition here is the same as smoothness (which we have not defined). But conics (assuming they are irreducible) are smooth.

PROOF 245. Let

$$X = P_1P_5 \cap P_2P_6$$
$$Y = P_2P_4 \cap P_3P_5$$
$$Z = P_1P_4 \cap P_3P_6$$

We claim that these lie on the same line. Let L_{AB} some equations of a line through A, B. Consider the cubic curves:

$$C_1 = L_{p_1p_4}L_{p_2p_6}L_{p_3p_5}$$
$$C_2 = L_{p_1p_5}L_{p_2p_4}L_{p_3p_6}$$

For every $\lambda \in k$, $C(\lambda) := C_1 + \lambda C_2$ is a cubic curve. $C(\lambda)$ passes through $p_1, ..., p_6, X, Y, Z$ for any λ.

There exists λ_0 such that $C(\lambda)$ passes through one more point on the conic. (This is immediate since we just take a point not on $p_1, ..., p_6$ and not lying on the lines, and we can solve.) Then look at the intersection of $C(\lambda_0)$ and the conic. But $C(\lambda_0)$ has degree 3 and conic has degree 2. They intersect at least at 7 points. By Bézout, they must have a common component. Thus, the conic is a component of $C(\lambda)$. So, $C(\lambda)$ is a union of a conic and a line, ℓ. Thus, $X, Y, Z \in \ell$.

□

Remark 246 (Intuition.). A different approach is dimension counting. The space of all cubic curves is $\mathbb{P}^9$. So, 9 generic points give a cubic. With 8, there should be a family of cubics. Taking two cubics going through all 8, there is a point of intersection. But taking another choice of conic, it will also intersect at this point.

So, if we look at the cubic (the three lines) going through the first 8 points $p_1, ..., p_6, X, Y$ and then another cubic (the union of the other 3 lines), they intersect at Z. But then from the above general theory, another cubic (union of conic and a line) must intersect at the same intersection point.

11.4 Problem Session - 11.9.2018

1.) Arrondo 5.17
  (a.) Don't want intersections that are not points; FINITE intersections;
  (b.) Bézout is typically line and something else; intersection theory gets complicated very quickly, so do the standard ones and move on. (e.g. intersection multiplicity has many different definitions; gets complicated very very quickly)
  (c.) Take the statement with two things and generalize to many stuff
2.) Intersection Multiplicity
  (a.) If we a an intersection of two curves defined by f, g, and take a local ring arou dd the intersection, then the intersection multiplicity is just $\dim O_z(f, g)$

[13] See for instance Paul Cohn, *Basic Algebra: Groups, Rings and Fields* and Berry, The Irrelevance of Algebraic Inequalities.

   (b.) This is where you'd want to pass to shceme theory; in addition to the topological structure given by Zariski, there is also a structure given by the *functions*, e.g. $V(x) = V(x^2)$. The second one allows for an error up to error term. This encodes the intersection multiplicity...
   (c.) Hilbert polynomial is somehow looking at these functions moving in the background
3.) Arithmetic Genus
   (a.) Take curve over $\mathbb{C}$. Then Arithmetic genus is just the number of the holes
   (b.) These stuff are defined via cohomology; there is also notion of the geometruc genus
   (c.) Computing Hilbert polynomial is like computing the Euler characteristic
4.) Problem 2 and 3:
   (a.) Just do low dimensional cases.
5.) Intersection Theory
   (a.) Fulton's book
   (b.) Chow's Lemma: when you have an algebraic variety intersecting, you can perturb it!
6.) Gathmann's book: Bézout application
7.) See Hartshorne
8.) Take curve $X \subseteq \mathbb{P}^3$ not contained in any proper linear subspace in $\mathbb{P}^3$
   (a.) If $\deg X$ is prime number, then $I(X)$ cannot be generated by (more than) two elements.
      i.)

Week 7. Degree of a Projective Set.

12 Lecture 12: Singular Points of Curves and More Applications of Bézout. -Thursday, 11.13.2018

1.) Bézout interacts very interestingly with cup products. (Additional supplemental problems.)
We present more applications of Bézout's Theorem.

12.1 Bézout's Theorem Application 2: Irreducible Curve Contained in Hyperplane.

Proposition 247. Let $X \subseteq \mathbb{P}^n$ be an irreducible curve. If $\deg X < n$, then X is contained in some hyperplane, $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.

In other words, if the curve is taking up the entire space, then it better be degree n.

Remark 248 (Rational Normal Curve). In particular, the rational normal curve $X := \nu_d(\mathbb{P}^1)$ with

$$\nu_n : \mathbb{P}^1 \to \mathbb{P}^n \tag{293}$$

$$[s:t] \mapsto [s^n : s^{n-1}t : ... : t^n] \tag{294}$$

then

$$X \cap V(a_0x_0 + ... + a_nx_n) = V(a_0s^n + a_1s^{n-1}t + ... + a_nt^n)$$

for any hyperplane $V(a_0x_0 + ... + a_nx_n)$. Thus, $\deg X = n$.

In fact, $\nu_n(\mathbb{P}^1)$ is the unique (up to $PGL_{n+1}(k) = \mathrm{Aut}(\mathbb{P}^n)$) degree n curve contained in any hyperplane. (Very useful!)

Proof 249. Suppose $X \subseteq \mathbb{P}^n$ with $\deg X < n$. Choose distinct points $p_1, ..., p_n \in X$. These line in some hyperplane $\mathbb{P}^{n-1} \simeq H \subseteq \mathbb{P}^n$.

Now

$$\dim(H \cap X) \leq \dim X = 1 \tag{295}$$

with equality iff $X \subseteq H$ since $H \cap X \subseteq X$. But since X is irreducible, $H \cap X = X$ so H contains X.

So, if H does not contain X, then $\dim(H \cap X) = 0$, so it is a finite set, and

$$\deg(X \cap H) = \sum_{p \in X \cap H} I_p(X, H) \geq n \tag{296}$$

since $|X \cap H| \geq n$. But this is a contradiction to the assumption that $\deg X < n$.

□

By iterating the above, a curve would be inside $\mathbb{P}^d$.

