



Lecture 3

Introduction of Group (群)

What you will learn in Lecture 3

3.1 Introduction of Group (群)

3.2 Permutation Group (置換群)

3.1 Introduction of Group (群)

Definition 3.1

A **group (群)** is an ordered pair (G, \circ) , where G is a nonempty set and \circ is a **binary operation** on G (i.e. $G \times G \rightarrow G$) such that the following properties hold:

(G1) (associative law) For all $a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$.

(G2) (existence of an identity) There exists identity element $e \in G$ such that for all $a \in G$, $a \circ e = a = e \circ a$.

(G3) (existence of an inverse) For all $a \in G$, there exists $b \in G$ such that $a \circ b = e = b \circ a$.

#(G4) (closure property) For all $a, b \in G$, the result of the operation, $a \circ b$, is also in G .

Thus, a group is a mathematical system (G, \circ) satisfying axioms (公理) G1, G2, G3 (and G4).

Example 3.1

Consider \mathbb{Z} , the set of integers, together with the binary operation $+$, where $+$ is the usual addition. We know that $+$ is closed and associative on \mathbb{Z} . Now $0 \in \mathbb{Z}$ and for all $a \in \mathbb{Z}$,

$$a + 0 = a = 0 + a.$$

So 0 is an identity.

Also, for all $a \in \mathbb{Z}$, $-a \in \mathbb{Z}$ and $a + (-a) = 0 = (-a) + a$.

That is, $-a$ is an inverse of a .

It now follows that $(\mathbb{Z}, +)$ satisfies axioms G1 to G3 (and G4), so $(\mathbb{Z}, +)$ is a group.

As in Example 2.7, we can show that $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are also groups, where $+$ is the usual addition.

3.1 Introduction of Group (群)

Theorem 3.1

Let (G, \circ) be a group.

- (i) There exists a **unique identity element** $e \in G$ such that $e \circ a = a = a \circ e$ for all $a \in G$.
- (ii) For all $a \in G$, there exists a **unique** $b \in G$ such that $a \circ b = e = b \circ a$.

Proof: (i) Now (G, \circ) is group. Therefore, by axiom G2, there exists $e \in G$ such that $e \circ a = a = a \circ e$ for all $a \in G$. Because (G, \circ) is a mathematical system, e is unique by Theorem 2.1.
(ii) Let $a \in G$. By axiom G3, there exists $b \in G$ such that $a \circ b = e = b \circ a$. Suppose there exists $c \in G$ such that $a \circ c = e = c \circ a$. We show that $b = c$.

$$\begin{aligned} b &= b \circ e \\ &= b \circ (a \circ c) \text{ (substituting } e = a \circ c) \\ &= (b \circ a) \circ c \text{ (using the associativity of } \circ) \\ &= e \circ c \text{ (because } b \circ a = e) \\ &= c. \end{aligned}$$

2019/7/1 Thus, b is unique.

Definition 3.2

Let (G, \circ) be a group. If for all $a, b \in G$,

$$a \circ b = b \circ a$$

then (G, \circ) is called **commutative group (可換群)** or **Abelian group (アーベル群)**. A group (G, \circ) is called **noncommutative** if it is **not commutative**.

3.1 Introduction of Group (群)

Example 3.2

Consider the group $(\mathbb{Z}, +)$ of Example 2.7. Because $a + b = b + a$ for all $a, b \in \mathbb{Z}$, it follows that $+$ is commutative. Hence, $(\mathbb{Z}, +)$ is a commutative group.

Similarly, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{R} \setminus \{0\}, \cdot)$, $(\mathbb{C} \setminus \{0\}, \cdot)$ are also commutative groups, where $+$ is the usual addition and \cdot is the usual multiplication.

3.1 Introduction of Group (群)

Cayley table

It is often convenient to describe a group in terms of an addition or multiplication table. Such a table is called a *Cayley table*.

Example 3.3

The integers mod n form a group under addition modulo n . Consider \mathbb{Z}_5 , consisting of the equivalence classes of the integers 0, 1, 2, 3, and 4. We define the group operation on \mathbb{Z}_5 by modular addition.

We write the binary operation on the group additively; that is, we write $m + n$. The element 0 is the identity of the group and each element in \mathbb{Z}_5 has an inverse. For instance, $2 + 3 = 3 + 2 = 0$. Table 3.1 is a Cayley table for \mathbb{Z}_5 . We can see \mathbb{Z}_5 is a group under the binary operation of addition mod n .

