```
In [1]:  1  import numpy as np
         2  import pandas as pd
         3  import matplotlib.pyplot as plt
         4  import math
         5  % matplotlib inline
```

1 (1)

$$P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

とすると、

$$P^{-1}JP = D$$

と対角化できる。

ここで、

$$x = \begin{bmatrix} p \\ q \end{bmatrix}$$

とし、

$$x = Pu$$

と置くと、

$$\dot{u} = P^{-1}\dot{x}$$
$$= P^{-1}Jx$$
$$= P^{-1}JPP^{-1}x$$
$$= Du$$

つまり、

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

として、

$$\dot{u_1} = iu_1$$

$$\dot{u_2} = -iu_2$$

$$u_1(0) = \frac{1}{2}$$

$$u_2(0) = \frac{1}{2}$$

と書ける。

よって、uについて数値的に考察すれば、

$$x = Pu$$

を用いて

$$x = \begin{bmatrix} p \\ q \end{bmatrix}$$

の挙動を数値的に考えていくことができる。

また、この方程式を解くと、解は

$$x = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$$

となる。

```
In [2]:  1  P = np.array([[1, 1], [-1j, 1j]])
         2
         3  def u2x(u_1_array, u_2_array):
         4      return np.array([np.dot(P, np.array([[u_1], [u_2]])) for u_1, u_2 in zip(u_1_array, u_2_array)])
         5
         6  def plot_x(x, title):
         7      plt.scatter(x[0, 0], x[0, 1], color='green', label='Start')
         8      plt.scatter(x[-1, 0], x[-1, 1], color='red', label='End')
         9      plt.plot(x[:, 0], x[:, 1], alpha = 0.7)
        10
        11      plt.legend()
        12      plt.title(title)
        13      plt.xlabel('p')
        14      plt.ylabel('q')
        15      plt.grid()
        16      plt.axis('equal')
        17      plt.show()
```

```
In [3]:  1  def f_1(x):
         2      return 1j * x
         3
         4  def f_2(x):
         5      return -1j * x
```

1.1 真の解

```
In [4]:  1  T = [i * 0.1 for i in range(1001)]
         2  x = np.array([[[math.cos(t)], [math.sin(t)]] for t in T])
         3
         4  plot_x(x, '真の解')
```


真の解
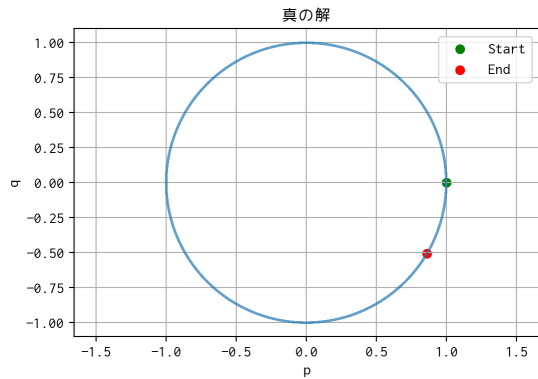
1.2 陽的Euler法

```
In [5]:   1  def explicit_euler_method(f, x_0, h = 0.1, T = 100):
          2      t = 0
          3      xs = [x_0]
          4      while t < T:
          5          xs.append(h * f(xs[-1]) + xs[-1])
          6          t += h
          7      return xs
          8
          9  u_1 = explicit_euler_method(f_1, 0.5)
         10  u_2 = explicit_euler_method(f_2, 0.5)
         11
         12  x = u2x(u_1, u_2)
         13
         14  plot_x(x, '陽的Euler法')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
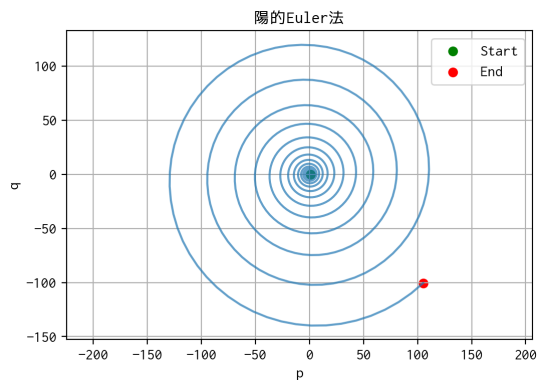
陽的Euler法

1.3 陰的Euler法