12.2 Bézout Application 3: Automorphisms of Projective Space.

Recall that the action of $GL_{n+1}(k)$ on $\mathbb{A}^{n+1}$ given by

$$GL_{n+1}k \to \text{Aut}(\mathbb{P}^n)GL_{n+1}k/\{\lambda \in k^*\} =: PGL_{n+1}k \tag{297}$$

Proposition 250. ψ is surjective, i.e.

$$\psi : PGL_{n+1}k \xrightarrow{\simeq} \text{Aut}(\mathbb{P}^n) \tag{298}$$

Remark 251. The corresponding theorem for $\mathbb{A}^n$ is false, since bijective morphism is not necessarily an isomorphism, e.g.

$$F : \mathbb{A}^2 \to \mathbb{A}^2 \tag{299}$$

$$(x, y) \mapsto (x, y + x^2) \tag{300}$$

Then F is an isomorphism of affine varieties with

$$F^{-1}(x, y) = (x, y - x^2) \tag{301}$$

But $F \notin GL_2k$ or $\mathbb{A}k$.

Remark 252. Morphisms on $\mathbb{P}^n$ is just a morphism on the affine charts.

This is a "rigidity property"; the input is that F is a polynomial, but the output is that it is linear.

PROOF 253. Let $F \in \text{Aut}(\mathbb{P}^n), n \geq 1$ be given. We must find $T \in PGL_{n+1}k$ such that $F = T$.

Let $H \subseteq \mathbb{P}^n$ be any hyperplane. Let L be a line not contained in H (so $L \cap H$ is a single point p, so $I_p(L, H) = 1$).

Now $F(L)$ is a curve, $F(H)$ is a hyperplane, and

$$I_{F(p)}(F(L), F(H)) = I_p(L, H) \tag{302}$$

(Note that this just follows from the isomorphism of the coordinate rings.)

Now from Bézout,

$$\begin{aligned}(\deg F(L))(\deg F(H)) &= \sum_{q \in F(L) \cap F(H)} I_q(F(L), F(H)) \\ &= \sum_{p \in L \cap H} I_{F(p)}(F(L), F(H)) \\ &= I_p(L, H) = 1\end{aligned}$$

So, $\deg(F(L)) = \deg(F(H)) = 1$. The latter implies $F(H) = V(G)$ for some linear polynomial G.

So F takes hyperplanes in $\mathbb{P}^n$ to hyperplanes in $\mathbb{P}^n$. So, there exists some linear map $B \in PGL_{n+1}k$ such that

$$B \circ F(\{x_0 = 0\}) = \{x_0 = 0\} \tag{303}$$

since $PGL_{n+1}k$ acts transitively on the set of hyperplanes.

So, $B \circ F : \mathbb{A}^n \to \mathbb{A}^n$.

Exercise. Any morphism from $\mathbb{A}^n$ to itself taking hyperplanes to hyperplanes is linear.

This implies there exists a linear map C for which $C = B \circ F$.

□

Remark 254. If $F \in \text{Aut}(\mathbb{P}^d)$ i.e. automorphism of the ambient space, then everything (degree etc) is preserved. We can just think of this as just sending coordinates $k[x_0, ..., x_n]$ to different coordinates $k[z_0, ..., z_n]$. The degree is extrinsic, so if you preserve the ambient space, then it is preserved.

12.3 Singular Points of Curves.

Here is a new topic.

Definition 255. The **local dimension of a variety** X **at a point** $z \in X$ is the maximal dimension of any irreducible component of X containing z.

Example 256. Think of a plane with a perpendicular line through it. Then the points on the plane including the point of intersection has local dimension 2 whereas on the line, it is 1.

Definition 257. A **point** $z \in X$ **is a singular point** if

$$I_p(X) = (G_1, ..., G_r) \tag{304}$$

with homogeneous polynomials $G_i \in k[x_0, ..., x_n]$ for which

$$\text{rank}\left(\frac{\partial G_i}{\partial x_j}\right) < r \tag{305}$$

where r is the local dimension of z at X.

The **point** z **is a smooth point of** X if it is not a singular point. The **algebraic set** X **is smooth** if it is smooth at every point.

Remark 258 (Zariski Localness.). $X \subseteq \mathbb{P}^n$ is smooth at z iff on every local patch U_i containing z such tat $U_i \cap X$ is smooth at z, and this is true iff there exists an affine patch U_i containing z such that $U_i \cap X$ is smooth at z.
So we can do the computation of the partials in the affine charts.

Remark 259. For $k = \mathbb{C}$ or $\mathbb{R}$, X is smooth at z iff (by Implicit Function Theorem) there exists a neighborhood $N_z \subseteq X$ such that $N_d \simeq k^d$ where d is the local dimension of X at z. So X is a smooth projective variety, and thus smooth manifold.

The technicality is stuff like the cuspodal cubic. This is homeomorphic to unit interval but not diffeomorphic, because of the singularity issues.

Implicit Function Theorem takes you from the infinitesimal T_xX (given by linear algebra) to the local in the neighborhood of a point (given by calculus). Global is the hardest.

If $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, and then DF_x is full rank iff F local diffeomorphism iff F is not the identity. From this we get projection.

Example 260. Take

$$(zy - x^2) = V(F) \subseteq \mathbb{P}^2 \tag{306}$$

is smooth with $F(x,y,z) := zy - x^2$. For this,

$$DF = (-2x, z, y) \tag{307}$$

is full rank unless $(0,0,0) \notin \mathbb{P}^2$.

Homework. (Euler's Identity) If Jacobian is no full rank, then the point automatically on the curve.

Example 261 (Nodal Cubic). Take $V(F)$ for $F(x,y,z) = y^2z - x^2z - x^3$. Then

$$DF = (-2xz - 3x^2, 2yz, y^2 - x^2) \tag{308}$$

This is full rank unless $y = 0$ or $z = 0$ and $x = \pm y$, so just the points $[0:0:1]$.

For U_z, $F(x,y) = y^2 - x^2 - x^3$ from which we get a nodal cubic.

In particular, $V(F)$ is smooth at $[0:1:0] \in\in \mathbb{P}^2$ (i.e. $\mathbb{P}^1$ at infinity), so the tangent line to $V(F)$ at $[0:1:0]$ is

$$\frac{\partial F}{\partial x}(0,1,0)x + \frac{\partial F}{\partial y}(0,1,0)y + \frac{\partial F}{\partial z}(0,1,0)z = 0 \tag{309}$$

Example 262. Take $V(F)$ for $F(x,y,z) = x^2z - y^3$ which is smooth except at $[0:0:1]$.

We now want to bound the number of singularities. (The question is still open for surfaces.)

Proposition 263. Let $X \subseteq \mathbb{P}^2$ be an irreducible curve of degree d, i.e. $X = V(F)$ where $F \in k[x,y,z]_{(d)}$. Then X has at most $\binom{d-1}{2}$ singular points.

Remark 264. This is intuitive in the sense that if we have a curve, we want to avoid intersecting at same point more than once. So, intuitively, there must be an upper bound to the number of intersections.

Example 265. For X is a line $d = 1$, there is not singular points.

