A Saddle Point Paradigm for Finite Element Analysis and Its Role in the DPG Methodology

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Funding: National Science Foundation, UT Austin Graduate School Continuing Fellowship

Collaborators: Marek Behr, Stefanie Elgeti, Federico Fuentes, Jay Gopalakrishnan, Philipp Knechtges, Sriram Nagaraj, Socratis Petrides, Nathan V. Roberts, Patrick Le Tallec, Ali Vaziri

The Institute for Computational Engineering and Sciences

The University of Texas at Austin

August 10, 2018

Outline

Introduction

- Babuška to Brezzi
- DPG methods
- DPG* methods

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Applications

- Poisson
- Viscoelastic fluids

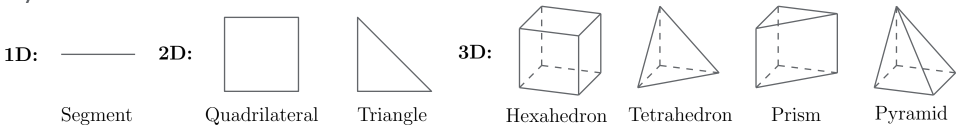
Prelude: Other Research

Shape Function Software Library

- Energy Spaces

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\nabla\times} H(\mathrm{div}) \xrightarrow{\nabla\cdot} L^2$$

- Hybrid Meshes



- High order, *anisotropic*, hierarchical shape functions
- Affine invariance to master element geometries
- *Sparse* stiffness matrices
- Orientation embeddings

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Shape Function Software Library

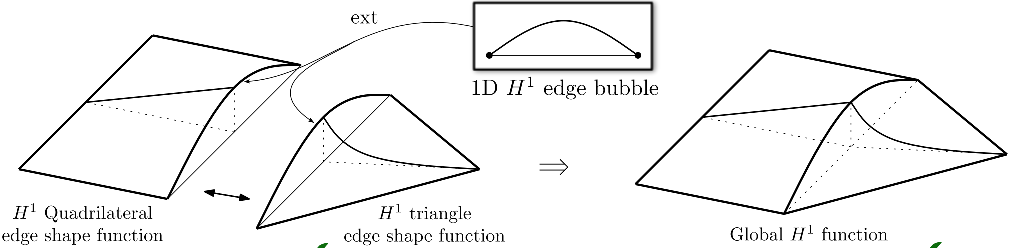
$$
\begin{tikzcd}
\text{3D:} & H^1 \arrow[r, "\nabla"] & H(\mathrm{curl}) \arrow[r, "\nabla\times"] & H(\mathrm{div}) \arrow[r, "\nabla\cdot"] & L^2 \\
\text{2D:} & H^1 \arrow[r, "\nabla"] \arrow[u, "\mathrm{ext}"] & H(\mathrm{curl}) \arrow[r, "\nabla\times"] \arrow[u, "\mathrm{ext}"] & L^2 \arrow[u, "\mathrm{ext}"] & \\
\text{1D:} & H^1 \arrow[r, "\nabla"] \arrow[u, "\mathrm{ext}"] & L^2 \arrow[u, "\mathrm{ext}"] & &
\end{tikzcd}
$$

Classification into *subtopologies*

1D: vertex and edge shape functions,

2D: vertex, edge, and face shape functions,

3D: vertex, edge, face and interior shape functions.

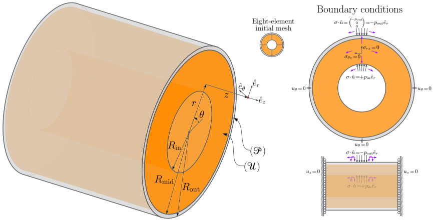


Compatibility: ✓

Global conformity: ✓

Structural Mechanics

Sheathed hose

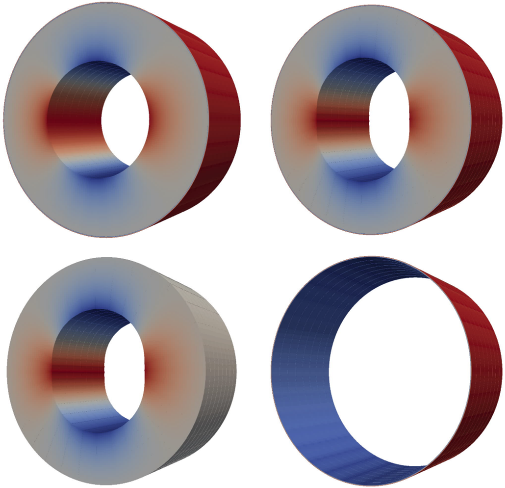


$\left(E_{\text{steel}} = 200\text{GPA}, E_{\text{rubber}} = 0.01\text{GPA}\right)$

Coupled formulations

- All coupled formulations are mutually compatible throughout the domain.

Circumferential stress: $\boldsymbol{\sigma}_{\theta\theta}$



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Fluid Mechanics

Oldroyd-B fluid

Conservation of mass and momentum:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\boldsymbol{\sigma} = \rho\,\mathbf{f} \quad \text{on } \Omega\times(0,T)\,.$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{on } \Omega\times(0,T)\,.$$

Constitutive law:

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\mu_S\,\varepsilon(\mathbf{u}) + \mathbf{T}\,,$$

where

$$\mathbf{T} + \lambda\,\mathfrak{L}_{\mathbf{u}}\mathbf{T} = 2\mu_P\,\varepsilon(\mathbf{u})\,.$$

Lie derivative:

$$\mathfrak{L}_{\mathbf{u}}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{T} - \left(\nabla\mathbf{u}\,\mathbf{T} - \mathbf{T}\,\nabla^{\mathsf{T}}\mathbf{u}\right)\,.$$

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In preparation, 2018.

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Quantity of interest

Drag coefficient

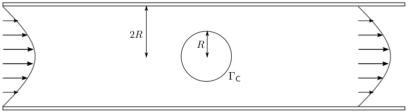
$$c_{\mathrm{D}}(\boldsymbol{\sigma}) = \frac{1}{\mu\bar{u}}\int_{\Gamma_C}(\boldsymbol{\sigma}\vec{n})\cdot\mathbf{e}_x\,\mathrm{d}s\,.$$

Γ_C: boundary of cylinder

$\hat{t}$: traction

$\mu = \mu_S + \mu_P$: viscosity

$\bar{u}$: average inflow velocity.



Confined cylinder domain.

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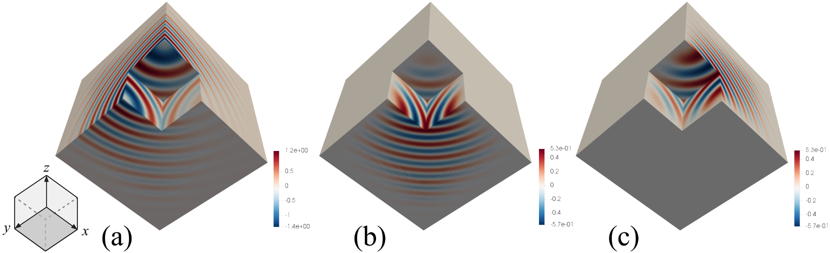
Wave Mechanics

- Acoustics $\quad -\Delta p - \omega^2 p = f$
- Electromagnetics $\quad \frac{1}{\mu}\nabla \times \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{J}^{\mathrm{imp}}$
- Elastodynamics $\quad -\boldsymbol{\nabla} \cdot (\mathsf{C} : \boldsymbol{\nabla} \boldsymbol{u}) - \rho\omega^2 \boldsymbol{u} = \boldsymbol{f}$



Electromagnetic wave scattering of the discrete electric field $\mathbf{E}$.

(a) The x-component; (b) the y-component; (c) the z-component. Only the real part of the solution is visualized.

A. Vaziri Astaneh, B. Keith, and L. Demkowicz.
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arXiv:1804.04496 (math.NA), 2018.

Discrete Least Squares

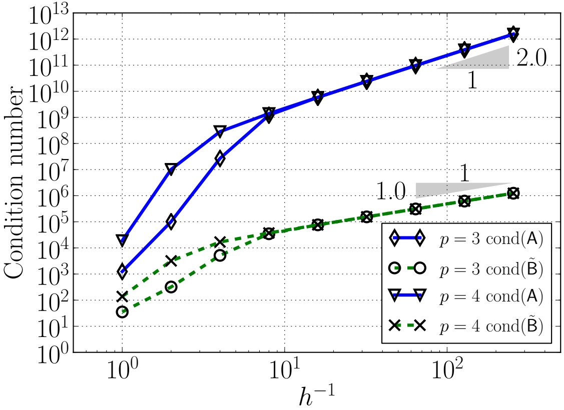
Two solution paradigms

Normal equations

$$\underbrace{B^T G^{-1} B}_{A}\, u = B^T G^{-1} f$$

Overdetermined system

$$\underbrace{L^{-1} B}_{\widetilde{B}}\, u = L^{-1} f$$



Acoustics with DPG near resonance

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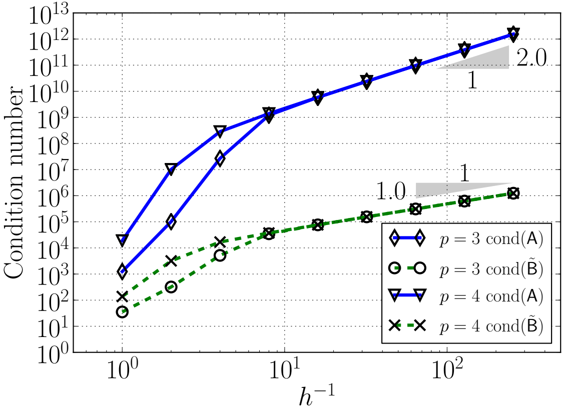
- ⋆ Static condensation for $\mathsf{L}^{-1}\mathsf{B}$ is explained in the article.
- ⋆ $\mathsf{L}^{-1}\mathsf{B}$ has $O(h^{-1})$ condition number!
- ⋆ Less round off error when handling $\mathsf{L}^{-1}\mathsf{B}$ directly.



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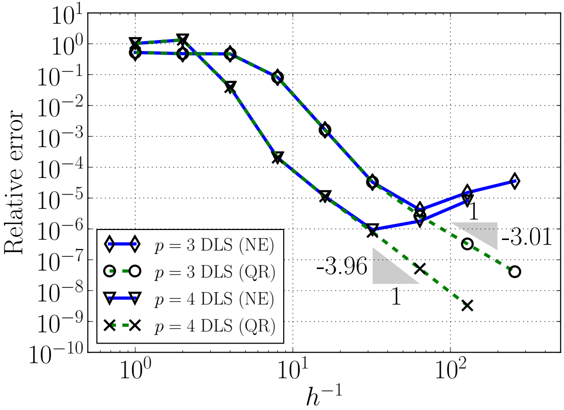

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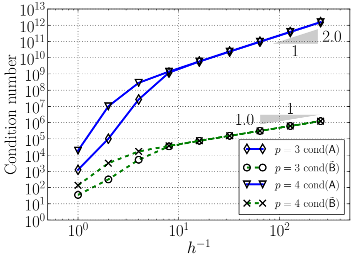
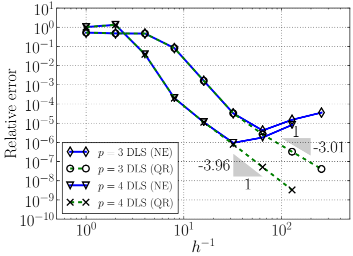
Summary





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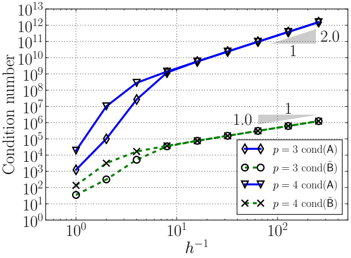

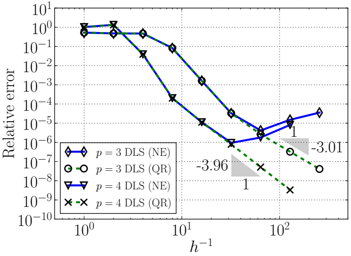

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Babuška to Brezzi

Let $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ be a continuous bilinear form.

