Lecture 4

Subgroup & Cyclic Group

& Lagrange’s Theorem

What you will learn in Lecture 4

4.1 Subgroup (部分群)

4.2 Cyclic Group (巡回群)

4.3 Lagrange's Theorem (ラグランジュの定理)

4.1 Subgroup (部分群)

4.1 Subgroup (部分群)

Subgroup

Let us consider the groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$, where $+$ is the usual addition of numbers, and note the following:

1. **Both these groups** have **the same binary operation**.
2. $\mathbb{Z}$ is **a subset of** $\mathbb{Q}$.

The same is true for the groups $(\mathbb{Z}, +)$ and $(\mathbb{R}, +)$; $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$; $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$.

4.1 Subgroup (部分群)

Subgroup

This leads us to the concept of a **subgroup**. Before formally defining subgroups, let us also note the following:

Let $(G,\circ)$ be **a group** and H be a **nonempty subset** of G. Then H is said to be **closed** under the **binary operation** $\circ$ if $a \circ b \in H$ for all $a, b \in H$.

Suppose H is **closed** under **the binary operation** $\circ$. Then the restriction of $\circ$ to $H \times H$ is a mapping from $H \times H$ into H. Thus, the binary operation $\circ$ defined on G induces **a binary operation on** H. We denote this induced binary operation on H by $\circ$ also. Thus, $(H,\circ)$ is a **mathematical system**.

It also follows that $\circ$ is **associative** as a binary operation on H, i.e., $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$.

If $(H,\circ)$ is a **group**, then we call H a **subgroup** of G.

More formally, we have the following definition.

Definition 4.1

Let $(G,\circ)$ be a group and H be a **nonempty subset** of G. If $(H,\circ)$ is a group, then $(H,\circ)$ is called a **subgroup** of $(G,\circ)$.

For **example**, consider the **rational number group** $(\mathbb{Q},+)$ and its **subgroups** $(\mathbb{Z},+)$.
Now the **identity elements** of both these groups is 0.

Next, let $a \in \mathbb{Z}$. Then $a \in \mathbb{Q}$.

Also, the **inverse of** a in $\mathbb{Z}$ **as well as in** $\mathbb{Q}$ **is** $-a$.
In other words, **the inverse of** a in $\mathbb{Z}$ **and the inverse of** a in $\mathbb{Q}$ **is the same.**

Trivial Subgroup

In general, we have the following result.

Theorem 4.1

Let $(G,\circ)$ be a group and $(H,\circ)$ be a subgroup of $(G,\circ)$.
(i) The **identity elements** of $(H,\circ)$ and $(G,\circ)$ are the same.
(ii) If $h \in H$, then **the inverse of** h in H and **the inverse of** h in G **is the same**.

Remark
If $(G,\circ)$ is a group, then $(\{e\},\circ)$ and $(G,\circ)$ are **subgroups** of $(G,\circ)$. These subgroups are called **trivial subgroup (自明な部分群)**.

Example 4.1

Consider the following list of groups.

(i) $(\{0\},+),(\mathbb{Z},+),(\mathbb{Q},+),(\mathbb{R},+),(\mathbb{C},+)$,

(ii) $(\{1\},\cdot),(\mathbb{Q}\backslash\{0\},\cdot),(\mathbb{R}\backslash\{0\},\cdot),(\mathbb{C}\backslash\{0\},\cdot)$,

where $+$ is the usual addition operation and $\cdot$ is the usual multiplication operation.

Each group is a subgroup of the group listed to its right.

For example, $(\mathbb{Z},+)$ is a subgroup of $(\mathbb{Q},+)$, $(\mathbb{R},+)$, and $(\mathbb{C},+)$, and $(\mathbb{R}\backslash\{0\},\cdot)$ is a subgroup of $(\mathbb{C}\backslash\{0\},\cdot)$.

Theorem 4.2

Let $(G,\circ)$ be a group and H be a **nonempty subset** of G. Then $(H,\circ)$ is a subgroup of $(G,\circ)$ **if and only if** for all $a, b \in H$, $ab^{-1} \in H$.

