



Lecture 11

Integral Domains (整域)

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11.1 Euclidean Domains

(ユークリッド整域)

11.1 Euclidean Domains (ユークリッド整域)

We have seen that both rings \mathbb{Z} and $F[x]$, where F is a field, have a **division algorithm**. In this lecture, we discuss the properties for **integral domains (整域)**.

Recall in Lecture 1, we defined Complement of Set as

Definition 1.4

Given two sets A and B , the **relative complement** of B in A , denoted by *the set difference* $A \setminus B$, is the set

$$A \setminus B = A \cap B' = \{x \mid x \in A, \text{ but } x \notin B\}.$$

Definition 11.1

A **Euclidean domain** is an **integral domain** $(E, +, \cdot)$ together with a function $v : E \setminus \{0\} \rightarrow \mathbb{Z}^\#$

(Here $\mathbb{Z}^\#$ denotes the set of nonnegative (非負の) integers) such that

(i) for all $a, b \in E$ with $b \neq 0$, there exist $q, r \in E$ such that $a = qb + r$, where either $r = 0$ or $v(r) < v(b)$

and

(ii) for all $a, b \in E \setminus \{0\}$, $v(a) \leq v(ab)$.

The function v is called a **Euclidean valuation (ユークリッド賦値)**.

11.1 Euclidean Domains (ユークリッド整域)

Example 11.1

Let $(E, +, \cdot)$ be a **Euclidean domain** with **Gaussian valuation** v .

(a) Show that $v(a) = v(-a)$ for all $a \in E \setminus \{0\}$.

(b) Show that for all $a \in E \setminus \{0\}$, $v(a) \geq v(1)$, where equality holds if and only if a is a **unit** in E .

(c) Let n be an integer such that $v(1) + n \geq 0$. Show that the function

$$v_n: E \setminus \{0\} \rightarrow \mathbb{Z}^\#$$

defined by $v_n(a) = v(a) + n$ for all $a \in E \setminus \{0\}$ is a **Euclidean valuation**.

Solution

(a) For all $a \in E \setminus \{0\}$, $v(a) = v((-1)(-a)) \geq v(-a) = v((-1)a) \geq v(a)$ according to definition 11.1.

Hence, $v(a) = v(-a)$ for all $a \in E \setminus \{0\}$.

(b) Let $a \in E \setminus \{0\}$. Now $v(a) = v(1a) \geq v(1)$. Suppose a is a **unit**. Then there exists an element $c \in E$ such that $ac = 1$. Thus, $v(1) = v(ac) \geq v(a)$. This implies that $v(a) = v(1)$. Conversely, suppose that $v(a) = v(1)$. Since $a \neq 0$, there exist $q, r \in E$ such that $1 = qa + r$, where $r = 0$ or $v(r) < v(1)$. Now $v(r) < v(1)$ is impossible. Hence, $r = 0$, showing that $1 = qa$. Thus, a is a unit.

(c) Let $a \in E \setminus \{0\}$. Then $v_n(a) = v(a) + n \geq v(1) + n \geq 0$. Hence, $v_n(a) \in \mathbb{Z}^\#$. Suppose $a, b \in E$ with $b \neq 0$. There exist $q, r \in E$ such that $a = qb + r$, where either $r = 0$ or $v(r) < v(b)$. Now $v(r) < v(b)$ implies that $v(r) + n < v(b) + n$. Thus, $v_n(r) < v_n(b)$. Also, for $a, b \in E \setminus \{0\}$, $v_n(ab) = v(ab) + n \geq v(a) + n = v_n(a)$. Therefore, v_n is a **Euclidean valuation** on E .

11.1 Euclidean Domains (ユークリッド整域)

Example 11.2

The ring \mathbb{Z} of integers can be considered a **Euclidean domain** with $v(a) = |a|, a \neq 0$.

11.1 Euclidean Domains (ユークリッド整域)

Theorem 11.1

If F is a field, then the polynomial ring $F[x]$ is a Euclidean domain.