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$\mathcal{U}$ and $\mathcal{V}$ Hilbert

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Let $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ be a continuous bilinear form.

$\mathcal{U}$ and $\mathcal{V}$ Hilbert

$\mathcal{U}_h \subset \mathcal{U}$ and $\mathcal{V}_h \subset \mathcal{V}$ finite-dimensional

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$\mathcal{U}$ and $\mathcal{V}$ Hilbert

$\mathcal{U}_h \subset \mathcal{U}$ and $\mathcal{V}_h \subset \mathcal{V}$ finite-dimensional

Problem:

Find $u_h \in \mathcal{U}_h$ satisfying $\qquad b(u_h, v_h) = F(v_h) \quad \forall\, v_h \in \mathcal{V}_h$

Stability

Babuška to Brezzi

Define $\mathcal{B}(u)(v) = b(u, v)$ for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$

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Define $\mathcal{B}(u)(v) = b(u, v)$ for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$

Assume $F \in (\operatorname{Null} \mathcal{B}')^{\perp}$

Stability

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Define $\mathcal{B}(u)(v) = b(u, v)$ for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$

Assume $F \in (\mathrm{Null}\,\mathcal{B}')^{\perp}$

The following continuous problem has a unique solution:

Find $u \in \mathcal{U}$ satisfying $\qquad b(u, v) = F(v) \quad \forall\, v \in \mathcal{V}$,

$\|u\|_{\mathcal{U}} \leq \gamma^{-1}\|F\|_{\mathcal{V}'}$, if and only if

$$\inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} \frac{b(u, v)}{\|u\|_{\mathcal{U}}\|v\|_{\mathcal{V}}} = \gamma > 0$$

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$$\inf_{u\in\mathcal{U}} \sup_{v\in\mathcal{V}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}\|v\|_{\mathcal{V}}} = \gamma > 0, \qquad \gamma = \|\mathcal{B}^{-1}\|^{-1}$$

Stability

Babuška to Brezzi

Continuous stability

Discrete stability

$$\inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{b(u,v)}{\|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}} = \gamma > 0 \quad \not\Rightarrow \quad \inf_{u_h \in \mathcal{U}_h} \sup_{v_h \in \mathcal{V}_h} \frac{b(u_h,v_h)}{\|u_h\|_{\mathcal{U}} \|v_h\|_{\mathcal{V}}} = \gamma_h > 0$$

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Q: How to satisfy the discrete $\inf$–$\sup$ condition

$$\sup_{v_h\in\mathcal{V}_h}\frac{b(u_h,v_h)}{\|v_h\|_{\mathcal{V}}}\geq\gamma_h\|u_h\|_{\mathcal{U}}\,,\quad\forall\,u_h\in\mathcal{U}_h \quad ?$$

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Q: How to satisfy the discrete inf–sup condition

$$\sup_{v_h\in\mathcal{V}_h} \frac{b(u_h,v_h)}{\|v_h\|_{\mathcal{V}}} \geq \gamma_h \|u_h\|_{\mathcal{U}}\,, \quad \forall\, u_h \in \mathcal{U}_h \quad ?$$

A: Fix $\mathcal{U}_h$ and increase the dimension of $\mathcal{V}_h$ until satisfied.

Stability

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$$\inf_{u_h \in \mathcal{U}_h} \sup_{v_h \in \mathcal{V}_h} \frac{b(u_h, v_h)}{\|u_h\|_{\mathcal{U}} \|v_h\|_{\mathcal{V}}} = \gamma_h > 0 \quad \text{where} \quad \dim \mathcal{V}_h > \dim \mathcal{U}_h$$

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$$\begin{aligned} (\varepsilon_h, v_h)_{\mathcal{V}} + b(u_h, v_h) &= F(v_h) && \forall\, v_h \in \mathcal{V}_h \\ b(w_h, \varepsilon_h) &= 0 && \forall\, w_h \in \mathcal{U}_h \end{aligned}$$

- Requires *two* inf–sup conditions

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- Requires *two* inf–sup conditions
- An inf–sup condition relating spaces $\mathcal{U}_h$ and $\mathcal{V}_h$ ✓
- An inf–sup condition *in the kernel* of the dual operator ✓

Stability

Babuška to Brezzi

$$\dim \mathcal{V}_h > \dim \mathcal{U}_h$$

Stability

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Dual problem:

Find $v_h \in \mathcal{V}_h$ satisfying $\quad b(w_h, v_h) = G(w_h) \quad \forall\, w_h \in \mathcal{U}_h$

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Q: The dual problem is now *underdetermined*. What to do?

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Q: The dual problem is now *underdetermined*. What to do?

A: Consider a different embedding in the saddle point system.

Stability

Babuška to Brezzi

Primal:

Find $u_h \in \mathcal{U}_h$ and $\varepsilon_h \in \mathcal{V}_h$:

$$\begin{aligned} (\varepsilon_h, v_h)_{\mathcal{V}} + b(u_h, v_h) &= F(v_h) && \forall\, v_h \in \mathcal{V}_h \\ b(w_h, \varepsilon_h) &= 0 && \forall\, w_h \in \mathcal{U}_h \end{aligned}$$

Dual:

Find $v_h \in \mathcal{V}_h$ and $\lambda_h \in \mathcal{U}_h$:

$$\begin{aligned} (v_h, z_h)_{\mathcal{V}} + b(\lambda_h, z_h) &= 0 && \forall\, z_h \in \mathcal{V}_h \\ b(w_h, v_h) &= G(w_h) && \forall\, w_h \in \mathcal{U}_h \end{aligned}$$

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\end{aligned}
$$

Dual:

Find $v_h \in \mathcal{V}_h$ and $\lambda_h \in \mathcal{U}_h$:

$$
\begin{aligned}
(v_h, z_h)_{\mathcal{V}} - b(\lambda_h, z_h) &= 0 && \forall\, z_h \in \mathcal{V}_h \\
b(w_h, v_h) &= G(w_h) && \forall\, w_h \in \mathcal{U}_h
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Implementation

General problem:

Find $v_h \in \mathcal{V}_h$ and $w_h \in \mathcal{U}_h$:

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\begin{aligned}
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Discretization:

$$\begin{bmatrix} \mathsf{G} & \mathsf{B} \\ \mathsf{B}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathsf{v} \\ \mathsf{w} \end{bmatrix} = \begin{bmatrix} \mathsf{f} \\ \mathsf{g} \end{bmatrix}$$

Schur complement:

$$\begin{aligned} \mathsf{B}^\mathsf{T}\mathsf{G}^{-1}\mathsf{B}\mathsf{w} &= \mathsf{B}^\mathsf{T}\mathsf{G}^{-1}\mathsf{f} - \mathsf{g} \\ \mathsf{v} &= \mathsf{G}^{-1}(\mathsf{f} - \mathsf{B}\mathsf{w}) \end{aligned}$$

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DPG: G is block-diagonal

DPG and DPG* comparison

Minimum norm derivation

Stiffness matrix:	Gram matrix:	Load vector:	Goal vector:
$\mathsf{B}_{ij} = b(u_j, v_i)$	$\mathsf{G}_{ij} = (v_j, v_i)_{\mathcal{V}}$	$\mathsf{f}_i = F(v_i)$	$\mathsf{g}_j = G(u_j)$

DPG and DPG* comparison

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Stiffness matrix: $\mathsf{B}_{ij} = b(u_j, v_i)$

Gram matrix: $\mathsf{G}_{ij} = (v_j, v_i)_{\mathcal{V}}$

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Goal vector: $\mathsf{g}_j = G(u_j)$

In DPG, we solve:

$$\mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{B}\mathsf{u} = \mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{f}$$

DPG and DPG* comparison

Minimum norm derivation

Stiffness matrix: $\mathsf{B}_{ij} = b(u_j, v_i)$

Gram matrix: $\mathsf{G}_{ij} = (v_j, v_i)_\mathcal{V}$

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In DPG, we solve:

$$\mathsf{B}^\mathsf{T}\mathsf{G}^{-1}\mathsf{B}\mathsf{u} = \mathsf{B}^\mathsf{T}\mathsf{G}^{-1}\mathsf{f}$$

In DPG*, we solve:

$$\mathsf{B}^\mathsf{T}\mathsf{G}^{-1}\mathsf{B}\mathsf{w} = \mathsf{g}$$

and then post-process w:

$$\mathsf{v} = \mathsf{G}^{-1}\mathsf{B}\mathsf{w}$$

Discrete least squares finite element methods

Cholesky factorization

$$G = LL^T .$$

Discrete least squares finite element methods

Cholesky factorization

$$\mathsf{G} = \mathsf{L}\mathsf{L}^{\mathsf{T}}.$$

Residual minimization problem

$$\mathsf{u} = \arg\min_{\mathsf{w}\in\mathbb{R}^N} (\mathsf{B}\mathsf{w} - \mathsf{f})^{\mathsf{T}}\mathsf{G}^{-1}(\mathsf{B}\mathsf{w} - \mathsf{f})$$

Discrete least squares finite element methods

Cholesky factorization

$$\mathsf{G} = \mathsf{L}\mathsf{L}^{\mathsf{T}}.$$

Residual minimization problem

$$\begin{aligned}\mathsf{u} &= \underset{\mathsf{w}\in\mathbb{R}^N}{\arg\min}\,(\mathsf{B}\mathsf{w}-\mathsf{f})^{\mathsf{T}}\mathsf{G}^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f}) \\ &= \underset{\mathsf{w}\in\mathbb{R}^N}{\arg\min}(\mathsf{B}\mathsf{w}-\mathsf{f})^{\mathsf{T}}(\mathsf{L}\mathsf{L}^{\mathsf{T}})^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f})\end{aligned}$$

Discrete least squares finite element methods

Cholesky factorization

$$G = LL^{\mathsf{T}}.$$

Residual minimization problem

$$\begin{aligned}
u &= \operatorname*{arg\,min}_{w\in\mathbb{R}^N} (Bw - f)^{\mathsf{T}} G^{-1} (Bw - f) \\
&= \operatorname*{arg\,min}_{w\in\mathbb{R}^N} (Bw - f)^{\mathsf{T}} (LL^{\mathsf{T}})^{-1} (Bw - f) \\
&= \operatorname*{arg\,min}_{w\in\mathbb{R}^N} \left(L^{-1}(Bw - f)\right)^{\mathsf{T}} L^{-1} (Bw - f)
\end{aligned}$$

