Lecture 3

Introduction of Group (群)

What you will learn in Lecture 3

3.1 Introduction of Group (群)

3.2 Permutation Group (置換群)

3.1 Introduction of Group (群)

Definition of group

Definition 3.1

A group (群) is an ordered pair $(G,\circ)$, where G is a nonempty set and $\circ$ is a binary operation on G (i.e. $G \times G \to G$) such that the following properties hold:

(G1) (associative law) For all $a, b, c \in G, a \circ (b \circ c) = (a \circ b) \circ c$.

(G2) (existence of an identity) There exists identity element $e \in G$ such that for all $a \in G$, $a \circ e = a = e \circ a$.

(G3) (existence of an inverse) For all $a \in G$, there exists $b \in G$ such that $a \circ b = e = b \circ a$.

#(G4) (closure property) For all $a, b \in G$, the result of the operation, $a \circ b$, is also in G.

Thus, a group is a mathematical system $(G,\circ)$ satisfying axioms (公理) G1, G2, G3 (and G4).

Example 3.1

Consider $\mathbb{Z}$, the set of integers, together with the binary operation $+$, where $+$ is the usual addition. We know that $+$ is closed and associative on $\mathbb{Z}$. Now $0 \in \mathbb{Z}$ and for all $a \in \mathbb{Z}$,

$$a + 0 = a = 0 + a.$$

So 0 is an identity.

Also, for all $a \in \mathbb{Z}$, $-a \in \mathbb{Z}$ and $a + (-a) = 0 = (-a) + a$.

That is, $-a$ is an inverse of a.

It now follows that $(\mathbb{Z}, +)$ satisfies axioms G1 to G3 (and G4), so $(\mathbb{Z}, +)$ is a group.

As in Example 2.7, we can show that $(\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$ are also groups, where $+$ is the usual addition.

Theorem 3.1

Let $(G,\circ)$ be a group.
(i) There exists a **unique identity element** $e \in G$ such that $e \circ a = a = a \circ e$ for all $a \in G$.
(ii) For all $a \in G$, there exists a **unique** $b \in G$ such that $a \circ b = e = b \circ a$.

Proof: (i) Now $(G,\circ)$ is group. Therefore, by axiom G2, there exists $e \in G$ such that $e \circ a = a = a \circ e$ for all $a \in G$. Because $(G,\circ)$ is a mathematical system, e is unique by Theorem 2.1.

(ii) Let $a \in G$. By axiom G3, there exists $b \in G$ such that $a \circ b = e = b \circ a$. Suppose there exists $c \in G$ such that $a \circ c = e = c \circ a$. We show that $b = c$.

$$\begin{aligned} b &= b \circ e \\ &= b \circ (a \circ c) \text{ (substituting } e = a \circ c\text{)} \\ &= (b \circ a) \circ c \text{ (using the associativity of } \circ\text{)} \\ &= e \circ c \text{ (because } b \circ a = e\text{)} \\ &= c. \end{aligned}$$

Thus, b is unique.

Abelian group

Definition 3.2

Let $(G,\circ)$ be a group. If for all $a, b \in G$,

$$a \circ b = b \circ a$$

then $(G,\circ)$ is called **commutative group (可換群)** or **Abelian group (アーベル群)**. A group $(G,\circ)$ is called **noncommutative** if it is **not commutative**.

Example 3.2
Consider the group $(\mathbb{Z}, +)$ of Example 2.7. Because $a + b = b + a$ for all $a, b \in \mathbb{Z}$, it follows that $+$ is commutative. Hence, $(\mathbb{Z}, +)$ is a commutative group.

Similarly, $(\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{Q}\backslash\{0\}, \cdot), (\mathbb{R}\backslash\{0\}, \cdot), (\mathbb{C}\backslash\{0\}, \cdot)$ are also commutative groups, where $+$ is the usual addition and $\cdot$ is the usual multiplication.

Cayley table

It is often convenient to **describe a group** in terms of an addition or multiplication table. Such a table is called a ***Cayley table***.

Example 3.3

The integers mod n form a group under addition modulo n. Consider $\mathbb{Z}_5$, consisting of the equivalence classes of the integers $0, 1, 2, 3$, and 4. We define the group operation on $\mathbb{Z}_5$ by modular addition.

