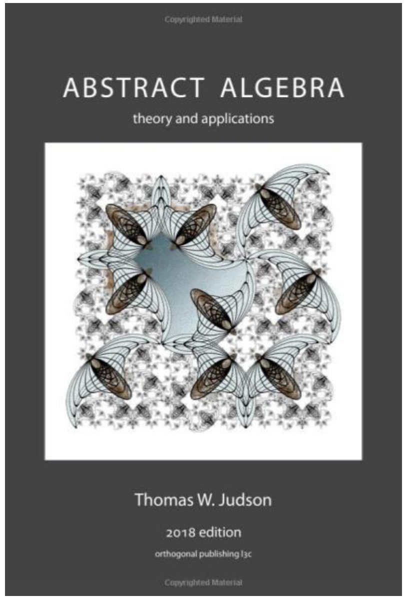
AY2019 Q2

MA08

Applied Algebra, 応用代数

(Abstract Algebra, 抽象代数学)

Class Information



Lectures: Period 1, 2 Monday and Period 1, 2 Thursday

Grades: 20% Assignments (10, Attendance >2/3)

30% Middle Examination

50% Final Examination

+5 Bonus Points from Quizzes

Office hours: Period 3, 4 Monday and Thursday; @研究棟#247C

Textbook: (教科書) [Eng] **Abstract Algebra Theory and Applications, 2018,** Thomas W. Judson etc.

Free online: http://abstract.ups.edu/download/aata-20180801-sage-8.2.pdf

参考書: [Eng] **Introduction to Abstract Algebra, 2007,** D. S. Malik, John N. Mordeson, M.K. Sen

Free online: https://www.researchgate.net/publication/238669835_MTH_581-582_Introduction_to_Abstract_Algebra

[Jap] 工学のための応用代数, 1999, 杉原 厚吉, 今井 敏行, 共立出版

Hint for Middle & Final Exams

90% $\left\{ \begin{array}{l} \textbf{Lecture Slides (Example, Definition, Theorem)} \\ \textbf{Assignments} \end{array} \right.$

10% Other questions

About Lecture Notes & Assignments

Please Check https://github.com/uoaworks/Applied-Algebra

before and after each lecture for slides and assignments.

What we will cover

Syllabus on course website

01 promenade to algebraic system

02 remainder of integer and polynomial

03 group(1): Lagrange theorem

04 group(2): quotient group and homomorphism theorem

05 group(3): analysis of group structure

06 applications of group

TEST 07 Mid-exam

08 ring and field(1): ideal, quotient ring

09 ring and field(2): polynomial ring

10 ring and field(3): reversible

11 application(1): quotient field and operator theory

12 ring and field(4): extension of field

13 application(2): M-sequence random number generation

14 application(3): error correct coding

Prerequisites

MA03 **Calculus I**

*MA01 Linear Algebra I

Notice: * is optional.

Important related courses:

NS03 Quantum Mechanics

TEST

Lecture 1

Promenade to Algebraic System

0.1 Why abstract algebra

0.1 Why Abstract Algebra

- **Abstract Algebra** is the study of **algebraic structures (代数構造)**.
- **Algebraic structures** include **groups (群), rings (環), fields (体),** modules, vector spaces, lattices, and algebras.

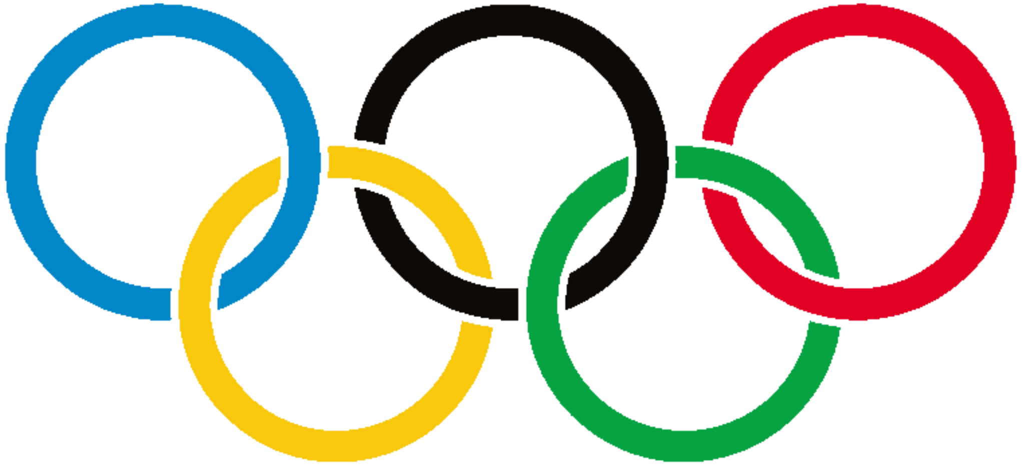
0.1 Why Abstract Algebra

The first time we hear the names of group, ring, field ...

Group (群)

Ring (環)

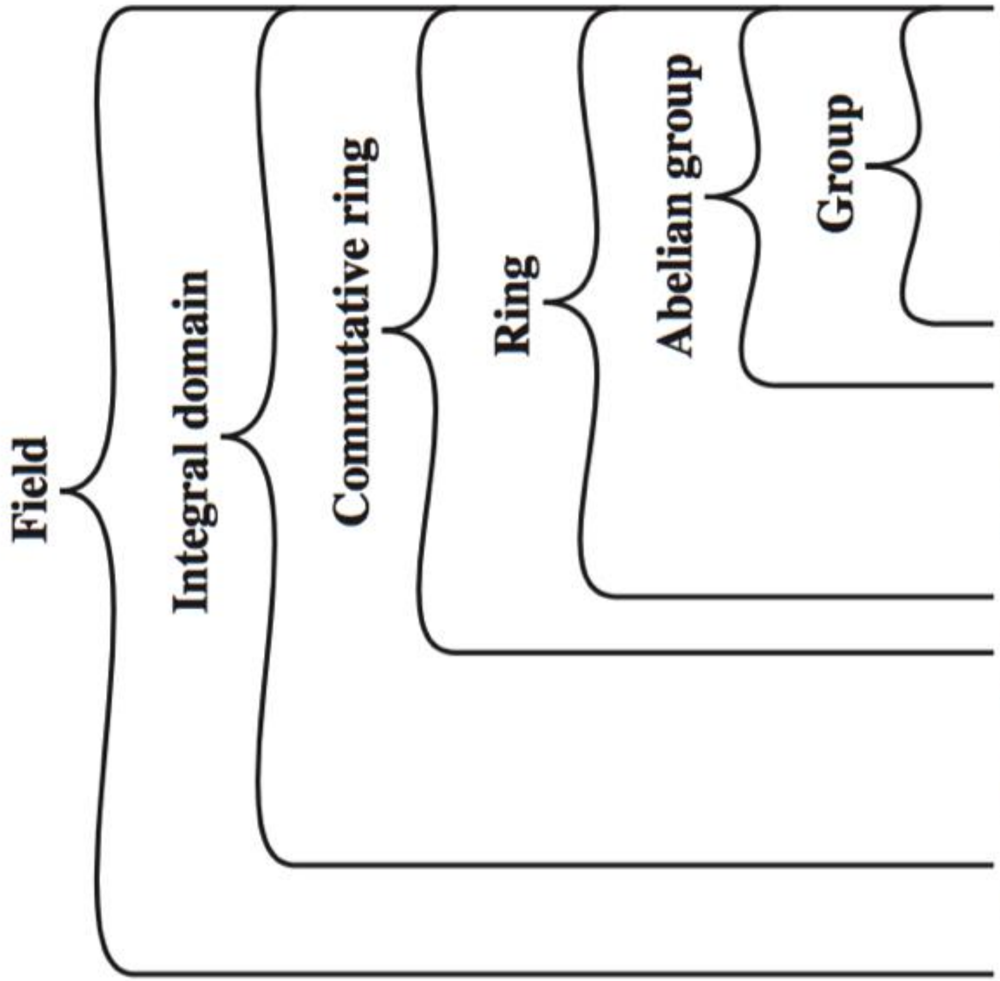
Field (体)