Discrete least squares finite element methods

Cholesky factorization

$$\mathsf{G} = \mathsf{L}\mathsf{L}^{\mathsf{T}}.$$

Residual minimization problem

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\mathsf{u} &= \arg\min_{\mathsf{w}\in\mathbb{R}^N} (\mathsf{B}\mathsf{w}-\mathsf{f})^{\mathsf{T}}\mathsf{G}^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f}) \\
&= \arg\min_{\mathsf{w}\in\mathbb{R}^N} (\mathsf{B}\mathsf{w}-\mathsf{f})^{\mathsf{T}}(\mathsf{L}\mathsf{L}^{\mathsf{T}})^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f}) \\
&= \arg\min_{\mathsf{w}\in\mathbb{R}^N} \left(\mathsf{L}^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f})\right)^{\mathsf{T}}\mathsf{L}^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f}) \\
&= \arg\min_{\mathsf{w}\in\mathbb{R}^N} \left\|\mathsf{L}^{-1}(\mathsf{B}\mathsf{w}-\mathsf{f})\right\|_2^2.
\end{aligned}$$

Discrete Least Squares

Two solution paradigms

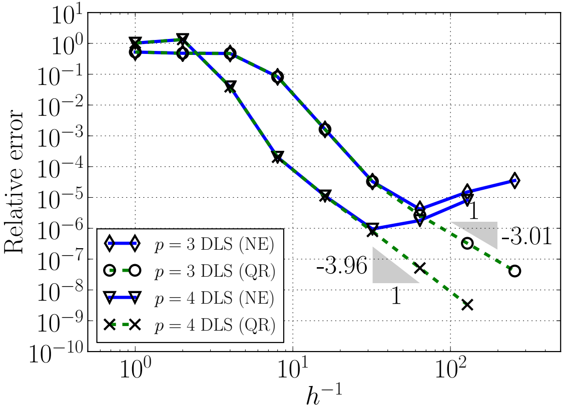
Normal equations

$$\mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{B}\mathsf{u} = \mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{f}$$

Overdetermined system

$$\mathsf{L}^{-1}\mathsf{B}\mathsf{u} = \mathsf{L}^{-1}\mathsf{f}$$

- ⋆ Static condensation for $\mathsf{L}^{-1}\mathsf{B}$ is explained in the article
- ⋆ $\mathsf{L}^{-1}\mathsf{B}$ has $O(h^{-1})$ condition number
- ⋆ Less round off error when handling $\mathsf{L}^{-1}\mathsf{B}$ directly



Acoustics with DPG near resonance

Highlights

- ⋆ DPG/DPG* always deliver SPD (HPD) stiffness matrices

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- ⋆ DPG/DPG* always deliver SPD (HPD) stiffness matrices
- ⋆ DPG/DPG* have built-in stability
- ⋆ DPG has a built-in error estimator
- ⋆ DPG* works for some problems without uniqueness

A priori error estimation

Continuous problem:

Find $\boldsymbol{v} \in \mathcal{V}$ and $\boldsymbol{w} \in \mathcal{U}$:

$$
\begin{aligned}
(\boldsymbol{v}, \nu)_{\mathcal{V}} + b(\boldsymbol{w}, \nu) &= F(\nu), \quad \nu \in \mathcal{V} \\
b(\mu, \boldsymbol{v}) \qquad\qquad\quad &= G(\mu), \quad \mu \in \mathcal{U}
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Discrete problem:

Find $v_h \in \mathcal{V}_h$ and $w_h \in \mathcal{U}_h$:

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Fortin operator:

Continuous operator $\Pi_h : \mathcal{V} \to \mathcal{V}_h$ such that

$$b(\mu, \nu - \Pi_h \nu) = 0 \qquad \forall\, \mu \in \mathcal{U}_h, \ \nu \in \mathcal{V}.$$

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Theorem (Brezzi)

Let $F \in \mathcal{V}'$ and $G \in \mathcal{U}'$. There is a constant C such that

$$\|v - v_h\|_{\mathcal{V}} + \|w - w_h\|_{\mathcal{U}} \leq C \left[\inf_{\nu \in \mathcal{V}_h} \|v - \nu\|_{\mathcal{V}} + \inf_{\mu \in \mathcal{U}_h} \|w - \mu\|_{\mathcal{U}} \right].$$

A priori error estimation: DPG

Continuous problem:

Find $\varepsilon \in \mathcal{V}$ and $\boldsymbol{u} \in \mathcal{U}$:

$$\begin{aligned} (\varepsilon, \nu)_{\mathcal{V}} + b(\boldsymbol{u}, \nu) &= F(\nu), \quad & \nu \in \mathcal{V} \\ b(\mu, \varepsilon) &= 0, \quad & \mu \in \mathcal{U} \end{aligned}$$

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Theorem

Let $F \in (\mathrm{Null}\,\mathcal{B}')^{\perp}$. *There is a constant* C *such that*

$$\|\varepsilon - \varepsilon_h\|_{\mathcal{V}} + \|u - u_h\|_{\mathcal{U}} \le C\left[\inf_{\nu \in \mathcal{V}_h} \|\varepsilon - \nu\|_{\mathcal{V}} + \inf_{\mu \in \mathcal{U}_h} \|u - \mu\|_{\mathcal{U}}\right].$$

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Recall:

$$b(u, v) = F(v) \qquad \forall\, v \in \mathcal{V} \qquad \Longrightarrow \qquad \varepsilon = 0.$$

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A priori error estimation: DPG*

Continuous problem:

Find $v \in \mathcal{V}$ and $\lambda \in \mathcal{U}$:

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DPG*: The discretization error is affected by λ!

A priori error estimation: DPG*

Poisson's equation (ultraweak)

Consider

$$-\Delta v = f$$

$$\Longrightarrow$$

Broken ultraweak formulation

$$\begin{aligned} b((u,\vec{q},\hat{u},\hat{q}_n),(v,\vec{p})) &= (\vec{q},\vec{p}-\operatorname{grad}_h v)_\Omega - (u,\operatorname{div}_h \vec{p})_\Omega \\ &+ \langle \hat{u},\vec{p}\cdot\vec{n}\rangle_h + \langle \hat{q}_n, v\rangle_h \end{aligned}$$

A priori error estimation: DPG*

Poisson's equation (ultraweak)

Consider

$$-\Delta v = f \quad \Longrightarrow$$

Broken ultraweak formulation

$$
\begin{aligned}
b((u, \vec{q}, \hat{u}, \hat{q}_n), (v, \vec{p}))
&= \big((u, \vec{q}), \mathcal{L}_h(v, \vec{p})\big)_\Omega \\
&\quad + \langle \hat{u}, \vec{p} \cdot \vec{n} \rangle_h + \langle \hat{q}_n, v \rangle_h
\end{aligned}
$$

where $\mathcal{L}_h(v, \vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h v)$.

A priori error estimation: DPG*

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where $\mathcal{L}_h(v,\vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h v)$.

DPG* formulation:

$$\begin{aligned} ((v,\vec{p}),(\nu,\vec{\tau}))_V - b((\lambda,\vec{\zeta},\hat{\lambda},\hat{\zeta}),(\nu,\vec{\tau})) &= 0 && \forall\, \nu,\vec{\tau} \\ b((\mu,\vec{\sigma},\hat{\mu},\hat{\sigma}_n),(v,\vec{p})) &= (f,\mu)_\Omega && \forall\, \mu,\vec{\sigma},\hat{\mu},\hat{\sigma}_n \end{aligned}$$

A priori error estimation: DPG*

Poisson's equation (ultraweak)

Proposition

Let $\|(v,\vec{p})\|_V^2 = \|\operatorname{grad}_h v\|_\Omega^2 + \|v\|_\Omega^2 + \|\operatorname{div}_h \vec{p}\|_\Omega^2 + \|\vec{p}\|_\Omega^2$. The solution components $\lambda, \vec{\zeta}, \hat{\lambda}, \hat{\zeta}$ of the system above can be characterized using the remaining solution components v, $\vec{p}$ and f as

$$
\begin{aligned}
\lambda &= f + e, & \hat{\lambda} &= e \\
\vec{\zeta} &= \vec{p} + \vec{r}, & \hat{\zeta} &= 2\vec{p}\cdot\vec{n} + \vec{r}\cdot\vec{n}
\end{aligned}
$$

where $e \in H_0^1(\Omega)$ satisfies the Dirichlet problem $-\Delta e = v + 2f$ and $\vec{r} = -\operatorname{grad} e$.

L. Demkowicz, J. Gopalakrishnan, and B. Keith.
The DPG-star method.
In preparation, 2018.

T. Führer.
Superconvergence in a DPG method for an ultra-weak formulation.
Comput. Math. Appl., 75(5):1705 -- 1718, 2018.

Sketch of proof

- Use the second line to characterize the solution $\upsilon, \vec{p}$.

Note that $b((\mu,\vec{\sigma},\hat{\mu},\hat{\sigma}_n),(\upsilon,\vec{p})) = (f,\mu)_\Omega$.

$$\Longrightarrow \qquad \begin{aligned} \big((\mu,\vec{\sigma}),\mathcal{L}_h(\upsilon,\vec{p})\big)_\Omega &= (f,\mu)_\Omega && \forall\, \mu,\vec{\sigma} \\ \langle \hat{\mu}, \vec{p}\cdot\vec{n}\rangle_h &= 0 && \forall\, \hat{\mu} \\ \langle \hat{\sigma}_n, \upsilon\rangle_h &= 0 && \forall\, \hat{\sigma}_n \end{aligned}$$

where $\mathcal{L}_h(\upsilon,\vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h \upsilon)$.

Sketch of proof

- Use the second line to characterize the solution $\upsilon, \vec{p}$.

Note that $b((\mu, \vec{\sigma}, \hat{\mu}, \hat{\sigma}_n), (\upsilon, \vec{p})) = (f, \mu)_\Omega$.

$$\implies \qquad \begin{aligned} \big((\mu, \vec{\sigma}), \mathcal{L}_h(\upsilon, \vec{p})\big)_\Omega &= (f, \mu)_\Omega && \forall\, \mu, \vec{\sigma} \\ \langle \hat{\mu}, \vec{p} \cdot \vec{n} \rangle_h &= 0 && \forall\, \hat{\mu} \\ \langle \hat{\sigma}_n, \upsilon \rangle_h &= 0 && \forall\, \hat{\sigma}_n \end{aligned}$$

where $\mathcal{L}_h(\upsilon, \vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h \upsilon)$.

- Show that this implies

$$\vec{p} - \operatorname{grad} \upsilon = \vec{0}, \qquad -\operatorname{div} \vec{p} = f.$$

L. Demkowicz, J. Gopalakrishnan, and B. Keith.
The DPG-star method.
In preparation, 2018.

T. Führer.
Superconvergence in a DPG method for an ultra-weak formulation.
Comput. Math. Appl., 75(5):1705 -- 1718, 2018.

