TEST

# Lecture 4

# Subgroup & Cyclic Group

# & Lagrange's Theorem

# What you will learn in Lecture 4

**4.1** Subgroup (部分群)

**4.2** Cyclic Group (巡回群)

**4.3** Lagrange's Theorem (ラグランジュの定理)

# 4.1 Subgroup (部分群)

## 4.1 Subgroup (部分群)

Subgroup

Let us consider the groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$, where $+$ is the usual addition of numbers, and note the following:

1. **Both these groups** have **the same binary operation**.
2. $\mathbb{Z}$ is a **subset of** $\mathbb{Q}$.

The same is true for the groups $(\mathbb{Z}, +)$ and $(\mathbb{R}, +)$; $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$; $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$.

# 4.1 Subgroup (部分群)

Subgroup

This leads us to the concept of a **subgroup**. Before formally defining subgroups, let us also note the following:

Let $(G,\circ)$ be **a group** and $H$ be a **nonempty subset** of $G$. Then $H$ is said to be **closed** under the **binary operation** $\circ$ if $a \circ b \in H$ for all $a, b \in H$.

Suppose $H$ is **closed** under **the binary operation** $\circ$. Then the restriction of $\circ$ to $H \times H$ is a mapping from $H \times H$ into $H$. Thus, the binary operation $\circ$ defined on $G$ induces **a binary operation on** $H$. We denote this induced binary operation on $H$ by $\circ$ also. Thus, $(H,\circ)$ is a **mathematical system**.

It also follows that $\circ$ is **associative** as a binary operation on $H$, i.e., $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$.

If $(H,\circ)$ is a **group**, then we call $H$ a **subgroup** of $G$.

# 4.1 Subgroup (部分群)

Subgroup

More formally, we have the following definition.

## Definition 4.1

Let $(G,\circ)$ be a group and $H$ be a **nonempty subset** of $G$. If $(H,\circ)$ is a group, then $(H,\circ)$ is called a **subgroup** of $(G,\circ)$.

For **example**, consider the **rational number group** $(\mathbb{Q},+)$ and its **subgroups** $(\mathbb{Z},+)$. Now the **identity elements** of both these groups is $0$.

Next, let $a \in \mathbb{Z}$. Then $a \in \mathbb{Q}$.

Also, the **inverse of** $a$ in $\mathbb{Z}$ **as well as in** $\mathbb{Q}$ **is** $-a$.
In other words, **the inverse of** $a$ in $\mathbb{Z}$ **and the inverse of** $a$ in $\mathbb{Q}$ **is the same.**

In general, we have the following result.

### Theorem 4.1

Let $(G,\circ)$ be a group and $(H,\circ)$ be a subgroup of $(G,\circ)$.
(i) The **identity elements** of $(H,\circ)$ and $(G,\circ)$ are the same.
(ii) If $h \in H$, then **the inverse of** $h$ in $H$ and **the inverse of** $h$ in $G$ **is the same**.

**Remark**
If $(G,\circ)$ is a group, then $(\{e\},\circ)$ and $(G,\circ)$ are **subgroups** of $(G,\circ)$. These subgroups are called **trivial subgroup (自明な部分群)**.

**Example 4.1**

Consider the following list of groups.

(i) $(\{0\},+), (\mathbb{Z},+), (\mathbb{Q},+), (\mathbb{R},+), (\mathbb{C},+)$,

(ii) $(\{1\},\cdot), (\mathbb{Q}\backslash\{0\},\cdot), (\mathbb{R}\backslash\{0\},\cdot), (\mathbb{C}\backslash\{0\},\cdot)$,

where $+$ is the usual addition operation and $\cdot$ is the usual multiplication operation.

Each group is a subgroup of the group listed to its right.

For example, $(\mathbb{Z},+)$ is a subgroup of $(\mathbb{Q},+)$, $(\mathbb{R},+)$, and $(\mathbb{C},+)$, and $(\mathbb{R}\backslash\{0\},\cdot)$ is a subgroup of $(\mathbb{C}\backslash\{0\},\cdot)$.

## Theorem 4.2

Let $(G,\circ)$ be a group and $H$ be a **nonempty subset** of $G$. Then $(H,\circ)$ is a subgroup of $(G,\circ)$ **if and only if** for all $a, b \in H$, $ab^{-1} \in H$.