```
In [6]:  1  def u_1_implicit_euler_method(x_0, h = 0.1, T = 100):
         2      t = 0
         3      xs = [x_0]
         4      while t < T:
         5          xs.append(xs[-1] / (1 - 1j * h))
         6          t += h
         7      return xs
         8
         9  def u_2_implicit_euler_method(x_0, h = 0.1, T = 100):
        10      t = 0
        11      xs = [x_0]
        12      while t < T:
        13          xs.append(xs[-1] / (1 + 1j * h))
        14          t += h
        15      return xs
        16
        17  u_1 = u_1_implicit_euler_method(0.5)
        18  u_2 = u_2_implicit_euler_method(0.5)
        19
        20  x = u2x(u_1, u_2)
        21
        22  plot_x(x, '陰的Euler法')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
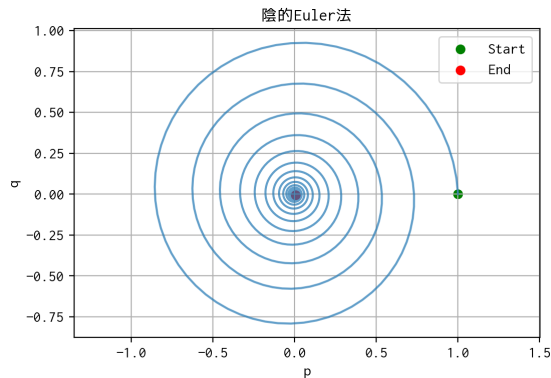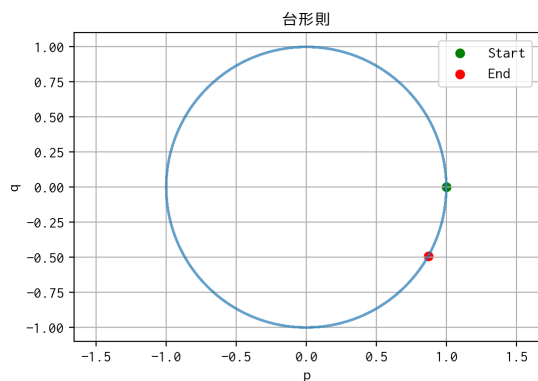

1.4 台形則

```
In [7]:  1  def u_1_trapezoidal_rule(x_0, h = 0.1, T = 100):
         2      t = 0
         3      xs = [x_0]
         4      while t < T:
         5          xs.append((2 + h * 1j) / (2 - h * 1j) * xs[-1])
         6          t += h
         7      return xs
         8
         9  def u_2_trapezoidal_rule(x_0, h = 0.1, T = 100):
        10      t = 0
        11      xs = [x_0]
        12      while t < T:
        13          xs.append((2 - h * 1j) / (2 + h * 1j) * xs[-1])
        14          t += h
        15      return xs
        16
        17  u_1 = u_1_trapezoidal_rule(0.5)
        18  u_2 = u_2_trapezoidal_rule(0.5)
        19
        20  x = u2x(u_1, u_2)
        21
        22  plot_x(x, '台形則')
```
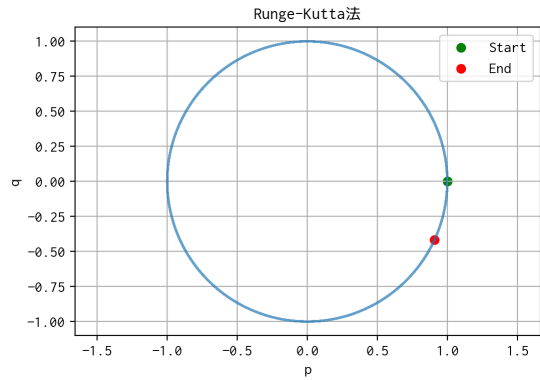
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

1.5 Runge-Kutta法

In [8]:
```python
def runge_kutta_method(f, x_0, h = 0.1, T = 100):
    t = 0
    xs = [x_0]
    while t < T:
        y_n = xs[-1]
        k_1 = f(y_n)
        k_2 = f(y_n + h / 2 * k_1)
        k_3 = f(y_n + h / 2 * k_2)
        k_4 = f(y_n + h * k_3)
        xs.append(xs[-1] + h * (k_1 / 6 + k_2 / 3 + k_3 / 3 + k_4 / 6))
        t += h
    return xs

u_1 = runge_kutta_method(f_1, 0.5)
u_2 = runge_kutta_method(f_2, 0.5)

x = u2x(u_1, u_2)

plot_x(x, 'Runge-Kutta法')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
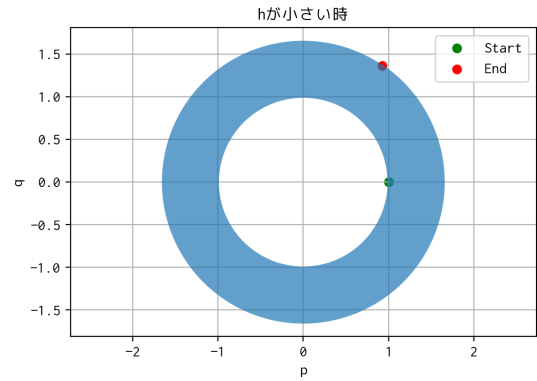 return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

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2 (2)

2.1 陽的Euler法