Sketch of proof

- Manipulate first term in first line:

$$
\begin{aligned}
((\upsilon, \vec{p}))_{\mathcal{V}} &= (\vec{p}, \vec{\tau})_\Omega + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega + (\operatorname{grad} \upsilon, \operatorname{grad} \nu)_\Omega \\
&= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_\Omega + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega + 2(\operatorname{grad} \upsilon, \operatorname{grad} \nu)_\Omega \\
&= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_\Omega + (f, -\operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega \\
&\qquad + 2 \sum_{K \in \Omega} \Big[\langle \vec{n} \cdot \operatorname{grad} \upsilon, \nu \rangle_{H^{1/2}(\partial K)} - (\Delta \upsilon, \nu)_K \Big] \\
&= b((f, \vec{p}, 0, 2\vec{p} \cdot \vec{n}), (\nu, \vec{\tau})) + (\upsilon + 2f, \nu)_\Omega
\end{aligned}
$$

Sketch of proof

- Manipulate first term in first line:

$$
\begin{aligned}
((\upsilon, \vec{p}))_{\mathcal{V}} &= (\vec{p}, \vec{\tau})_\Omega + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega + (\operatorname{grad} \upsilon, \operatorname{grad} \nu)_\Omega \\
&= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_\Omega + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega + 2(\operatorname{grad} \upsilon, \operatorname{grad} \nu)_\Omega \\
&= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_\Omega + (f, -\operatorname{div} \vec{\tau})_\Omega + (\upsilon, \nu)_\Omega \\
&\qquad + 2 \sum_{K \in \Omega} \Big[\langle \vec{n} \cdot \operatorname{grad} \upsilon, \nu \rangle_{H^{1/2}(\partial K)} - (\Delta \upsilon, \nu)_K \Big] \\
&= b((f, \vec{p}, 0, 2\vec{p} \cdot \vec{n}), (\nu, \vec{\tau})) + (\upsilon + 2f, \nu)_\Omega
\end{aligned}
$$

- Define e, $\vec{r}$, $\hat{e}$, and $\hat{r}$ such that

$$
b((e, \vec{r}, \hat{e}, \hat{r}), (\nu, \vec{\tau})) = (\upsilon + 2f, \nu)_\Omega \qquad \Longrightarrow \qquad -\Delta e = \upsilon + 2f
$$

A priori error estimation: DPG*

Poisson´s equation

Bramble-Hilbert argument:

Corollary

Let $\vec{\upsilon}_h = (\upsilon_h, \vec{p}_h) \in \mathcal{V}_h$ *and* $\vec{\lambda}_h = (\lambda_h, \vec{\zeta}_h, \hat{\lambda}_h, \hat{\zeta}_h) \in \mathcal{U}_h$ *be the DPG* solutions.*
Let $e \in H_0^1(\Omega)$ *satisfy* $-\Delta e = \upsilon + 2f$*. Then*

$$\|\vec{\upsilon} - \vec{\upsilon}_h\|_{\mathcal{V}} + \|\vec{\lambda} - \vec{\lambda}_h\|_{\mathcal{U}} \leq Ch^s \big(\|\upsilon\|_{H^{s+2}(\Omega)} + \|e\|_{H^{s+2}(\Omega)} \big)$$

for all $1/2 < s < p + 1$.

A priori error estimation: DPG*

Poisson's equation

Manufactured solution 1: $\upsilon = \sin(\pi x)\sin(\pi y), \qquad \Omega = (0,1)^2$

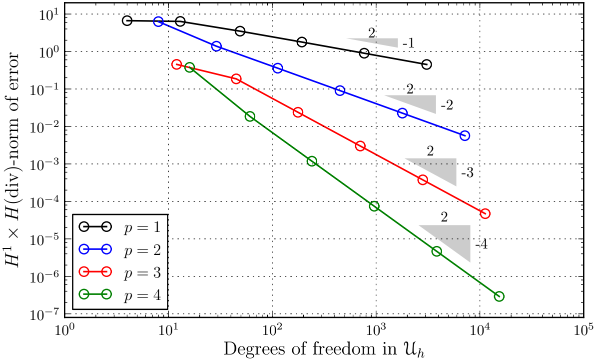


Figure: Uniform h-refinements

A priori error estimation: DPG*

Poisson's equation

Manufactured solution 2: $\upsilon = 1$

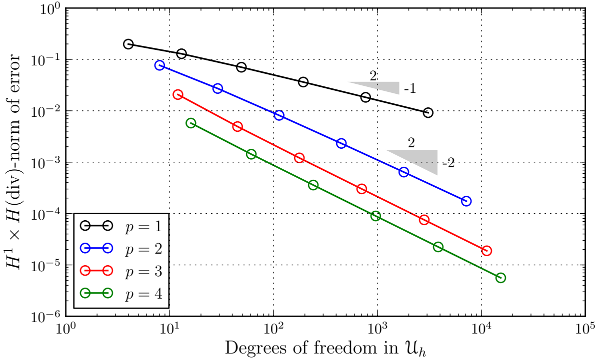


Figure: Uniform h-refinements

A priori error estimation: DPG*

Poisson's equation

Manufactured solution 2: $u = 1$

$$\boxed{-\Delta e = 1 \text{ in } \Omega, \quad e|_{\partial\Omega} = 0}$$

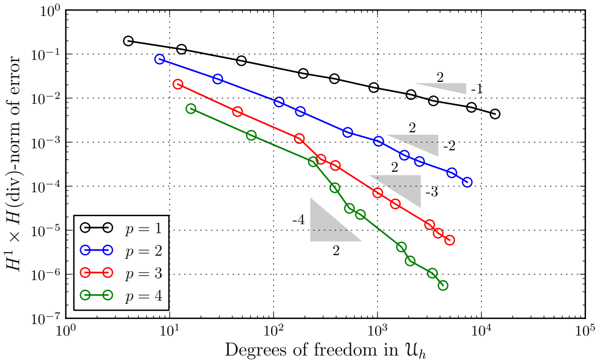


Figure: *Adaptive h*-refinements

A priori error estimation: DPG*

Poisson´s equation

Manufactured solution 2: $\quad \upsilon = 1$

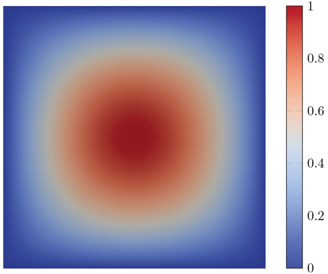
$$\boxed{-\Delta e = 1 \text{ in } \Omega, \quad e\big|_{\partial\Omega} = 0}$$

A priori error estimation: DPG*

Poisson's equation

Manufactured solution 2: $\upsilon = 1$

$$\boxed{-\Delta\lambda = 1 \text{ in } \Omega, \quad \lambda|_{\partial\Omega} = 0}$$



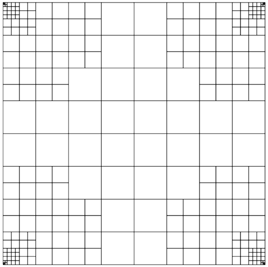


Figure: Left: The solution component λ when $\upsilon = 1$. Right: The corresponding mesh (ten refinements).

Duality

DPG:

(1) $(\varepsilon, z)_{\mathcal{V}} + b(u, z) = F(z)$

(2) $b(w, \varepsilon) = 0$

DPG*:

(3) $(v, z)_{\mathcal{V}} - b(\lambda, z) = 0$

(4) $b(w, v) = G(w)$

DPG:

(1) $(\varepsilon, z)_{\mathcal{V}} + b(u, z) = F(z)$

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(3) $(v, z)_{\mathcal{V}} - b(\lambda, z) = 0$

(4) $b(w, v) = G(w)$

Observe:

$G(u - u_h) = b(u - u_h, v)$ by (4)

DPG:

(1) $(\varepsilon, z)_{\mathcal{V}} + b(u, z) = F(z)$

(2) $b(w, \varepsilon) = 0$

DPG*:

(3) $(v, z)_{\mathcal{V}} - b(\lambda, z) = 0$

(4) $b(w, v) = G(w)$

Observe:

$$
\begin{aligned}
G(u - u_h) &= b(u - u_h, v) && \text{by (4)} \\
&= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)}
\end{aligned}
$$

DPG:

$$(5)\quad (\varepsilon_h, z_h)_V + b(u_h, z_h) = F(z_h)$$
$$(6)\quad b(w_h, \varepsilon_h) = 0$$

DPG*:

$$(7)\quad (v_h, z_h)_V - b(\lambda_h, z_h) = 0$$
$$(8)\quad b(w_h, v_h) = G(w_h)$$

Observe:

$$\begin{aligned} G(u - u_h) &= b(u - u_h, v) && \text{by (4)} \\ &= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_V - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \end{aligned}$$

DPG:

(9) $(\varepsilon - \varepsilon_h, z_h)_{\mathcal{V}} + b(u - u_h, z_h) = 0$

(10) $b(w_h, \varepsilon - \varepsilon_h) = 0$

DPG*:

(11) $(v - v_h, z_h)_{\mathcal{V}} - b(\lambda - \lambda_h, z_h) = 0$

(12) $b(w_h, v - v_h) = 0$

Observe:

$$G(u - u_h) = b(u - u_h, v) \qquad \text{by (4)}$$

$$= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda, \varepsilon - \varepsilon_h) \qquad \text{by (3)}$$

DPG:

(9) $(\varepsilon - \varepsilon_h, z_h)_{\mathcal{V}} + b(u - u_h, z_h) = 0$

(10) $b(w_h, \varepsilon - \varepsilon_h) = 0$

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Observe:

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&= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \\
&= b(u - u_h, v - z_h) + (v - z_h, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda - w_h, \varepsilon - \varepsilon_h) && \text{by (9-10)}
\end{aligned}
$$

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&= b(u - u_h, v - z_h) + (v - z_h, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda - w_h, \varepsilon - \varepsilon_h) && \text{by (9-10)} \\
&= b(u - u_h, v - z_h) - (v - z_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - w_h, \varepsilon_h) && \varepsilon = 0
\end{aligned}
$$

DPG:

(9) $(\varepsilon - \varepsilon_h, z_h)_\mathcal{V} + b(u - u_h, z_h) = 0$

(10) $b(w_h, \varepsilon - \varepsilon_h) = 0$

DPG*:

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&= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_\mathcal{V} - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \\
&= b(u - u_h, v - z_h) + (v - z_h, \varepsilon - \varepsilon_h)_\mathcal{V} - b(\lambda - w_h, \varepsilon - \varepsilon_h) && \text{by (9-10)} \\
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&= b(u - u_h, v - v_h) - (v - v_h, \varepsilon_h)_\mathcal{V} + b(\lambda - \lambda_h, \varepsilon_h)
\end{aligned}
$$

DPG:

$$(9) \quad (\varepsilon - \varepsilon_h, z_h)_\mathbb{V} + b(u - u_h, z_h) = 0$$
$$(10) \quad b(w_h, \varepsilon - \varepsilon_h) = 0$$

DPG*:

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G(u - u_h) &= b(u - u_h, v) && \text{by (4)} \\
&= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_\mathbb{V} - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \\
&= b(u - u_h, v - z_h) + (v - z_h, \varepsilon - \varepsilon_h)_\mathbb{V} - b(\lambda - w_h, \varepsilon - \varepsilon_h) && \text{by (9-10)} \\
&= b(u - u_h, v - z_h) - (v - z_h, \varepsilon_h)_\mathbb{V} + b(\lambda - w_h, \varepsilon_h) && \varepsilon = 0 \\
&= b(u - u_h, v - v_h) - (v - v_h, \varepsilon_h)_\mathbb{V} + b(\lambda - \lambda_h, \varepsilon_h) && \\
&= b(u - u_h, v - v_h) && \text{by (11)}
\end{aligned}$$

DPG:

$$(9)\quad (\varepsilon - \varepsilon_h, z_h)_{\mathcal{V}} + b(u - u_h, z_h) = 0$$
$$(10)\quad b(w_h, \varepsilon - \varepsilon_h) = 0$$

DPG*:

$$(11)\quad (v - v_h, z_h)_{\mathcal{V}} - b(\lambda - \lambda_h, z_h) = 0$$
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Observe:

$$
\begin{aligned}
G(u - u_h) &= b(u - u_h, v) && \text{by (4)}\\
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&= b(u - u_h, v - z_h) - (v - z_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - w_h, \varepsilon_h) && \varepsilon = 0\\
&= b(u - u_h, v - v_h) - (v - v_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - \lambda_h, \varepsilon_h) &&\\
&= b(u - u_h, v - v_h) && \text{by (11)}\\
&= b(u - w_h, v - v_h) && \text{by (12)}
\end{aligned}
$$