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 3.1

Definition 3.3

A group (G, \circ) is called a **finite group (有限群)** if G has **only a finite number of elements**. The **order (位数)** of a group (G, \circ) , which is written as $|G|$, is **the number of elements of G** .

Definition 3.4

A **group with an infinite number of elements** is called an **infinite group (無限群)**.

*Definition 3.5

A **semigroup (半群)** is an ordered pair (M, \circ) , where M is a nonempty set and \circ is an associative binary operation on M .

Thus, a semigroup is a mathematical system with one binary operation such that the binary operation is associative.

Remark

For any group (G, \circ) , the binary operation \circ is associative. Therefore, every group (G, \circ) is a semigroup.

Notice: * mark is optional study material. It will be not included in middle and final examinations.

*Theorem 3.2

A semigroup (M, \circ) is a **group** if and only if

- (i) there exists $e \in M$ such that $e \circ a = a$ for all $a \in M$, (i.e., e is a **left identity**),
- (ii) for all $a \in M$ there exists $b \in M$ such that $b \circ a = e$, (i.e., every element has a **left inverse**).

3.2 Permutation Group (置換群)

Definition 3.6

For any **nonempty set** S , a **one-to-one** and **onto** mapping $\pi: S \rightarrow S$ is called a **permutation (置換)** of S .

Example 3.4

- (i) Let S be a nonempty set. Define $\pi: S \rightarrow S$ by $\pi(x) = x$ for all $x \in S$. Then π is one-to-one function of S onto S . Thus, π is a permutation of S . Note that π is called the identity permutations and is, usually, denoted by i_S or e .
- (ii) Let $S = \{a, b, c\}$. Define $\alpha: S \rightarrow S$ such that $\alpha(a) = b, \alpha(b) = a$, and $\alpha(c) = c$. By the definition of α it follows that α is one-to-one function of S onto S . Thus, α is a permutation of S .
- (iii) Consider \mathbb{R} , the set of real numbers. Define $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ by $\alpha(x) = 3x + 5$ for all $x \in \mathbb{R}$. It can be shown that α is a one-to-one function of \mathbb{R} onto \mathbb{R} . Thus, α is a permutation of \mathbb{R} .
Similarly, if $\beta: \mathbb{R} \rightarrow \mathbb{R}$ by $\beta(x) = x^3$ for all $x \in \mathbb{R}$. It can be shown that β is a one-to-one function of \mathbb{R} onto \mathbb{R} . Thus, β is a permutation of \mathbb{R} .

Definition 3.7

A group (G, \circ) is called a **permutation group (置換群)** on a **nonempty set S** if the elements of G are permutations of S and the operation \circ is the composition of two functions.

Example 3.5

Let $S = \{1, 2\}$. Define $\alpha : S \rightarrow S$ such that $\alpha(1) = 1, \alpha(2) = 2$.

Then α is a one-to-one function of S onto S , so α is a permutation of S .

Next define $\beta : S \rightarrow S$ such that $\beta(1) = 2$ and $\beta(2) = 1$.

Then β is a one-to-one function of S onto S , so β is a permutation of S .

Let $G = \{\alpha, \beta\}$. Then (G, \circ) is a group, where \circ is the composition of functions.

Note that on this set S , α and β are the only permutations on S .

Moreover, α is the identity permutation and $\beta^{-1} = \beta$.

Let $I_n = \{1, 2, \dots, n\}, n \geq 1$. Let π be a permutation on I_n . Then

$$\pi = \{(1, \pi(1)), (2, \pi(2)), \dots, (n, \pi(n))\}.$$

Recall that a function $f : S \rightarrow S$ is a subset of $S \times S$. By introducing two-row notation, we have

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$$

Example 3.6

(1) Let $n = 4$ and π be the permutation on I_4 defined by $\pi(1) = 2, \pi(2) = 4, \pi(3) = 3$, and $\pi(4) = 1$. Then using the two-row notation we can write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

(2) Let $n = 7$ and π and σ be two permutations on I_7 defined by

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 7 & 2 & 5 \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 1 & 7 & 6 & 4 \end{pmatrix}$$

Let us compute $\pi \circ \sigma$. Now by the definition of the composition of functions

$$(\pi \circ \sigma)(i) = \pi(\sigma(i))$$

for all $i \in I_7$. Thus,

$$(\pi \circ \sigma)(1) = \pi(\sigma(1)) = \pi(2) = 3$$

$$(\pi \circ \sigma)(2) = \pi(\sigma(2)) = \pi(5) = 7$$

and so on.