```
In [9]:   1  u_1 = explicit_euler_method(f_1, 0.5, 0.001, 1000)
          2  u_2 = explicit_euler_method(f_2, 0.5, 0.001, 1000)
          3
          4  x = u2x(u_1, u_2)
          5
          6  plot_x(x, 'hが小さい時')
```
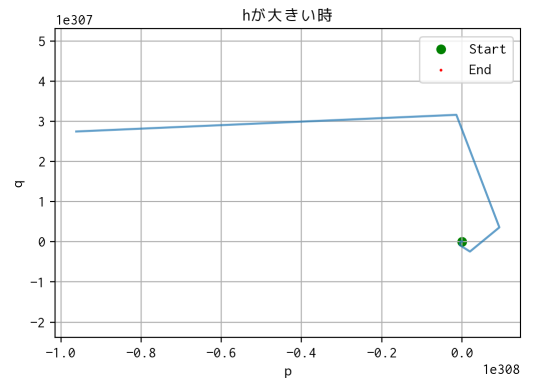
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

発散のスピードは小さくなっているが、やはり発散するような軌道を描いている。

```
In [10]:  1  u_1 = explicit_euler_method(f_1, 0.5, 3, 10000)
          2  u_2 = explicit_euler_method(f_2, 0.5, 3, 10000)
          3
          4  x = u2x(u_1, u_2)
          5
          6  plot_x(x, 'hが大きい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/matplotlib/ticker.py:1914: RuntimeWarning: overflow encountered in multiply
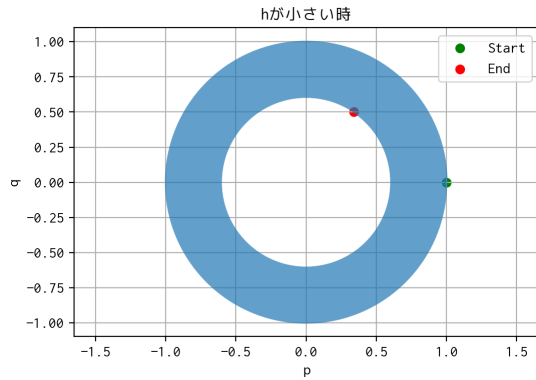 steps = self._extended_steps * scale

非常に早く発散し、overflowがおきている。

2.2 陰的Euler法

In [11]:
```python
u_1 = u_1_implicit_euler_method(0.5, 0.001, 1000)
u_2 = u_2_implicit_euler_method(0.5, 0.001, 1000)

x = u2x(u_1, u_2)

plot_x(x, 'hが小さい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

hが小さい時

収束スピードは遅いが、原点に収束しているような軌道を描いている。

In [12]:
```python
u_1 = u_1_implicit_euler_method(0.5, 3)
u_2 = u_2_implicit_euler_method(0.5, 3)

x = u2x(u_1, u_2)

plot_x(x, 'hが大きい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
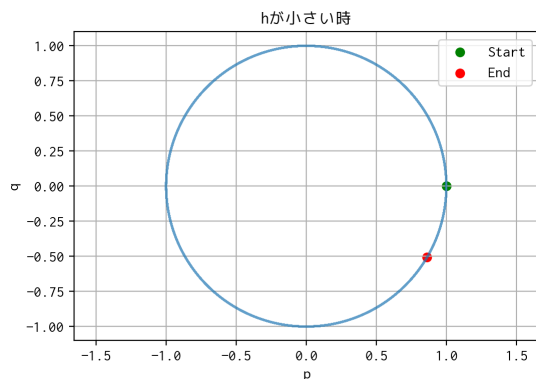 return array(a, dtype, copy=False, order=order, subok=True)
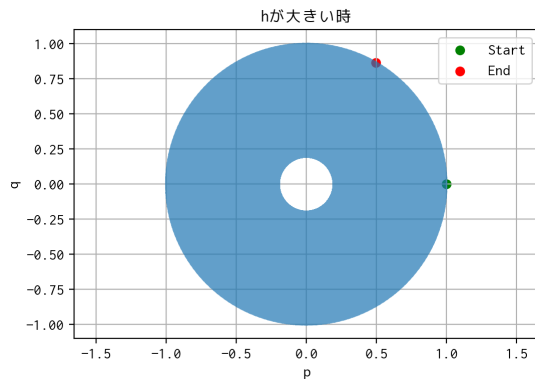/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