Example 266. For X is an irreducible conic $d = 2$, by our classification result, there is no singular points.

Example 267. For $d = 3$, there is at most one singular point.

Remark 268. The statement is false for reducible curves, for instance the usual example $V(xy)$.

We will prove this in the next lecture. The idea is just a dimension counting argument. Look at all the intersection points, and assume statement is false. If we then chose some other arbitrary points, then we can find degree d polynomial which goes through all the points. By Beézout, if we have a singularity, the multiplicity is at least 2. But this gives us an upper bound because Bézout gives us an upper bound. With some additional work we show that the bound is sharp.

13 Lecture 13: Singular Points and Real Algebraic Geometry -Thursday, 11.15.2018

Announcements.

1.) Look at the new supplementary problems.

13.1 Singular Point on Curves.

Proposition 269. Let $X \subseteq \mathbb{P}^2$ be an irreducible curve of degree d. Then X has at most $\binom{d-1}{2}$ singular points.

Remark 270. This is probably true for $\mathbb{F}_p$ for p sufficiently large...

Example 271. Irreducible conics do not have singular points. Note that we need irreducibility since we can have $V(xy)$.

Example 272. The only cubics which have singular points are cuspidal cubic and nodal cubic (we will prove this).

PROOF 273. We can assume $d \geq 3$. Assume for contradiction that the statement is false. Then there exists distinc singular points $a_1, ..., a_{\binom{d-1}{2}+1}$ of X. Pick $d-3$ other points $b_1, ..., b_{d-3}$ on X [14] and take

$$S := \{a_i\} \cup \{b_j\} \tag{310}$$

so that $|S| = \binom{d}{2} - 1$.

(*Step 1.*) **Proposition.** Let $d \geq 1$. Given any $\binom{d+2}{2} - 1$ distinct point sin $\mathbb{P}^2$, and there exists a curve of degree d passing through them.

So for instance, given 5 points, there is a conic going through, and given 9 points, we have a cubic. The proof of this is an easy dimension count (i.e. same as in the existence of conic through 5 points proof) left for homework 5.

So the above proposition says in our case that there exists a curve $Y \subseteq \mathbb{P}^2$ with degree at most $d-2$ and passing through all points in S.

(*Step 2.*) Bézout [15] (this is where we need $\mathbb{P}^2$ rather than $\mathbb{P}^n$) then says that

$$d(d-2) = (\deg X)(\deg Y) = \sum_{p \in X \cap Y} I_p(X, Y) \tag{311}$$

(*Step 3.*) If $X, Y \subseteq \mathbb{P}^2$ are curves, and $p \in X \cap Y$ and if p is a singular point of X, then $I_p(X, Y) \geq 2$. (See HW 5 for proof.)

Intersection multiplicity 2 just means there are two tangents. So we can just think of X intersecting itself already.

Now we are assuming that X is singular at each a_i also $S \subseteq X \cap Y$, so

[14] These points are not necessary, but we do this to get a better bound.

[15] The key idea for Bézout is to take the points with the key property, and then take curve going through them.

$$\begin{aligned}\sum_{p\in X\cap Y} I_p(X,Y) &\geq \sum_{s\in S} I_p(X,Y)\\ &\geq 2\left(\binom{d-1}{2}+1\right)+d-3\\ &= d(d-2)+1 > d(d-2)\end{aligned}$$

This contradicts our hypothesis on the number of intersections.

□

13.2 Real Algebraic Curves: Harnack's Theorem and (First Half of) Hilbert's 16th Problem

All of number theory is just complex algebraic geometry. In contrast, real algebraic geometry is a fundamental thing that Newton thought about...

We will consider $\mathbb{P}^2$ over the field $\mathbb{R}$. (We will just say $\mathbb{P}^2$.) Of course this is just given by the stereographic projection on the real unit 2 sphere. This is of course just $\mathbb{S}^2$ with the antipodal points identified. In particular, this is a smooth 1-manifold.

Let $C \subseteq \mathbb{P}^2$ be any smooth curve. So C is just a compact 1-manifold (compact because it is just a zero set), i.e.

$$C = \gamma_1 \cup \ldots \cup \gamma_r \tag{312}$$

for some r where each γ_i is a loop (i.e an embedding of the 1-sphere into $\mathbb{P}^2$). We require them to be disjoint because we do not want self intersection.

Example 274. $V(F) \subseteq \mathbb{P}^2$, for instance $F(x,y,z) = y^2z - x^3$ and if we graph in the chart $\mathbb{A}_{z=1}$, then we get the nodal cubic. Then inside the loop and between the tails, the function is negative and positive on the other two regions.

For $G(x,y) = y^2 - x^2 - x^3$ has constant sign on each connected component of $\mathbb{R}^2 \setminus V(F)$ (just by continuity of G).

Now let's try perturbing it! Take $\epsilon > 0$ small and consider $V(G-\epsilon)$. Then we get one connected component.

If we instead have $V(G+\epsilon)$ where we get a smooth curve with 2 irreducible components.

Example 275. For $V(F) \subseteq \mathbb{A}^2$ with $F(x,y) = (x^2+2y^2-1)(2x^2+y^2-1)$. Here $\deg F = 4$.

Now let's look at $V(F-\epsilon)$. We then get nested loops.

Now let's look at $V(F+\epsilon)$. We then get no nested loops and with two components.

The logic behind all of these is to look at the value of F on each region and to see whether it increases or decreases and see how the components get connected.

See more examples on HW5!!!

We will now introduce a definition (which we will rarely have to worry about).

Definition 276. A **loop** $\gamma \subseteq \mathbb{P}^2$ **is 1-sided ("even") (resp. 2-sided ("odd"))** if $\mathbb{P}^2 \setminus \gamma$ has 1 component (resp. 2 component).

Example 277.

Remark 278. Observe that any disjoint union $\gamma_1 \cup ... \cup \gamma_r$ of loops in $\mathbb{P}^2$ gas at most one 1-sided loops.

Proof. The complement of a one-sided loop is homeomorphic to a open 2-disk, so now apply Jordan Separation Theorem.

Proposition 279 (Harnack's Theorem (1876)). An irreducible, smooth curve $\mathbb{P}^2$ of degree d (so $C = V(F)$ for $F \in \mathbb{R}[x, y, z]_d$) has at most $\binom{d-1}{2} + 1$ loops.

Example 280 (Small d.). For small values of d:
1.) $d = 1$ for which it is certainly true.
2.) $d = 2$, then we get an irreducible conic which has 1 loop.
3.) $d = 3$ then there are at most 2 loops as we saw above.
4.) $d = 4$ there is at most 4 loops as we saw above.