We write the binary operation on the group additively; that is, we write $m + n$. The element 0 is the identity of the group and each element in $\mathbb{Z}_5$ has an inverse. For instance, $2 + 3 = 3 + 2 = 0$. Table 3.1 is a **Cayley table** for $\mathbb{Z}_5$. We can see $\mathbb{Z}_5$ is a **group** under the binary operation of addition mod n.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 3.1

Finite Group & Infinite Group

Definition 3.3

A group $(G,\circ)$ is called a **finite group (有限群)** if G has **only a finite number of elements**. The **order (位数) of a group** $(G,\circ)$, which is written as $|G|$, is **the number of elements of** G.

Definition 3.4

A **group** with **an infinite number of elements** is called an **infinite group (無限群)**.

Semigroup

*Definition 3.5

A **semigroup (半群)** is an **ordered pair** $(M,\circ)$, where M is a **nonempty set** and $\circ$ is an **associative binary operation** on M.

Thus, a **semigroup** is a **mathematical system** with one **binary operation** such that the **binary operation** is **associative**.

Remark

For any **group** $(G,\circ)$, the **binary operation** $\circ$ is **associative**. Therefore, every **group** $(G,\circ)$ is a **semigroup**.

Notice: * mark is optional study material. It will be not included in middle and final examinations.

*Theorem 3.2

A **semigroup** $(M,\circ)$ is a **group** if and only if

(i) there exists $e \in M$ such that $e \circ a = a$ for all $a \in M$, (i.e., e is a **left identity**),

(ii) for all $a \in M$ there exists $b \in M$ such that $b \circ a = e$, (i.e., every element has a **left inverse**).

3.2 Permutation Group (置換群)

Definition 3.6

For any **nonempty set** S, a ***one-to-one*** and ***onto*** mapping $\pi: S \to S$ is called a ***permutation* (置換)** of S.

Example 3.4

(i) Let S be a nonempty set. Define $\pi : S \to S$ by $\pi(x) = x$ for all $x \in S$. Then π is one-to-one function of S onto S. Thus, π is a permutation of S. Note that π is called the identity permutations and is, usually, denoted by i_S or e.

(ii) Let $S = \{a, b, c\}$. Define $\alpha : S \to S$ such that $\alpha(a) = b, \alpha(b) = a$, and $\alpha(c) = c$. By the definition of α it follows that is α is one-to-one function of S onto S. Thus, α is a permutation of S.

(iii) Consider $\mathbb{R}$, the set of real numbers. Define $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = 3x + 5$ for all $x \in \mathbb{R}$. It can be shown that α is a one-to-one function of $\mathbb{R}$ onto $\mathbb{R}$. Thus, α is a permutation of $\mathbb{R}$.
Similarly, if $\beta : \mathbb{R} \to \mathbb{R}$ by $\beta(x) = x^3$ for all $x \in \mathbb{R}$. It can be shown that β is a one-to-one function of $\mathbb{R}$ onto $\mathbb{R}$. Thus, β is a permutation of $\mathbb{R}$.

Permutation group

Definition 3.7

A **group** $(G,\circ)$ is called a **permutation group (置換群)** on a **nonempty set** S if the elements of G are permutations of S and the operation $\circ$ is the composition of two functions.

Example 3.5

Let $S = \{1,2\}$. Define $\alpha : S \to S$ such that $\alpha(1) = 1, \alpha(2) = 2$.
Then α is a one-to-one function of S onto S, so α is a permutation of S.

Next define $\beta : S \to S$ such that $\beta(1) = 2$ and $\beta(2) = 1$.
Then β is a one-to-one function of S onto S, so β is a permutation of S.

Let $G = \{\alpha, \beta\}$. Then $(G,\circ)$ is a group, where $\circ$ is the composition of functions.

Note that on this set S, α and b are the only permutations on S.
Moreover, α is the identity permutation and $\beta^{-1} = \beta$.