DPG:

(1) $(\varepsilon, z)_{\mathcal{V}} + b(u, z) = F(z)$

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DPG*:

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&= b(u - u_h, v - z_h) - (v - z_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - w_h, \varepsilon_h) && \varepsilon = 0\\
&= b(u - u_h, v - v_h) - (v - v_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - \lambda_h, \varepsilon_h) &&\\
&= b(u - u_h, v - v_h) && \text{by (11)}\\
&= b(u - w_h, v - v_h) && \text{by (12)}\\
&= F(v - v_h) && \text{by (1)}
\end{aligned}
$$

DPG:

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&= b(u - u_h, v) + (v, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \\
&= b(u - u_h, v - z_h) + (v - z_h, \varepsilon - \varepsilon_h)_{\mathcal{V}} - b(\lambda - w_h, \varepsilon - \varepsilon_h) && \text{by (9-10)} \\
&= \boxed{b(u - u_h, v - z_h) - (v - z_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - w_h, \varepsilon_h)} && \varepsilon = 0 \\
&= b(u - u_h, v - v_h) - (v - v_h, \varepsilon_h)_{\mathcal{V}} + b(\lambda - \lambda_h, \varepsilon_h) && \\
&= \boxed{b(u - u_h, v - v_h)} && \text{by (11)} \\
&= \boxed{b(u - w_h, v - v_h)} && \text{by (12)} \\
&= F(v - v_h) && \text{by (1)}
\end{aligned}
$$

Aubin-Nitsche

Corollary

The following crude upper bounds hold:

$$b(u-u_h, v-v_h) \lesssim \begin{cases} \|u-u_h\|_{\mathcal{U}} \displaystyle\inf_{z_h \in \mathcal{V}_h,\, w_h \in \mathcal{U}_h} \Big(\|v - z_h\|_{\mathcal{V}} + \|\lambda - w_h\|_{\mathcal{U}} \Big) \\ \|v - v_h\|_{\mathcal{V}} \displaystyle\inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} \end{cases}$$

Aubin-Nitsche

Theorem

Suppose that there is a positive function $c_1(h)$ that goes to 0 as $h \to 0$ satisfying

$$\inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} \leq c_1(h) \|F\|_{\mathcal{V}'}.$$

Then the error in the DPG solution component u_h satisfies*

$$G(u - u_h) \leq c_0(h) \|G\|_{\mathcal{U}'} \|\mathcal{B}\| \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} .$$

- Can be used to demonstrate accelerated convergence of DPG* solution components)

A priori error estimation: DPG*

Poisson's equation

Manufactured solution 2: $\upsilon = 1$

$$\boxed{-\Delta\lambda = 1 \text{ in } \Omega, \quad \lambda|_{\partial\Omega} = 0}$$

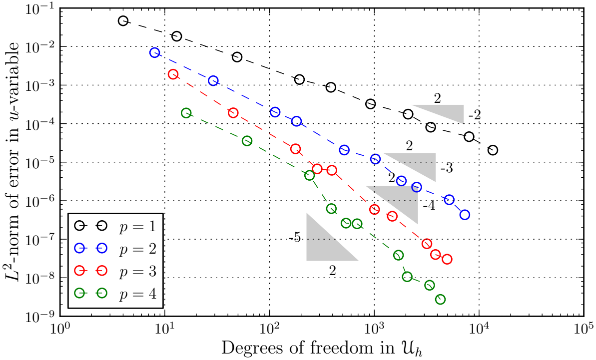

Figure: *Adaptive* h-refinements.

A posteriori error estimation

A posteriori error estimation: DPG

DPG minimizes the error in the *energy* norm

$$|||w|||_{\mathcal{U}} := \|\mathcal{B}w\|_{\mathcal{V}'} \qquad \forall\, w \in \mathcal{U}$$

C. Carstensen, L. Demkowicz, and J. Gopalakrishnan.
A posteriori error control for DPG methods.
SIAM J. Numer. Anal., 52(3):1335--1353, 2014.

A posteriori error estimation: DPG

DPG minimizes the error in the *energy* norm

$$|||w|||_{\mathcal{U}} := \|\mathcal{B}w\|_{\mathcal{V}'} \qquad \forall\, w \in \mathcal{U}$$

The energy norm $|||\cdot|||_{\mathcal{U}}$ is equivalent to the norm $\|\cdot\|_{\mathcal{U}}$

$$\|\mathcal{B}^{-1}\|^{-1}\|w\|_{\mathcal{U}} \leq |||w|||_{\mathcal{U}} \leq \|\mathcal{B}\|\|w\|_{\mathcal{U}}$$

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A posteriori error estimator for broken test spaces

$$|||u - w|||_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2$$

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A posteriori error estimator for broken test spaces

$$|||u - w|||_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2 = \sum_{K\in\mathcal{T}} \|\mathcal{B}w - F\|_{\mathcal{V}(K)'}^2$$

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A posteriori error estimator for broken test spaces

$$|||u - w|||_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2 = \sum_{K\in\mathcal{T}} \|\mathcal{B}w - F\|_{\mathcal{V}(K)'}^2 \simeq \sum_{K\in\mathcal{T}} \Big(\underbrace{\|\mathcal{B}w - F\|_{\mathcal{V}_h(K)'}}_{:=\eta_K(w)}\Big)^2$$

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The energy norm $|||\cdot|||_{\mathcal{U}}$ is equivalent to the norm $\|\cdot\|_{\mathcal{U}}$

$$\|\mathcal{B}^{-1}\|^{-1}\|w\|_{\mathcal{U}} \leq |||w|||_{\mathcal{U}} \leq \|\mathcal{B}\|\|w\|_{\mathcal{U}}$$

A posteriori error estimator for broken test spaces

$$|||u - w|||^2_{\mathcal{U}} = \|\mathcal{B}w - F\|^2_{\mathcal{V}'} = \sum_{K\in\mathcal{T}} \|\mathcal{B}w - F\|^2_{\mathcal{V}(K)'} \simeq \sum_{K\in\mathcal{T}} \Big(\underbrace{\|\mathcal{B}w - F\|_{\mathcal{V}_h(K)'}}_{:=\eta_K(w)}\Big)^2$$

$$\boxed{\eta(u_h)^2 = \sum_{K\in\mathcal{T}} \eta_K(u_h)^2}$$

where

$$\boxed{\eta_K(u_h) = \sup_{z_h\in\mathcal{V}_h(K)} \frac{b(u_h, z_h) - F(z_h)}{\|z_h\|_{\mathcal{V}}}}$$

C. Carstensen, L. Demkowicz, and J. Gopalakrishnan.
A posteriori error control for DPG methods.
SIAM J. Numer. Anal., 52(3):1335–1353, 2014.

A posteriori error estimation: DPG*

Let $\mathcal{P} : \mathcal{V} \to \mathrm{Null}\,\mathcal{B}'$ be the orthogonal projection.

A posteriori error estimation: DPG*

Let $\mathcal{P} : \mathcal{V} \to \mathrm{Null}\,\mathcal{B}'$ be the orthogonal projection.

Claim. Identity for the error in the dual problem:

$$\|\boldsymbol{z}\|_{\mathcal{V}}^2 = |||\mathcal{B}'\boldsymbol{z}|||_{\mathcal{U}'}^2 + \|\mathcal{P}\boldsymbol{z}\|_{\mathcal{V}}^2 \qquad \forall\, \boldsymbol{z} \in \mathcal{V}$$

A posteriori error estimation: DPG*

Let $\mathcal{P} : \mathcal{V} \to \mathrm{Null}\,\mathcal{B}'$ be the orthogonal projection.

Claim. Identity for the error in the dual problem:

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Proof:

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Moreover, $\|\mathcal{B}^{-1}\|^{-1}\|\cdot\|_{\mathcal{U}} \leq |||\cdot|||_{\mathcal{U}} \leq \|\mathcal{B}\|\|\cdot\|_{\mathcal{U}}$, so

$$\|\mathcal{B}\|^{-1}\|\mathcal{B}'v_h - G\|_{\mathcal{U}'}^2 \leq \|v - v_h\|_{\mathcal{V}}^2 \leq \|\mathcal{B}^{-1}\|\|\mathcal{B}'v_h - G\|_{\mathcal{U}'}^2$$

Residual bounds

Theorem

Let $w \in \mathcal{U}$ and $z \in \mathcal{V}$ be arbitrary. Then the following upper bound holds:

$$|b(u - w, v - z)| \leq \|\mathcal{B}^{-1}\| \|\mathcal{B}w - F\|_{\mathcal{V}'} \|\mathcal{B}'z - G\|_{\mathcal{U}'} \,.$$

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Proof:

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$$\begin{aligned} |b(u - w, v - z)| &= |b(u - w, v - z - z_0)| \\ &\leq \|\mathcal{B}w - F\|_{\mathcal{V}'} \|v - z - z_0\| \\ &= \|\mathcal{B}w - F\|_{\mathcal{V}'} \big(\|\mathcal{P}(v - z - z_0)\|_{\mathcal{V}}^2 + \|\|\mathcal{B}'z - G\|\|_{\mathcal{U}'}^2\big)^{1/2} \,. \end{aligned}$$

- Set $z_0 = \mathcal{P}(v - z) \in \mathrm{Null}\,\mathcal{B}'$.

Algorithm and Marking Strategies

. . . Solve–Estimate–Estimate–Mark–Refine. . .

Adaptive mesh refinement

1. Solve for u_h and/or v_h

- Mark all elements $K \in \mathcal{T}$ such that $\theta \cdot \widetilde{\eta}_{\max} \leq \widetilde{\eta}_K$, $\theta \in (0, 1)$
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5. Refine all marked elements and construct a new mesh

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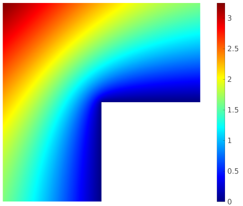
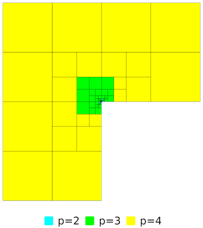
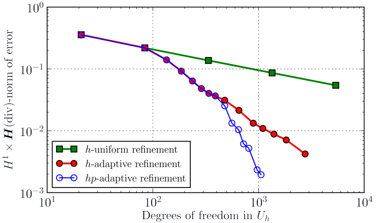
Applications

hp-adaptive mesh refinement in DPG* methods



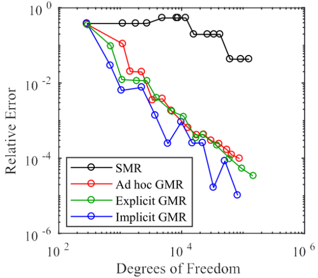




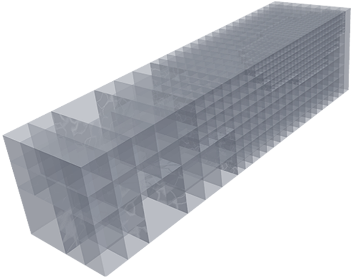
Singular solution and hp-adaptive mesh refinements with a DPG* method for Poisson's equation.

L. Demkowicz, J. Gopalakrishnan, and B. Keith.
The DPG-star method.
In preparation, 2018.

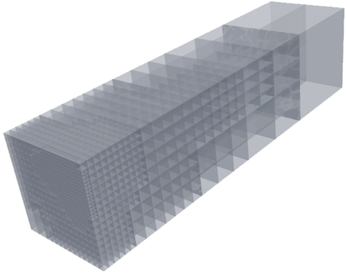
Goal-Oriented Adaptive Mesh Refinement (AMR) in 3D



The error in the average flux.