Remark 281 (Historical Remark). Russian math papers were like 3 pages for a long time because paper was expensive...

Proposition 282. For $d \geq 1$ there exists a smooth irreducible curve $C \subseteq \mathbb{P}^2$ of degree d realizing Harnack's bounds. (Such curves are called **M-curves** or **Harnack curves**).

Proof: HW 5. (You can google the proof.)

Remark 283 (Riemann Surfaces). The real curve $V(F)$ is fixed by the involution acting on the Riemann surface. So in this sense, Riemann surfaces *with involution* is the same as these real curves. For instance, if we had a three holed donut and cut it with a plane through the centers of the donuts, then we get curve which is union of three connected components.

Remark 284 (Historical Remark: Hilbert and Why He Cared About This.). Hilbert was looking at vector fields in the plane given by polynomials (via approximation). What kind of limiting cycles can you get from ODEs (second part of Hilbert's 16th problem).

PROOF 285. The proof is the same as the number of singularities proof we just looked at. Philosophically, we are somehow just "detecting points" so it makes sense why they are the same... Before, the theorem was in complex curves, and here it is real curves. (So maybe we can get one from the other...)

Take $C = V(F)$ where $F \in \mathbb{R}[x, y, z]_{(d)}$.

Step 1. If the theorem is false, there are at least $\binom{d-1}{2} + 2$ distinct loops, at least $\binom{d-1}{2} + 1$ being 2-sided.

Pick $p_1, ..., p_{\binom{d-1}{2}+1}$ distinct points, one on each even loop. Choose distinct $a_1, ..., a_{d-3}$ other points on another loop.

Let

$$S := \{p_i\} \cup \{q_j\} \tag{313}$$

so $|S| = \binom{d}{2} - 1$.

Step 2. Now by dimension count (see HW 5), there exists $G \in \mathbb{R}[x, y, z]_{(d-2)}$ such that $S \subseteq Y := V(F)$.

Now if these curves go into the interior of the curves, then it either crosses the curves twice or has intersection of multiplicity 2.

Since $\deg F = d > d - 2 = \deg Y$ where X, Y have no common component since X is irreducible.

So now we can apply Bézout to get

$$\begin{aligned} d(d-2) &= \deg C \cdot \deg Y \\ &= \sum_{p \in C \cap Y} I_p(C, Y) \\ &\geq \sum_{p_i} I_{p_i}(C, Y) + \sum_{q_j} I_{q_j}(C, Y) \\ &\geq 2 \cdot \left(\binom{d-2}{2} + 1 \right) + (d-3) = d(d-2) + 1 \end{aligned}$$

Above, we used: for all i either $I_{p_i}(C, Y) = 2$ or $I_{p_i}(C, Y) = 1$ but since loop γ_i is 2-sided, there exists $\tilde{p}_i \neq p_i \in \gamma_i$ such that $\tilde{p}_i \in C \cap Y$.

The above contradicts our hypothesis.

□

Remark 286. Peter Sarnack says 4% of all curves are Harnack.

Proposition 287. If X is a (real planar irreducible smooth) quartic has > 2 ovals, then none of them are inside any other.

PROOF 288. Suppose assertion is false. Take a point p inside inner oval and take q in the other. Then take a line $\ell = \overline{pq}$. Then

$$\sum_{p \in X \cap \ell} I_p(X, \ell) = (\deg X)(\deg \ell) = 4 \tag{314}$$

But the LHS must be at least 5 from a picture.

□

Remark 289. The above implies that for degree 4 curves, there can be at most 4 components and we only have 6 patterns.

For degree 6, this is more complicated.

13.3 Problem Session - Friday, 11.16.2018

1.) Perturbation: Local algebra
  (a.) Chow's lemma
  (b.)
2.) Problem 1.
  (a.) Make suitable choice when higher power
  (b.) Look at normal vectors or something
  (c.) $n \geq 2$
3.) Problem 2.
  (a.) Do it for monomial to make it cleaner
4.) Problem 3
  (a.) Problem 3: Use problem 4.
  (b.) Most basic example $V(zy - x^2)$ and (y) in $\mathbb{P}^2$
  (c.)
5.) Problem 4.
  (a.) Take $p \in X \cap Y$ singular for X
  (b.) $I_p(X, Y) \geq 2$ with $X = V(f)$ and $Y = V(g)$ line.
  (c.) Equivalently, we get $f(p) = g(p) = 0$ and $\nabla f(p) = 0$
  (d.)
6.) Problem 5
  (a.) b. consider d lines in $\mathbb{P}^n$ in affine chart.
  (b.) a. Induction on the number of components since each component has reducible thing. For irreducible case, we have $\binom{d-1}{2}$.
  (c.) There can be at most d irreducible components since each factor corresponds to an irreducible component.
  (d.) Just take
$$\binom{\alpha}{2} + \binom{\beta}{2} + \alpha\beta \leq \binom{\alpha+\beta}{2} \tag{315}$$
  In fact this gives equality. This gives induction step.
7.) Problem 6
  (a.) Don't do Chebychev polynomials
8.) From lecture:
  (a.) The nodal cubic is irreducible
9.) See Gathmann ch.13 for Farb lectures (even the examples)
10.) Projection from a Linear Space ... a nice transformations (most standard birational morphism in AG)
  (a.) Birational morphism: $\mathbb{P}^n \to \mathbb{P}^{n+1}$; birational geometry
  (b.) Every rational normal curve intersects a hypersurface
  (c.) Consider $\mathbb{P}^n$ and $E = (L_1, ..., L_d)$ and take the morphism
$$\mathbb{P}^n \setminus E \to \mathbb{P}^{n-d-1} \tag{316}$$
  given by the map
$$x \mapsto (L_1(x) : ... : L_d(x)) \tag{317}$$
  This is the projection from E.
  (d.) Canonical morphism $E = \{p\}$: consider cuspidal cubic. take line connecting the cusp point and points on the curve. This projects the curve onto $\mathbb{P}^{n-1}$
  (e.) Take $X \subseteq \mathbb{P}^n$ take
$$\pi : \mathbb{P}^n \to \mathbb{P}^{n-1} \tag{318}$$
$$H' = \pi^{-1}(H) \leftarrow H \tag{319}$$
  where H is a hyperplane. We can then work inductively. From there, we get one point (which we started with) and $n-1$ points.
  (f.) See Shafarevich p.53

Week 8. Classification of the Cubic.

14 Lecture 14: Classification of the Planar Cubic -Tuesday, 11.20.2018

14.1 Classification of the Planar Cubic

Note that this is the classification of *all* cubics.

