Lecture 5

Lagrange’s Theorem & Quotient Group & Homomorphism (準同型) and Isomorphism (同型) of Groups

What you will learn in Lecture 5

5.1 Lagrange's Theorem (ラグランジュの定理)

5.2 Normal Subgroup (正規部分群) & Quotient Group (商群)

5.3 Homomorphism (準同型) and Isomorphism (同型) of Groups

5.1 Lagrange's Theorem

(ラグランジュの定理)

5.1 Lagrange's Theorem (ラグランジュの定理)

In the last section, **we noted that the order of a subgroup of a finite cyclic group divides the order of the group** (Corollary 4.2).

We will learn that this is a special case of a general result, called **Lagrange's theorem**, i.e., **the order of a subgroup of a finite group divides the order of the group**.

History:
Lagrange proved this result in 1770, long before the creation of group theory, while working on the permutations of the roots of a polynomial equation. Lagrange's theorem is a basic theorem of **finite group theory** and is considered by some to be the most important result in finite group theory.

Left and Right Cosets

Definition 5.1

Let $(H,\circ)$ be **a subgroup of a group** $(G,\circ)$ and $a \in G$. The sets $aH = \{ah \mid h \in H\}$ and $Ha = \{ha \mid h \in H\}$ are called the **left and right cosets (左剰余類と右剰余類) of H in G**, respectively. The element a is called **a representative** of aH **and** Ha.

If G is **commutative**, then of course we have $aH = Ha$.

Observe that $eH = H = He$ and that $a = ae \in aH$ and $a = ea \in Ha$.

Example 5.1 Exhibit the left cosets and the right coset of the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$.

Solution:

Due to the notation here is additive, so the left coset of $3\mathbb{Z}$ containing m is $m + 3\mathbb{Z}$. We **first** take $m = 0$, and obtain

$$3\mathbb{Z} = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$$

It is one of the left cosets, which contains 0.

Next, we find other left cosets. Now select an element of $\mathbb{Z}$ not in $3\mathbb{Z}$, for example, 1, and find the left coset containing it. We have

$$1 + 3\mathbb{Z} = \{\dots, -8, -5, -2, 1, 4, 7, 10, \dots\}$$

These two left cosets $3\mathbb{Z}$ and $1 + 3\mathbb{Z}$ still do not yet exhaust all elements of $\mathbb{Z}$. For example, 2 is in neither of them. **Then** we find the left coset containing 2 is

$$2 + 3\mathbb{Z} = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$$

It is clear that these three left cosets we have found do exhaust $\mathbb{Z}$, so they constitute $\mathbb{Z}$ by three left cosets of $3\mathbb{Z}$.

Example 5.2 Consider the symmetric group S_3 (Example 3.7).

(1)
$$H = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

is a subgroup of S_3.

We now compute the left and right cosets of H in S_3. The left cosets of H in S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H = H$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H = \boxed{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H} =$$

$(1\ 2)H$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

and the right cosets of H in are

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = H$$

and

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right\}.$$

Thus, for all $a \in S_3$, $aH = Ha$.

5.1 Lagrange's Theorem (ラグランジュの定理)

Left and Right Cosets

(2) $$H' = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$$

is also a subgroup of S_3.

Now we compute the left and right cosets of H' in S_3. The left cosets of H' in S_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H' = H',$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\},$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$$

and the right cosets of H' in S_3 are

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H',$$

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right\},$$

and

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}.$$

We see that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' \neq H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Thus, the left and right cosets of H' in S_3 are not the same.

There are some interesting phenomena happening in the above example.

- We see that all left and right cosets of H in S_3 have the same number of elements, namely, 3; that there are the same number of distinct left cosets of H in S_3 as of right cosets, namely, 2; that the set of all left cosets and the set of all right cosets form partitions of S_3; and, finally, that $3 \cdot 2$ equals the order of S_3.
- Similar statements hold for the subgroup H'. We show, in the results to follow, that these phenomena hold in general.

5.1 Lagrange's Theorem (ラグランジュの定理)

The following theorem tells us **when two left (right) cosets are equal.** It is a result that is used often in the study of groups.

Theorem 5.1

Let $(H,\circ)$ be a **subgroup** of a **group** $(G,\circ)$ and $a, b \in G$. Then
(i) $aH = bH$ if and only if $b^{-1}a \in H$.
(ii) $Ha = Hb$ if and only if $ab^{-1} \in H$.

Theorem 5.2

Let $(H,\circ)$ be a **subgroup** of a **group** $(G,\circ)$. Then for all $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$ (i.e., two left cosets are either equal or they are disjoint). Similar result also satisfied for two right cosets.