Solution-oriented AMR.



Goal-oriented AMR

B. Keith, A. Vaziri Astaneh, and L. Demkowicz.
Goal-oriented adaptive mesh refinement for non-symmetric functional settings.
arXiv:1711.01996 (math.NA), 2017.

Fluid Mechanics

Oldroyd-B fluid

Conservation of mass and momentum:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\boldsymbol{\sigma} = \rho\,\mathbf{f} \quad \text{on } \Omega\times(0,T)\,.$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{on } \Omega\times(0,T)\,.$$

Constitutive law:

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\mu_S\,\varepsilon(\mathbf{u}) + \mathbf{T}\,,$$

where

$$\mathbf{T} + \lambda\,\mathfrak{L}_{\mathbf{u}}\mathbf{T} = 2\mu_P\,\varepsilon(\mathbf{u})\,.$$

Lie derivative:

$$\mathfrak{L}_{\mathbf{u}}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{T} - \left(\nabla\mathbf{u}\,\mathbf{T} - \mathbf{T}\,\nabla^{\mathsf{T}}\mathbf{u}\right)\,.$$

B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr, and L. Demkowicz.
An ultraweak DPG method for viscoelastic fluids.
J. Non-Newton. Fluid Mech., 247:107--122, 2017.

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Quantity of interest

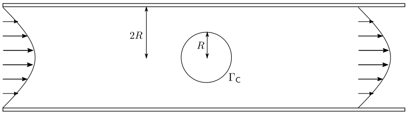
Drag coefficient

$$c_{\mathrm{D}}(\boldsymbol{\sigma}) = \frac{1}{\mu\bar{u}}\int_{\Gamma_C}(\boldsymbol{\sigma}\vec{n})\cdot\mathbf{e}_x\,\mathrm{d}s\,.$$

Γ_C: boundary of cylinder

$\mu = \mu_S + \mu_P$: viscosity

$\bar{u}$: average inflow velocity.



Confined cylinder domain.

B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr, and L. Demkowicz.
An ultraweak DPG method for viscoelastic fluids.
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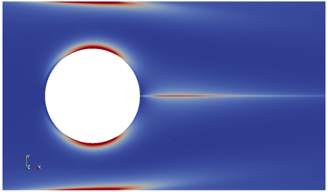
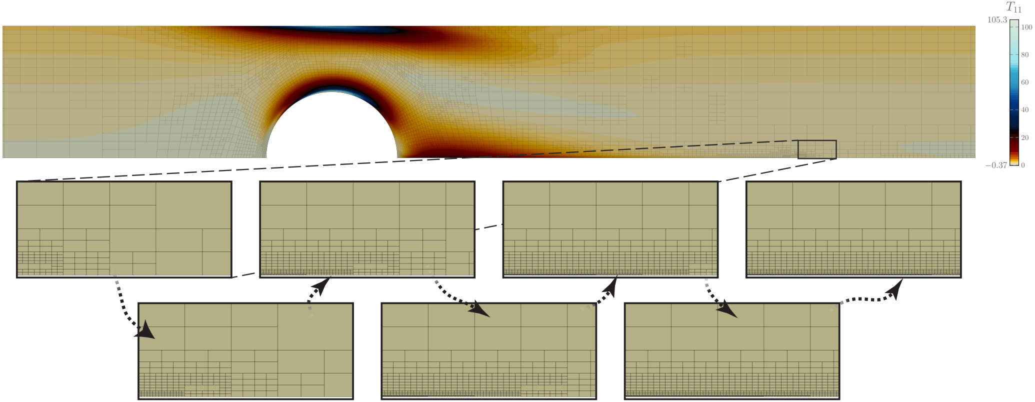


Close-up of rescaled T_{11}-component from $\lambda = 0.9$.

Fluid Mechanics

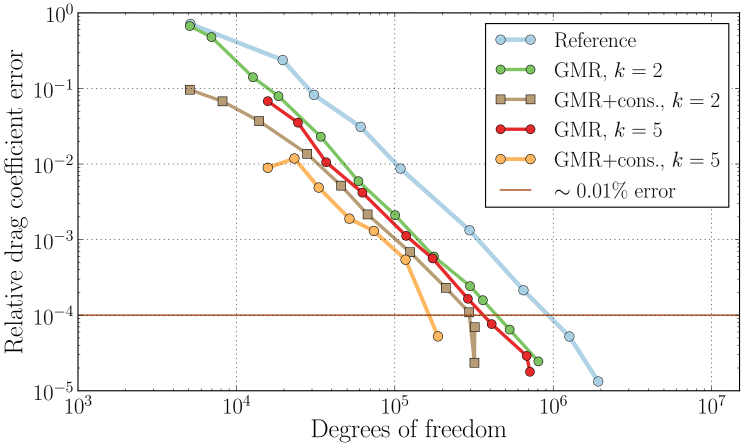
Solution-oriented adaptive mesh refinement



Close-up of rescaled T_{11}-component from $\lambda = 0.7$.

B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr, and L. Demkowicz.
An ultraweak DPG method for viscoelastic fluids.
J. Non-Newton. Fluid Mech., 247:107--122, 2017.

Goal-oriented adaptivity in nonlinear problems



The error in the drag coefficient of the confined cylinder ($\lambda = 0.4$).

B. Keith, N. V. Roberts, and L. Demkowicz.
Goal-oriented adaptive mesh refinement with a DPG method for viscoelastic fluids.
In preparation, 2018.

A Posteriori Error Estimation

DPG* Methods

- $\mathcal{T}$: regular subdivision of $\Omega \subset \mathbb{R}^2$ into triangles

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Theorem (Demkowicz, Gopalakrishnan, & K.)

Assume $\Omega \subset \mathbb{R}^2$ with sufficiently regular boundary. If $\vec{v}_h = (v_h, \vec{p}_h) \in \mathcal{V}_h$ satisfies

$$b(\vec{w}_h, \vec{v}_h) = G(\vec{w}_h) \qquad \forall\, \vec{w}_h \in \mathcal{U}_h\,,$$

then $\exists\; C_2 > C_1 > 0$, independent of the element sizes h_K, such that

$$C_1 \eta^*(\vec{v}_h) \leq \|\vec{v} - \vec{v}_h\|_{\mathcal{V}} \leq C_2 \eta^*(\vec{v}_h)\,,$$

where

$$\eta^*(\vec{v}_h)^2 = \big\|\mathcal{L}_h \vec{v}_h - f\big\|_\Omega^2 + \sum_{E \in \mathcal{E}_{\mathrm{int}}} h_E \big\|[\![\vec{p}_h]\!]\big\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}} h_E \big\|[\![v_h]\!]\big\|_{H^1(E)}^2$$

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$$\|\upsilon - \upsilon_h\|_{\mathcal{V}}^2 \eqsim \|\mathcal{B}'\upsilon_h - G\|_{\mathcal{U}'}^2$$

- Decompose supremum into three terms:

$$\left(\sup_{\vec{u}\in\mathcal{U}} \frac{b(\vec{u},\vec{\upsilon}_h) - G(\vec{u})}{\|\vec{u}\|_{\mathcal{U}}} \right)^2 = \left(\sup_{(u,\vec{q})\in L^2(\Omega)\times L^2(\Omega)^d} \frac{((u,\vec{q}),\mathcal{L}_h\vec{\upsilon}_h - f)_\Omega}{\|(u,\vec{q})\|_\Omega} \right)^2$$
$$+ \left(\sup_{\hat{u}\in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \vec{p}_h\cdot\vec{n}\rangle_h}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}} \right)^2$$
$$+ \left(\sup_{\hat{q}_n\in H^{-1/2}(\mathcal{S})} \frac{\langle \hat{q}_n, \upsilon_h\rangle_h}{\|\hat{q}_n\|_{H^{-1/2}(\partial\mathcal{T})}} \right)^2$$

Sketch of the Proof

Term 1:

- $$\sup_{(u,\vec{q})\in L^2(\Omega)\times L^2(\Omega)^d} \frac{\big((u,\vec{q}),\mathcal{L}_h\vec{v}_h - f\big)_\Omega}{\big\|(u,\vec{q})\big\|_\Omega} = \big\|\mathcal{L}_h v_h - f\big\|_\Omega$$

Sketch of the Proof

Term 1:

- $$\sup_{(u,\vec{q})\in L^2(\Omega)\times L^2(\Omega)^d} \frac{((u,\vec{q}),\mathcal{L}_h\vec{v}_h - f)_\Omega}{\|(u,\vec{q})\|_\Omega} = \|\mathcal{L}_h v_h - f\|_\Omega$$

Term 2*:

- $$\sup_{\hat{u}\in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \vec{p}_h\cdot\vec{n}\rangle_h}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}} = \sup_{u\in H_0^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h\cdot\vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

Sketch of the Proof

Term 1:

- $$\sup_{(u,\vec{q})\in L^2(\Omega)\times L^2(\Omega)^d} \frac{((u,\vec{q}),\mathcal{L}_h\vec{v}_h - f)_\Omega}{\|(u,\vec{q})\|_\Omega} = \|\mathcal{L}_h v_h - f\|_\Omega$$

Term 2*:

- $$\sup_{\hat{u}\in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \vec{p}_h\cdot\vec{n}\rangle_h}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}} = \sup_{u\in H_0^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h\cdot\vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

Term 3*:

- $$\sup_{\hat{q}_n\in H^{-1/2}(\mathcal{S})} \frac{\langle \hat{q}_n, v_h\rangle_h}{\|\hat{q}_n\|_{H^{-1/2}(\partial\mathcal{T})}} = \sup_{\vec{q}\in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), v_h\rangle_h}{\|\vec{q}\|_{H(\mathrm{div},\Omega)}}$$

*requires a lemma

Sketch of the Proof

Theorem (Demkowicz, Gopalakrishnan, K.)

If $\int_{\partial K} \hat{p}_K [\![\vec{p}_h]\!] = 0$ for all $\hat{p}_K \in P^1_{\mathrm{C}}(\partial K)$ then

$$\sup_{\mu \in H^1_0(\Omega)} \frac{\langle \mathrm{tr}\mu, \vec{p}_h \cdot \vec{n} \rangle_h^2}{\|\mu\|^2_{H^1(\Omega)}} \eqsim \sum_{E \in \mathcal{E}} h_E \big\| [\![\vec{p}_h]\!] \big\|^2_{L^2(E)} .$$

Theorem (Demkowicz, Gopalakrishnan, K.)

If $\int_E [\![v_h]\!] = 0$ for all $E \in \mathcal{E}$ then

$$\sup_{\vec{q} \in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), v_h \rangle_h^2}{\|\vec{q}\|^2_{H(\mathrm{div},\Omega)}} \eqsim \sum_{E \in \mathcal{E}} h_E \big\| [\![v_h]\!] \big\|^2_{H^1(E)} \eqsim \sum_{E \in \mathcal{E}} h_E^{-1} \big\| [\![v_h]\!] \big\|^2_{L^2(E)} .$$

Sketch of the Proof

Theorem (Regular decompositions)

Given any $\vec{q} \in H(\mathrm{div}, \Omega)$, there exist $\varphi \in H^1(\Omega)$ and $\vec{\psi} \in H^1(\Omega)$ such that $\vec{q} = \mathrm{curl}(\varphi) + \vec{\psi}$ and

$$\|\varphi\|_{H^1(\Omega)} + \|\vec{\psi}\|_{H^1(\Omega)} \lesssim \|\vec{q}\|_{H(\mathrm{div},\Omega)} \,.$$

In this situation, $\vec{q} = \mathrm{curl}(\varphi) + \vec{\psi}$ is called a *regular decomposition* of $\vec{q}$.