Here is the set up. Take $F \in k[x_0, x_1, x_2]_{(d)},\ d \geq 1$. If $F = F_1^{d_1}...F_r^{d_r}$ for $d = \sum d_i$. The **curve** $C = V(F)$ is a formal sum

$$C = d_1 c_1 + ... + d_r c_r \tag{320}$$

where $c_i = V(F_i)$ are the irreducible components of C.

Example 290. Take $V(x_0^m) = mV(x_0)$. Then theere exists a bijections between plane curves in $\mathbb{P}^2$ and the $\mathbb{P}(k[x_0, x_1, x_2]_{(d)}) \simeq \mathbb{P}^{\binom{d+1}{2}-1}$.

Now $PGL_3\mathbb{C}$ acts on both sides of the respective bijection by the action

$$(x_0, x_1, x_2) \mapsto A(x_0, x_1, x_2) \tag{321}$$

for $A \in GL_3\mathbb{C}$ and $[A] \in \mathrm{Aut}\mathbb{P}^2$.

Remark 291 (Rep Theory). Study of polynomials and representations are related. Consider for instance

$$\mathrm{Sym}^d(\mathbb{C}^n)^* =\simeq \mathbb{C}[x_0, ...x_{n-1}]_{(d)} \tag{322}$$

Recall the following definition.

Definition 292. Two curves $C = V(F), C' = V(F')$ are **projectively equivalent** if there exists $\varphi \in PGL_3k$ such that $\varphi(C) = C'$, i.e.

$$F'([x_0 : x_1 : x_2]) = F(\varphi([x_0 : x_1 : x_2])) \tag{323}$$

and so we have the bijection between degree d curves in $\mathbb{P}^2$ mod projective equivalence and PGL_3k orbits in $\mathbb{P}(k[x_0, x_1, x_2])_{(d)})$.

Remark 293. In rep theory, we try to decompose into irreducibles. Studying orbits is something we do in AG, but it is VERY connected to rep theory...

Let k be algebraic closed from now on. Recall the following classification result:

1.) All degree 1 curve $C \subseteq \mathbb{P}^2$ is projectively equivalent to $V(x_0)$.
2.) Any degree 2 curve in $\mathbb{P}^2$ is projectively equivalent to exactly one of
  (a.) Double line $V(x_0^2)$
  (b.) Lines $V(x_0^2 + x_1^2)$
  (c.) Smooth conic $V(x_0^2 + x_1^2 + x_2^2)$

Remark 294 (Classification of Quartics). Smooth cases are hardest, and degenerate cases are easiest. For the latter, the degree of the polynomial is smaller.

Try looking at the DUAL!! It is so much easier.

So now we consider the classification of conics in $\mathbb{P}^2$.

Take $C = V(F)$ for $F \in k[x_0, x_1, x_2]_{(3)}$. The possible cases are

1.) $F = F_1F_2F_3$ for $\deg F_i = 1$
2.) $F = F_1F_2$ and $\deg F_i = i, i = 1, 2$
3.) F is irreducible

There are two big possible cases.

Case 1.) Union of Three Lines. We first show the following classification result.

Proposition 295. Let C be a union of 3 lines. Then C is projectively equivalent to exactly one of

1.) $C = V(x_0x_1x_2)$ (intersect in three distinct points)
2.) $C = V(x_0x_1(x_0 + x_1))$ (all intersect at one point)
3.) $C = V(x_0^2x_1)$ (two lines with one double line)
4.) $C = V(x_0^3)$ (triple lines)

PROOF 296. Consider $C = \ell_1 \cup \ell_2 \cup \ell_3$ where ℓ_i is just plane P_i in k^3. We just have the cases:

Case 1. ℓ_i are distinct and $\ell_1 \cap \ell_2 \cap \ell_3 = \emptyset$. Here is a sketch. We can use the fact that GL_3k act on k^3. By linear algebra $GL_3\mathbb{C}$ act transitively on the set of such triples of planes in k^3.

The hypothesis implies that for $P_i = V_i^{\perp}$, there is an element in GL_3k whose column vectors are V_i.

Another approach is to take the set $(\mathbb{P}^2)^v$ of lines in $\mathbb{P}^2$ which (by projective duality) is isomorphic to $\mathbb{P}^2$. Then the three lines ℓ_i in $\mathbb{P}^2$ correspond to three points ℓ_i^v in $\mathbb{P}_i^v$, and this correspondence respects the PGL_3k actions. Note that this isomorphism takes lines to points and points to lines.

So the theorem is equivalent to determining the triples of points P_i in $(\mathbb{P}^2)^v \simeq \mathbb{P}^2$ mod PGL_3k. From this duality, we have:

1.) P_1, P_2, P_3 are noncolinear and distinct
2.) P_1, P_2, P_3 are distinct and are collinear
3.) $P_1 = P_2 \neq P_3$, then $P_1 = [1 : 0 : 0], P_3 = [0 : 1 : 0]$
4.) $P_1 = P_2 = P_3$

We can also do this using rep theory. Consider the triples of points $(p_1, p_2, p_3) \in (\mathbb{P}^2)^3$. Then we have the action $GL_3\mathbb{C}$ acting on $\mathbb{C}^3$ from which we get the representation V. This gives $GL_3\mathbb{C}$ acting on $V \oplus V \oplus V$ Under projection, this induces the action $PGL_3\mathbb{C}$ acting on $\mathbb{P}(V)$. So we get the action $PGL_3\mathbb{C}$ on $\mathbb{P}(V) \times \mathbb{P}(V) \times \mathbb{P}(V) \simeq (\mathbb{P}^2)^3$. Under this, the invariant subvarieties are just (p, p, p) for $p \in \mathbb{P}(V)$ which is just $\mathbb{P}^2$.

$\square$

Case 2.) C is a union of an irreducible conic and lines.

Proposition 297. Let $C = C_0 + L$ for irreducible conic C_0 and line L. Then C is exactly one of:

1.) $C_1 = V((x_0x_2 - x_1^2)x_1)$ intersecting at $[1 : 0 : 0], [0 : 0 : 1]$
2.) $C_2 = V((x_0x_2 - x_1^2)x_0)$ intersecting at $[0 : 0 : 1]$ with multiplicity 2

We know a priori from Bézout that either there are two points of intersection or one point of intersection with multiplicity 2. For the first case, we apply projective transformation to get the above form *while preserving the conic.*

PROOF 298. Here is the outline:

1.) Classification of irreducible conics
2.) Apply Bézout to get circle and line
3.) Take 2 cases. Use line ar transformations to normalize

□

Case 3.1) C is irreducible and singular.