0.1 Why Abstract Algebra

理論の土台となる約束事。前提事項。
（証明の必要は無い。）

Algebra Structure (代数構造)	Component Axioms (公理)	Explanation of Axiom



(A1) Closure under addition:	If a and b belong to S, then $a + b$ is also in S
(A2) Associativity of addition:	$a + (b + c) = (a + b) + c$ for all a, b, c in S
(A3) Additive identity:	There is an element 0 in R such that $a + 0 = 0 + a = a$ for all a in S
(A4) Additive inverse:	For each a in S there is an element $-a$ in S such that $a + (-a) = (-a) + a = 0$
(A5) Commutativity of addition:	$a + b = b + a$ for all a, b in S
(M1) Closure under multiplication:	If a and b belong to S, then ab is also in S
(M2) Associativity of multiplication:	$a(bc) = (ab)c$ for all a, b, c in S
(M3) Distributive laws:	$a(b + c) = ab + ac$ for all a, b, c in S $(a + b)c = ac + bc$ for all a, b, c in S
(M4) Commutativity of multiplication:	$ab = ba$ for all a, b in S
(M5) Multiplicative identity:	There is an element 1 in S such that $a1 = 1a = a$ for all a in S
(M6) No zero divisors:	If a, b in S and $ab = 0$, then either $a = 0$ or $b = 0$
(M7) Multiplicative inverse:	If a belongs to S and a 0, there is an element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Figure 4.2 Groups, Ring, and Field

0.1 Why Abstract Algebra

The **three main areas** that were to give rise to **group theory** are:

- **geometry** at the beginning of the 19th Century,

  # studied properties invariant

- **number theory** at the end of the 18th Century,

  # studied modular arithmetic

- **the theory of algebraic equations** at the end of the 18th Century

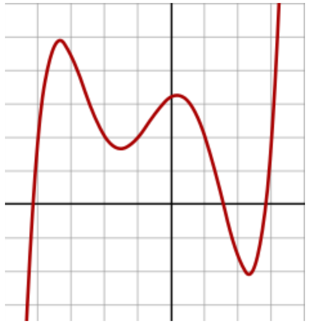
  # lead to the study of permutations (置換).

Reference: https://www-history.mcs.st-and.ac.uk/HistTopics/Development_group_theory.html

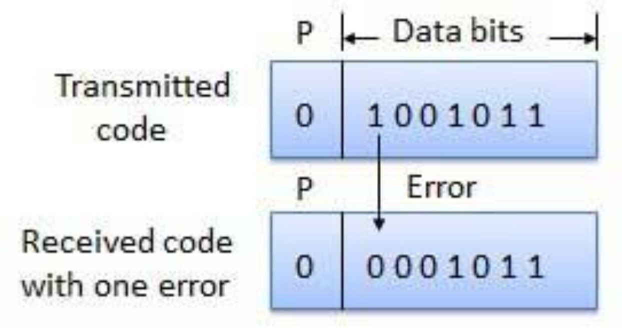
0.1 Why Abstract Algebra

Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$



Error Correction



Rubik's Cube

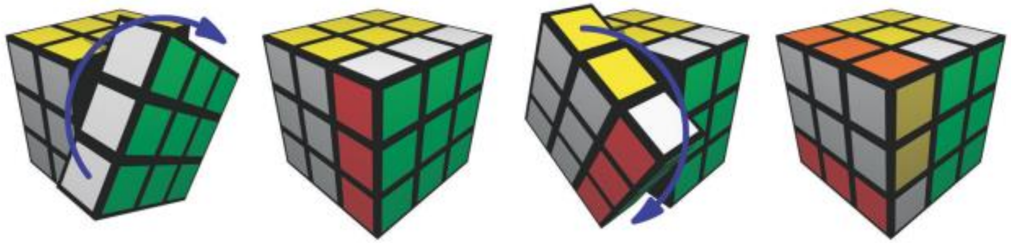
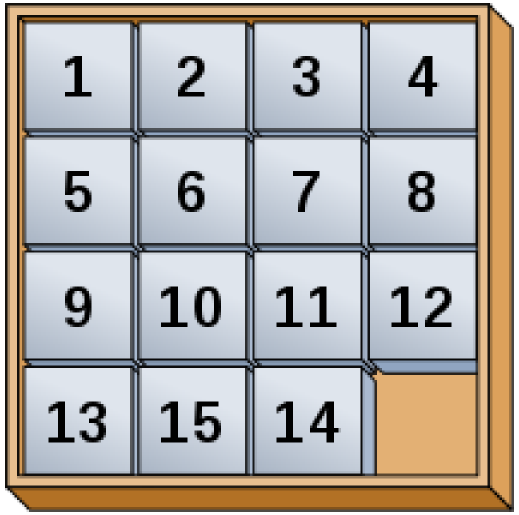
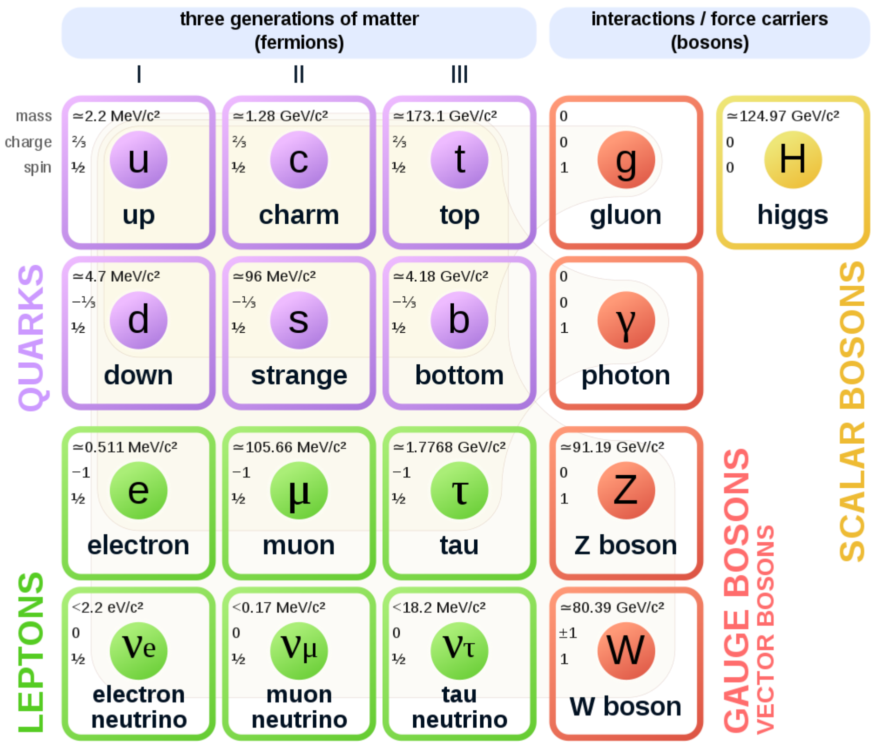


Figure 1.2. The leftmost cube shows the green face rotating 90 degrees clockwise; the next cube shows the result of that move. The third cube shows the white face rotating 90 degrees clockwise; the final cube shows the result of that move.