**Proof:**

Suppose $(H,\circ)$ is a subgroup of $(G,\circ)$. Let $a, b \in H$. Because $(H,\circ)$ is a subgroup, it is a group. Therefore, $b \in H$ implies that $b^{-1} \in H$. Thus, $ab^{-1} \in H$ because $H$ is closed under the binary operation.

Conversely, suppose $H$ is a nonempty subset of $G$ such that $a, b \in H$ implies $ab^{-1} \in H$. Because $H \neq \emptyset$, there exists $a \in H$. Now $a, a^{-1} \in H$. Therefore, $e = aa^{-1} \in H$, i.e., $H$ contains the identity. Next, let $b \in H$. Then $e, b \in H$, implies that $b^{-1} = eb^{-1} \in H$. Thus, every element of $H$ has an inverse in $H$.

To show that $H$ is closed under the binary operation, let $a, b \in H$. Then $a, b^{-1} \in H$. Thus, $ab = a(b^{-1})^{-1} \in H$.

Hence, $H$ is closed under the binary operation. From the statements preceding Definition, associativity holds for $H$. Hence, $(H,\circ)$ is a group, so $(H,\circ)$ is subgroup of $(G,\circ)$.

## **Example 4.2** Find all subgroups of $(\mathbb{Z}, +)$.

**Solution:** Let $(H, +)$ be a subgroup of $(\mathbb{Z}, +)$. Suppose $H \neq \{0\}$.

Let $a$ be a nonzero element of $H$. Then $-a \in H$. Since either $a$ or $-a$ is a positive integer, $H$ contains a positive integer. With the help of the principle of well-ordering, we can show that $H$ contains a smallest positive integer. Let $a$ be the smallest positive integer in $H$. We claim that $H = \{na \mid n \in \mathbb{Z}\}$.

Now $na \in H$ for all $n \in \mathbb{Z}$ and so $\{na \mid n \in \mathbb{Z}\} \subseteq H$. On the other hand, let $b \in H$.

By the division algorithm, there exist $c$ and $r$ in $\mathbb{Z}$ such that $b = ca + r$, where $0 \leq r < a$. Suppose $r \neq 0$. Then $r = b - ca \in H$. Thus, $H$ contains a positive integer smaller than $a$, a contradiction.

Hence, $r = 0$ and so $b = ca \in \{na \mid n \in \mathbb{Z}\}$.

This implies that $H \subseteq \{na \mid n \in \mathbb{Z}\}$. Thus, $H = \{na \mid n \in \mathbb{Z}\}$ for some $a \in \mathbb{Z}$. Also, for all $n \in \mathbb{Z}$, the set $T = \{nm \mid m \in \mathbb{Z}\} = n\mathbb{Z}$ generate a subgroup of $(\mathbb{Z}, +)$.

Hence, $(n\mathbb{Z}, +)$, $n = 0, 1, 2, \ldots$ are the subgroups of $(\mathbb{Z}, +)$.

## Definition 4.2

Let $H$ and $L$ be **nonempty subsets** of $G$ from a **group** $(G,\circ)$.
The product of $H$ and $L$ is defined to be the set
$HL = \{hl \mid h \in H, l \in L\}$.

**Example** 4.3

Consider the group of symmetries of the square.

Let $H = \{r_{360}, d_1\}$ and $L = \{r_{360}, h\}$. Then H and $L$ are subgroups of $G$. Now

$$HL = \{r_{360}r_{360},\ r_{360}h,\ d_1r_{360},\ d_1h\} = \{r_{360},\ h,\ d_1,\ r_{90}\}.$$

Now $hd_1 = r_{270} \notin HL$, so $HL$ is **not closed** under the binary operation.
Hence, $HL$ is **not a subgroup** of the symmetries of the square. Also, note that

$$LH = \{r_{360}r_{360},\quad r_{360}d_1,\quad hr_{360},\quad hd_1\} = \{r360,\quad d1,\quad h,\quad r_{270}\},$$

And $\langle H \cup L\rangle = \{r_{360}, r_{90}, r_{180}, r_{270}, h, v, d_1, d_2\}$.