Proposition 299. Let $C \subseteq \mathbb{P}^2$ be irreducible and singular. Then C has precisely 1 singularity and is projectively equivalent to exactly one of:

1.) (Nodal Cubic.) $V(x_1^2x_2 - x_0^3 - x_0^2x_2)$ with the cusp at $[0:0:1]$
2.) (Cuspidal Cubic.) $V(x_2x_1^2 - x_0^3)$ whose cusp is at $[0:0:1]$

PROOF 300. C has exactly 1 singularity since if not, by Bézout

$$3 = \sum_{p \in \ell \cap C} I_p(\ell, C) \geq 2 + 2 \geq 4 \tag{324}$$

which is a contradiction.

Now wlog (i.e by projective transformation) take singular point of C to be $[0:0:1]$. Thus $C = V(F)$.

Now

$$F(x_0, x_1, x_2) = x_1 q(x_0, x_1) + bx_0^3 + cx_0^2 + dx_0x_1^2 + ex_1^3 \tag{325}$$

with $q \neq 0$ else F would factor into 3 linear factors.

Now $F(0,0,1) = 0$ and

$$\frac{\partial F}{\partial x}(0,0,1) = \frac{\partial F}{\partial y}(0,0,1) = \frac{\partial F}{\partial z}(0,0,1) = 0 \tag{326}$$

since $[0:0:1]$ is a singular point. Using these constraints, we claim that

$$q(x_0, x_1) = \ell_0(x_0, x_1)\ell_1(x_0, x_1) \tag{327}$$

where ℓ_i is linear. Here, there are two cases: either $\ell_0 = \lambda \ell_i$ for $\lambda \in k^*$ and $\ell_0 \neq \lambda \ell_1$.

We then make some clever substitution (that we will not go into detail).

□

Case 3.2) Classification of Smooth Planar Cubics.

This is very hard. It is rich enough that we can do a whole course on this. They pop up everywhere. (This connects to modding out the complex plane by the fundamental parallelogram.)

Assume $k = \mathbb{C}$.

Fact. Let C be a smooth cubic in $\mathbb{P}^2_{\mathbb{C}}$ homeomorphic (or diffeomorphic) to a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Example 301 (Weierstrass Curves.). Consider curves of the form $C_{g_1,g_2} = V(F),\ g_1, g_2 \in \mathbb{C}$ where

$$F(x_0, x_1, x_2) = x_0 x_2^2 - 4x_1^3 + g_2 x_1 x_2^2 + g_3 x_0^3 \tag{328}$$

In classical books (which is given by just a change of variables of the above),

$$y^2 = 4x^3 - g_2 x - g_3 \tag{329}$$

Let $\Delta(g_2, g_3) = g_2^3 - 27g_3^2$ which is the discriminant of the curve.

Pertaining to the above example, we have the following.

Proposition 302. The following are equivalent:

1.) C_{g_2,g_3} is smooth
2.) $\Delta(g_2, g_3) \neq 0$
3.) $4x^3 - g_2 x - g_3$ has 3 distinct roots.

For next time: the list of cubics is NOT FINITE. Instead, we have a family which is given by the J-invariant.

Week 9. 27 Lines on a Cubic.

15 Lecture 15: Classification of the Planar Cubic II and the 27 Lines on the Cubic. -Tuesday, 11.27.2018

15.1 Classification of Smooth Cubics in $\mathbb{P}^2$.

Proposition 303. Assume k is algebrabraically closed. Then any smooth cubic $C \subseteq \mathbb{P}^2$ is projectively equivalent to a curve with affine equation

$$y^2 = 4x^3 - g_2x - g_3 \qquad g_2, g_3 \in k \tag{330}$$

i.e. the projective closyure of the curve

$$V(y^2 - Q(x)) \subseteq \mathbb{A}^2 \tag{331}$$

where

$$Q(x) = (x - r_1)(x - r_2)(x - r_3) \tag{332}$$

for $r_1 + r_2 + r_3 = 0$ and

$$C_{g_2,g_3} := V(y^z - 4x^3 + g_2xz^2 + g_3z^3) \tag{333}$$

Some other facts (from homework 7):

1.) C_{g_2,g_3} is smooth iff the discriminant $g_2^3 - 27g_3^2 \neq 0$
2.) Uniqueness: C_{g_2,g_3} is projectively equivalent to $C_{g_2',g_3'}$ iff there exists $\lambda \in k^*$ such that $(g_2', g_3') = (\lambda^2 g_2, \lambda^3 g_3)$

Upshot: there exists a bijection (up to automorphism which we destroy via a base point) between smooth cubics in $\mathbb{P}^2$ (i.e. a 9 dimensional moduli space) mod PGL_3k and the set $(g_2, g_3) \in (\mathbb{C}^*)^2$ with $g_2^3 - 27g_3^2 \neq 0$.

15.2 27 Lines on a Cubic.

This is what we will do for the remainder of the course. This is the first theorem of modern algebraic geometry.

Take k algebraically closed.

Definition 304. A **cubic surface** S is a degree 3 hypersurface in $\mathbb{P}^3$, i.e. $S = V(F)$ with $F \in k[x_0, x_1, x_2, x_3]_{(3)}$.

Remark 305. Why do we not study other things in $\mathbb{P}^2$ (e.g. quadrics)? Every quadric is $\mathbb{P}^1 \times \mathbb{P}^1$ via Segre embedding.

Recall that S is smooth if for all $p \in S$, $DF(p)$ has maximal rank (i.e. locally, it is a graph of a function). In other words, at every point, there is some partial which does not vanish.

Also recall: a line $L \subseteq \mathbb{P}^3$ is $L = V(F_1, F_2)$ for some $F_1, F_2 \in k[x_0, x_1, x_2, x_3]_{(1)}$ linearly independent is in bijection with 2 dimensional subspace of k^4 under the mapping $k^4 \setminus \{0\} \to \mathbb{P}^3$.

Example 306. Recall that the Fermat cubic is given by $S := V(x_0^3 + x_1^3 + x_2^3 + x_3^3)$. S is smooth since the only way all the partials vanish is if all the coordinates vanish (which cannot happen in projective space).

This contains the line $x_0 = -x_1, x_2 = -x_3$.

Here is the main theorem.

Proposition 307. Let k be algebraically closed. Then any smooth cubic surface S in $\mathbb{P}^3$ contains precisely 27 lines.

Remark 308. The very hard thing about this is that you can't *find* the lines. (In fact, we can't find the lines in terms of radicals (Harris, 1975))...

Note that if we restrict over the *reals*, we do not always get 27 lines.