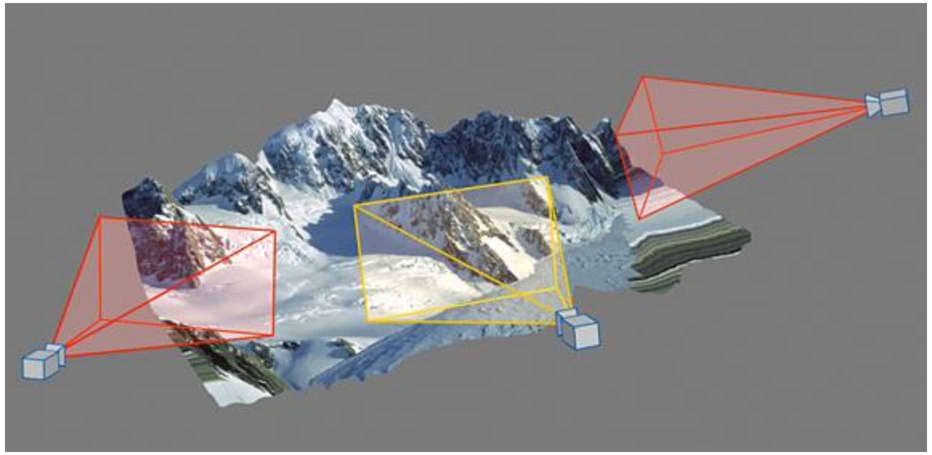
Fifteen Puzzle



Standard Model of Elementary Particles



3D Mapping in Computer Vision



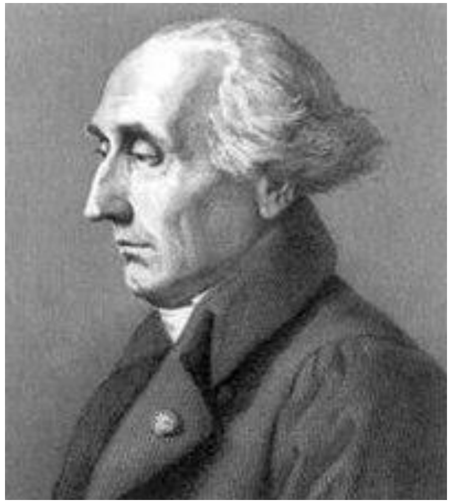
0.1 Why Abstract Algebra

Birth of Group Theory (群論)

General Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

They reveal Non-existence of a formula that is comprised of these coefficients, arithmetic operations and roots for general quantic equation



Joseph-Louis Lagrange (ラグランジュ)
France, 1736 ~ 1813

Évariste Galois (ガロア)
France, 1811 ~ 1832

Niels Henrik Abel (アーベル)
Norway, 1802 ~ 1829

0.1 Why Abstract Algebra

Lagrange **firstly studied Permutations** (置換) in his 1770 paper on the theory of algebraic equations. Lagrange's main object was to find out why cubic (三次) and quartic (四次) equations (方程式) could be solved algebraically.

Cauchy played a major role in **developing the theory of permutations**. His first paper on the subject was in 1815 but at this stage Cauchy is motivated by permutations of roots of equations.

Abel, in 1824, **gave the first accepted proof of the insolubility of the quantic equation (五次方程式)**, and he used the existing ideas on permutations of roots but little new in the development of group theory.

Galois in 1831 was the first to really understand that the algebraic solution of an equation **was related to the structure of a group** le groupe of permutations related to the equation. By 1832 **Galois had discovered that special subgroups (部分群) (now called normal subgroups) are fundamental**. He calls **the decomposition of a group into cosets (剰余類) of a subgroup a proper decomposition if the right and left coset decompositions coincide**. Galois then shows that the non-abelian simple group of smallest order has order 60.

Galois' work was not known until Liouville published Galois' papers in 1846. Liouville saw clearly the connection between Cauchy's theory of permutations and Galois' work. However Liouville failed to grasp that the importance of Galois' work lay in the group concept.

Reference: https://www-history.mcs.st-and.ac.uk/HistTopics/Development_group_theory.html

0.2 A Short Note on Proofs

Introducing Axioms

In studying **abstract mathematics**, we take what is called an **axiomatic** approach; that is, we **take a collection of objects** S and **assume some rules about their structure**.

These rules are called **axioms (公理).** Using the axioms for S, we wish to derive other information about S by using logical arguments.

Statements

A **statement (言明, 命題)** in logic or mathematics is **an assertion (主張)** that **is either true (真) or false (偽)**. Consider the following examples:

1. ~~$3 + 56 - 13 + 8/2.$~~
2. All cats are black.
3. $2 + 3 = 5.$
4. $2x = 6$ exactly when $x = 4.$
5. If $ax^2 + bx + c = 0$ and $a \neq 0$, then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
6. ~~$x^3 - 4x^2 + 5x - 6.$~~

All examples except the first and last examples are statements, and **must be either true or false.**

Hypothesis and Conclusion

- **A mathematical proof is nothing more than a convincing argument about the accuracy of a statement.**

Let us examine different types of statements. A statement could be as simple as " $10/5 = 2$"

however, **mathematicians are usually interested in more complex statements** such as **"If p, then q," where p and q are both statements**.

Here p is called the **hypothesis** and q is known as the **conclusion**.

0.2 A Short Note on Proofs

Proof

Consider the following **statement**: **If** $ax^2 + bx + c = 0$ **and** $a \neq 0$, **then**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here the **hypothesis** is $ax^2 + bx + c = 0$ **and** $a \neq 0$; the **conclusion** is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that the **statement** says nothing about whether or not the hypothesis is true. However, if this entire statement is true and we can show that $ax^2 + bx + c = 0$ with $a \neq 0$ is true, then the conclusion *must* be true. A proof of this statement might simply be a series of equations:

Proof:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \qquad \because a \neq 0 \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}$$

Proposition and Theorem

If we can prove a statement true or false, then that statement is called a proposition (命題).

A proposition of major importance is called a theorem (定理).

