Lecture 9

Introduction of Rings (環) & Fields (体)

9.1 Rings (環) & Fields (体)

9.1 Rings (環) & Fields (体)

Definition 9.1 (1)

A **ring (環)** is an ordered triple $(R, +, \cdot)$ such that R is a **nonempty** set and $+$ and $\cdot$ are **two binary operations** (which we call addition and multiplication) **on** R (i.e. $\underbrace{R \times R \rightarrow R}_{\text{Closure Property}}$) satisfying the following axioms.

(R1) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$. (Addition is associative.)
(R2) There exists an element 0 in R such that $a + 0 = a$ for all $a \in R$. (Identity element exists for Addition.)
(R3) For all $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0$. (Inverse element exists for Addition.)
(R4) $a + b = b + a$ for all $a, b \in R$. (Addition is commutative/abelian.)
(R5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$. (Multiplication is associative.)
(R6) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$. (Left Distributive Law)
(R7) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$. (Right Distributive Law)

Definition 9.1 (2)

A **ring (環)** is an ordered triple $(R, +, \cdot)$ such that R is a **nonempty** set and $+$ and $\cdot$ are **two binary operations** (which we call addition and multiplication) **on** R (i.e. $\underbrace{R \times R \to R}_{\text{Closure Property}}$) satisfying the following axioms.

(New R1) $(R, +)$ is an abelian group.
(New R2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$. (Multiplication is associative.)
(New R3) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$. (Left Distributive Law)
(New R4) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$. (Right Distributive Law)

9.1 Rings (環) & Fields (体)

During the development of the theory of rings, we will use the following conventions.

1. Multiplication is assumed to be performed before addition.
2. We write ab for $a \cdot b$.
3. We write $a - b$ for $a + (-b)$.
4. We refer to a ring $(R, +, \cdot)$ as a ring R.

9.1 Rings (環) & Fields (体)

Example 9.1

We are well aware that **axioms** for a **ring** hold in **any subset of the complex numbers that is a group under addition** and that is **closed under multiplication**. It can be shown that $(\mathbb{Z},+,\cdot)$, $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, and $(\mathbb{C},+,\cdot)$ are rings.

Notice: The ring $(\mathbb{Z},+,\cdot)$ is called the **ring of integers**.
This ring plays an important role in the study of ring theory. One of the basic problems in ring theory is to determine rings, which satisfy the same type of properties as the ring of integers.

Example 9.2

Recall that in group theory, $n\mathbb{Z}$ is the cyclic subgroup of $\mathbb{Z}$ under addition consisting of all integer multiples of the integer n.

For $r, s \in \mathbb{Z}$, we have $nr, ns \in n\mathbb{Z}$, since $(nr)(ns) = n(nrs)$, we see that $n\mathbb{Z}$, is closed under multiplication.

The associative and distributive laws which hold in $\mathbb{Z}$ then assure us that $(n\mathbb{Z}, +, \cdot)$ is a ring.

Theorem 9.1

Let R be a ring and $a, b, c \in R$. Then

(i) $a0 = 0a = 0$,

(ii) $a(-b) = (-a)b = -(ab)$,

(iii) $(-a)(-b) = ab$,

(iv) $a(b-c) = ab - ac$ and $(b-c)a = ba - ca$.

Proof

9.1 Rings (環) & Fields (体)

Definition 9.2

Let R be a **ring**. An element $e \in R$ is called an **identity element** if $ea = a = ae$ for all $a \in R$.

Definition 9.3

A ring R is called a **ring with identity (単位元持つ環)** if it has an **identity element**.

Example The ring $(\mathbb{Z}, +, \cdot)$ of integers is a ring with identity. The integer 1 is the identity element of $\mathbb{Z}$.

Definition 9.4

A ring R for which $ab = ba$ for all $a, b \in R$ is called a **commutative ring (可換環)**.

Definition 9.5

A nonzero element a in a ring R is called a **zero divisor (零因子)** if there is a nonzero element b in R such that $ab = 0$.

9.1 Rings (環) & Fields (体)

Definition 9.6

A **commutative ring** R **with identity** is called an **integral domain (整域)** if, for every $a, b \in R$ such that $ab = 0$, either $a = 0$ or $b = 0$. Namely, R has no **zero divisor**.

Definition 9.7

Let R be a **ring with identity**. An element $a \in R$ and $a \neq 0$ is called a **unit (単元)** (or an **invertible element (可逆元)**) if there exists $\mathrm{b} = a^{-1} \in R$ such that $aa^{-1} = e = a^{-1}a$.