This example shows that in general the product of subgroups need not be a subgroup.

# 4.1 Subgroup (部分群)

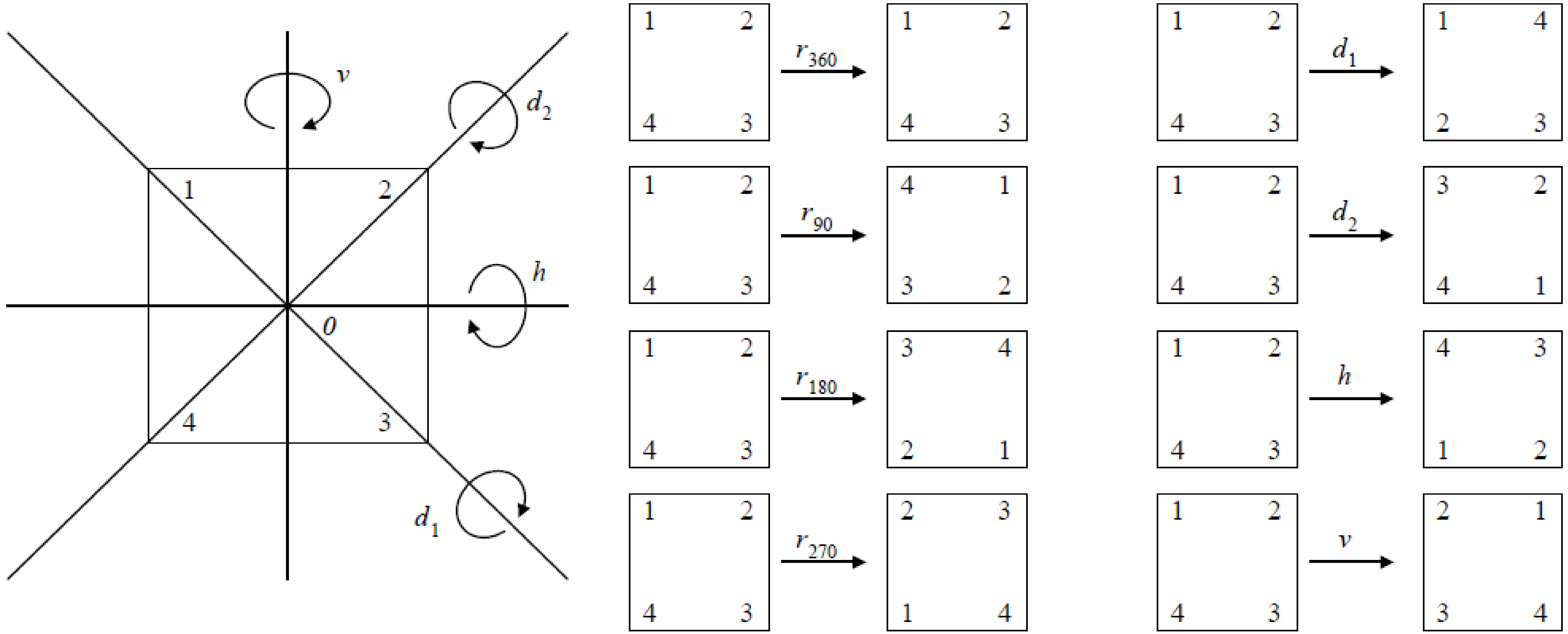
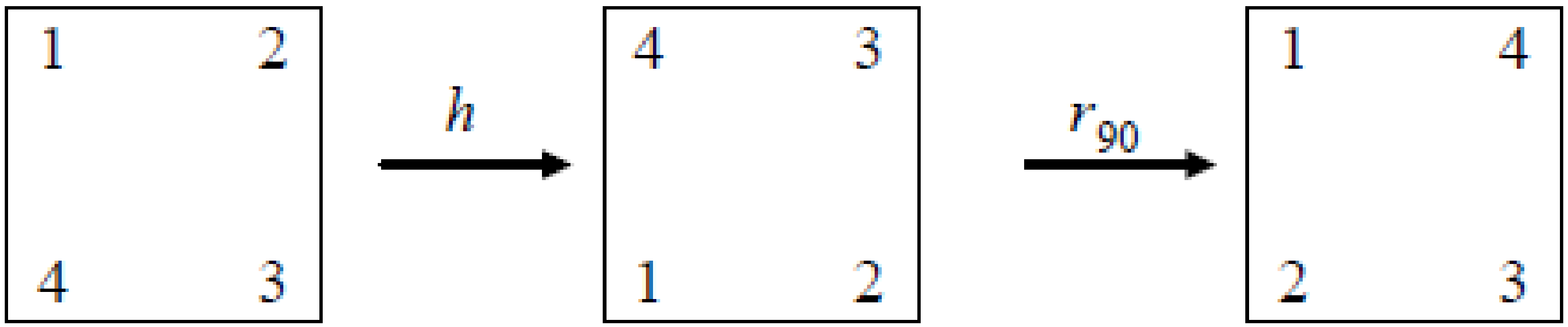