Sometimes instead of proving a theorem or proposition all at once, **we break the proof (証明) down into modules; that is, we prove several supporting propositions, which are called lemmas (補題),** and **use the results of these propositions to prove the main result.**

If we can prove a proposition or a theorem, we will often, with very little effort, be able to derive other related propositions called corollaries (系).

What you will learn in Lecture 1

1.1 Sets (集合)

1.2 Relation (関係), Mapping (写像) and Permutation (置換)

1.3 Division (除法), Quotient (商), Remainder (剰余), Great Common Divisor (最大公約数)

1.4 Equivalence Classes (同値類)

1.1 Sets (集合)

1.1 Sets (集合)

We will denote sets by capital letters, such as X.

- A ***set*** X is a well-defined collection of objects
(集合とは、well-defined なモノの集まりのことである。);
that is, it is defined that we can determine for any given object x **whether or not** x belongs to the set X.
- The objects that belong to a set are called its ***elements* (要素)** or ***members***.

Given a set X, we use the notation

$x \in X$ to mean x is an element of X

$x \notin X$ to mean x is **not** an element of X

For example, for the set $X = \overbrace{\{1, 2, 3\}}^{\text{collection of objects}}$, we have $1 \in X$ and $4 \notin X$.

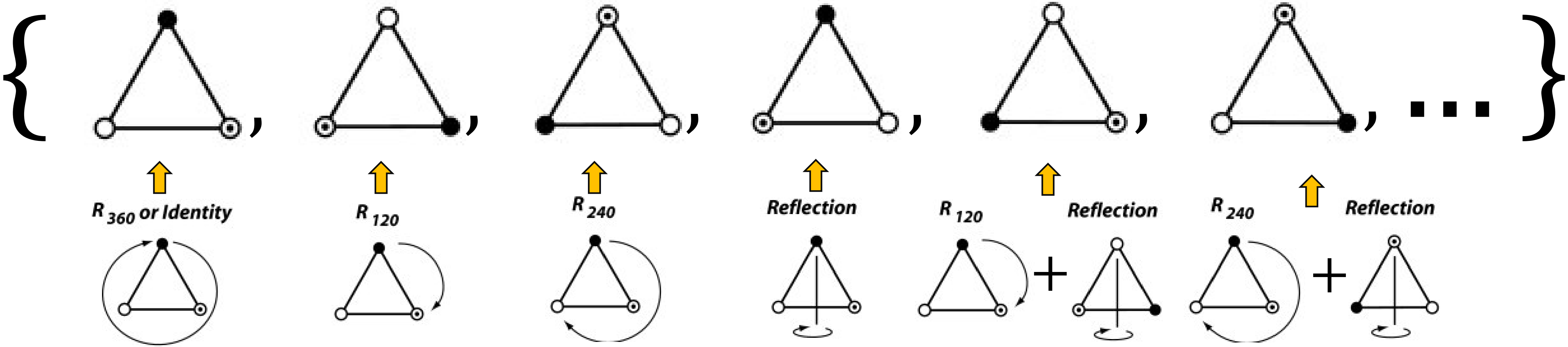
object, i.e. element or member

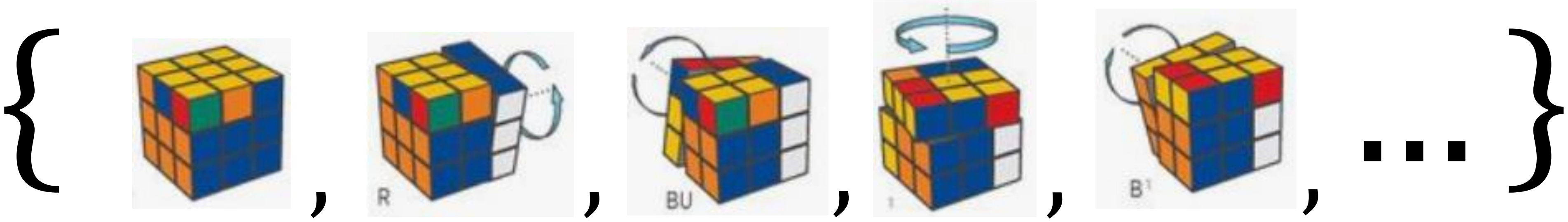
1.1 Sets (集合)

Example of Sets

$\{1, 2, 3, 4, 5, \ldots\}$

$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$





1.1 Sets (集合)

“well-defined”

- A ***set*** X is a **well-defined** **collection of objects**;
  that is, it is defined that we can determine for any given object x **whether or not** x belongs to the set X.

A set X is **well-defined (ちゃんと定義になっている)**, meaning that if X is a set and x is some object, then either x is definitely in X, denoted by $x \in X$, or x is definitely not in X, denoted by $x \notin X$.

Thus, we should never say, “Consider the set S of **some** positive integers,”, because it is **not definite** whether $2 \in S$ or $2 \notin S$.
On the other hand, we can consider the set of all **even** positive integers, because every positive integer is definitely either even or odd. Thus $2 \in S$ and $3 \notin S$.

Notation of Set

A set is usually specified by

(1) listing all of its elements inside a pair of braces

$$X = \{x_1, x_2, \ldots, x_n\}$$

for a set containing elements $x_1, x_2, \ldots, x_n$;

(2) stating the characterizing property that determines whether or not an object x belongs to the set

$$X = \{x \mid x \text{ satisfies } P(x)\} \quad \text{or} \quad X = \{x : x \text{ satisfies } P(x)\}$$

if each x in X satisfies a certain property $P(x)$.

The notation $X = \{x \mid x \text{ satisfies } P(x)\}$ is often called "set-builder notation"

Important Sets

If S is the set of even positive integers, we can describe S by writing either

$$S = \{2, 4, 6, \dots\} \text{ or } S = \{x \mid x \text{ is an even integer and } x > 0\}$$

We write $2 \in S$ when we want to say that 2 is in the set S, and $3 \notin S$ to say that 3 is not in the set S.

- Some of the more **important sets** that we will consider are the following:

$$\mathbb{N} = \{n \mid n \text{ is a natural number}\} = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{z \mid z \text{ is an integer}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Q} = \{r \mid r \text{ is a rational number}\} = \{p/q \mid p, q \in \mathbb{Z} \text{ where } q \neq 0\}$$

$$\mathbb{R} = \{x \mid x \text{ is a real number}\}$$

$$\mathbb{C} = \{z \mid z \text{ is a complex number}\}$$

Subset & Proper Subset

A set A is said to be a **subset (部分集合)** of a set S if every element of A is an element of S. In this case, we write $A \subseteq S$ and say that A is contained in S.

If $A \subseteq S$, but $A \neq S$, then we write $A \subset S$ and say that A is properly contained in S. Or we say A is a **proper subset (真部分集合)** of S.

As an example, we have $\{1,2,3\} \subseteq \{1,2,3\}$ and $\{1,2\} \subset \{1,2,3\}$.

Equal Sets

If every element of A is a element of B and every element of B is a element of A, then we say that A and B are the **same** or **equal**.

In this case, we write $A = B$.

Then, we can easily have the following theorem

Theorem 1.1

Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Empty Set

The **empty set (空集合)** or **null set** is the **set with no elements.**

- We usually denote the **empty set** by $\emptyset$.
- For any set A, we have $\emptyset \subseteq A$.