Definition 9.8

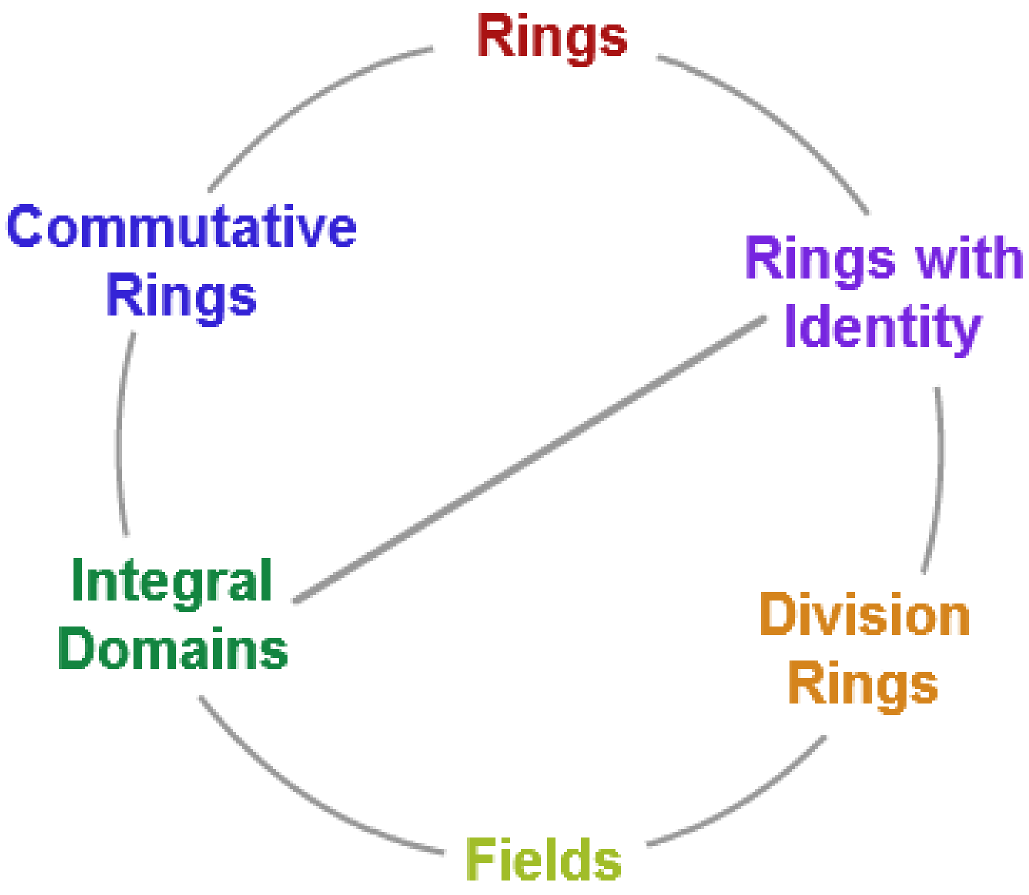
A ring R with identity is called a **division ring (可除環)** (**skew-field (斜体)**) if every nonzero element of R is a **unit**.

Definition 9.9

A commutative division ring R is called a **field (体)**.

9.1 Rings (環) & Fields (体)

The relationship among rings, integral domains, division rings, and fields is shown in the following Figure



Example 9.3

Because multiplication of numbers is commutative, it follows that $(\mathbb{Z}, +,\cdot)$, $(\mathbb{Q}, +,\cdot)$, $(\mathbb{R}, +,\cdot)$, and $(\mathbb{C}, +,\cdot)$ are commutative rings.

Example 9.4

Consider $2\mathbb{Z}$, the ring of even integers. In $2\mathbb{Z}$, there does not exist any element e such that $ex = x = xe$ for all $x \in 2\mathbb{Z}$. Hence, $2\mathbb{Z}$, is a ring without identity.

Example 9.5
The ring of even integers $2\mathbb{Z}$ is a commutative ring, without identity, and without zero divisors.
Thus, $2\mathbb{Z}$ is not an integral domain.

Example 9.6
The ring of integers $\mathbb{Z}$ is an **integral domain**.