PROOF 309 (For $k = \mathbb{C}$ via topology.). The Fermat cubic contains precisely 27 lines. We show that the number of lines is constant in the connected component. (Or as Riemann calls it, this is the "method of continuity"...)

(*Step 1.* (Fermat Cubic.)) After performing a projective linear transformation, any line $L \subseteq \mathbb{P}^3$ is of the form

$$L = V(x_0 - a_2x_2a_3x_3, x_1 - b_2x_2 - b_3x_3,) \qquad a_i, b_i \in \mathbb{C} \tag{334}$$

So $L \subseteq S$ iff

$$(a_2x_2 + a_3x_3)^3 + (b_2x_2 + b_3x_3)^3 + x_2^3 + x_3^3 = 0 \tag{335}$$

with equality *as polynomials in four variables.* But this is true iff each coefficient vanishes. So,

$$a_2^3 + b_2^3 = -1 \tag{336}$$
$$a_3^3 + b_3^3 = -1 \tag{337}$$
$$a_2^2a_3 = -b_2^2b_3 \tag{338}$$
$$a_2a_3^2 = -b_2b_3^2 \tag{339}$$

We claim that $a_i = 0$ or $b_j = 0$. If we assume for contradiction that it is not, we can look at 338 squared and its quotient with 339. Thus, wlog take $a_2 = 0$. So,

$$b_2^3 = -1, \; b_3 = 0, \; a_3^3 = -1 \tag{340}$$

We can now get a line in S by setting

$$b_2 = -w^j, \; a_3 = -w^k \qquad 0 \le j, k \le 2 \tag{341}$$

with $w = e^{2\pi i/3}$. We can then permute the variables to get the lines in S:

$$V(x_0 + w^k x_3, x_1 w^j x_2), \; V(x_0 + w^k x_2, x_3 w^j x_1), \; V(x_0 + w^k x_1, x_3 w^j x_2) \tag{342}$$

and there are 9 choices each. So, there are 27 distinct lines total.

Note that $S(\mathbb{R})$ contains 3 real lines.

(*Step 2.* (Parameter Space.)) Consider the space of all cubic surfaces $S \subseteq \mathbb{P}^3$. This is just the homogeneous cubics in 4 variables up to scaling by $\mathbb{C}^*$ which is just $\mathbb{P}^{19}$. (And for rep theory, this is $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^4)$.)

Proposition. Let $\Sigma \subseteq \mathbb{P}^1 9$ be the set of singular cubic surfaces (i.e. not smooth cubics). This is a (, singular, of huge degree) hypersurface in $\mathbb{P}^{19}$.

Proof of Proposition. A cubic $F \in k[x_0, ..., x_3]_{(3)}$ is $F = \sum c_I x^I$ where $I = (\alpha_0, ..., \alpha_3)$, $\sum \alpha_i = 3$, $0 \leq \alpha_i \leq 3$, $x^I = x_0^{\alpha_0} ..., x_3^{\alpha_3}$ and $c_I \in k$.

Now $V(F)$ is singlar at $p := (x_0, ..., x_3)$ iff F and all the partials of F vanish at p.

Thus, $V(F)$ has *some* singular point iff the 5 polynomials given by F and the partials of F have a common root!

The proposition now follows from the following (we need the field to be algebraically closed here):

Proposition. (Sylvester) Let $G_1, ..., G_r \in \mathbb{C}[x_0, ..., x_n]$, $d_i := \deg G_i$. For all $n \geq 1$ and for all $\{d_1, ..., d_r\}$ there exists a polynomial (the **(multivariable) resultant**)

$$\text{Res} := \text{Res}_{n,d_1,...,d_r} \in \mathbb{C}[...] \tag{343}$$

depending only on n and $\{d_1, ..., d_r\}$ so that $G_1, ..., G_r$ have a common root iff $\text{Res}(a_1, ..., a_d, b_1, ..., b_{d_2}, ...) = 0$.

This generalizes the notion of discriminant. We blackbox this result.

Here is a corollary of the proposition.

Proposition. In the usual topology, the space $P^{19} \setminus \Sigma$ of all smooth cubic surfaces $S \subseteq \mathbb{P}^3$ is path connected.

Proof of Corollary. Σ has complex codimension 1 and real codimension 2. (Singular points have real codimension greater than 4.)

So locally, we have $\mathbb{R}^{38} \setminus \mathbb{R}^{36}$. □

□

16 Lecture 16: 27 Lines on the Cubic II. -Thursday, 11.29.2018

16.1 27 Lines on the Cubic (Continued.)

Recall:

Proposition 310. Every smooth cubic surface $S \subseteq \mathbb{P}^3$ contains exactly 27 lines.

PROOF 311. Here is the outlines:

1.) True for the Fermat cubic
2.) Let $U \subseteq \mathbb{P}^{19}$ be the set of smooth cubic surface where $\mathbb{P}^{19}$ is just the set of all cubic surfaces
   (a.) **Prop.** U is a complement of some hypersurface Σ, i.e. a quasi-projective set
   (b.) **Cor.** U is path connected
3.) (Incidence Variety.) **Recall.** The lines in $\mathbb{P}^3$ are in bijection with 2-dimensional subspaces in $\mathbb{C}^4$, i.e. the the Grassmannian $\mathbb{G}(2, 4)$

Definition 312. The **incidence variety** of a lines on smooth cubic surface is

$$M := \{(S, L) : S \text{ smooth cubic surface } L \subseteq S \text{ is a line}\} \subseteq U \times \mathbb{G}(2, 4) \tag{344}$$

where $M \xrightarrow{\pi} U$ and $\pi(S, L) = S$ so $\pi^{-1}(S)$ are the set of lines in S.

Key Lemma.

- M is Zariski closed in $U \times \mathbb{G}(2,4)$
- In the classical topology of $\mathbb{C}^n$, M is locally $U \times \mathbb{G}(2,4)$ the graph of a C^1 function $U \to \mathbb{G}(2,4)$.

Remark 313. In any reasonable space (e.g. locally compact etc.), a local diffeomorphism is a covering map.

Now if we assume lemma for now:

4.) (Endgame: Method of Continuity.) By Step 3, $\left|\pi^{-1}(S)\right|$ is locally constant function on U. But from step 2, $\left|\pi^{-1}(S)\right|$ is constant on U. By Step 1, we get $\left|\pi^{-1}(S)\right| = 27$.

Proof of Key Lemma. Will compute $D\pi$ and apply IFT (i.e. invertible Jacobian implies local diffeomorphism). The goal is to show that $\Delta\pi_m$ is invertible for all $m \in M$.