Permutation group

Let $I_n = \{1, 2, \dots, n\}, n \geq 1$. Let π be a permutation on I_n. Then

$$\pi = \{(1, \pi(1)), (2, \pi(2)), \dots, (n, \pi(n))\}.$$

Recall that a function $f : S \to S$ is a subset of $S \times S$. By introducing two-row notation, we have

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$$

Example 3.6

(1) Let $n = 4$ and π be the permutation on I_4 defined by $\pi(1) = 2,\ \pi(2) = 4, \pi(3) = 3,\ $ and $\pi(4) = 1.$
Then using the two-row notation we can write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Permutation group

(2) Let $n = 7$ and π and σ be two permutations on I_7 defined by

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 7 & 2 & 5 \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 1 & 7 & 6 & 4 \end{pmatrix}$$

Let us compute $\pi \circ \sigma$. Now by the definition of the composition of functions

$$(\pi \circ \sigma)(i) = \pi(\sigma(i))$$

for all $i \in I_7$. Thus,

$$(\pi \circ \sigma)(1) = \pi\big(\sigma(1)\big) = \pi(2) = 3$$

$$(\pi \circ \sigma)(2) = \pi\big(\sigma(2)\big) = \pi(5) = 7$$

and so on.

Permutation group

From this, it is clear that when determining, say, $(\pi \circ \sigma)(1)$, we start with σ and finish with π and read as follows: 1 goes to 2 (under σ) and 2 goes to 3 (under π), so 1 goes to 3 (under $\pi \circ \sigma$).

We can exhibit this in the following form:

$$\begin{array}{lll} 1 \xrightarrow{\sigma} 2 \xrightarrow{\pi} 3 & \qquad & 1 \xrightarrow{\pi\circ\sigma} 3 \\ 2 \xrightarrow{\sigma} 5 \xrightarrow{\pi} 7 & & 2 \xrightarrow{\pi\circ\sigma} 7 \\ 3 \xrightarrow{\sigma} 3 \xrightarrow{\pi} 4 & & 3 \xrightarrow{\pi\circ\sigma} 4 \\ 4 \xrightarrow{\sigma} 1 \xrightarrow{\pi} 1 & & 4 \xrightarrow{\pi\circ\sigma} 1 \\ 5 \xrightarrow{\sigma} 7 \xrightarrow{\pi} 5 & & 5 \xrightarrow{\pi\circ\sigma} 5 \\ 6 \xrightarrow{\sigma} 6 \xrightarrow{\pi} 2 & & 6 \xrightarrow{\pi\circ\sigma} 2 \\ 7 \xrightarrow{\sigma} 4 \xrightarrow{\pi} 6 & & 7 \xrightarrow{\pi\circ\sigma} 6. \end{array}$$

Thus,

$$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}$$

Example 3.7

In this example, we describe S_3, i.e., the set of all permutations on $I_3 = \{1, 2, 3\}$.

From equilateral triangle example in Lecture 2, we know that the number of one-to-one functions of I_3 onto I_3 is $3! = 6$. Thus, $|S_3| = 6$. Let e denote the identity permutation on I_3, i.e.,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Permutation group

Let α be a nonidentity permutation on I_3. Let us see some of the choices for α.

Suppose $\alpha(1) = 1$. If $\alpha(2) = 2$, then we must have $\alpha(3) = 3$ because α is a permutation.

In this case, we see that $\alpha = e$, a contradiction. Thus, we must have $\alpha(2) = 3$ and $\alpha(3) = 2$, i.e., we define this α as

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Permutation group

In a similar manner, we can show that the other four permutations on I_3 are

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

For closure property, it can check that, for example

$$\alpha_2 \circ \alpha_4 = \alpha_1$$

Hence, we can write

$$S_3 = \{e, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

Theorem 3.3

(i) $(S_n, \circ)$ is a **group** for any positive integer $n \geq 1$.

(ii) If $n \geq 3$, then $(S_n, \circ)$ is **noncommutative**.

(iii) $|S_n| = n!$

Symmetric Group

Definition 3.8

The **symmetric group (对称群)** S_n is a group with $n!$ elements, where the binary operation is the composition of maps.

The group $(S_n, \circ)$ is called the **symmetric group** on I_n.

k-cycle & Transposition (互換)

Definition 3.9

Let π be an element of S_n. Then π is called a k-**cycle**, written as $(i_1 \ \ i_2 \ \ \cdots \ \ i_k)$, if

$$\pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_{k-1} & i_k \\ i_2 & i_3 & \cdots & i_k & i_1 \end{pmatrix},$$

i.e., $\pi(i_j) = i_{j+1}, j = 1, 2, \ldots, k-1, \pi(i_k) = i_1$, and $\pi(a) = a$ for any other element of I_n.

