



Lecture 12

Extension Fields (体の拡大)

12.1 Extension Fields (体の拡大)

12.2 Simple Field

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It is natural to ask whether or not some field F is contained in a larger field. We think of the rational numbers, which reside inside the real numbers, while in turn, **the real numbers live inside the complex numbers**. We can also study the fields between \mathbb{Q} and \mathbb{R} and inquire as to the nature of these fields.

More specifically if we are given a field F and a polynomial $p(x) \in F[x]$, we can ask whether or not we can find a field E containing F such that $p(x)$ factors into linear factors over $E[x]$. **For example, if we consider the polynomial**

$$p(x) = x^4 - 5x^2 + 6$$

In $\mathbb{Q}[x]$, then $p(x)$ factors as $(x^2 - 2)(x^2 - 3)$. However, both of these factors are irreducible in $\mathbb{Q}[x]$. **If we wish to find a root of $p(x)$, we must go to a larger field. Certainly the field of real numbers will work, since**

$$p(x) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3})$$

It is possible to find a smaller field in which $p(x)$ has a root, namely

$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$$

We wish to be able to compute and study such fields for arbitrary polynomials over a field F .

12.1 Extension Fields (体の拡大)

Definition 12.1

A field E is an extension field of a field F if F is a subfield of E . We write $F \subseteq E$. The field F is called the **base field**.

For example,

\mathbb{R} is an extension field of \mathbb{Q} ,

\mathbb{C} is an extension field of both \mathbb{R} and \mathbb{Q} ,

$\mathbb{Q}(\sqrt{2})$ is an extension field of \mathbb{Q} .

12.1 Extension Fields (体の拡大)

Theorem 12.1 (Kronecker's Theorem or Fundamental Theorem of Field Theory)

Let F be a field and $f(x)$ is a polynomial in $F[x]$ of degree at least 1.
Then **there exists an extension field E of F and an $\tau \in E$ such that $f(\tau) = 0$.**

Proof (See page 356 of the textbook,)

12.1 Extension Fields (体の拡大)

Definition 12.2

Let E be an extension field of a field F . For $\tau \in E$, if $f(\tau) = 0$ for some nonzero $f(x) \in F[x]$, we say τ is **algebraic (代数的) over F** . Otherwise, τ is **transcendental (超越的) over F** .

An extension field E of a field F is an algebraic extension of F if every element in E is algebraic over F .

Example 12.1

- (1) Because $\sqrt{2}$ is a root of $x^2 - 2 \in \mathbb{Q}[x]$, so $\sqrt{2}$ is algebraic over \mathbb{Q} .
- (2) i is algebraic over \mathbb{Q} because i is a root of $x^2 + 1 \in \mathbb{Q}[x]$.

12.1 Extension Fields (体の拡大)

Example 12.2

We will show that $\sqrt{1 + \sqrt{3}}$ is algebraic over \mathbb{Q} .

If $\tau = \sqrt{1 + \sqrt{3}}$, then $\tau^2 = 1 + \sqrt{3}$.

Hence $\tau^2 - 1 = \sqrt{3}$ and $(\tau^2 - 1)^2 = 3$.

Since $\tau^4 - 2\tau^2 - 2 = 0$, it must be true that τ is a root of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

12.2 Simple Extension (単純拡大)

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Definition 12.3

An element of \mathbb{C} that is algebraic over \mathbb{Q} is an **algebraic number** (代数的数).

A **transcendental number** (超越数) is an element of \mathbb{C} that is transcendental over \mathbb{Q} .

12.2 Simple Extension (単純拡大)

Consider the **extension field** \mathbb{R} of \mathbb{Q} . We know that $\sqrt{2}$ is **algebraic over** \mathbb{Q} , being a **root** of $x^2 - 2$. We know that $\sqrt{2}$ is also a **root** of $x^3 - 2x$ and of $x^4 - 3x^2 + 2 = (x^2 - 2)(x^2 - 1)$.

Both these other polynomials having $\sqrt{2}$ as a root were multiples of $x^2 - 2$. The next theorem shows that this is an illustration of a general situation.

12.2 Simple Extension (単純拡大)

Theorem 12.2

Let E be an extension field of F , and let $\tau \in E$, where τ is algebraic over F . Then there is an irreducible polynomial $p(x) \in F[x]$ such that $p(\tau) = 0$. This irreducible polynomial $p(x)$ is uniquely determined up to a constant factor in F and is a polynomial of minimal degree $\deg p(x) \geq 1$ in $F[x]$ having τ as a root. If $f(x) = 0$ for $f(x) \in F[x]$, with $f(x) \neq 0$, then $p(x)$ divides $f(x)$.

By multiplying by a suitable constant in F , we can assume that the coefficient of the highest power of x appearing in $p(x)$ of the Theorem 12.2 is 1. Such a polynomial having 1 as the coefficient of the highest power of x appearing is a **monic polynomial** (モニック多項式).