Sketch of the Proof

Theorem (Demlow, Hirani, Schöberl)

Let $E \in \mathcal{E}$, $u \in H^1(\Omega)$, and $\vec{q} \in H(\mathrm{div}, \Omega)$ be arbitrary. Let h_E denote the length of the edge E and let $\varphi \in H^1(\Omega)$ and $\vec{\psi} \in H^1(\Omega)$ belong to a *regular decomposition* $\vec{q} = \mathrm{curl}(\varphi) + \vec{\psi}$. There exist *commuting* quasi-interpolation operators $\mathcal{I} : H^1(\Omega) \to P^1_C(\mathcal{T})$ and $\vec{\mathcal{I}} : H(\mathrm{div}, \Omega) \to \mathcal{RT}^0(\mathcal{T})$, such that $\mathrm{curl} \circ \mathcal{I} = \vec{\mathcal{I}} \circ \mathrm{curl}$ and the following inequalities hold:

(5) $$\|u - \mathcal{I}u\|^2_{L^2(E)} \lesssim h_E \|u\|^2_{H^1(\Omega_E)} ,$$

(6) $$\|\varphi - \mathcal{I}\varphi\|^2_{L^2(E)} + \|(\vec{\psi} - \vec{\mathcal{I}}\vec{\psi}) \cdot \vec{n}\|^2_{L^2(E)} \lesssim h_E \|\vec{q}\|^2_{H(\mathrm{div},\Omega_E)} .$$

Each of the statements above hold with $H^1(\Omega)$ replaced by $H^1_0(\Omega)$, $H^1(\Omega)$ replaced by $H^1_0(\Omega)$, and $H(\mathrm{div}, \Omega)$ replaced by $H_0(\mathrm{div}, \Omega)$. In this case, $\mathcal{I} : H^1_0(\Omega) \to P^1_{C,0}(\mathcal{T})$ and $\vec{\mathcal{I}} : H_0(\mathrm{div}, \Omega) \to \mathcal{RT}^0_0(\mathcal{T})$.

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Sketch of the Proof

$$\sup_{u\in H^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h \cdot \vec{n} \rangle_h}{\|u\|_{H^1(\Omega)}}$$

Sketch of the Proof

$$\sup_{u\in H^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h\cdot\vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

- Let $u_h = \mathcal{I}u$. Then

$$\langle u - u_h, \vec{p}_h \cdot \vec{n}\rangle_h = \sum_{K\in\mathcal{T}} (u - u_h, \vec{p}_h \cdot \vec{n}_K)_{\partial K} = \sum_{E\in\mathcal{E}} (u - u_h, [\![\vec{p}_h]\!])_E$$

Sketch of the Proof

$$\sup_{u\in H^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h\cdot\vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

- Let $u_h = \mathcal{I}u$. Then

$$\langle u - u_h, \vec{p}_h \cdot \vec{n}\rangle_h = \sum_{K\in\mathcal{T}} (u - u_h, \vec{p}_h \cdot \vec{n}_K)_{\partial K} = \sum_{E\in\mathcal{E}} (u - u_h, [\![\vec{p}_h]\!])_E$$

- For each edge E,

$$(u - u_h, [\![\vec{p}_h]\!])_E \leq \|u - \mathcal{I}u\|_E \|[\![\vec{p}_h]\!]\|_E \lesssim h_K^{1/2} \|u\|_{H^1(\Omega_E)} \|[\![\vec{p}_h]\!]\|_E$$

where Ω_E denotes the patch of elements associated with the edge E.

Sketch of the Proof

$$\sup_{u\in H^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h \cdot \vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

- Let $u_h = \mathcal{I}u$. Then

$$\langle u - u_h, \vec{p}_h \cdot \vec{n}\rangle_h = \sum_{K\in\mathcal{T}} (u - u_h, \vec{p}_h \cdot \vec{n}_K)_{\partial K} = \sum_{E\in\mathcal{E}} (u - u_h, [\![\vec{p}_h]\!])_E$$

- For each edge E,

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where Ω_E denotes the patch of elements associated with the edge E.

- Upper bound follows from manipulating the supremum

Sketch of the Proof

$$\sup_{u\in H^1(\Omega)} \frac{\langle \mathrm{tr}(u), \vec{p}_h\cdot\vec{n}\rangle_h}{\|u\|_{H^1(\Omega)}}$$

- Let $u_h = \mathcal{I}u$. Then

$$\langle u - u_h, \vec{p}_h \cdot \vec{n}\rangle_h = \sum_{K\in\mathcal{T}} (u - u_h, \vec{p}_h \cdot \vec{n}_K)_{\partial K} = \sum_{E\in\mathcal{E}} (u - u_h, [\![\vec{p}_h]\!])_E$$

- For each edge E,

$$(u - u_h, [\![\vec{p}_h]\!])_E \leq \|u - \mathcal{I}u\|_E \|[\![\vec{p}_h]\!]\|_E \lesssim h_K^{1/2} \|u\|_{H^1(\Omega_E)} \|[\![\vec{p}_h]\!]\|_E$$

where Ω_E denotes the patch of elements associated with the edge E.

- Upper bound follows from manipulating the supremum
- Lower bound follows from Verfürth's bubble function technique

Sketch of the Proof

$$\sup_{\vec{q}\in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), v_h\rangle_h}{\|\vec{q}\|_{H(\mathrm{div},\Omega)}}$$

- Gauss & Stokes:

$$\langle(\mathrm{curl}\,\phi)\cdot\vec{n}_K, v\rangle_{\partial K} = (\mathrm{curl}\,\phi, \mathrm{grad}\,v)_K + (\mathrm{div}\,\mathrm{curl}\,\phi, v)_K = \langle\phi, (\mathrm{grad}\,v)\cdot\vec{t}_K\rangle_{\partial K}$$

Sketch of the Proof

$$\sup_{\vec{q}\in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), \upsilon_h\rangle_h}{\|\vec{q}\|_{H(\mathrm{div},\Omega)}}$$

- Gauss & Stokes:

$$\langle (\mathrm{curl}\,\phi)\cdot \vec{n}_K, \upsilon\rangle_{\partial K} = (\mathrm{curl}\,\phi, \mathrm{grad}\,\upsilon)_K + (\mathrm{div}\,\mathrm{curl}\,\phi, \upsilon)_K = \langle \phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K\rangle_{\partial K}$$

- Let $\sigma_h = \vec{\mathcal{I}}\,\mathrm{curl}\,\phi = \mathrm{curl}\,\mathcal{I}\phi$. Then

$$\begin{aligned}\langle (\phi - \sigma_h)\cdot \vec{n}_K, \upsilon\rangle_{\partial K} &= \langle \phi - \mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K\rangle_{\partial K} \\ &= \sum_{K\in\mathcal{T}} (\phi - \mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K)_{\partial K} \\ &= \sum_{E\in\mathcal{E}} (\phi - \mathcal{I}\phi, [\![(\mathrm{grad}\,\upsilon)\cdot \vec{t}_K]\!])_E\end{aligned}$$

Sketch of the Proof

$$\sup_{\vec{q}\in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), \upsilon_h\rangle_h}{\|\vec{q}\|_{H(\mathrm{div},\Omega)}}$$

- Gauss & Stokes:

$$\langle(\mathrm{curl}\,\phi)\cdot\vec{n}_K, \upsilon\rangle_{\partial K} = (\mathrm{curl}\,\phi, \mathrm{grad}\,\upsilon)_K + (\mathrm{div}\,\mathrm{curl}\,\phi, \upsilon)_K = \langle\phi, (\mathrm{grad}\,\upsilon)\cdot\vec{t}_K\rangle_{\partial K}$$

- Let $\sigma_h = \vec{\mathcal{I}}\,\mathrm{curl}\,\phi = \mathrm{curl}\,\mathcal{I}\phi$. Then

$$\begin{aligned}\langle(\phi-\sigma_h)\cdot\vec{n}_K, \upsilon\rangle_{\partial K} &= \langle\phi-\mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot\vec{t}_K\rangle_{\partial K}\\ &= \sum_{K\in\mathcal{T}}(\phi-\mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot\vec{t}_K)_{\partial K}\\ &= \sum_{E\in\mathcal{E}}(\phi-\mathcal{I}\phi, [\![(\mathrm{grad}\,\upsilon)\cdot\vec{t}_K]\!])_E\end{aligned}$$

- Manipulate as before to get upper bound

Sketch of the Proof

$$\sup_{\vec{q}\in H(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\vec{q}), \upsilon_h\rangle_h}{\|\vec{q}\|_{H(\mathrm{div},\Omega)}}$$

- Gauss & Stokes:

$$\langle (\mathrm{curl}\,\phi)\cdot \vec{n}_K, \upsilon\rangle_{\partial K} = (\mathrm{curl}\,\phi, \mathrm{grad}\,\upsilon)_K + (\mathrm{div}\,\mathrm{curl}\,\phi, \upsilon)_K = \langle \phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K\rangle_{\partial K}$$

- Let $\sigma_h = \vec{\mathcal{I}}\,\mathrm{curl}\,\phi = \mathrm{curl}\,\mathcal{I}\phi$. Then

$$\begin{aligned}
\langle (\phi - \sigma_h)\cdot \vec{n}_K, \upsilon\rangle_{\partial K} &= \langle \phi - \mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K\rangle_{\partial K} \\
&= \sum_{K\in\mathcal{T}} (\phi - \mathcal{I}\phi, (\mathrm{grad}\,\upsilon)\cdot \vec{t}_K)_{\partial K} \\
&= \sum_{E\in\mathcal{E}} (\phi - \mathcal{I}\phi, [\![(\mathrm{grad}\,\upsilon)\cdot \vec{t}_K]\!])_E
\end{aligned}$$

- Manipulate as before to get upper bound
- Lower bound from Verfürth

Summary

- Early work and applications
- DPG/DPG* paradigm
- *A priori* error analysis
  - DPG error controlled by regularity of solution
  - DPG* error controlled by regularity of solution and Lagrange multiplier
- Duality
- *A posteriori* error control
- Goal-oriented adaptivity

CSEM objectives

Area A:

- Analysis of the saddle point paradigm
- *A priori* and *a posteriori* error analysis of DPG* methods
- Goal-oriented adaptive mesh refinement

Area B:

- Implement finite element methods and compute *a posteriori* error indicators

Area C:

- Analysis of viscoelastic fluid flow

CSEM objectives

Area A:

- Analysis of the saddle point paradigm
- *A priori* and *a posteriori* error analysis of DPG* methods
- Goal-oriented adaptive mesh refinement
- Well-posedness of multiple/coupled variational formulations (linear elasticity)

Area B:

- Implement finite element methods and compute *a posteriori* error indicators
- ESEAS shape functions library
- New assembly algorithm for DPG methods

Area C:

- Analysis of viscoelastic fluid flow
- Structural mechanics and wave propagation problems

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Thank you!