3.2 Permutation Group (置換群)

Permutation group

From this, it is clear that when determining, say, $(\pi \circ \sigma)(1)$, we start with σ and finish with π and read as follows: 1 goes to 2 (under σ) and 2 goes to 3 (under π), so 1 goes to 3 (under $\pi \circ \sigma$).

We can exhibit this in the following form:

$$\begin{array}{ll} 1 \xrightarrow{\sigma} 2 \xrightarrow{\pi} 3 & 1 \xrightarrow{\pi \circ \sigma} 3 \\ 2 \xrightarrow{\sigma} 5 \xrightarrow{\pi} 7 & 2 \xrightarrow{\pi \circ \sigma} 7 \\ 3 \xrightarrow{\sigma} 3 \xrightarrow{\pi} 4 & 3 \xrightarrow{\pi \circ \sigma} 4 \\ 4 \xrightarrow{\sigma} 1 \xrightarrow{\pi} 1 & 4 \xrightarrow{\pi \circ \sigma} 1 \\ 5 \xrightarrow{\sigma} 7 \xrightarrow{\pi} 5 & 5 \xrightarrow{\pi \circ \sigma} 5 \\ 6 \xrightarrow{\sigma} 6 \xrightarrow{\pi} 2 & 6 \xrightarrow{\pi \circ \sigma} 2 \\ 7 \xrightarrow{\sigma} 4 \xrightarrow{\pi} 6 & 7 \xrightarrow{\pi \circ \sigma} 6. \end{array}$$

Thus,

$$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 1 & 5 & 2 & 6 \end{pmatrix}$$

Example 3.7

In this example, we describe S_3 , i.e., the set of all permutations on $I_3 = \{1, 2, 3\}$.

From equilateral triangle example in Lecture 2, we know that the number of one-to-one functions of I_3 onto I_3 is $3! = 6$. Thus, $|S_3| = 6$. Let e denote the identity permutation on I_3 , i.e.,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Let α be a nonidentity permutation on I_3 . Let us see some of the choices for α .

Suppose $\alpha(1) = 1$. If $\alpha(2) = 2$, then we must have $\alpha(3) = 3$ because α is a permutation.

In this case, we see that $\alpha = e$, a contradiction. Thus, we must have $\alpha(2) = 3$ and $\alpha(3) = 2$, i.e., we define this α as

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

In a similar manner, we can show that the other four permutations on I_3 are

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

For closure property, it can check that, for example

$$\alpha_2 \circ \alpha_4 = \alpha_1$$

Hence, we can write

$$S_3 = \{e, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

Theorem 3.3

- (i) (S_n, \circ) is a **group** for any positive integer $n \geq 1$.
- (ii) If $n \geq 3$, then (S_n, \circ) is **noncommutative**.
- (iii) $|S_n| = n!$

Definition 3.8

The **symmetric group** (对称群) S_n is a group with $n!$ elements, where the binary operation is the composition of maps.

The group (S_n, \circ) is called the **symmetric group** on I_n .

Definition 3.9

Let π be an element of S_n . Then π is called a k -cycle, written as $(i_1 \ i_2 \ \cdots \ i_k)$, if

$$\pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_{k-1} & i_k \\ i_2 & i_3 & \cdots & i_k & i_1 \end{pmatrix},$$

i.e., $\pi(i_j) = i_{j+1}, j = 1, 2, \dots, k-1, \pi(i_k) = i_1$, and $\pi(a) = a$ for any other element of I_n .

Notice: When $k = 2$, a k -cycle is called a **transposition (互換)**.

For **example**, consider the cycle notation $(3 \ 4)$ for $\pi \in S_4$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (3 \ 4)$$

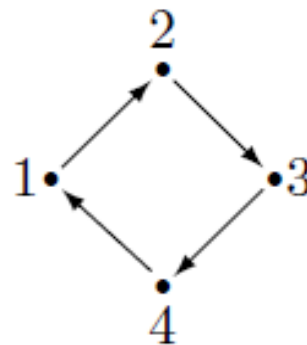
Example 3.8

Consider the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

It permutes the elements 1, 2, 3, and 4 cyclically, then we have the cycle notation $\alpha = (1\ 2\ 3\ 4)$.