Figure. Rigid motions of a square in symmetry

# 4.1 Subgroup (部分群)

$$r_{90} \circ h$$



# 4.1 Subgroup (部分群)

The complete operation table for the operation $\circ$ is as following

| $\circ$ | $r_{360}$ | $r_{90}$ | $r_{180}$ | $r_{270}$ | $h$ | $v$ | $d_1$ | $d_2$ |
|---|---|---|---|---|---|---|---|---|
| $r_{360}$ | $r_{360}$ | $r_{90}$ | $r_{180}$ | $r_{270}$ | $h$ | $v$ | $d_1$ | $d_2$ |
| $r_{90}$ | $r_{90}$ | $r_{180}$ | $r_{270}$ | $r_{360}$ | $d_1$ | $d_2$ | $v$ | $h$ |
| $r_{180}$ | $r_{180}$ | $r_{270}$ | $r_{360}$ | $r_{90}$ | $v$ | $h$ | $d_2$ | $d_1$ |
| $r_{270}$ | $r_{270}$ | $r_{360}$ | $r_{90}$ | $r_{180}$ | $d_2$ | $d_1$ | $h$ | $v$ |
| $h$ | $h$ | $d_2$ | $v$ | $d_1$ | $r_{360}$ | $r_{180}$ | $r_{270}$ | $r_{90}$ |
| $v$ | $v$ | $d_1$ | $h$ | $d_2$ | $r_{180}$ | $r_{360}$ | $r_{90}$ | $r_{270}$ |
| $d_1$ | $d_1$ | $h$ | $d_2$ | $v$ | $r_{90}$ | $r_{270}$ | $r_{360}$ | $r_{180}$ |
| $d_2$ | $d_2$ | $v$ | $d_1$ | $h$ | $r_{270}$ | $r_{90}$ | $r_{180}$ | $r_{360}$ |

In the following theorem, we give a necessary and sufficient condition for the product of subgroups to be a subgroup.

## Theorem 4.3

Let $(H,\circ)$ and $(L,\circ)$ be subgroups of a group $(G,\circ)$. Then $(HL,\circ)$ is a subgroup of $(G,\circ)$ **if and only if** $HL = LH$.

## Corollary 3.2

If $(H,\circ)$ and $(L,\circ)$ are subgroups of **a commutative group** $(G,\circ)$, then $(HL,\circ)$ is **a subgroup of** $(G,\circ)$.

# 4.2 Cyclic Group (巡回群)

# 4.2 Cyclic Group (巡回群)



In the previous section, we introduced the notion of a subgroup generated by a set. **Groups that are generated by a single element, called cyclic groups**, are of special importance. Cyclic groups are easier to study than any other group.

## Definition 4.3

A group $(G,\circ)$ is called a **cyclic group** if there exists $a \in G$ such that

$$G = \langle a \rangle$$

where $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$.

Let $G = \langle a \rangle$ defines a cyclic group and $b, c \in G$. Then $b = a^n$ and $c = a^m$ for some $n, m \in Z$. Now

$$bc = a^n a^m = a^{n+m} = a^{m+n} = a^m a^n = cb.$$

This shows that $G$ is **commutative**. **Hence, every cyclic group is commutative**. We record this result in the following theorem.

## Theorem 4.4

Every cyclic group is commutative.