Proof (See page 185 of Ref. Textbook, *Introduction of Abstract Algebra*)

11.1 Euclidean Domains (ユークリッド整域)

Example 11.3

Any field can be considered as a **Euclidean domain** with $v(a) = 1$ for all $a \neq 0$.

$$(a = (ab^{-1})b + 0.)$$

11.1 Euclidean Domains (ユークリッド整域)

Definition 11.2

The subset $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ of the complex numbers is called the set of Gaussian integers (ガウス整数).



Carl Gauss (German mathematician, 1777-1885) was the first to study $\mathbb{Z}[i]$ and hence in his honor $\mathbb{Z}[i]$ is called the ring of Gaussian integers.

Theorem 11.2

The set $\mathbb{Z}[i]$ of Gaussian integers is a subring of \mathbb{C} . The units of $\mathbb{Z}[i]$ are ± 1 and $\pm i$.

Proof (See page 186 of Ref. Textbook, *Introduction of Abstract Algebra*)

11.1 Euclidean Domains (ユークリッド整域)

Theorem 11.3

The ring $\mathbb{Z}[i]$ of Gaussian integers becomes a Euclidean domain when we let the function,

$f: \mathbb{Z}[i] \setminus \{0\} \rightarrow \mathbb{Z}^{\#}$

defined by $f(a + bi) = (a + bi)(a - bi) = a^2 + b^2$ for all $a, b \in \mathbb{Z}$, serve as a Euclidean valuation function v .

Proof (See page 186 of Ref. Textbook, Malik, *Introduction to Abstract Algebra*)

11.1 Euclidean Domains (ユークリッド整域)

We now consider the ideals of a Euclidean domain.

Recall that an **ideal** I of a ring R is called a **principal ideal** if $I = \langle a \rangle = \{ar : r \in R\}$ for some $a \in I$.

Definition 11.3

Let R be a commutative ring with identity. If every ideal of R is a principal ideal, then R is called a **principal ideal ring**. An integral domain which is also a principal ideal ring is called a **principal ideal domain (PID)** (単項イデアル整域).

Theorem 11.4

Every Euclidean domain is a principal ideal domain (PID).

Proof (See page 186 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

11.1 Euclidean Domains (ユークリッド整域)

Theorem 11.5

Let R be a commutative ring with identity. The following conditions are equivalent.

- (i) R is a field.
- (ii) $R[x]$ is a Euclidean domain.
- (iii) $R[x]$ is a PID.

Proof (See page 187 of Ref. Textbook, Malik, Introduction of Abstract Algebra)

Corollary 11.1

$\mathbb{Z}[x]$ is not a PID.

Proof

Now \mathbb{Z} is a commutative ring with identity. Since \mathbb{Z} is not a field, $\mathbb{Z}[x]$ is not a PID by Theorem 11.5.

11.2 Factorization (分解) in Integral Domains

11.2 Factorization (分解) in Integral Domains

Definition 11.4

Let R be a commutative ring and $a, b \in R$ be such that $a \neq 0$. If there exists $c \in R$ such that $b = ac$, then a is said to **divide** b or a is said to be a **divisor** of b and we write $a \mid b$.

Definition 11.5

Let R be a commutative ring with identity. **A nonzero element $a \in R$ is said to be an associate (同伴) of a nonzero element $b \in R$ if $a = ub$ for some unit $u \in R$.**

11.2 Factorization (分解) in Integral Domains

Example 11.4

- (i) In \mathbb{Z} , 1 and -1 are the only **units**. For every $a \in \mathbb{Z}$ and $a \neq 0$, we know a and $-a$ are **associates**.
- (ii) In $\mathbb{Z}[i]$, $1, -1, i, -i$ are the **only units**. Thus, $1 + i, -1 - i, -1 + i, 1 - i$ are all **associates** of $1 + i$.