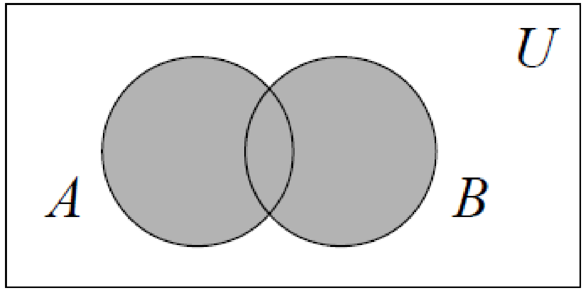
Union & Intersection

Definition 1.1

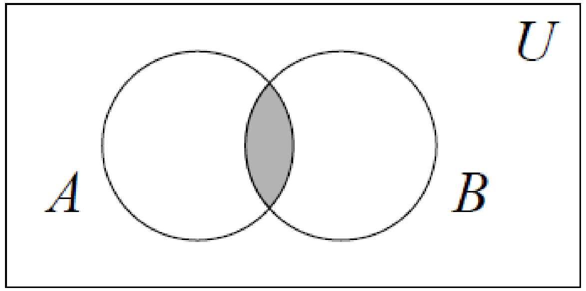
The **union (和集合)** of two sets A and B, written $A \cup B$, is defined to be the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Definition 1.2

The **intersection (積集合, 共通部分)** of two sets A and B, written $A \cap B$, is defined to be the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.



$A \cup B$



$A \cap B$

Union & Intersection

Example 1.1

Let A be the set $\{1, 2, 3, 4\}$ and B be the set $\{3, 4, 5, 6\}$.

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{3, 4\}$

If C is the set $\{5, 6\}$, then $A \cup C = \{1, 2, 3, 4, 5, 6\}$ while $A \cap C = \emptyset$.

Notice:

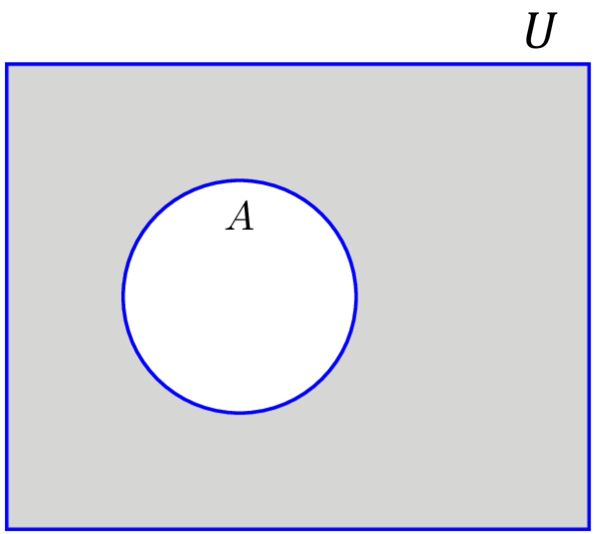
Two sets A and B are called **disjoint** exactly when $A \cap B = \emptyset$.

Complement of Set

Definition 1.3

For any set $A \subset U$, where U is **universal set (全体集合)**, we define the ***complement (補集合)*** of A, denoted by A', to be the set

$$A' = \{x \mid x \in U \text{ and } x \notin A\}.$$

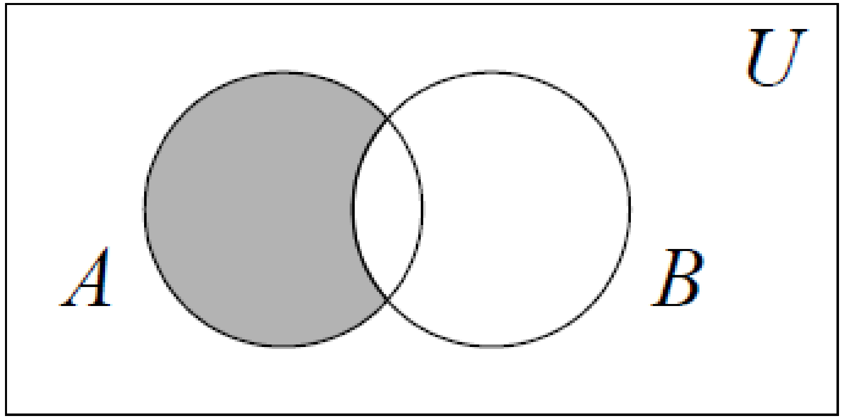


Complement of Set

Definition 1.4

Given two sets A and B, the **relative complement** of B in A, denoted by *the set difference* $A\backslash B$, is the set

$$A\backslash B = A \cap B' = \{x \mid x \in A, \ \text{but } x \notin B\}.$$



$A\backslash B$

Complement of Set

Example 1.2

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A\backslash \mathrm{B} = \{1, 2\}$

Remark

Let P and Q be statements.

Throughout the class, we will encounter questions in which we will be asked to show that P if and only if Q;

that is, show that statement P is true if and only if statement Q is true.

In situations like this,

- we first assume that statement P is true and show that statement Q is true.
- Then we assume that statement Q is true and show that statement P is true.

The statement P if and only if Q is also equivalent to the statement: if P, then Q, and if Q, then P.

1.2 Relation (関係), Mapping (写像) and Permutation (置換)

Definition 1.5

Let A and B be **nonempty sets (空でない集合)** and $x \in A$, $y \in B$.
The **Cartesian cross product (Cartesian product, 直積集合)** of A and B, written $A \times B$, is defined to be the set

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

where (x, y) is called the **ordered pair (順序対)**.

Example 1.3

If $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \emptyset$,
then $A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$.
and $A \times C = \emptyset$

Cartesian Product

We define the **Cartesian product of** n **sets** to be

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid \underline{a_i \in A_i \; for \; i = 1, \dots, n}\}.$$

$a_1 \in A_1, a_2 \in A_2, \dots,$ and $a_n \in A_n$

If $A_1 = A_2 = \cdots = A_n$, we often write A^n for $A \times \cdots \times A$ (where *A* would be written n times).

For example, the set $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ consists of all of 3-tuples of real numbers.

Definition 1.6

A **binary relation (二項關係)** or simply a **relation** $\mathcal{R}$ from a set A to a set B is a **subset** of $A \times B$.

Let $\mathcal{R}$ be a relation from a set A into a set B.

If $(x, y) \in \mathcal{R}$, we write $x\mathcal{R}y$ or $\mathcal{R}(x) = y$.

If $x\mathcal{R}y$, then sometimes we say that x is related to y (or y is in relation with x) with respect to $\mathcal{R}$ or simply x is related to y.

Mapping or Function

Definition 1.7

Let A and B be nonempty sets. A special type of **relation** $f \subset A \times B$ from A into B is called a **function (関数)** (or **mapping (写像)**) from A into B if

(i) $\mathcal{D}(f) = A$, i.e. the ***domain* (定義域)** of f is the set A;

(ii) for all $(x, y), (x', y') \in f$, we have that $x = x'$ implies $y = y'$.