Proof:

Suppose H is a subgroup of G. Let $a, b \in H$. Because H is a subgroup, it is a group. Therefore, $b \in H$ implies that $b^{-1} \in H$. Thus, $ab^{-1} \in H$ because H is closed under the binary operation.
Conversely, suppose H is a nonempty subset of G such that $a, b \in H$ implies $ab^{-1} \in H$.
Because $H \neq \emptyset$, there exists $a \in H$. Now a, $a^{-1} \in H$. Therefore, $e = aa^{-1} \in H$, i.e., H contains the identity.
Next, let $b \in H$. Then $e, b \in H$, implies that $b^{-1} = eb^{-1} \in H$. Thus, every element of H has an inverse in H.
To show that H is closed under the binary operation, let $a, b \in H$. Then $a, b^{-1} \in H$. Thus, $ab = a(b^{-1})^{-1} \in H$.
Hence, H is closed under the binary operation. From the statements preceding Definition, associativity holds for H. Hence, H is a group, so H is subgroup of G.

Example 4.2 Find all subgroups of $(\mathbb{Z}, +)$.

Solution: Let $(H, +)$ be a subgroup of $(\mathbb{Z}, +)$. Suppose $H \neq \{0\}$.

Let a be a nonzero element of H. Then $-a \in H$. Since either a or $-a$ is a positive integer, H contains a positive integer. With the help of the principle of well-ordering, we can show that H contains a smallest positive integer. Let a be the smallest positive integer in H. We claim that $H = \{na \mid n \in \mathbb{Z}\}$.

Now $na \in H$ for all $n \in \mathbb{Z}$ and so $\{na \mid n \in \mathbb{Z}\} \subseteq H$. On the other hand, let $b \in H$.
By the division algorithm, there exist c and r in $\mathbb{Z}$ such that $b = ca + r$, where $0 \leq r < a$. Suppose $r \neq 0$. Then $r = b - ca \in H$. Thus, H contains a positive integer smaller than a, a contradiction.

Hence, $r = 0$ and so $b = ca \in \{na \mid n \in \mathbb{Z}\}$.
This implies that $H \subseteq \{na \mid n \in \mathbb{Z}\}$. Thus, $H = \{na \mid n \in \mathbb{Z}\}$ for some $a \in \mathbb{Z}$. Also, for all $n \in \mathbb{Z}$, the set $T = \{nm \mid m \in \mathbb{Z}\} = n\mathbb{Z}$ generate a subgroup of $(\mathbb{Z}, +)$.

Hence, $(n\mathbb{Z}, +)$, $n = 0, 1, 2, \ldots$ are the subgroups of $(\mathbb{Z}, +)$.

Definition 4.2

Let H and L be **nonempty subsets** of G from a **group** $(G,\circ)$.
The product of H and L is defined to be the set
$HL = \{hl \mid h \in H, l \in L\}$.

Example 4.3
Consider the group of symmetries of the square.

Let $H = \{r_{360}, d_1\}$ and $L = \{r_{360}, h\}$. Then H and L are subgroups of G. Now

$$HL = \{r_{360}r_{360},\ r_{360}h,\ d_1r_{360},\ d_1h\} = \{r_{360},\ h,\ d_1,\ r_{90}\}.$$

Now $hd_1 = r_{270} \notin HL$, so HL is **not closed** under the binary operation.
Hence, HL is **not a subgroup** of the symmetries of the square. Also, note that

$$LH = \{r_{360}r_{360},\quad r_{360}d_1,\quad hr_{360},\quad hd_1\} = \{r360,\quad d1,\quad h,\quad r_{270}\},$$

And $\langle H \cup L\rangle = \{r_{360}, r_{90}, r_{180}, r_{270}, h, v, d_1, d_2\}$.

This example shows that in general the product of subgroups need not be a subgroup.