5.1 Lagrange's Theorem (ラグランジュの定理)

Number of Distinct Cosets

Definition 5.2

Let $(H,\circ)$ be a subgroup of a group $(G,\circ)$. Then the number of distinct (相異なる) left (or right) cosets, written as $[G:H]$, of H in G is called the index of H in G.

Theorem 5.3 (Lagrange's Theorem)

Let $(H,\circ)$ be a **subgroup** of a **finite group** $(G,\circ)$. Then **the order of** $(H,\circ)$ **divides the order of** $(G,\circ)$. In particular,

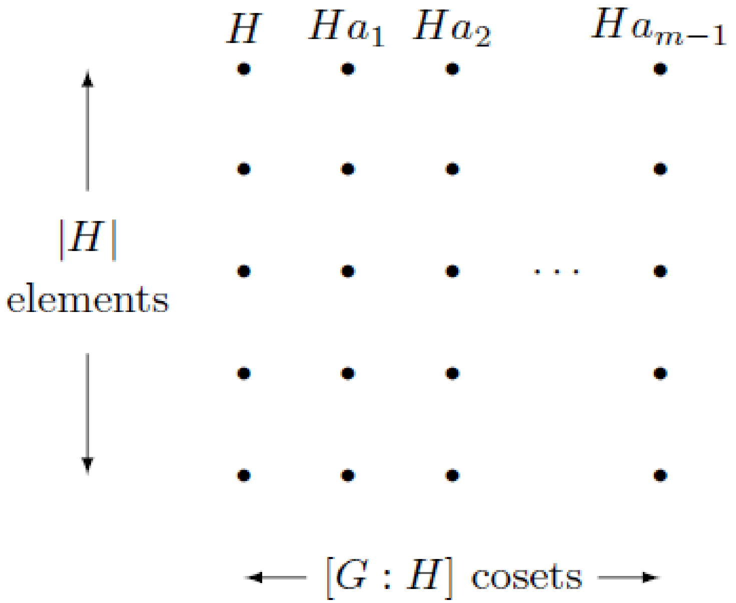
$$|G| = [G : H]|H|.$$

Proof:

Suppose that $[G : H] = m$. Every element of G is in a coset of H, and Theorem 4.8 tells us we can decompose G into a union of m pairwise disjoint cosets:

$$G = H \cup Ha_1 \cup Ha_2 \cup \cdots \cup Ha_{m-1}$$

But each of these cosets has $|H|$ elements. Thus, there must be $[G : H]|H|$ elements in G altogether.



Theorem 5.4

Every group of prime order is cyclic.

Proof:

Let group $(G,\circ)$ be of prime order p, and let a be an element of G different from the identity element e.

Then the cyclic subgroup $(\langle a\rangle,\circ)$ of $(G,\circ)$ generated by a has at least two elements, a and e.

But by Lagrange’s Theorem, the order $m \geq 2$ of $\langle a\rangle$ must divide the prime p.

Thus we must have $m = p$ and $\langle a\rangle = G$, so $(G,\circ)$ is cyclic.

5.2 Normal Subgroup (正規部分群) &
Quotient Group (商群)

Normal Subgroup

Definition 5.3

Let $(G,\circ)$ be a group. A subgroup $(H,\circ)$ of $(G,\circ)$ is said to be a **normal subgroup (正規部分群) (or invariant subgroup)** of G if $aH = Ha$ for all $a \in G$.

Example 5.3 Let $(G,\circ)$ be an **abelian group**. Every subgroup $(H,\circ)$ of $(G,\circ)$ is a normal subgroup.

Since $gh = hg$ for all $g \in G$ and $h \in H$, it will always be the case that $gH = Hg$.

Example 5.4 Let $(H,\circ)$ be the subgroup of S_3 consisting of elements (1) and (1 2). Since

$(1\ 2\ 3)H = \{(1\ 2\ 3), (1\ 3)\}$ and $H(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3)\}$;

$(H,\circ)$ cannot be a normal subgroup of S_3.

However, the subgroup $(N,\circ)$, consisting of the permutations $e, (1\ 2\ 3)$, and $(1\ 3\ 2)$, is normal since the cosets of N are

$$N = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$$

$$(1\ 2)N = N(1\ 2) = \{(1\ 2), (1\ 3), (2\ 3)\}$$

Example 5.5

Recall Example 5.2. $(H,\circ)$ is a normal subgroup of S_3. Consider $h = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in H$. Then

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ h = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

and

$$h \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ h \neq h \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

even though

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H = H \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Theorem 5.5

Let $(H,\circ)$ be a subgroup of a group $(G,\circ)$. Then $(H,\circ)$ is a **normal subgroup** of $(G,\circ)$ **if and only if** for all $a \in G$, $aHa^{-1} \subseteq H$.