Example 11.5

In the polynomial ring $F[x]$ over a field F , the units form the set $F \setminus \{0\}$. A **nonconstant polynomial** $f(x)$ has $uf(x)$ for an **associate**, where u is a **unit** in F .

11.2 Factorization (分解) in Integral Domains

Definition 11.6

Let R be a commutative ring with identity.

- (i) An element p of R is called **irreducible (既約な)** if p is nonzero and a **nonunit** and $p = ab$ with $a, b \in R$ implies that either a or b is a **unit**. An element p of R is called **reducible** if p is **not irreducible**.
- (ii) An element p of R is called **prime** if p is **nonzero and a nonunit**, and if whenever $p \mid ab$, $a, b \in R$, then either p divides a or p divides b .
- (iii) Two elements a and b of R are called **relatively prime** if their **only common divisors are units**.

From the definition of an irreducible element, it follows that the **only divisors of an irreducible element p are the associates of p and the unit elements of R** .

The converse of this result does not always hold in a commutative ring with identity.

11.2 Factorization (分解) in Integral Domains

Theorem 11.6

Let R be an integral domain and $p \in R$ be such that p is nonzero and a nonunit. Then p is irreducible if and only if the only divisors of p are the associates of p and the unit elements of R .

Proof (See page 194 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

Example 11.6

In \mathbb{Z} , 1 and -1 are the only units, and therefore 2 is divisible by ± 1 and ± 2 . It follows that 2 is not divisible by any other integer. Therefore, 2 is an irreducible element.

Suppose now $2 \mid ab$ and 2 does not divide a for some $a, b \in \mathbb{Z}$. Since 2 does not divide a , a is an odd integer and so $\gcd(2, a) = 1$.

Therefore, there exist $c, d \in \mathbb{Z}$ such that $1 = 2c + ad$. Thus, $b = 2bc + abd$. Since $2 \mid ab$ and $2 \mid 2bc$, it follows that $2 \mid b$. Hence, 2 is prime.

11.2 Factorization (分解) in Integral Domains

Example 11.7

Consider the integral domain

$$\mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}$$

Let us show that $3 = 3 + 0i\sqrt{5} \in \mathbb{Z}[i\sqrt{5}]$ is **irreducible, but not prime**. Suppose $3 = (a + bi\sqrt{5})(c + di\sqrt{5})$ in $\mathbb{Z}[i\sqrt{5}]$. Then $3 = \bar{3} = \overline{(a + bi\sqrt{5})(c + di\sqrt{5})} = (a - bi\sqrt{5})(c - di\sqrt{5})$. Hence, $9 = (a^2 + 5b^2)(c^2 + 5d^2)$. Since a, b, c, d are integers, the previous equality implies that

$$a^2 + 5b^2 = 3 \text{ and } c^2 + 5d^2 = 3 \quad (11.1)$$

or

$$a^2 + 5b^2 = 1 \text{ and } c^2 + 5d^2 = 9 \quad (11.2)$$

or

$$a^2 + 5b^2 = 9 \text{ and } c^2 + 5d^2 = 1 \quad (11.3)$$

Clearly there do not exist integers a, b, c, d satisfying Eqs. (11.1). The first equation of Eqs. (11.2) implies that $b = 0$ and $a = \pm 1$. Thus, it follows that $a + bi\sqrt{5}$ is a unit. Similarly, the second equation of Eqs. (11.3) implies that $c + di\sqrt{5}$ is a unit. Hence, 3 is irreducible. Now $3 \mid 6$ and $6 = (1 + i\sqrt{5})(1 - i\sqrt{5})$. Suppose $3 \mid (1 + i\sqrt{5})$. Then $1 + i\sqrt{5} = 3(a + bi\sqrt{5})$ for some $a, b \in \mathbb{Z}$. This implies that $3a = 1$, a contradiction, since the equation $3a = 1$ has no solution in \mathbb{Z} .