When (ii) is satisfied by a relation f, we say that f is **well defined** or **single-valued**.

Domain & Range

We can also say that for every element in A, f assigns a unique element in B. We usually write $f: A \to B$ or $A \xrightarrow{f} B$.
Instead of writing down ordered pairs $(a, b) \in A \times B$, we write $f(a) = b$ or $f: a \mapsto b$.

The set A is called the ***domain (定義域)*** of f, denote by $\mathcal{D}(f) = A$.
The set $f(A)$ is called the ***range (値域)*** or ***image*** of f, defined by

$$f(A) = \{f(a) \mid a \in A\} \subset B.$$

Notice that here $f(A)$ is a subset of B.

We can think of the elements in the function's domain as input values and the elements in the function's range as output values.

Domain & Range

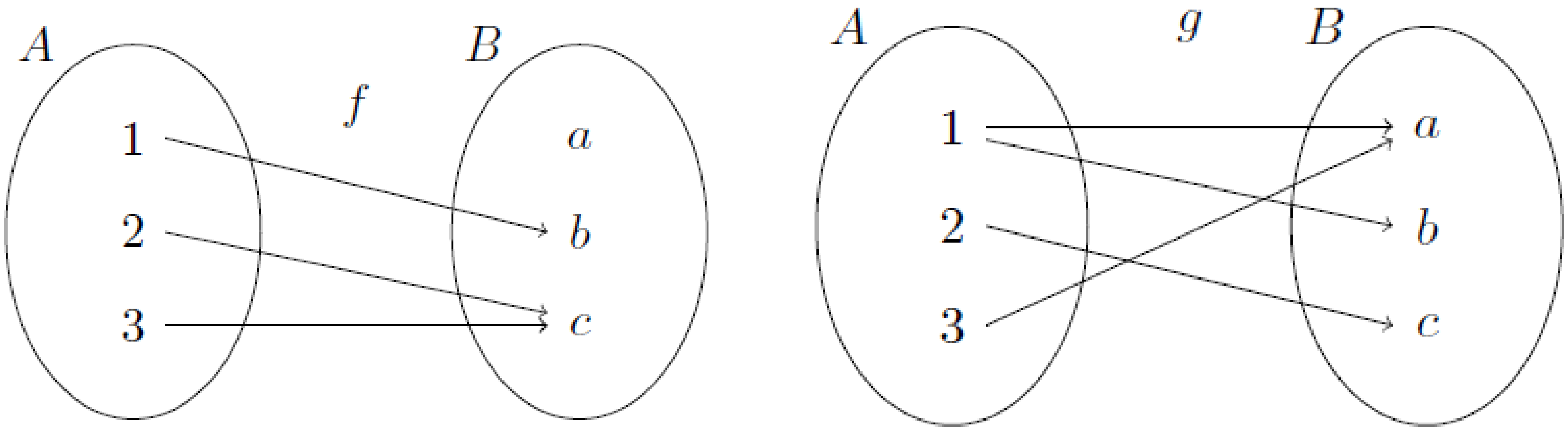


Figure 1.1 Mappings and relations

Example 1.4

Suppose $A = \{1,2,3\}$, and $B = \{a, b, c\}$. In Figure 1.1 we define relations f and g from A to B. The relation f is a mapping, but g is not, because $1 \in A$ **is not assigned to a unique element in** B; that is, $g(1) = a$ and $g(1) = b$.

Definition 1.8

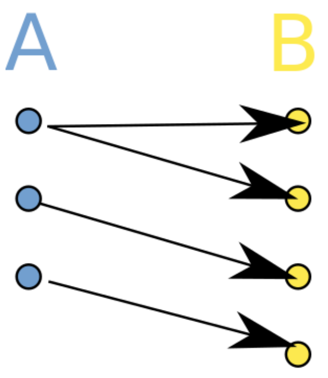
Let f be a function from a set A to a set B. Then

(i) f is called ***one-to-one (一対一)*** or ***injective (単射)*** if for all $x, x' \in A$, we have that $f(x) = f(x')$ implies $x = x'$.

(ii) f is called ***onto*** or ***surjective (全射)*** B (or f maps A onto B) if $f(A) = B$.

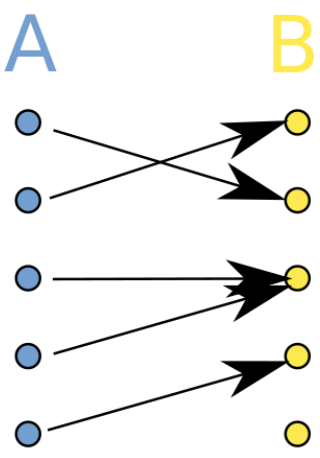
(iii) A map is called ***bijective (全単射、あるいは双射)*** if both ***one-to-one*** and ***onto***.

1.2 Relation, Mapping and Permutation



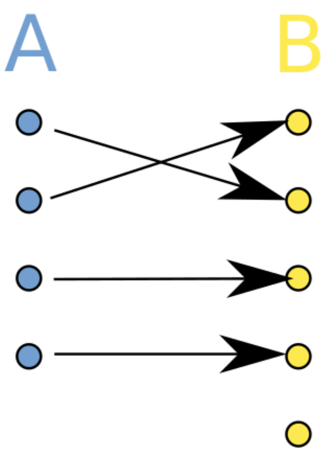
NOT a Function

A has many B



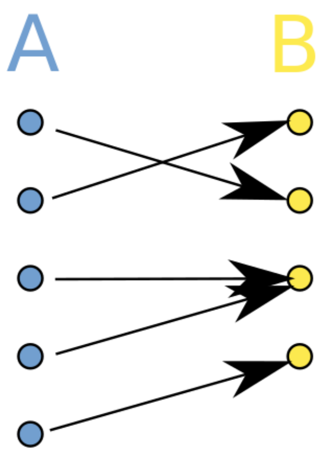
General Function

B can have many A



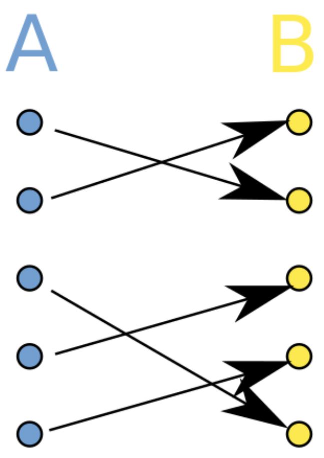
Injective
(not surjective)

B can't have many A



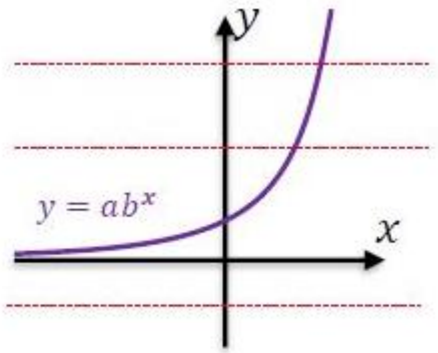
Surjective
(not injective)