9.1 Rings (環) & Fields (体)

Example 9.6 Let R denote the set of all functions $f: \mathbb{R} \to \mathbb{R}$. Define $+,\cdot$ on R by for all $f, g \in R$ and for all $a \in R$,

$$(f+g)(a) = f(a) + g(a),$$
$$(f \cdot g)(a) = f(a)g(a).$$

- From the definition of + and ·, it follows that + and · are binary operations on R. Let $f, g, h \in R$. Then for all $a \in R$, we have by using the associativity of R that

$$\begin{aligned}\big((f+g)+h\big)(a) &= (f+g)(a) + h(a) \\ &= \big(f(a)+g(a)\big) + h(a) \\ &= f(a) + \big(g(a)+h(a)\big) \\ &= f(a) + (g+h)(a) \\ &= (f+(g+h))(a)\end{aligned}$$

Thus, $(f+g)+h = f+(g+h)$. This shows that + is associative.

In a similar way, we can show that the other properties of a ring hold for R by using the fact that they hold for R. Thus, $(R, +, \cdot)$ is a ring.

- We note that the function $i_0 : \mathbb{R} \to \mathbb{R}$, where $i_0(a) = 0$ for all $a \in R$, is the additive identity of R and the element $i_1 \in R$, where $i_1(a) = 1$ for all $a \in R$, is the identity of R. Also, for all $f, g \in R$ and for all $a \in R$, $(f \cdot g)(a) = f(a)g(a) = g(a)f(a) = (g \cdot f)(a)$.

Thus, for all $f, g \in R$, $f \cdot g = g \cdot f$. Consequently, $(R, +, \cdot)$ is a commutative ring with identity.

Example 9.7

Consider $\mathbb{Z}$, the ring of integers. Let $a \in \mathbb{Z}$ be such that $a \neq 0$, $a \neq 1$, and $a \neq -1$.

Now $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$. That is, the multiplicative inverse of a is $\frac{1}{a}$.

However, $\frac{1}{a} \notin \mathbb{Z}$. (For example, the multiplicative inverse of 2 is $\frac{1}{2} \notin \mathbb{Z}$.).

It follows that $\mathbb{Z}$ is not a field. Note that in $\mathbb{Z}$, the only invertible elements are 1 and -1.

Definition 9.10

A ring R is called a **finite ring** if R has only a **finite number of elements**; otherwise R is called an **infinite ring**.

9.1 Rings (環) & Fields (体)

*Theorem 9.2

A **finite commutative ring** R with **more than one element** and **without zero divisors** is a **field**.

Proof: Page 136 of Ref. Textbook

*Corollary 9.1

Every **finite integral domain** is a **field**.

*Corollary 9.2

Let n be a positive integer. Then $\mathbb{Z}_n$ is a **field** if and only if n is prime.

9.1 Rings (環) & Fields (体)

Definition 9.11

Let $(R,+,\cdot)$ be a ring. Let S be a **subset** of R. Then $(S,+,\cdot)$ is called a **subring** of $(R,+,\cdot)$ if
(i) $(S,+)$ is a **subgroup** of $(R,+)$ and
(ii) for all $x,y \in S, x \cdot y \in S$.

Remark When S and R are **fields**, S is called a **subfield** of R.

Theorem 9.3

Let S be a ring. A **nonempty subset** S of R is a **subring** of R if and only if $x - y \in S$ and $xy \in S$ for all $x, y \in S$.

Proof

- First suppose that S is a subring of R. Then S is a ring. Hence, for all $x,y \in R, x-y, xy \in S$.
- Conversely, suppose $x - y \in S$ and $xy \in S$ for all $x, y \in S$. Because $x - y \in S$ for all $x, y \in S$, $(S,+)$ is a subgroup of $(R,+)$ by Theorem 4.2. By the hypothesis, $xy \in S$ for all $x, y \in S$. Hence, S is a subring of R.

Example 9.8

Let $m\mathbb{Z} = \{mn : n \in \mathbb{Z}\}$, where m is an integer greater than 1. That is, $m\mathbb{Z}$ is the set of integer multiples of m.

We claim that $m\mathbb{Z}$ is a subring of $\mathbb{Z}$.

For if $ma, mb \in m\mathbb{Z}$, then

$$ma - mb = m(a - b) \in m\mathbb{Z}$$

and so $m\mathbb{Z}$ is **closed under subtraction**. Similarly,

$$(ma)(mb) = m(mab) \in m\mathbb{Z}$$

and so $m\mathbb{Z}$ is **closed under multiplication**.

*Theorem 9.4

Let F be a **field**. A **nonempty subset** U of F is a **subfield** of F **if and only if**

(i) U contains **more than one element**,

(ii) $x - y, xy \in U$ for all $x, y \in U$, and

(iii) $x^{-1} \in U$ for all $x \in U$, $x \neq 0$.