4.1 Subgroup (部分群)

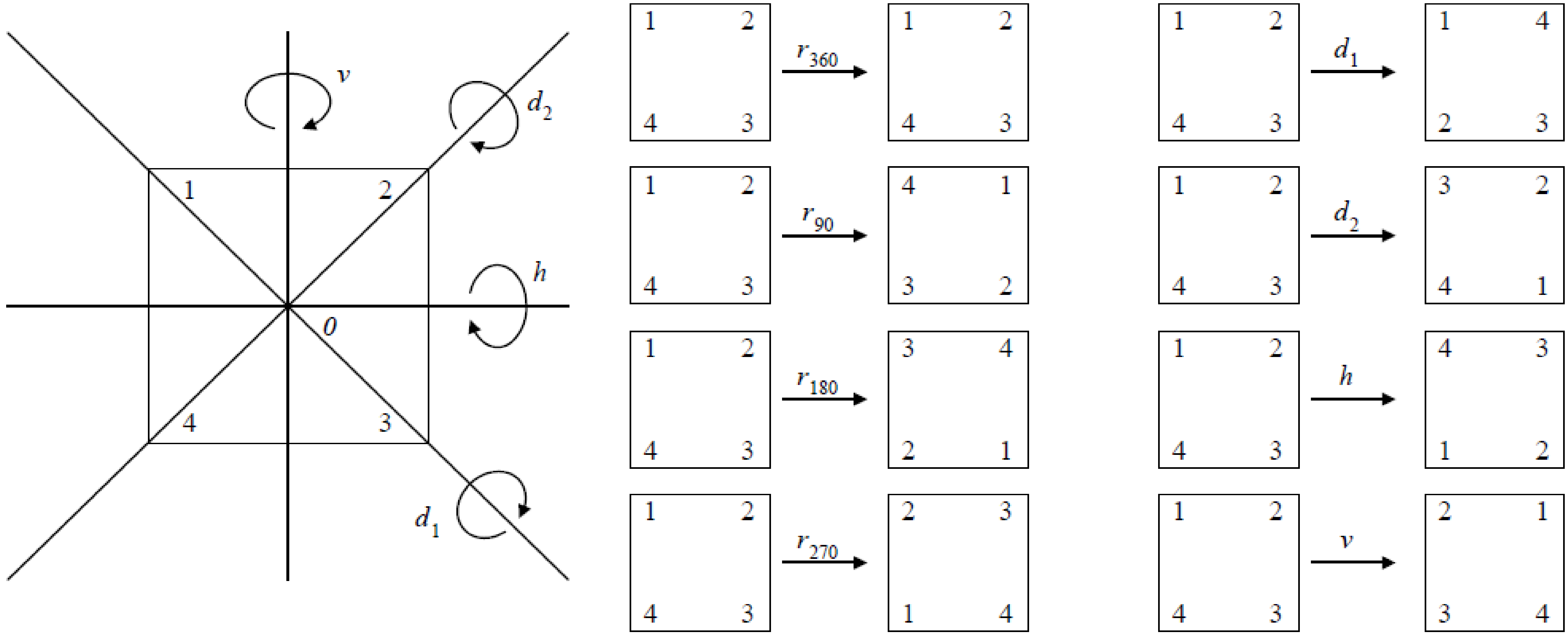
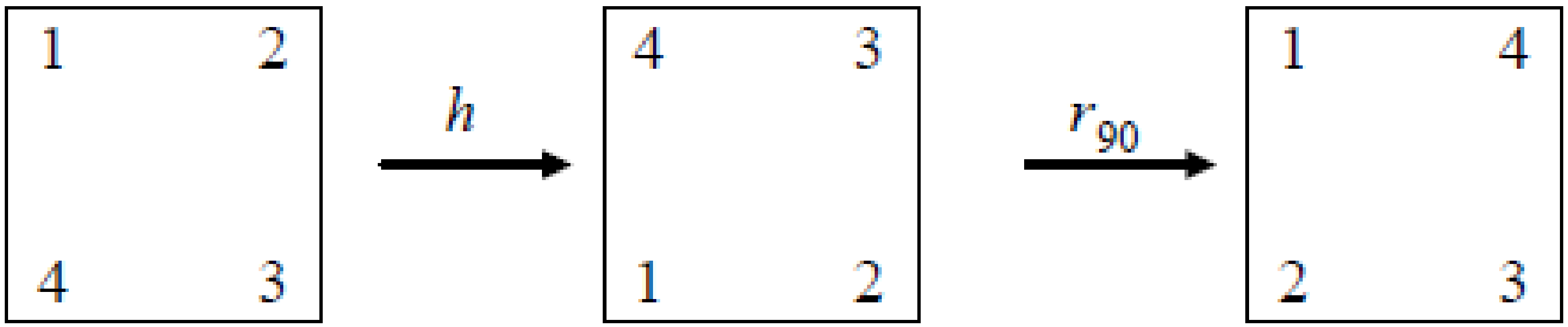


Figure. Rigid motions of a square in symmetry

4.1 Subgroup (部分群)

$$r_{90} \circ h$$



4.1 Subgroup (部分群)

The complete operation table for the operation ∘ is as following

$\circ$	r_{360}	r_{90}	r_{180}	r_{270}	h	v	d_1	d_2
r_{360}	r_{360}	r_{90}	r_{180}	r_{270}	h	v	d_1	d_2
r_{90}	r_{90}	r_{180}	r_{270}	r_{360}	d_1	d_2	v	h
r_{180}	r_{180}	r_{270}	r_{360}	r_{90}	v	h	d_2	d_1
r_{270}	r_{270}	r_{360}	r_{90}	r_{180}	d_2	d_1	h	v
h	h	d_2	v	d_1	r_{360}	r_{180}	r_{270}	r_{90}
v	v	d_1	h	d_2	r_{180}	r_{360}	r_{90}	r_{270}
d_1	d_1	h	d_2	v	r_{90}	r_{270}	r_{360}	r_{180}
d_2	d_2	v	d_1	h	r_{270}	r_{90}	r_{180}	r_{360}

4.1 Subgroup (部分群)

Subgroup

In the following theorem, we give a necessary and sufficient condition for the product of subgroups to be a subgroup.