### Example 4.4

(i) $(\mathbb{Z}, +)$ is a cyclic group because $\mathbb{Z} = \langle 1 \rangle$.

(ii) $(\{na \mid n \in \mathbb{Z}\}, +)$ is a cyclic group, where $a$ is any fixed element of $\mathbb{Z}$.

**Example 4.5**

Consider the set $G = \{e, a, b, c\}$. Define $\circ$ on $G$ by means of the following operation table.

| $\circ$ | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

From the multiplication table, it follows that $(G,\circ)$ is a **commutative** group. **However, $\boldsymbol{G}$ is NOT a cyclic group** because
$\langle e \rangle = \{e\}, \langle a \rangle = \{e, a\}, \langle b \rangle = \{e, b\}$, and $\langle c \rangle = \{e, c\}$
and each of these subgroups is properly contained in $G$. $G$ is known as the **Klein 4-group** (クラインの四元群).

# 4.2 Cyclic Group (巡回群)

## Theorem 4.5

Let $\langle a \rangle$ be a finite cyclic group of order $n$.
Then $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$.

## Theorem 4.6

Every subgroup of a cyclic group is cyclic.

## Corollary 3.2

Let $G = \langle a \rangle$ be **a cyclic group of order** $n$, $n > 1$, and $H$ be **a proper subgroup of** $G$.
Then $H = a^k$ for some integer $k$ such that $k$ divides $n$ and $k > 1$.
Furthermore, the order $|H|$ **divides** $n$.

# 4.3 Lagrange's Theorem

# (ラグランジュの定理)

# 4.3 Lagrange’s Theorem (ラグランジュの定理)

In the last section, we noted that the order of a subgroup of a finite cyclic group divides the order of the group (Corollary 4.2).

We will learn that this is a special case of a general result, called Lagrange’s theorem, i.e., the order of a subgroup of a finite group divides the order of the group.

**History:**
**Lagrange proved this result in 1770, long before the creation of group theory, while working on the permutations of the roots of a polynomial equation. Lagrange’s theorem** is a basic theorem of **finite group theory** and is considered by some to be the most important result in finite group theory.

## Definition 4.4

Let $H$ be **a subgroup of a group** $G$ and $a \in G$. The sets $aH = \{ah \mid h \in H\}$ and $Ha = \{ha \mid h \in H\}$ are called the **left and right cosets (左剰余類と右剰余類) of** $H$ **in** $G$, respectively. The element $a$ is called **a representative** of $aH$ **and** $Ha$.

If $G$ is **commutative**, then of course we have $aH = Ha$.
Observe that $eH = H = He$ and that $a = ae \in aH$ and $a = ea \in Ha$.

**Example 4.6** Consider the symmetric group $S_3$ (Example 3.7).

(1)
$$H = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

is a subgroup of $S_3$.

We now compute the left and right cosets of $H$ in $S_3$. The left cosets of $H$ in $S_3$ are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H = H$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H =$$
$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

**Left and Right Cosets**

and the right cosets of $H$ in are

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = H$$

and

$$H\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right\}.$$

Thus, for all $a \in S_3$, $aH = Ha$.

# 4.3 Lagrange's Theorem (ラグランジュの定理)

Left and Right Cosets

(2)
$$H' = \left\{ e, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$$

is also a subgroup of $S_3$.

Now we compute the left and right cosets of $H'$ in $S_3$. The left cosets of $H'$ in $S_3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} H' = H',$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\},$$

and

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} H' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} H' = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$$

and the right cosets of $H'$ in $S_3$ are

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = H',$$

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right\},$$

and

$$H'\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right\}.$$

We see that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} H' \neq H'\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Thus, the left and right cosets of $H'$ in $S_3$ are not the same.

There are some interesting phenomena happening in the above example.

- We see that all left and right cosets of $H$ in $S_3$ have the same number of elements, namely, 3; that there are the same number of distinct left cosets of $H$ in $S_3$ as of right cosets, namely, 2; that the set of all left cosets and the set of all right cosets form partitions of $S_3$; and, finally, that $3 \cdot 2$ equals the order of $S_3$.
- Similar statements hold for the subgroup $H'$. We show, in the results to follow, that these phenomena hold in general.