Every B has some A

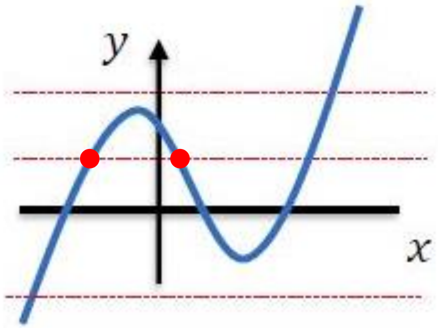


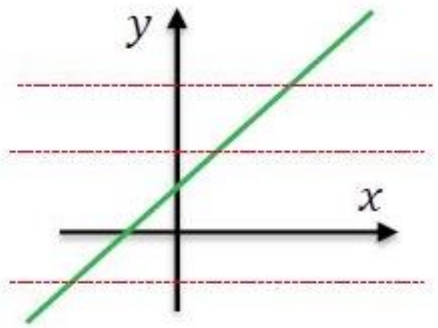
Bijective
(injective, surjective)

A to B, perfectly



Negative y is not used





Example 1.5

Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$. For the following relation between A and B given as a subset of $A \times B$, decide whether it is a function mapping A into B. If it is a function, decide whether it is ***one-to-one*** and whether it is ***onto*** B.

	is function	is ***one-to-one***	is ***onto*** B
(a) {(1,4),(2,4),(3,6)}	Yes	No	No
(b) {(1,4),(2,6),(3,2)}	Yes	Yes	Yes
(c) {(1,4),(1,6),(2,2)}	No		

Composition of Mapping

Given two functions, we can construct a new function by using the range of the first function as the domain of the second function.

Definition 1.9

Let $f: A \to B$ and $g: B \to C$ be mappings.
Define the ***composition of mapping (合成写像)*** of f and g from A to C, by $(g \circ f)(x) = g(f(x))$.

Composition of Mapping

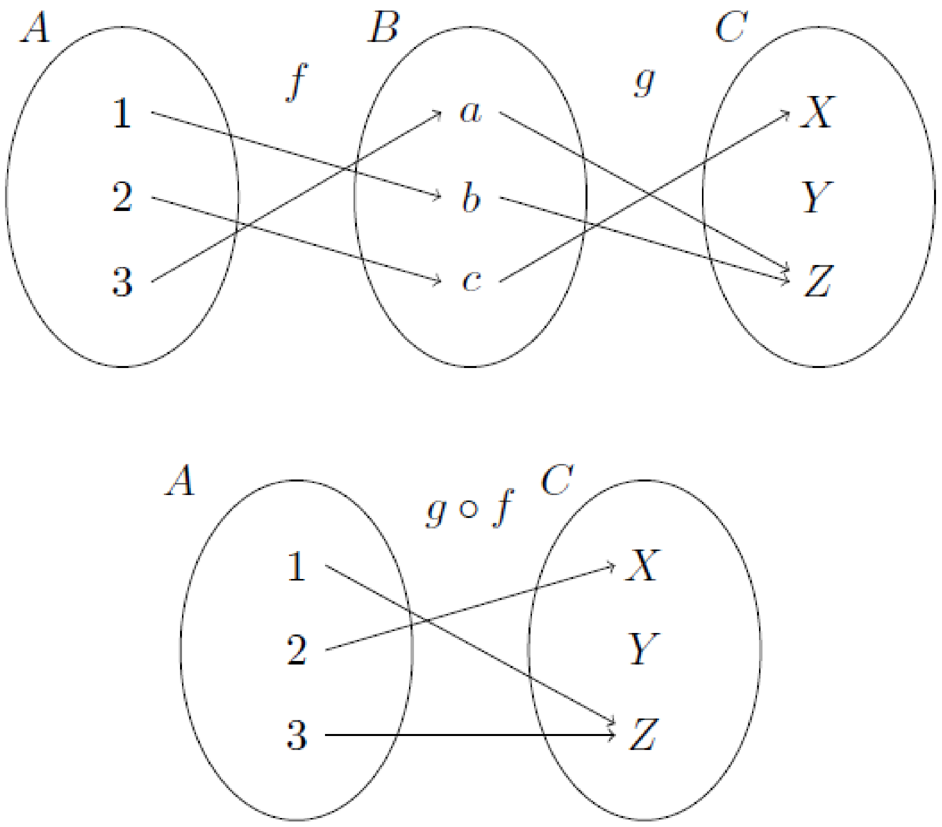


Figure 1.2 Composition of maps

Example 1.6 Consider the functions $f: A \to B$ and $g: B \to C$ that are defined in Figure 1.2 (top). The composition of these functions, $g \circ f: A \to C$, is defined in Figure 1.2 (bottom).

Composition of Mapping

Example 1.7

Let $f(x) = x^2$ and $g(x) = 2x + 5$. Then

$$(f \circ g)(x) = f(g(x)) = (2x + 5)^2 = 4x^2 + 20x + 25$$

and

$$(g \circ f)(x) = g(f(x)) = 2x^2 + 5$$

In general, order (順序) makes a difference; that is, in most cases

$f \circ g \neq g \circ f$.

Composition of Mapping

Example 1.8

Sometimes it has the case that $f \circ g = g \circ f$. Let $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$. Then

$$(f \circ g)(x) = f\big(g(x)\big) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

and

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^3) = \sqrt[3]{(x^3)} = x$$

Example 1.9

Suppose that $S = \{1, 2, 3\}$. Define a map $\pi: S \to S$ by

$\pi(1) = 2, \pi(2) = 1, \pi(3) = 3.$

This is a *bijective* map. An alternative way to write π is

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

*This notation is due to Cauchy and is called the **two-row notation**.

For any set S, a ***one-to-one*** and ***onto*** mapping $\pi: S \to S$ is called a ***permutation* (置换)** of S.

Identity Mapping & Invertible

If S is any set, we will use id_S or id to denote the ***identity mapping* (恒等写像)** from S to itself. Define this map by $id_S(s) = s$ for all $s \in S$.

Definition 1.10

Let A and B be sets and $f: A \to B$.
(i) f is called **left invertible (左可逆)** if there exists $g: B \to A$ such that $g \circ f = id_A$.
(ii) f is called **right invertible (右可逆)** if there exists $h: B \to A$ such that $f \circ h = id_B$.
A function $f: A \to B$ is called **invertible (可逆)** if f is **both left and right invertible**.

Invertible

Example 1.10

The function $f(x) = x^3$ has **inverse** $f^{-1}(x) = g(x) = \sqrt[3]{x}$.

Because **left invertible**

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^3) = \sqrt[3]{(x^3)} = x = id_X$$

and **right invertible**

$$(f \circ g)(x) = f\big(g(x)\big) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x = id_X$$

Invertible

Theorem 1.2

A mapping is **invertible** (可逆) if and only if it is *bijective* (全単射), i.e. both *one-to-one* and *onto*.

1.3 Division (除法), Quotient (商), Remainder (剰余), Great Common Divisor (最大公約数)

Theorem 1.3 Division Algorithm

Let x and y be integers, with $y \neq 0$. Then there exist unique integers q and r such that $x = yq + r$ where $0 \leq r < |y|$.

The integer q is called the **quotient** of x and y on dividing x by y

and

the integer r is called the **remainder** of x and y on dividing x by y.