12.2 Simple Extension (単純拡大)

Definition 12.4

Let E be an extension field of a field F , and let $\alpha \in E$ be algebraic over F . The unique **monic polynomial** $p(x)$ having the property described in Theorem 12.2 is the **irreducible polynomial for α over F** and will be denoted by $\text{irr}(\alpha, F)$. The **degree of irreducible polynomial $\text{irr}(\alpha, F)$** is the degree of α over F , denoted by $\text{deg}(\alpha, F)$.

12.2 Simple Extension (単純拡大)

Example 12.3

We know that $\text{irr}(\sqrt{2}, \mathbb{Q}) = x^2 - 2$. Referring to Example 12.2, we see that for $\tau = \sqrt{1 + \sqrt{3}}$ in \mathbb{R} , τ is a root of $x^4 - 2x^2 - 2$, which is in $\mathbb{Q}[x]$. Since $x^4 - 2x^2 - 2$ is irreducible over \mathbb{Q} , we see that

$$\text{irr}\left(\sqrt{1 + \sqrt{3}}, \mathbb{Q}\right) = x^4 - 2x^2 - 2$$

Thus $\sqrt{1 + \sqrt{3}}$ is algebraic of degree 4 over \mathbb{Q} .

12.2 Simple Extension (単純拡大)

Definition 12.5

An extension field E of a field F is a **simple extension** of F if $E = F(\tau)$ for some $\tau \in E$.

Theorem 12.3

Let E be a **simple extension** $F(\tau)$ of a field F , and let τ be algebraic over F . Let the degree of $\text{irr}(\tau, F)$ be at least 1. Then every element of E can be uniquely expressed in the form

$$\beta = b_0 + b_1\tau + b_2\tau^2 + \cdots + b_{n-1}\tau^{n-1}$$

where the b_i are in F .

12.2 Simple Extension (単純拡大)

*Theorem 12.4

Let $f(x) \in F[x]$ and let $f(x)$ be of degree 2 or 3. Then $f(x)$ is **reducible** over F if and only if it has a root in F .

Example 12.4 The polynomial in $p(x) = x^2 + x + 1$ in $\mathbb{Z}_2[x]$ is irreducible over \mathbb{Z}_2 by Theorem 12.4, since neither element 0 nor element 1 of \mathbb{Z}_2 is a root of $p(x)$. By Theorem 12.1, we know that there is an extension field E of \mathbb{Z}_2 containing a root τ of $x^2 + x + 1$. By theorem 12.3, it has as elements $0 + 0\tau, 1 + 0\tau, 0 + 1\tau$, and $1 + 1\tau$, that is $0, 1, \tau$ and $1 + \tau$. This gives us *a new finite field of four elements*! The addition and multiplication tables for this field are shown in Table 12.1 and Table 12.2. For example, to compute $(1 + \tau)(1 + \tau)$, we observe that since $p(\tau) = \tau^2 + \tau + 1 = 0$, then

$$\tau^2 = -\tau - 1 =_2 \tau + 1$$

Therefore

$$(1 + \tau)(1 + \tau) = 1 + 2\tau + \tau^2 =_2 1 + \tau^2 = 1 + (\tau + 1) =_2 \tau$$

+	0	1	τ	$1 + \tau$
0	0	1	τ	$1 + \tau$
1	1	0	$1 + \tau$	τ
τ	τ	$1 + \tau$	0	1
$1 + \tau$	$1 + \tau$	τ	1	0

Table 12.1

\cdot	0	1	τ	$1 + \tau$
0	0	0	0	0
1	0	1	τ	$1 + \tau$
τ	0	τ	$1 + \tau$	1
$1 + \tau$	0	$1 + \tau$	1	τ

Table 12.2

12.3 Algebra Closure

12.3 Algebra Closure

Theorem 12.5

Let E be an extension field of F . The set of elements in E that are algebraic over F form a field.

Corollary 12.1

The set of all algebraic numbers forms a field; that is, the set of all complex numbers that are algebraic over \mathbb{Q} makes up a field.

Definition 12.6

Let E be a field extension of a field F . We define the **algebraic closure** (代数的閉包) of a field F in E to be the field consisting of all elements in E that are algebraic over F . A field F is algebraically closed if every nonconstant polynomial in $F[x]$ has a root in F .

12.3 Algebra Closure

Theorem 12.6 (Fundamental Theorem of Algebra)

The field of complex numbers is algebraically closed.

Review for Lecture 12

- Extension Fields (体の拡大)
- Simple Field
- Algebraic number (代数的数) and Transcendental number (超越数)
- Algebra Closure

Assignment

Please Check <https://github.com/uoaworks/Applied-Algebra>

References

- [1] Thomas W. Judson etc. Abstract Algebra Theory and Applications, 2018
- [2] D. S. Malik, John N. Mordeson, M.K. Sen, Introduction to Abstract Algebra, 2007
- [3] (おすすめ) 松本 眞, 代数系への入門, <http://www.math.sci.hiroshima-u.ac.jp/~m-mat/TEACH/daisu-nyumon2014.pdf>
- [4] Wikipedia
- [5] Materials from internet.