By changing coordinates, we can assume

$$L = V(x_2, x_3) \tag{345}$$

Then in affine coordinates on $\mathbb{G}(2,4)$, the neighborhood of L in $\mathbb{G}(2,4)$ is spanned by the rows of

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & a_2 & a_3 \end{pmatrix} \tag{346}$$

where $(a_2, a_3, b_2, b_3) \leftrightarrow L$.

So the set of lines in a neighborhood of L in $\mathbb{G}(2,4)$ is in correspondence with (a_2, a_3, b_2, b_3).

On U, we have coordinates

$$\vec{c}_F := (c_I) \in \mathbb{P}^{19} \qquad F = \sum c_I x^I \tag{347}$$

Now a point of $\mathbb{G}(2,4) \times U$ is of the form

$$((a_2, a_3), (b_2, b_3), \vec{c}_F) =: (a, b, c) \tag{348}$$

Such a point lies in M iff $F : \mathbb{C}^4 \to \mathbb{C}$ vanishes the 2-plane in $\mathbb{C}^4$ corresponding to the line given by (a, b). This is true iff

$$F(s(1, 0, a_2, a_3) + t(0, 1, b_2, b_3)) = 0 \qquad s, t \in \mathbb{C} \tag{349}$$

This is true iff

$$\sum_I c_I s^{\alpha_0} t^{\alpha_1} (sa_2 + tb_2)^{\alpha_2} (sa_3 + tb_3)^{\alpha_3} = 0 \qquad \forall s, t \in \mathbb{C} \tag{350}$$

But via binomial theorem

$$\sum_{i=0}^{3} s^i t^{3-i} G_i(a, b, c) = 0 \qquad \forall s, t \tag{351}$$

where G_0, G_1, G_2, G_3 are polynomials in a_2, a_3, b_2, b_3. Therefore,

$$G_i(a, b, c) = 0 \tag{352}$$

for each $i = 0, 1, 2, 3$. This only shows that M is *locally* given by polynomials. This proves the first part of the key lemma.

For the second part, the claim is that all 4 equations $F_i, i = 0, 1, 2, 3$ uniquely determine the line (a_2, a_3, b_2, b_3) in terms of $\vec{c}_F$.

By the Inverse Function Theorem, it suffices to compute the Jacobian

$$D\pi_(a, b, \vec{c}_F) = \left(\frac{\partial F_i}{\partial d_j}\right) \tag{353}$$

with $(d_i) = (a_2, a_3, b_2, b_3)$ is invertible at 0. Proof is Homework 8.

□

Remark 314. Supplementary Problem: Any smooth quartic has 28 bitangent.

Remark 315. Read: immersion and submersion theorem from Guillemin and Pollack (to realy understand Implicit Function Theorem).

16.2 Second Proof of 27 Lines of the Cubic.

Here is another approach: find the 27 lines given 1 line.

Proposition 316. Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface.

1.) For all $p \in S$, there exists at least 3 lines in S through P. If there exists 2 or 3 of these, then they are coplanar.
2.) Every hyperplane $H \subseteq \mathbb{P}^3$, $H \cap S$ is exactly one of
   (a.) An irreducible cubic curve
   (b.) $C + L$ for an irreducible conic C and a line L
   (c.) 3 lines $L_1 + L_2 + L_3$

PROOF 317. Suffice to show cubic. Just need to exclude the triple line and line with double line.

□

Proposition 318. Given line $\ell \subseteq S$ for a smooth cubic surface S, there exists 5 distinct pairs of lines (ℓ_i, ℓ_j) such that

1.) $\ell \cup \ell_i \cup \ell_i'$ is contained in plane in $\mathbb{P}^3$
2.) $(\ell_i \cup \ell_i') \cap (\ell_j \cap \ell_j') = \emptyset$ for all $i \neq j$

PROOF 319. Wlog, take $\ell = V(x_2, x_3)$. Any plane through ℓ is of the form $V(\mu x_2 - \lambda x_3),\ \mu, \lambda \in \mathbb{C}$. Take wlog $\mu \neq 0$ so that we have the linear plane in $\mathbb{P}^3$:

$$H_\lambda := V(x_2 - \lambda x_3) \tag{354}$$

So,

$$\begin{aligned} S \cap H_\lambda &= V(F|_{H_\lambda}) \\ &= x_3 Q(x_0, x_1, x_2) \end{aligned}$$

where

$$Q(x_0, x_1, x_2) = a(\lambda, 1)x_0^2 + b(\lambda, 1)x_0x_1 + c(\lambda, 1)x_1^2 + ... \tag{355}$$

Proposition 320. A conic in $\mathbb{A}^2$ given by

$$F(x_0, x_1) = ax_0^2 + bx_0x_1 + ... \tag{356}$$

is singular iff $\Delta(a, b, c, d, e, f) = 0$ for the discriminant Δ of the family of conics where

$$\Delta(a, b, c, d, e, f) = \det \begin{pmatrix} a & b & c \\ b & c & e \\ d & e & f \end{pmatrix} \tag{357}$$

□

16.3 Problem Session - 11.30.2018

Homework 8.

1.) 1.
  (a.) a.) Pick any equation;
  (b.) Consider automorphism, i.e. third roots of unity
  (c.) See Gathmann
2.) 2.
  (a.) Consider
$$x_0 + x_1 + ... + x_4 = 0 \tag{358}$$
$$x_0^3 + x_1^3 + ... + x_4^3 = 0 \tag{359}$$
  (b.) Try plugging general line.
3.) 3.
  (a.) b.) Just show for local; not sure if it is true for global.
  (b.) c.) Fermat hypersurface
    i.) There is probably a nice trick...
    ii.) Use: Fermat quintic threefold
4.) 4.
  (a.) p.92 in Gathmann (version 2 pdf); clearer in class
  (b.) Recall that we have
$$\sum s^i t^{3-i} F_i(a, b, c) = 0 \tag{360}$$
5.) 5.
  (a.) Use discriminant theorem for conics we did in class.
  (b.) Just a computation
  (c.) the 1-parameter family does not give all the lines
6.) 6.
  (a.) Just send the lines using projective transformations
  (b.) In $\mathbb{P}^3$, $Q = V(f)$ is a quadric. Take the line $L \subseteq \mathbb{P}^3$. L is disjoint from Q. $L \subseteq Q$ iff $L \cap Q$ contains 3 points. (Method from Reu paper.)

Part II
Notes.

The purpose of these notes is to pull together different ideas from the course in an attempt to reach a cohesive, thematic understanding of the content of the course. These "notes" differ from the preceding lecture notes in that the content is not strictly from the lectures, and there are more explanations and ideas taken from books (such as Arrondo's notes and Fulton's book).