Notice: When $k = 2$, a k-cycle is called a **transposition (互換)**.

For **example**, consider the cycle notation (3 4) for $\pi \in S_4$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (3\ 4)$$

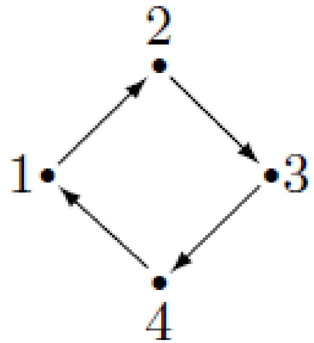
k-cycle & Transposition (互換)

Example 3.8
Consider the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

It permutes the elements 1, 2, 3, and 4 cyclically, then we have the cycle notation $\alpha = (1\ 2\ 3\ 4)$.

And the following picture suggests:



Permutation and Transposition

Theorem 3.4

Any **permutation (置換)** π of S_n can be expressed as **a product of disjoint cycles.**

Example 3.9

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}$$

Using cycle notation, we can write

$$\sigma = (1\ 6\ 2\ 4)$$

$$\tau = (1\ 3)(4\ 5\ 6)$$

$$\sigma\tau = (1\ 3\ 6)(2\ 4\ 5)$$

$$\tau\sigma = (1\ 4\ 3)(2\ 5\ 6)$$

Permutation and Transposition

Corollary 3.1

Let $n \geq 2$. Any **permutation (置換)** π of S_n can be expressed as a **product** of **transpositions (互換)**.

The cycle $(i_1 \ \ i_2 \ \ \cdots \ \ i_k) = (i_1 \ \ i_n)(i_1 \ \ i_{n-1}) \cdots (i_1 \ \ i_3)(i_1 \ \ i_2)$

For **example**, the **Fifteen Puzzle**

http://lorecioni.github.io/fifteen-puzzle-game/

Permutation and Transposition

Example 3.10

Consider the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$

Let us write it as a product of disjoint cycles.
First, 1 is moved to 6 and then 6 to 1, giving the cycle (1,6).
Then 2 is moved to 5, which is moved to 3, which is moved to 2, giving the cycle (2 5 3).
This process takes care of all elements except 4, which is left fixed. Thus

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix} = (1\ 6)(2\ 5\ 3) = (2\ 5\ 3)(1\ 6)$$

Notice: Multiplication of disjoint cycles is **commutative**, so **the order of the factors** (1 6) and (2 5 3) **is not important**.

According to Corollary 3.1:

$$\alpha = (1\ 6)(2\ 5\ 3) = (1\ 6)(2\ 3)(2\ 5)$$

Even Permutation & Odd Permutation

Definition 3.10

Let $\pi \in S_n$. If π is a **product of** an even number of **transpositions (互換)**, then π is called an even permutation (偶置換); otherwise π is called an odd permutation (奇置換).

Corollary 3.2

Let $\pi \in S_n$ be a k-**cycle**. Then π is an even permutation if and only if k is odd.

Proof.

Let $\pi = (1\ \ 2\ \ \cdots\ \ k)$. Then $\pi = (1\ \ k) \circ (1\ \ k-1) \circ \cdots \circ (1\ \ 2)$, i.e., π is a product of $k-1$ transposition. If π is an even permutation then $k-1$ is even, so k is odd. On the other hand, if k is odd, then $k-1$ is even, so π is an even permutation. This completes the proof.

Even Permutation & Odd Permutation

Example 3.11

From Example 3.7,

(1)use cycle notation to express the permutation

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

(2)Determine the permutation is an even permutation or odd permutation.

Solution:

(1) $\alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2)$

(2) $\alpha_5 = (1\ 3\ 2) = (1\ 3)(1\ 2)$ is even permutation.

Review for Lecture 3

- Definition of Group (群)
- Commutative Group (可換群) or Abelian Group (アーベル群)
- Finite Group (有限群) and Infinite Group (無限群)
- Order (位数) of a group
- Permutation Group (置換群)
- Symmetry Group (対称群)
- k-cycle & Transposition (互換)
- Even Permutation (偶置換) and Odd Permutation (奇置換)

Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

References

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