Theorem 4.3

Let $(H,\circ)$ and $(L,\circ)$ be subgroups of a group $(G,\circ)$. Then $(HL,\circ)$ is a subgroup of $(G,\circ)$ **if and only if** $HL = LH$.

Corollary 3.2

If $(H,\circ)$ and $(L,\circ)$ are subgroups of **a commutative group** $(G,\circ)$, then $(HL,\circ)$ is **a subgroup of** $(G,\circ)$.

4.2 Cyclic Group (巡回群)

4.2 Cyclic Group (巡回群)

In the previous section, we introduced the notion of a subgroup generated by a set. **Groups that are generated by a single element, called cyclic groups**, are of special importance. Cyclic groups are easier to study than any other group.

Definition 4.3

A group $(G, \circ)$ is called a **cyclic group** if there exists $a \in G$ such that

$$G = \langle a \rangle$$

where $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$.

Let $G = \langle a \rangle$ defines a cyclic group and $b, c \in G$. Then $b = a^n$ and $c = a^m$ for some $n, m \in Z$. Now

$$bc = a^n a^m = a^{n+m} = a^{m+n} = a^m a^n = cb.$$

This shows that G is **commutative**. **Hence, every cyclic group is commutative**. We record this result in the following theorem.

Theorem 4.4

Every cyclic group is commutative.

Example 4.4

(i) $(\mathbb{Z}, +)$ is a cyclic group because $\mathbb{Z} = \langle 1 \rangle$.

(ii) $(\{na \mid n \in \mathbb{Z}\}, +)$ is a cyclic group, where a is any fixed element of $\mathbb{Z}$.

Example 4.5

Consider the set $G = \{e, a, b, c\}$. Define $\circ$ on G by means of the following operation table.

$\circ$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

From the multiplication table, it follows that $(G,\circ)$ is a **commutative** group. **However, G is NOT a cyclic group** because $\langle e\rangle = \{e\}, \langle a\rangle = \{e, a\}, \langle b\rangle = \{e, b\}$, and $\langle c\rangle = \{e, c\}$ and each of these subgroups is properly contained in G. G is known as the **Klein 4-group** (クラインの四元群).

4.2 Cyclic Group (巡回群)

Theorem 4.5

Let $\langle a \rangle$ be a finite cyclic group of order n.
Then $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$.

Theorem 4.6

Every subgroup of a cyclic group is **cyclic.**

Corollary 3.2

Let $G = \langle a \rangle$ be **a cyclic group of order** n, $n > 1$, and H be **a proper subgroup of** G.
Then $H = a^k$ for some integer k such that k divides n and $k > 1$.
Furthermore, the order $|H|$ **divides** n.

4.3 Lagrange’s Theorem

(ラグランジュの定理)

4.3 Lagrange's Theorem (ラグランジュの定理)

In the last section, **we noted that the order of a subgroup of a finite cyclic group divides the order of the group** (Corollary 4.2).

We will learn that this is a special case of a general result, called **Lagrange's theorem**, i.e., **the order of a subgroup of a finite group divides the order of the group**.

History:

Lagrange proved this result in 1770, long before the creation of group theory, while working on the permutations of the roots of a polynomial equation. Lagrange's theorem is a basic theorem of **finite group theory** and is considered by some to be the most important result in finite group theory.

Definition 4.4

Let H be a subgroup of a group G and $a \in G$. The sets $aH = \{ah \mid h \in H\}$ and $Ha = \{ha \mid h \in H\}$ are called the **left and right cosets (左剰余類と右剰余類) of H in G**, respectively. The element a is called **a representative** of aH **and** Ha.

If G is **commutative**, then of course we have $aH = Ha$.
Observe that $eH = H = He$ and that $a = ae \in aH$ and $a = ea \in Ha$.

Example 4.6 Consider the symmetric group S_3 (Example 3.7).

(1)
$$H = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$
is a subgroup of S_3.

We now compute the left and right cosets of H in S_3. The left cosets of H in S_3 are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H = H$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H =$$
$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

and the right cosets of H in are

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = H$$

and

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right\}.$$

Thus, for all $a \in S_3$, $aH = Ha$.