Proof:
First suppose that H is a normal subgroup of G. Let $a \in G$.
We now show that $aHa^{-1} \subseteq H$. Let $aha^{-1} \in aHa^{-1}$, where $h \in H$. Since H is a normal subgroup of G, $aH = Ha$. Also, since $ah \in aH$, we have $ah \in Ha$ and so $ah = h'a$ for some $h' \in H$.
Thus, $aha^{-1} = h' \in H$. Hence, $aHa^{-1} \subseteq H$.
Conversely, suppose $aHa^{-1} \subseteq H$ for all $a \in G$. Let $a \in G$.
We show that $aH = Ha$. Let $ah \in aH$, where $h \in H$. Now $aha^{-1} \in aHa^{-1}$ and so $aha^{-1} \in H$. Thus, $aha^{-1} = h'$ for some $h' \in H$. This implies that $ah = h'a \in Ha$. Therefore, $aH \subseteq Ha$. Similarly, we can show that $Ha \subseteq aH$. Hence, $aH = Ha$.
Consequently, H is a normal subgroup of G.

Theorem 5.6

Let $(H,\circ)$ and $(L,\circ)$ be **normal subgroups of a group** $(G,\circ)$. Then

(i) $H \cap L$ **leads to a normal subgroup of** $(G,\circ)$,

(ii) $HL = LH$ **leads to a normal subgroup of** $(G,\circ)$,

(iii) $\langle H \cup L \rangle = HL$.

Definition 5.4

Let $(G,\circ)$ be a group and $(H,\circ)$ be a **normal subgroup** of $(G,\circ)$. The **group** $(G/H,\circ)$ is called the **quotient group (商群)** (or **factor group**) of G by H.

Theorem 5.7

Let $(H,\circ)$ be a **normal subgroup** of a group $(G,\circ)$. Denote the set of all left cosets $\{aH \mid a \in G\}$ by G/H and define $\circ$ on G/H by for all $aH, bH \in G/H$,

$$(aH) \circ (bH) = abH$$

Then $(G/H,\circ)$ is a **quotient group (商群)**.

Example 5.6

The normal subgroup of S_3 in Example 3.7,

$$H = \{e, (1\ 2\ 3), (1\ 3\ 2)\}.$$

The cosets of H in S_3 are H and $(1\ 2)H$ from Example 5.2.

The quotient group S_3/H has the following operation table by $aH, bH \in S_3/H$ in Theorem 5.7. (Here $aH = H$, $bH = (1\ 2)H$)

	H	$(1\ 2)H$
H	H	$(1\ 2)H$
$(1\ 2)H$	$(1\ 2)H$	H

Quotient Subgroup

Example 5.7

Consider the normal subgroup $3\mathbb{Z}$ of $\mathbb{Z}$ (Example 5.1). The cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ are

$$0+3\mathbb{Z}=\{\ldots,-3,0,3,6,\ldots\}$$
$$1+3\mathbb{Z}=\{\ldots,-2,1,4,7,\ldots\}$$
$$2+3\mathbb{Z}=\{\ldots,-1,2,5,8,\ldots\}$$

The group $\mathbb{Z}/3\mathbb{Z}$ is given by the operation table below.

$+$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$0+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$1+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$
$2+3\mathbb{Z}$	$2+3\mathbb{Z}$	$0+3\mathbb{Z}$	$1+3\mathbb{Z}$

 Quotient Subgroup

In general, the subgroup $(n\mathbb{Z},\circ)$ of $(\mathbb{Z},\circ)$ is **normal**. The cosets of $\mathbb{Z}/n\mathbb{Z}$ are

$$\begin{gathered} n\mathbb{Z} \\ 1 + n\mathbb{Z} \\ 2 + n\mathbb{Z} \\ \vdots \\ (n-1) + n\mathbb{Z}. \end{gathered}$$

The sum of the cosets $k + \mathbb{Z}$ and $l + \mathbb{Z}$ is $k + l + \mathbb{Z}$.
Notice that we have written our cosets additively, because the group operation is integer addition.

Quotient Subgroup

Example 5.8 Consider $\mathbb{Z}_8$ (see Example 2.7) and let $H = \{[0], [4]\}$. Then $(H, \circ)$ is a normal subgroup of $(\mathbb{Z}_8, \circ)$. Now $|H| = 2$ and $|\mathbb{Z}_8| = 8$. Thus, $|\mathbb{Z}_8/H| = \frac{|Z8|}{|H|} = 4$. Hence, $\mathbb{Z}_8/H$ has four elements. We know

$$[0] + H = H = [4] + H,$$

$$[1] + H = \{[1], [5]\} = [5] + H,$$

$$[2] + H = \{[2], [6]\} = [6] + H,$$

and

$$[3] + H = \{[3], [7]\} = [7] + H.$$

Hence, $\mathbb{Z}_8/H = \{[0] + H, [1] + H, [2] + H, [3] + H\}$.