hが大きい時
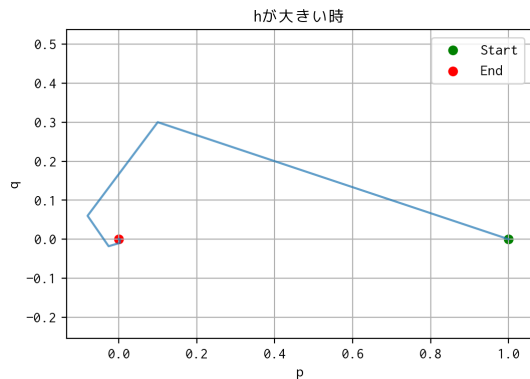
非常に早く原点に収束している

2.3 台形則

In [13]:
```python
u_1 = u_1_trapezoidal_rule(0.5, 0.0001)
u_2 = u_2_trapezoidal_rule(0.5, 0.0001)

x = u2x(u_1, u_2)

plot_x(x, 'hが小さい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

hが小さい時

hが小さい時は安定した正しい軌道を与える。

In [14]:
```python
u_1 = u_1_trapezoidal_rule(0.5, 10, 10000)
u_2 = u_2_trapezoidal_rule(0.5, 10, 10000)

x = u2x(u_1, u_2)

plot_x(x, 'hが大きい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
 return array(a, dtype, copy=False, order=order)

hが大きくてグラフが見にくいので、描画メソッドを編集する(plotではなく、scatterにする)

In [15]:
```python
def scatter_x(x, title):
    plt.scatter(x[:, 0], x[:, 1], alpha = 0.7)
    plt.scatter(x[0, 0], x[0, 1], color='yellow', label='Start')
    plt.scatter(x[-1, 0], x[-1, 1], color='red', label='End')

    plt.legend()
    plt.title(title)
    plt.xlabel('p')
    plt.ylabel('q')
    plt.grid()
    plt.axis('equal')
    plt.show()
```

In [16]:
```python
scatter_x(x, 'hが大きい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
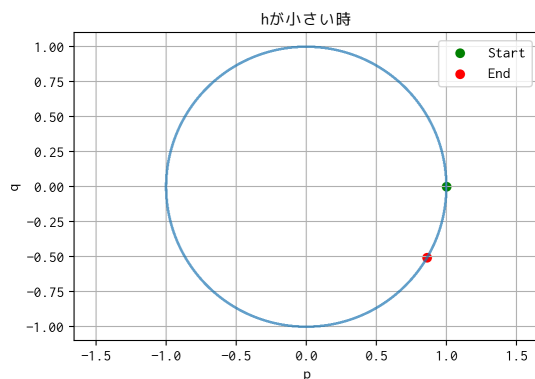 return array(a, dtype, copy=False, order=order, subok=True)
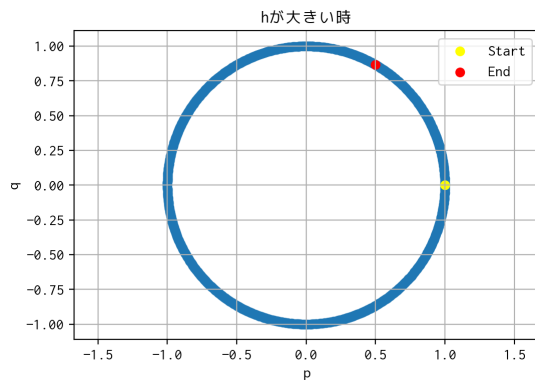
台形則では、hは大きい時も正しい軌道になることが確認できた。

2.4 Runge-Kutta法

In [17]:
```python
u_1 = runge_kutta_method(f_1, 0.5, 0.0001)
u_2 = runge_kutta_method(f_2, 0.5, 0.0001)

x = u2x(u_1, u_2)

plot_x(x, 'hが小さい時')
```
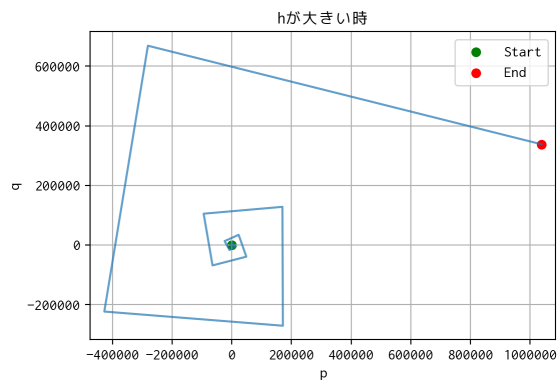
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

hを小さくしても(1)の状態と変わらない

In [18]:
```
1  u_1 = runge_kutta_method(f_1, 0.5, 3)
2  u_2 = runge_kutta_method(f_2, 0.5, 3)
3
4  x = u2x(u_1, u_2)
5
6  plot_x(x, 'hが大きい時')
```

/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:544: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order, subok=True)
/Users/uedatomohiro/.pyenv/versions/anaconda3-5.1.0/lib/python3.6/site-packages/numpy/core/numeric.py:492: ComplexWarning: Casting complex values to real discards the imaginary part
  return array(a, dtype, copy=False, order=order)

hを十分大きくすると(h = 3)発散してしまっている。