4.3 Lagrange's Theorem (ラグランジュの定理)

Left and Right Cosets

(2) $$H' = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$$

is also a subgroup of S_3.

Now we compute the left and right cosets of H' in S_3. The left cosets of H' in S_3

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H' = H',$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\},$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$$

4.3 Lagrange's Theorem (ラグランジュの定理)

and the right cosets of H' in S_3 are

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H',$$

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right\},$$

and

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}.$$

We see that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' \neq H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Thus, the left and right cosets of H' in S_3 are not the same.

Left and Right Cosets

There are some interesting phenomena happening in the above example.

- We see that all left and right cosets of H in S_3 have the same number of elements, namely, 3; that there are the same number of distinct left cosets of H in S_3 as of right cosets, namely, 2; that the set of all left cosets and the set of all right cosets form partitions of S_3; and, finally, that $3 \cdot 2$ equals the order of S_3.
- Similar statements hold for the subgroup H'. We show, in the results to follow, that these phenomena hold in general.

4.3 Lagrange's Theorem (ラグランジュの定理)

The following theorem tells us **when two left (right) cosets are equal.** It is a result that is used often in the study of groups.

Theorem 4.7

Let H be a **subgroup** of a **group** G and $a, b \in G$. Then
(i) $aH = bH$ if and only if $b^{-1}a \in H$.
(ii) $Ha = Hb$ if and only if $ab^{-1} \in H$.

Theorem 4.8

Let H be a **subgroup** of a **group** G. Then for all $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$ (i.e., two left cosets are either equal or they are disjoint). Similar result also satisfied for two right cosets.

Number of Distinct Cosets

Definition 4.5

Let H be a **subgroup** of a **group** G. Then **the number of distinct (相異なる) left (or right) cosets**, written as $[G:H]$, of H in G **is called the index of H in G**.

Theorem 4.9 (Lagrange's Theorem)

Let H be a **subgroup** of a **finite group** G. Then **the order of** H **divides the order of** G. In particular,

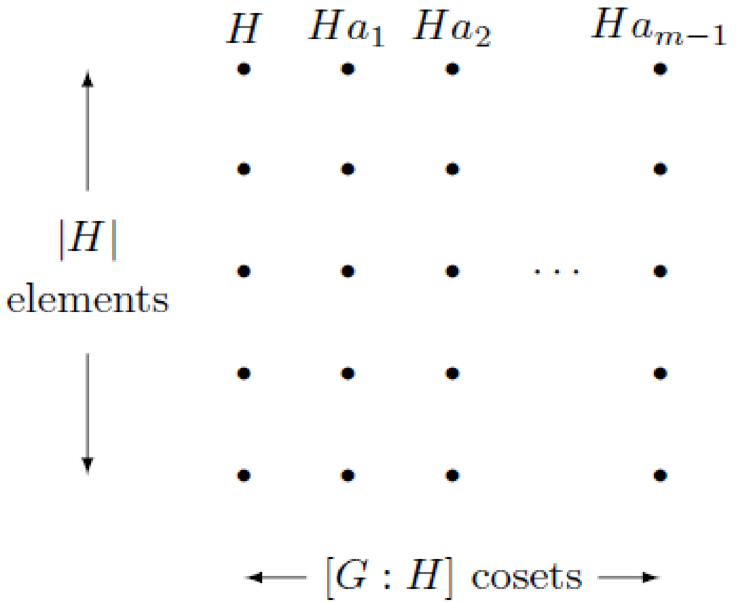
$$|G| = [G : H]|H|.$$

Proof:

Suppose that $[G : H] = m$. Every element of G is in a coset of H, and Theorem 4.8 tells us we can decompose G into a union of m pairwise disjoint cosets:

$$G = H \cup Ha_1 \cup Ha_2 \cup \cdots \cup Ha_{m-1}$$

But each of these cosets has $|H|$ elements. Thus, there must be $[G : H]|H|$ elements in G altogether.



4.3 Lagrange's Theorem (ラグランジュの定理)

Lagrange's Theorem

Theorem 4.10

Let H and L be **finite subgroups** of a **group** G. Then

$$|HL| = \frac{|H||L|}{|H \cap L|}$$

Review for Lecture 4

- Subgroup (部分群)
- Trivial Subgroup (自明な部分群)
- Cyclic Group (巡回群)
- Left and Right Cosets (左剰余類と右剰余類)
- Lagrange’s Theorem (ラグランジュの定理)

Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

References

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