And the following picture suggests:



Theorem 3.4

Any permutation (置換) π of S_n can be expressed as a product of disjoint cycles.

Example 3.9

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix}$$

Using cycle notation, we can write

$$\sigma = (1\ 6\ 2\ 4)$$

$$\tau = (1\ 3)(4\ 5\ 6)$$

$$\sigma\tau = (1\ 3\ 6)(2\ 4\ 5)$$

$$\tau\sigma = (1\ 4\ 3)(2\ 5\ 6)$$

Corollary 3.1

Let $n \geq 2$. Any permutation (置換) π of S_n can be expressed as a product of transpositions (互換).

The cycle $(i_1 \ i_2 \ \cdots \ i_k) = (i_1 \ i_n)(i_1 \ i_{n-1}) \cdots (i_1 \ i_3)(i_1 \ i_2)$

For example, the Fifteen Puzzle



<http://lorecioni.github.io/fifteen-puzzle-game/>

Example 3.10

Consider the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$

Let us write it as a product of disjoint cycles.

First, 1 is moved to 6 and then 6 to 1, giving the cycle (1,6).

Then 2 is moved to 5, which is moved to 3, which is moved to 2, giving the cycle (2 5 3).

This process takes care of all elements except 4, which is left fixed. Thus

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix} = (1\ 6)(2\ 5\ 3) = (2\ 5\ 3)(1\ 6)$$

Notice: Multiplication of disjoint cycles is commutative, so the order of the factors (1 6) and (2 5 3) is not important. But this may not be true for transpositions, check α_4 and α_5 in Example 3.11.

According to Corollary 3.1:

$$\alpha = (1\ 6)(2\ 5\ 3) = (1\ 6)(2\ 3)(2\ 5)$$

Definition 3.10

Let $\pi \in S_n$. If π is a product of an **even number** of transpositions (互換), then π is called an **even permutation** (偶置換); otherwise π is called an **odd permutation** (奇置換).

Corollary 3.2

Let $\pi \in S_n$ be a k -**cycle**. Then π is an **even permutation** if and only if k is **odd**.

Proof.

Let $\pi = (1\ 2\ \cdots\ k)$. Then $\pi = (1\ k) \circ (1\ k-1) \circ \cdots \circ (1\ 2)$, i.e., π is a product of $k-1$ transposition. If π is an even permutation then $k-1$ is even, so k is odd. On the other hand, if k is odd, then $k-1$ is even, so π is an even permutation. This completes the proof.

3.2 Permutation Group (置換群)

Even Permutation & Odd Permutation

Example 3.11

From Example 3.7,

(1) use cycle notation to express the permutation

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

(2) determine the permutations are even permutation or odd permutation.

Solution:

$$(1) \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2) \quad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$$

(2) $\alpha_5 = (1\ 3\ 2) = (1\ 2)(1\ 3)$ is even permutation.

$\alpha_4 = (1\ 2\ 3) = (1\ 3)(1\ 2)$ is even permutation.

Notice: $\alpha_2 = (1\ 2)$ $\alpha_3 = (1\ 3)$

$$(\alpha_2 \circ \alpha_3)(1) = \alpha_2(\alpha_3(1)) = \alpha_2(3) = 3 \quad (\alpha_3 \circ \alpha_2)(1) = \alpha_3(\alpha_2(1)) = \alpha_3(2) = 2$$

$$(\alpha_2 \circ \alpha_3)(2) = \alpha_2(\alpha_3(2)) = \alpha_2(2) = 1 \quad (\alpha_3 \circ \alpha_2)(2) = \alpha_3(\alpha_2(2)) = \alpha_3(1) = 3$$

$$(\alpha_2 \circ \alpha_3)(3) = \alpha_2(\alpha_3(3)) = \alpha_2(1) = 2 \quad (\alpha_3 \circ \alpha_2)(3) = \alpha_3(\alpha_2(3)) = \alpha_3(3) = 1$$

Review for Lecture 3

- Definition of Group (群)
- Commutative Group (可換群) or Abelian Group (アーベル群)
- Finite Group (有限群) and Infinite Group (無限群)
- Order (位数) of a group
- Permutation Group (置換群)
- Symmetry Group (対称群)
- k -cycle & Transposition (互換)
- Even Permutation (偶置換) and Odd Permutation (奇置換)

Assignment

Please Check <https://github.com/uoaworks/Applied-Algebra>

References

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