Hence, 3 does not divide $(1 + i\sqrt{5})$. Similarly, 3 does not divide $(1 - i\sqrt{5})$. Thus, 3 is not prime.

11.2 Factorization (分解) in Integral Domains

Every field is also an integral domain; however, there are many integral domains that are not fields. For example, the integers \mathbb{Z} form an integral domain but not a field. A question that naturally arises is how we might associate an integral domain with a field.

There is a natural way to construct the rationals \mathbb{Q} from the integers: the rationals can be represented as formal quotients of two integers. The rational numbers are certainly a field. In fact, it can be shown that the rationals are the smallest field that contains the integers.

Given an integral domain D , our question now becomes how to construct a smallest field F containing D . We will do this in the same way as we constructed the rationals from the integers.

11.2 Factorization (分解) in Integral Domains

Let's introduce the Fundamental Theorem of Arithmetic (算術の基本定理).

Theorem 11.7 (Fundamental Theorem of Arithmetic)

Let n be an integer such that $n > 1$. Then

$$n = p_1 p_2 \cdots p_k$$

where p_1, p_2, \dots, p_k are primes (not necessarily distinct).

Furthermore, this factorization is unique; that is, if

$$n = q_1 q_2 \cdots q_l$$

then $k = l$ and the q_i 's are just the p_i 's rearranged.

Proof (See page 13 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

11.2 Factorization (分解) in Integral Domains

We study those integral domains in which an analogue of the **Fundamental Theorem of Arithmetic** holds.

Definition 11.7

A nonzero nonunit element a of an integral domain D is said to have a **factorization (分解)** if a can be expressed as

$$a = p_1 p_2 \cdots p_k$$

where p_1, p_2, \dots, p_k are **irreducible elements** of D . The expression $p_1 p_2 \cdots p_k$ is called a **factorization** of a .

Definition 11.8

An integral domain D is called a **factorization domain (FD)** if every nonzero nonunit element has a factorization.

11.2 Factorization (分解) in Integral Domains

In an integral domain D every nonzero element $a \in D$ is always divisible by the associates of a and the units of D . These are called the **trivial factors** of a . **All other factors (if any) of a are called nontrivial.** For example, ± 2 and ± 3 are **nontrivial factors** of 6 in \mathbb{Z} .

In the following lemma, we show that a nonzero nonunit element that has no factorization as a product of irreducible elements can be expressed as a product of any number of nontrivial factors.

Lemma 11.1

Let D be an integral domain. Let a be a nonzero nonunit element of D such that a **does not have a factorization**. Then for **every positive integer n** , there exist **nontrivial factors** $a_1, a_2, \dots, a_n \in D$ of a such that $a = a_1 a_2 \cdots a_n$.

11.2 Factorization (分解) in Integral Domains

Theorem 11.8

Let D be an integral domain with a function $f: D \setminus \{0\} \rightarrow \mathbb{Z}^{\#}$ such that for all $a, b \in D \setminus \{0\}$, $f(ab) \geq f(b)$, where equality holds if and only if a is a unit. Then D is a **FD**.

Proof (See page 199~200 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

11.2 Factorization (分解) in Integral Domains

Example 11.8

Consider the integral domain $\mathbb{Z}[i]$. Define

$$f: \mathbb{Z}[i] \setminus \{0\} \rightarrow \mathbb{Z}^{\#}$$

by $f(a + bi) = a^2 + b^2$ for all $a + bi \in \mathbb{Z}[i]$.

It is easy to verify that $a + bi$ is a **unit** if and only if $f(a + bi) = 1$.

Let $a + bi, c + di$ be two nonzero elements of $\mathbb{Z}[i]$.

$$\begin{aligned} \text{Then } f((a + bi)(c + di)) &= f((ac - bd) + (ad + bc)i) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2 \\ &= a^2(c^2 + d^2) + b^2(c^2 + d^2) \\ &= (a^2 + b^2)(c^2 + d^2) \\ &\geq (c^2 + d^2) \\ &= f(c + di) \end{aligned}$$

where the equality holds if and only if $f(a + bi)$ is a **unit**.