# 4.3 Lagrange's Theorem (ラグランジュの定理)

The following theorem tells us **when two left (right) cosets are equal.** It is a result that is used often in the study of groups.

**Theorem 4.7**

Let $H$ be a **subgroup** of a **group** $G$ and $a, b \in G$. Then
(i) $aH = bH$ if and only if $b^{-1}a \in H$.
(ii) $Ha = Hb$ if and only if $ab^{-1} \in H$.

**Theorem 4.8**

Let $H$ be a **subgroup** of a **group** $G$. Then for all $a, b \in G$, either $aH = bH$ or $aH \cap bH = \emptyset$ (i.e., two left cosets are either equal or they are disjoint). Similar result also satisfied for two right cosets.

Number of Distinct Cosets

## Definition 4.5

Let $H$ be a **subgroup** of a **group** $G$. Then **the number of distinct (相異なる) left (or right) cosets**, written as $[G:H]$, of $H$ in $G$ is called **the index of $H$ in $G$**.

# 4.3 Lagrange's Theorem (ラグランジュの定理)

## Theorem 4.9 (Lagrange's Theorem)

Let $H$ be a **subgroup** of a **finite group** $G$. Then **the order of** $H$ **divides the order of** $G$. In particular,

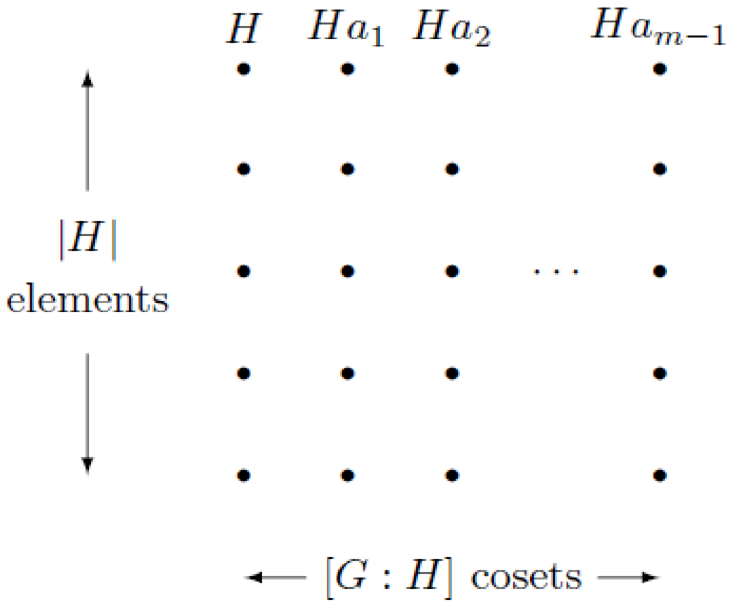
$$|G| = [G : H]|H|.$$

**Proof:**

Suppose that $[G : H] = m$. Every element of $G$ is in a coset of $H$, and Theorem 4.8 tells us we can decompose $G$ into a union of $m$ pairwise disjoint cosets:

$$G = H \cup Ha_1 \cup Ha_2 \cup \cdots \cup Ha_{m-1}$$

But each of these cosets has $|H|$ elements. Thus, there must be $[G : H]|H|$ elements in $G$ altogether.



## Theorem 4.10

Let $H$ and $L$ be **finite subgroups** of a **group** $G$. Then

$$|HL| = \frac{|H||L|}{|H \cap L|}$$

# Review for Lecture 4

- Subgroup (部分群)
- Trivial Subgroup (自明な部分群)
- Cyclic Group (巡回群)
- Left and Right Cosets (左剰余類と右剰余類)
- Lagrange's Theorem (ラグランジュの定理)

# Assignment

Please Check https://github.com/uoaworks/Applied-Algebra

# References


[1] Thomas W. Judson etc. Abstract Algebra Theory and Applications, 2018

[2] D. S. Malik, John N. Mordeson, M.K. Sen, Introduction to Abstract Algebra, 2007

[3] **(おすすめ)** 松本 眞, 代数系への入門, http://www.math.sci.hiroshima-u.ac.jp/~m-mat/TEACH/daisu-nyumon2014.pdf

[4] Wikipedia

[5] Materials from internet.