5.3 Homomorphism (準同型) and Isomorphism (同型) of Groups

Definition 5.5

A **homomorphism (準同型)** between groups $(G_1, \circ)$ and $(G_2, *)$ is a function (or map) $f : G_1 \to G_2$ such that

$$f(a \circ b) = f(a) * f(b).$$

for all $a, b \in G_1$.

(Here $\circ$ and $*$ are two binary operations.)

Example 5.9

Let $(\mathbb{Z}, +)$ and $(G, \cdot)$ be groups and $g \in G$.

Define a function $f: \mathbb{Z} \to G$ by $f(n) = g^n$.

Then f is a group homomorphism, since

$f(m+n) = g^{m+n} = g^m g^n = f(m)f(n)$:

This homomorphism maps $\mathbb{Z}$ onto the **cyclic subgroup** of G generated by g.

Example 5.10
We define a circle group $(T,\circ)$ consists of all **complex numbers** $z \in \mathbb{C}$ such that $|z| = 1$.

We can define a **homomorphism** f from the additive group of real numbers $\mathbb{R}$ to T by

$f: \theta \mapsto \cos\theta + i\sin\theta.$

Indeed, for $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} f(\alpha+\beta) &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= f(\alpha)f(\beta): \end{aligned}$$

Theorem 5.11

Let f be a **homomorphism** of a group $(G_1, \circ)$ into a group $(G_2, *)$.

Then

(i) $f(e_1) = e_2$.

(ii) $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G_1$.

(iii) If H_1 is a subgroup of G_1, then $f(H_1) = \{f(h) \mid h \in H_1\}$ is a subgroup of G_2.

(iv) If G_1 is commutative, then $f(G_1)$ is commutative.

Kernel

*Definition 5.6

Let f **be a homomorphism of a group** $(G_1,\circ)$ **into a group** $(G_2,*)$.

The **kernel** of f, written $Ker\ f$, is defined to be the set

$$Ker\ f = \{a \in G_1 \mid f(a) = e_2\}.$$

*Theorem 5.12

Let f **be a homomorphism of a group** $(G_1,\circ)$ **into a group** $(G_2,*)$.

Then $(Ker\ f,\circ)$ is a **normal subgroup** of $(G_1,\circ)$.

Notice: * mark is optional material. It will not be included in both middle and final examinations.

Isomorphism

*Definition 5.7

A **homomorphism** f **of a group** $(G_1,\circ)$ **into a group** $(G_2,*)$ is called an **isomorphism (同型)** of G_1 onto G_2 if f is **one-to-one** and **onto** G_2. In this case, we write $G_1 \simeq G_2$ and say that G_1 and G_2 are **isomorphic**.

Isomorphism

***Example 5.11** Let $(G,\circ)$ be a cyclic group with generator g.

Define a map $f: \mathbb{Z} \to G$ by $n \mapsto g^n$. This map is a **surjective homomorphism** since

$f(m+n) = g^{m+n} = g^m g^n = f(m)f(n)$:

Clearly f is **onto**. If $|g| = m$, then $g^m = e$.

Hence, $ker\, f = m\mathbb{Z}$ and $\mathbb{Z}/ker\, f = \mathbb{Z}/m\mathbb{Z} \simeq G$.

On the other hand, if the order of g is **infinite**, then $ker\, f = 0$ and f is an **isomorphism** of G and $\mathbb{Z}$.

Hence, **two cyclic groups** are **isomorphic** exactly when they have the **same order**.

Up to **isomorphism**, the only **cyclic groups** are $\mathbb{Z}$ and $\mathbb{Z}_n$.

*Theorem 5.13

Every **finite cyclic group** of **order** n is **isomorphic** to $(\mathbb{Z}_n, +_n)$ and every **infinite cyclic group** is **isomorphic** to $(\mathbb{Z}, +)$.

Review for Lecture 5

- Lagrange’s Theorem (ラグランジュの定理)
- Normal Subgroup (正規部分群)
- Quotient Group (商群)
- Homomorphism (準同型) of Groups
- Isomorphism (同型) of Groups

Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

References

[1] Thomas W. Judson etc. Abstract Algebra Theory and Applications, 2018

[2] D. S. Malik, John N. Mordeson, M.K. Sen, Introduction to Abstract Algebra, 2007

[3] **(おすすめ)** 松本 眞, 代数系への入門, http://www.math.sci.hiroshima-u.ac.jp/~m-mat/TEACH/daisu-nyumon2014.pdf

[4] Wikipedia

[5] Materials from internet.

Appendix (付録)

*Theorem

Let $(H,\circ)$ and $(L,\circ)$ be **finite subgroups** of a **group** $(G,\circ)$. Then the **order**

$$|HL| = \frac{|H||L|}{|H \cap L|}$$