Hence, $\mathbb{Z}[i]$ is a **FD**.

11.2 Factorization (分解) in Integral Domains

Definition 11.9

An integral domain D is called a **unique factorization domain (UFD)** (一意分解整域) if the following two conditions hold in D :

(i) every nonzero nonunit element of D can be expressed as

$$a = p_1 p_2 \cdots p_k$$

where p_1, p_2, \dots, p_k are irreducible elements of D

and

(ii) if $a = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_l$ are two factorizations of a as a finite product of irreducible elements of D ,

then $k = l$ and there is a permutation σ of $\{1, 2, \dots, k\}$ such that p_i and $q_{\sigma(i)}$ are associates for all $i = 1, 2, \dots, k$.

From the above definition, it follows that an integral domain D is a **UFD** if and only if D is a *FD* and every nonzero nonunit element of D is uniquely expressible (apart from unit factors and order of the factors) as a finite product of irreducible elements.

11.2 Factorization (分解) in Integral Domains

Theorem 11.9

In a unique factorization domain (UFD), every irreducible element is prime.

Proof (See page 201 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

Theorem 11.10

A factorization domain (FD) D is a UFD if and only if every irreducible element of D is a prime element.

Proof (See page 201 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

Theorem 11.11

A Euclidean domain is a unique factorization domain (UFD).

Proof (See page 202 of Ref. Textbook, Malik, *Introduction of Abstract Algebra*)

11.2 Factorization (分解) in Integral Domains

Example 11.9

Consider the integral domain $\mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}$. Define

$$f: \mathbb{Z}[i\sqrt{5}] \setminus \{0\} \rightarrow \mathbb{Z}^{\#}$$

by

$$f(a + bi\sqrt{5}) = a^2 + 5b^2$$

We can show that $a + bi\sqrt{5}$ is a unit if and only if $f(a + bi\sqrt{5}) = 1$. Let $a + bi\sqrt{5}, c + di\sqrt{5}$ be two nonzero elements of $\mathbb{Z}[i\sqrt{5}]$.

Then $f((a + bi\sqrt{5})(c + di\sqrt{5})) = f((ac - 5bd) + i(ad + bc)\sqrt{5}) = (ac - 5bd)^2 + 5(ad + bc)^2 = (a^2 + 5b^2)(c^2 + 5d^2) \geq (c^2 + 5d^2) = f(c + di\sqrt{5})$, where equality holds if and only if $f(a + bi\sqrt{5}) = 1$, i.e., if and only if $a + bi\sqrt{5}$ is a unit. Hence, $\mathbb{Z}[i\sqrt{5}]$ is a **FD** by Theorem 11.8. In Example 11.7, we showed that 3 is an irreducible element. Now $3 \mid (2 + i\sqrt{5})(2 - i\sqrt{5})$. Suppose $3 \mid (2 + i\sqrt{5})$. Then $2 + i\sqrt{5} = 3(m + ni\sqrt{5})$ for some $m + ni\sqrt{5} \in \mathbb{Z}[i\sqrt{5}]$. This implies $2 = 3m$ and $1 = 3n$, which is impossible for integers m and n . Therefore, 3 does not divide $(2 + i\sqrt{5})$. Similarly, 3 does not divide $(2 - i\sqrt{5})$. Thus, 3 is not prime in $\mathbb{Z}[i\sqrt{5}]$. Hence, $\mathbb{Z}[i\sqrt{5}]$ is not a **UFD** by Theorem 11.9.

Review for Lecture 11

- Euclidean Domain (ユークリッド整域)
- Gaussian Integers (ガウス整数)
- Associate (同伴)
- Fundamental Theorem of Arithmetic (算術の基本定理)
- Unique Factorization Domain (UFD) (一意分解整域)

Assignment

Please Check <https://github.com/uoaworks/Applied-Algebra>

References

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