Pricing the knock-out option of SABR model transform Hyperbolic Brownian Motion

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1 The Problem

1 Introduction

In this paper , we consider pricing ths knock-out barrier option of SABR model transform Hyperbolic Brownian Motion.

SABR model

The SABR model is a stochastic volatility model , and it is a two factor model with the dynamics give by a system of two stochastic differential equation.The state variables of the model can be though of as the forward price of an asset , and volatility parameter.The dynamics of the forword in the SABR model is given by :

$$dF_t = \Sigma_t C(F_t) dW_t$$
$$d\Sigma_{t]} = \mathrm{v}\Sigma_t dZ_t$$

where W_t and Z_T are Brownian motions with $E[dW_t dZ_t] = \rho d_t$. is the forword rate process , and Wt and Zt are Brownian Motions with $E[dW_t dZ_t] = \rho d_t$
where the correlation ρ is assumed constant.

Barrier option

Barrier option is a type of option , if the rate is not the underlying exceeds a certain price , it is nearing the Maturity. On the other hand , if the rate is the underlying exceeds a certain price , the right occurs or the right disaooear.
In this paper , in the case of the right disappear. It calls knock-out barrier option . This barrier option solved by symmetrization .

1-1 SABRmodel における knock out barrier option について

Let (F_t, Σ_t) be the solution of the stochastic differential equation :

$$dF_t = \Sigma_t C(F_t) dW_t$$
$$d\Sigma_{t]} = \text{v}\Sigma_t dZ_t$$

where W_t and Z_T are Brownian motions with $E[dW_t dZ_t] = \rho d_t$.

Let S be a unique (weak) solution to the following stochastic differential equation ;
$dS_t = V_t \sigma_1(S_t)(\rho dW_t^1 + \sqrt{1+\rho^2} dW_t^2) + rS_t dt$
$dV_t = \sigma_2(V_t) dW_t^1 + \mu(V_t) dt$
where $\sigma_i = 1, 2$ and μ are smooth functions on the half line, and μ is a constant in [-1,1] , and (W^1, W^2) is a standerd 2-dim Brownian motions definded on a filered probability space. Let K be a positive constant smaller than V_0 and
$\pi_K^1 = \pi_K^{spot} := \inf\{t > 0 : V_t < K\}$
$\pi_K^2 = \pi_K^{integ} := \inf\{t > 0 : \int_0^t V_s ds > K\}$
and
$\pi_K^3 = \pi_K^{abinteg} := \inf\{t > 0 : \frac{1}{t} \int_0^t V_s ds > K\}$
We are interested in numerical calculation of the following quantities:
$\pi^i(K, T, F) := \mathbb{E}[F(S_T) 1_{\{}\pi_K^i < T\}] for i = 1, 2, 3,$
which stand for the price of timer options, the timer being spot volatility for i=1, the integrated volatility for $= 2$, and the time-average of the integrated volatility for i=3 .Here T ¿ 0 means the maturity,and $F \in D([0, \infty))$ is the pay-off function, of the options.

2 Symmetrization for the spot timer

We apply the symmetrization technique proposed in [1] to calculate (2). Instead of explaining it in a general situation, we describe in detail of the case $i = 1$, where the technique works properly.
Put

$$\Sigma(x, y) := \begin{pmatrix} \rho y \sigma_1(x) & \sqrt{1-\rho^2} y \sigma_1(x) \\ \sigma_2(y) & 0 \end{pmatrix} \text{ and define}$$

Then there exist a right-continuous non-decreasing function F on $\mathbb{R}$ such that $0 \leq$ F ≤ 1 and a subsequence (n_i) such that
$\lim_{i \to \infty} F_{n_i}(x) = F(x)$ at every point of continuity F.
<u>Proof</u>

Let $C = (c_1, c_2, \cdots)$ be a countable dense set of $\mathbb{R}$.
対角線論法によって、
$n_i = n_{ii}$ とおくとき。

$$H(c) = \lim_{i\to\infty} F_{n_i}(c), \quad c \in \text{C. が成り立つ。}$$

Obviously

$$0 \leq H \leq 1, \ \ \text{H is non-decreasing function on C.}$$

ここで、

$$F(x) = \inf_{c>x} H(c) \text{ とおくと、F is non-decreasing function on } \mathbb{R}.$$

よって残りは、

$$\limsup_{i\to\infty} F_{n_i}(x) \leq F(x) \leq \liminf_{i\to\infty} F_{n_i}(x)$$

を示せばよい。

$\underline{\limsup_{i\to\infty} F_{n_i}(x) \leq F(x)}$
$\forall c > x$ に対して、
$F_{n_j}(x) \leq H(c)$ より、

$$\limsup_{j\to\infty} F_{n_j}(x) \leq \limsup F_{n_j}(c) = \lim_{j\to\infty} F_{n_j}(c) = H(c)$$

従って、

$$\limsup_{j\to\infty} F_{nj}(x) \leq \inf_{c>x} H(c) = F(x)$$

$\underline{F(x) \leq \liminf_{i\to\infty} F_{n_i}(x)}$
$\forall \epsilon > 0, c > x$ に対して、
$F(x-\epsilon) = \inf_{c>x-\epsilon} H(c) \leq \inf_{c\in(x-\epsilon,x)} H(x)$
ここで、$H(x) = \liminf_{i\to\infty} F_{n_i}(c) \leq \liminf_{i\to\infty} F_{n_i}(x)$ より、
$F(x-\epsilon) \leq \inf_{c\in()x-\epsilon,x} H(c) \leq \lim_{i\to\infty} F_{n_i}(x)$
F の left-continuous 性より、$\epsilon \nearrow 0$ として、

$F(x) \leq \lim_{i\to\infty} F_{n_i}(x)$ を得る。