9.2 Ideals (イデアル) &

Quotient Rings (剰余環, 商環)

9.2 Ideals (イデアル) & Quotient Rings (剰余環)

In this section, we introduce the notions of **ideals** and **quotient rings**. These concepts are analogous to **normal subgroups** and **quotient groups**.

Definition 9.12

Let R be a **ring**. Let I be a **nonempty subset** of R.
(i) I is called a left ideal of R if for all $a, b \in I$ and for all $r \in R$, $a - b \in I$, $ra \in I$.
(ii) I is called a right ideal of R if for all $a, b \in I$ and for all $r \in R$, $a - b \in I$, $ar \in I$.
(iii) I is called a (two-sided) ideal of R if I is both a left and a right ideal of R.

From the definition of a left (right) ideal, it follows that if I is a left (right) ideal of R, then I is a subring of R. Also, if R is a commutative ring, then every left ideal is also a right ideal and every right ideal is a left ideal.
Thus, for commutative rings every left or right ideal is an ideal.

9.2 Ideals (イデアル) & Quotient Rings (剰余環)

Example 9.9

Let R be a ring. The subsets $\{0\}$ and R of R are **(left, right) ideals**. **These ideals** are called **trivial ideals (自明なイデアル)**. **All other (left, right) ideals** are called **nontrivial**.

An ideal I of a ring R is called a **proper ideal (真のイデアル)** if $I \neq R$.

Example 9.10

We see that $n\mathbb{Z}$ is an ideal in the ring $\mathbb{Z}$ since we know it is a subring, and $s(nm) = (nm)s = n(ms) \in n\mathbb{Z}$ for all $s \in \mathbb{Z}$.

9.2 Ideals (イデアル) & Quotient Rings (剰余環)

Example 9.11
If a is any element in a commutative ring R with identity, then the set

$$\langle a \rangle = \{ar : r \in \mathrm{R}\}$$

is an **ideal** in R. Certainly, $\langle a \rangle$ is nonempty since both $0 = a0$ and $a = a1$ are in $\langle a \rangle$. The sum of two elements r, r' in $\langle a \rangle$ is again in $\langle a \rangle$ since $ar + ar' = a(r + r')$. The inverse of ar is $-ar = a(-r) \in \langle a \rangle$. Finally, if we multiply an element $ar \in \langle a \rangle$ by an arbitrary element $s \in R$, we have $s(ar) = a(sr)$. Therefore, $\langle a \rangle$ satisfies the definition of an ideal.

If R is a commutative ring with identity, then an ideal of the form $\langle a \rangle = \{ar : r \in \mathrm{R}\}$ is called a **principal ideal (主イデアル)**.

Theorem 9.5

Every ideal in the ring of integers $\mathbb{Z}$ is a **principal ideal**.

9.2 Ideals (イデアル) & Quotient Rings (剰余環)

Example 9.12 Find all ideals of $\mathbb{Z}$.

Solution:

We know that the subrings of $\mathbb{Z}$ are the subsets $n\mathbb{Z}, n = 0, 1, 2, \ldots$.

Let us now show that these subrings are precisely the ideals of $\mathbb{Z}$.

If I is an ideal of $\mathbb{Z}$, then I is a subring of $\mathbb{Z}$, so $I = n\mathbb{Z}$ for some nonnegative integer n.

Now, let $I = n\mathbb{Z}$ (n is a nonnegative integer). Then I is a subring. If $r \in \mathbb{Z}$, then $rI = r(n\mathbb{Z}) = n(r\mathbb{Z}) \subseteq n\mathbb{Z} = I$. Similarly, $Ir \subseteq I$. Hence, I is an ideal of $\mathbb{Z}$.

9.2 Ideals (イデアル) & Quotient Rings (剰余環)

Definition 9.5

If R is a **ring** and I is an **ideal** of R, then the **ring** $(R/I, +, \cdot)$ is called the **quotient ring** of R by I.

Remark

The **quotient ring** R/I can also be realized by observing the $(I, +)$ is a normal subgroup of $(R, +)$ because the latter group is commutative.
Hence, if R/I denotes the set of all cosets $x + I = \{x + a \mid a \in I\}$ for all $x \in R$, then $(R/I, +)$ is a commutative group, where

$$(x + I) + (y + I) = (x + y) + I$$

for all $x + I, y + I \in R/I$. Now define multiplication on R/I by $(x + I) \cdot (y + I) = xy + I$ for all $x + I, y + I \in R/I$. Then $(R/I, +, \cdot)$ forms a **ring**.

Review for Lecture 9

- Rings
- Integral domain (整域)
- Fields
- Subring & Subfield
- Ideal
- Quotient Rings

Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

References

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