



Lecture 6

Homomorphism (準同型) and Isomorphism (同型) of Groups

What you will learn in Lecture 6

6.1 Homomorphism (準同型) and Isomorphism (同型) of Groups

6.2 Solutions/Hints of Assignments

6.3 Exercises

6.4 Quiz 1

6.1 Homomorphism (準同型) and Isomorphism (同型) of Groups

Definition 6.1

A homomorphism (準同型) between groups (G_1, \circ) and $(G_2, *)$ is a function (or map) $f : G_1 \rightarrow G_2$ such that

$$f(a \circ b) = f(a) * f(b).$$

for all $a, b \in G_1$.

(Here \circ and $*$ are two binary operations.)

Example 6.1

Let $(\mathbb{Z}, +)$ and (G, \cdot) be groups and $g \in G$.

Define a function $f: \mathbb{Z} \rightarrow G$ by $f(n) = g^n$.

Then f is a group homomorphism, since

$$f(m + n) = g^{m+n} = g^m g^n = f(m)f(n):$$

This homomorphism (準同型) maps \mathbb{Z} onto the cyclic subgroup of G generated by g .

Example 6.2

We define a circle group (T, \circ) consists of all complex numbers $z \in \mathbb{C}$ such that $|z| = 1$.

We can define a homomorphism f from the additive group of real numbers \mathbb{R} to T by

$$f: \theta \mapsto \cos \theta + i \sin \theta.$$

Indeed, for $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} f(\alpha + \beta) &= \cos(\alpha + \beta) + i \sin(\alpha + \beta) \\ &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \\ &= (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= f(\alpha)f(\beta) \end{aligned}$$

Theorem 6.1

Let f be a homomorphism of a group (G_1, \circ) into a group $(G_2, *)$.

Then

(i) $f(e_1) = e_2$.

(ii) $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G_1$.

(iii) If H_1 is a subgroup of G_1 , then $f(H_1) = \{f(h) \mid h \in H_1\}$ is a subgroup of G_2 .

(iv) If G_1 is commutative, then $f(G_1)$ is commutative.

*Definition 6.2

A homomorphism f of a group (G_1, \circ) into a group $(G_2, *)$ is called an **isomorphism (同型)** of G_1 onto G_2 if f is **one-to-one** and **onto** G_2 . In this case, we write $G_1 \simeq G_2$ and say that G_1 and G_2 are **isomorphic**.

Notice: * mark is optional material. It will not be included in both middle and final examinations.

***Example 6.3**

Let us show that the mathematical structure $\langle \mathbb{R}, + \rangle$ with operation the usual addition is **isomorphic** to the structure $\langle \mathbb{R}^+, \cdot \rangle$ where \cdot is the usual multiplication. (Here \mathbb{R}^+ denotes the **set of positive numbers of \mathbb{R}** .)

1. We have to somehow convert an operation of addition to multiplication. Recall from $a^{b+c} = (a^b)(a^c)$ that addition of exponents corresponds to multiplication of two quantities.

Thus we try defining $f: \mathbb{R} \rightarrow \mathbb{R}^+$ by $f(x) = e^x$ for $x \in \mathbb{R}$.

Note that $e^x > 0$ for all $x \in \mathbb{R}$, so indeed $f(x) \in \mathbb{R}^+$.

2. If $f(x) = f(y)$ then $e^x = e^y$. Taking the natural logarithm, we see that $x = y$, so f is indeed **one-to-one**.

3. If $r \in \mathbb{R}^+$, then $\ln(r) \in \mathbb{R}$ and $f(\ln(r)) = e^{\ln(r)} = r$. Thus f is **onto \mathbb{R}^+** .

4. For $x, y \in \mathbb{R}$, we have $f(x+y) = e^{x+y} = e^x \cdot e^y = f(x) \cdot f(y)$. Thus we see that f is indeed an **isomorphism**.

Solutions/Hints of Assignments

Exercises

Quiz 1

Review for Lecture 6

- Homomorphism (準同型) of Groups
- *Isomorphism (同型) of Groups

Assignment

Please Check <https://github.com/uoaworks/Applied-Algebra>

References

- [1] Thomas W. Judson etc. Abstract Algebra Theory and Applications, 2018
- [2] D. S. Malik, John N. Mordeson, M.K. Sen, Introduction to Abstract Algebra, 2007
- [3] (おすすめ) 松本 眞, 代数系への入門, <http://www.math.sci.hiroshima-u.ac.jp/~m-mat/TEACH/daisu-nyumon2014.pdf>
- [4] Wikipedia
- [5] Materials from the internet.

*Theorem

Let $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$. Then

1. The composition of mappings is associative; that is, $(h \circ g) \circ f = h \circ (g \circ f)$;
2. If f and g are both *one-to-one*, then the mapping $g \circ f$ is *one-to-one*;
3. If f and g are both *onto*, then the mapping $g \circ f$ is *onto*;
4. If f and g are *bijective*, then so is $g \circ f$.

*Definition

Let f be a homomorphism of a group (G_1, \circ) into a group $(G_2, *)$.
The **kernel (核)** of f , written $Ker f$, is defined to be the set

$$Ker f = \{a \in G_1 \mid f(a) = e_2\}.$$

*Theorem

Let f be a homomorphism of a group (G_1, \circ) into a group $(G_2, *)$.
Then $(Ker f, \circ)$ is a normal subgroup of (G_1, \circ) .

Notice: * mark is optional material. It will not be included in both middle and final examinations.

*Example

Let (G, \circ) be a cyclic group with generator g .

Define a map $f: \mathbb{Z} \rightarrow G$ by $n \mapsto g^n$. This map is a **surjective (onto) homomorphism**

since $f(m + n) = g^{m+n} = g^m g^n = f(m)f(n)$

Clearly f is **onto**. If $|g| = m$, then $g^m = e$.

Hence, $\ker f = m\mathbb{Z}$ and $\mathbb{Z}/\ker f = \mathbb{Z}/m\mathbb{Z} \simeq G$.

On the other hand, if the order of g is **infinite**, then $\ker f = 0$ and f is an **isomorphism** of G and \mathbb{Z} .

Hence, **two cyclic groups** are **isomorphic** exactly when they have the **same order**.

Up to **isomorphism**, the only **cyclic groups** are \mathbb{Z} and \mathbb{Z}_n .

*Theorem

Every finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +_n)$ and every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.