Definition 1.11

Let $x, y \in \mathbb{Z}$ with $x \neq 0$. Then x is said to divide y or x is a **divisor (約数)** (or **factor (因子)**) of y, written $x|y$, provided there exists $q \in \mathbb{Z}$ such that $y = qx$. When x does not divide y, we sometimes write $x \nmid y$.

Definition 1.12

Let $x, y \in \mathbb{Z}$. A nonzero integer c is called a **common divisor (公約数)** of x and y if $c|x$ and $c|y$.

Definition 1.13

A nonzero integer d is called a **greatest common divisor** (gcd) (最大公約数) of the integers x and y if
(i) $d|x$ and $d|y$,
(ii) for all $c \in Z$ if $c|x$ and $c|y$, then $c|d$.

Theorem 1.4

Let $x, y \in \mathbb{Z}$ with either $x \neq 0$ and $y \neq 0$. Then x and y have a positive greatest common divisor $\gcd(x, y) = d$. Moreover, there exist integers $s, t \in \mathbb{Z}$ such that

$$\gcd(x, y) = d = sx + ty.$$

Example 1.11

Consider the gcd(45, 126).

$$126 = 2 \cdot 45 + 36$$
$$45 = 1 \cdot 36 + 9$$
$$36 = 4 \cdot 9 + 0$$

Thus, $\gcd(45, 126) = 9$

Also,

$$\begin{aligned} 9 &= 45 - 1 \cdot 36 \\ &= 45 - 1 \cdot [126 - 2 \cdot 45] \\ &= 3 \cdot 45 + (-1) \cdot 126. \end{aligned}$$

Here s = 3 and t = −1.

Definition 1.14

(i) An integer $p > 1$ is called **prime (素数)** if the only **divisors** of p are ± 1 and $\pm p$.
(ii) Two integers x and y are called **relatively prime (互いに素)** if $\gcd(x, y) = 1$.

1.4 Equivalence Classes (同値類)

1.4 Equivalence Classes (同值類)

Definition 1.15

Let $\mathcal{R}$ be a binary relation on a set X, i.e. $\mathcal{R} \subset X \times X$. Then $\mathcal{R}$ is called
(i) ***reflexive (反射的)*** if for all $x \in X$, $(x, x) \in \mathcal{R}$ (i.e. $x\mathcal{R}x$);
(ii) ***symmetric (对称的)*** if for all $x, y \in X$, $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$ (i.e. $x\mathcal{R}y$ implies $y\mathcal{R}x$);
(iii) ***Transitive (推移的)*** if for all $x, y, z \in X$, (x, y) and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$ (i.e. $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$).

Definition 1.16

A binary relation $\mathcal{R}$ on a set X is called an **equivalence relation (同值關係)** on X if $\mathcal{R}$ is **reflexive, symmetric, and transitive**.

Definition 1.17

Let $\mathcal{R}$ be an **equivalence relation** on a set A. For all $x \in A$, let $[x]$ denote the set

$$[x] = \{y \in A \mid y\mathcal{R}x\}.$$

The set $[x]$ is called the equivalence class (with respect to $\mathcal{R}$) of x.

Theorem 1.5

Let $\mathcal{R}$ be an equivalence relation on the set A. Then
(i) for all $x \in A$, $[x]\emptyset$,
(ii) if $y \in [x]$, then $[x] = [y]$, where $x, y \in A$,
(iii) for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$,
(iv) $A = \cup_{x \in A} [x]$, i.e., A is the union of all equivalence classes with respect to $\mathcal{R}$.

1.4 Equivalence Classes (同値類)

Let r and s be two integers and suppose that $n \in \mathbb{N}$.

We say that r is ***congruent (合同)*** to s ***modulo (剰余演算)*** n, or r is **congruent** to s **mod** n, if $r - s$ is evenly divisible by n; that is, $r - s = nk$ for some $k \in \mathbb{Z}$.

In this case we write $r \equiv s \pmod{n}$.

Example 1.12

$\mathbf{41 \equiv 17}$ **(mod** $\mathbf{8}$**) since** $\mathbf{41 - 17 = 24}$ **is divisible by** $\mathbf{8}$**.**

We claim that congruence modulo n forms an **equivalence relation** of $\mathbb{Z}$.

Certainly any integer <u>r **is equivalent to itself**</u> since <u>$r - r = 0$</u> is divisible by <u>n</u>.

1.4 Equivalence Classes (同値類)

We will now show that the relation is <u>symmetric</u>. If $r \equiv s \pmod{n}$, then $r - s = -(s - r)$ is divisible by n. So $s - r$ is divisible by n and $s \equiv r \pmod{n}$. Now suppose that $r \equiv s \pmod{n}$ and $s \equiv t \pmod{n}$. Then there exist integers k and m such that $r - s = kn$ and $s - t = mn$.

To show <u>transitivity</u>, it is necessary to prove that $r - t$ is divisible by n. However,

$$r - t = r - s + s - t = kn + mn = (k + l)n$$

and so $r - t$ is divisible by n.

1.4 Equivalence Classes (同値類)

If we consider the equivalence relation established by the integers modulo 3, then

$$[0] = \{\dots, -3, 0, 3, 6, \dots\}$$
$$[1] = \{\dots, -2, 1, 4, 7, \dots\}$$
$$[2] = \{\dots, -1, 2, 5, 8, \dots\}$$

Notice that $[0] \cup [1] \cup [2] = \mathbb{Z}$ and also that the sets are disjoint. The sets $[0]$, $[1]$, and $[2]$ form a partition of the integers.

Notice:
The integers modulo n are **a very important example** in the **study of abstract algebra** and will become quite useful in our investigation of **various algebraic structures** such as **groups** and **rings**.

Review for Lecture 1

- Sets
- Relation, Mapping
- Invertible
- Quotient, Remainder, Greatest Common Divisor
- Equivalence Classes
- Modulo

Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

References

[1] Thomas W. Judson etc. Abstract Algebra Theory and Applications, 2018

[2] D. S. Malik, John N. Mordeson, M.K. Sen, Introduction to Abstract Algebra, 2007

[3] **(おすすめ)** 松本 眞, 代数系への入門, http://www.math.sci.hiroshima-u.ac.jp/~m-mat/TEACH/daisu-nyumon2014.pdf

[4] Wikipedia

[5] Materials from internet.

Appendix (付録)

Definition

For any set X, the **power set** of X, written $\mathcal{P}(X)$, is defined to be the set $\{A \mid A \text{ is a subset of } X\}$.

Example

Let $X = \{1, 2, 3\}$. Then

$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Here $\mathcal{P}(X)$ has 2^3 elements.

Appendix (付録)

Theorem

Let $f: A \to B$, $g: B \to C$, and $h: C \to D$. Then

1. The composition of mappings is associative; that is, $(h \circ g) \circ f = h \circ (g \circ f)$;
2. If f and g are both ***one-to-one***, then the mapping $g \circ f$ is ***one-to-one***;
3. If f and g are both ***onto***, then the mapping $g \circ f$ is ***onto***;
4. If f and g are ***bijective***, then so is $g \circ f$.