Extra slides

The ultraweak form of Poisson's problem

Strong form:

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma} &= f\,, \\
\boldsymbol{\sigma} - \operatorname{grad} u &= 0
\end{aligned}
$$

The ultraweak form of Poisson's problem

Strong form:

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= f\,, \\ \boldsymbol{\sigma} - \operatorname{grad} u &= 0 \end{aligned}$$

(Ultra)weak form:

$$\begin{aligned} (\boldsymbol{\sigma}, \operatorname{grad} v)_\Omega &= f\,, && \forall\, v \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (u, \operatorname{div} \boldsymbol{\tau})_\Omega &= 0\,, && \forall\, \boldsymbol{\tau} \end{aligned}$$

The ultraweak form of Poisson's problem

Strong form:

$$-\operatorname{div}\boldsymbol{\sigma} = f\,,$$
$$\boldsymbol{\sigma} - \operatorname{grad} u = 0$$

(Ultra)weak form:

$$(\boldsymbol{\sigma}, \operatorname{grad} v)_\Omega = f\,, \qquad \forall\, v$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (u, \operatorname{div}\boldsymbol{\tau})_\Omega = 0\,, \qquad \forall\, \boldsymbol{\tau}$$

Broken ultraweak form:

$$(\boldsymbol{\sigma}, \operatorname{grad} v)_{\mathcal{T}} - \langle \hat{\sigma}_n, v\rangle_{\partial\mathcal{T}} = f\,, \qquad \forall\, v$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{T}} + (u, \operatorname{div}\boldsymbol{\tau})_{\mathcal{T}} - \langle \hat{u}, \boldsymbol{\tau}\cdot\hat{n}\rangle_{\partial\mathcal{T}} = 0\,, \qquad \forall\, \boldsymbol{\tau}$$

The ultraweak form of Poisson's problem

Strong form:

$$
\begin{aligned}
-\operatorname{div}\boldsymbol{\sigma} &= f\,,\\
\boldsymbol{\sigma} - \operatorname{grad} u &= 0
\end{aligned}
$$

(Ultra)weak form:

$$
\begin{aligned}
(\boldsymbol{\sigma}, \operatorname{grad} v)_\Omega &= f\,, && \forall\, v\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (u, \operatorname{div}\boldsymbol{\tau})_\Omega &= 0\,, && \forall\, \boldsymbol{\tau}
\end{aligned}
$$

Broken ultraweak form:

$$
\begin{aligned}
(\boldsymbol{\sigma}, \operatorname{grad} v)_{\mathcal{T}} - \langle \hat{\sigma}_n, v\rangle_{\partial\mathcal{T}} &= f\,, && \forall\, v\\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{T}} + (u, \operatorname{div}\boldsymbol{\tau})_{\mathcal{T}} - \langle \hat{u}, \boldsymbol{\tau}\cdot\hat{n}\rangle_{\partial\mathcal{T}} &= 0\,, && \forall\, \boldsymbol{\tau}
\end{aligned}
$$

N.B.

- The functions u and $\boldsymbol{\sigma}$ are discontinuous

$$
u \in L^2(\Omega) \qquad \boldsymbol{\sigma} \in (L^2(\Omega))^3 \qquad \mathfrak{u} = (u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n)
$$

- All QoI can be written

$$
g(\mathfrak{u}) = \int_\Omega g_1 \cdot u + \int_\Omega \boldsymbol{g}_2 \cdot \boldsymbol{\sigma} + \int_{\partial\Omega} \hat{g}_1 \cdot \hat{u} + \int_{\partial\Omega} \hat{g}_2 \cdot \hat{\sigma}_n
$$

Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\mathbf{u}) = \int_\Omega u(\boldsymbol{x})\, g_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

where

$$g_1(\boldsymbol{x}) = \begin{cases} 1 & x \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Manufactured solution

$$u^{\text{man}}(x, y, z) = f(x/4)\, f(y)\, f(z)$$

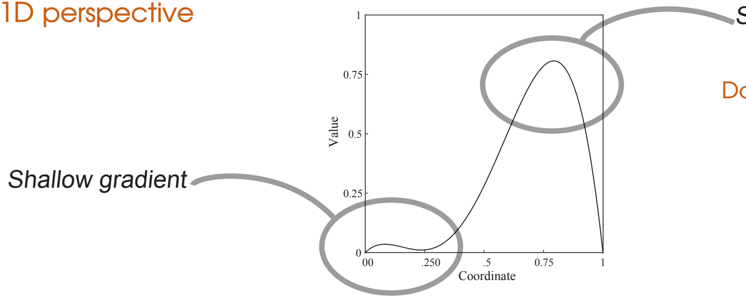
where

$$f(x) = x(1-x)\big((x/4) + (1-4x)^2\big)$$

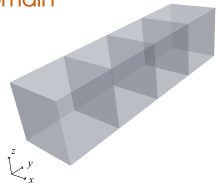
and

$$\Omega = [0,4] \times [0,1] \times [0,1]$$

1D perspective



Domain



Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\boldsymbol{u}) = \int_\Omega u(\boldsymbol{x})\, g_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

where

$$g_1(\boldsymbol{x}) = \begin{cases} 1 & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Manufactured solution

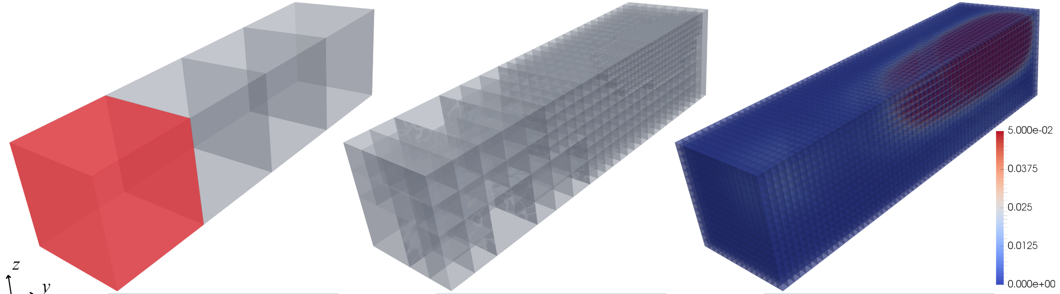
$$u^{\text{man}}(x, y, z) = f(x/4) f(y) f(z)$$

where

$$f(x) = x(1 - x)\big((x/4) + (1 - 4x)^2\big)$$

and

$$\Omega = [0, 4] \times [0, 1] \times [0, 1]$$



a) Initial mesh & QoI *Four elements*

b) Energy AMR *12 refinements*

c) Exact solution

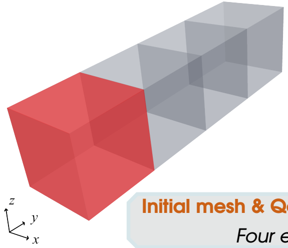
Goal-oriented error estimation and adaptivity

Quantity of interest

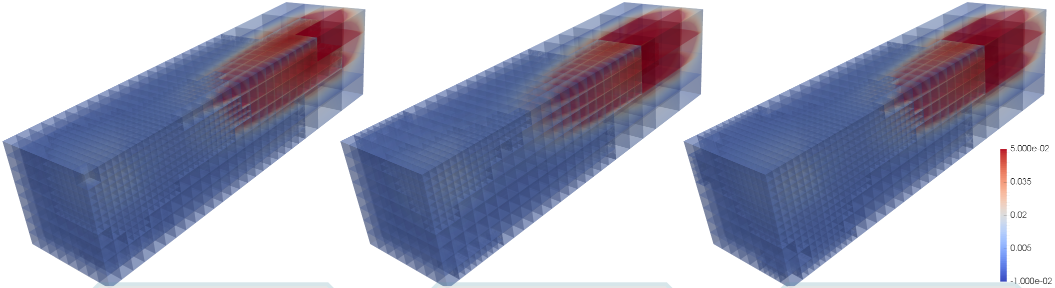
$$g(\mathbf{u}) = \int_\Omega u(\boldsymbol{x})\, g_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

where

$$g_1(\boldsymbol{x}) = \begin{cases} 1 & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Initial mesh & QoI
Four elements



Strategy 1 *Ad hoc*

Strategy 2 *Explicit*

Strategy 3 *Implicit*

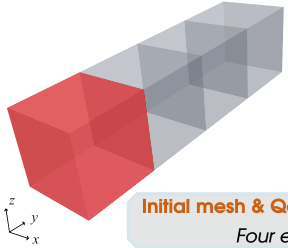
Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\mathbf{u}) = \int_\Omega u(\boldsymbol{x})\, g_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

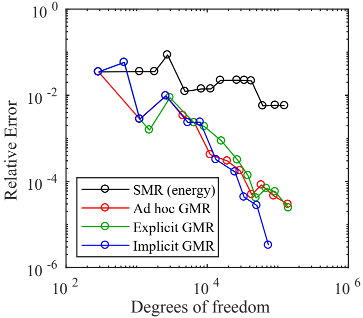
where

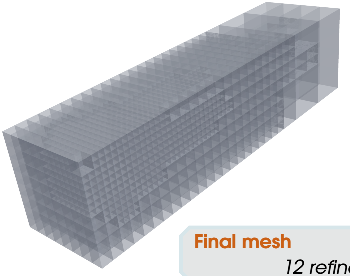
$$g_1(\boldsymbol{x}) = \begin{cases} 1 & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Initial mesh & QoI

Four elements



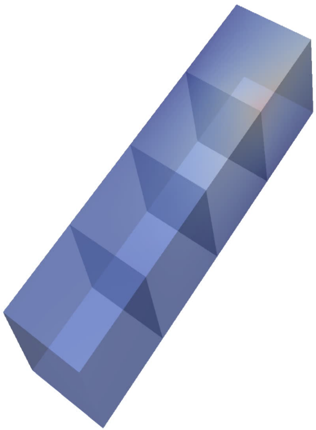


Final mesh

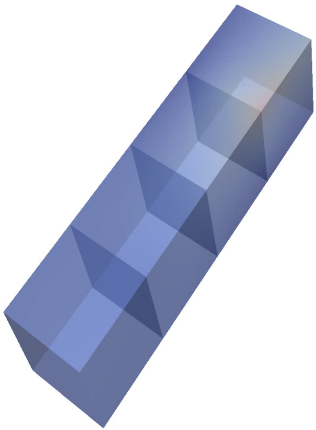
12 refinements

Goal-oriented error estimation and adaptivity

Energy strategy



Goal-oriented strategy



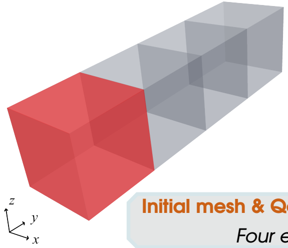
Goal-oriented error estimation and adaptivity

Quantity of interest (flux)

$$g(\mathbf{u}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{g}_2(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

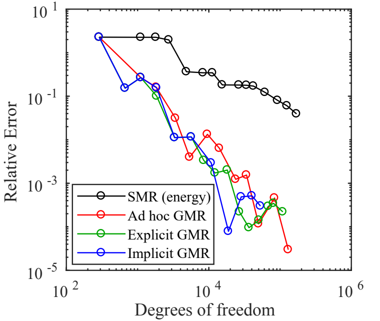
where

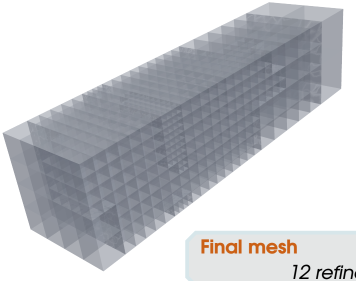
$$\boldsymbol{g}_2(\boldsymbol{x}) = \begin{cases} [1,0,0]^\mathsf{T} & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Initial mesh & QoI

Four elements





Final mesh

12 refinements

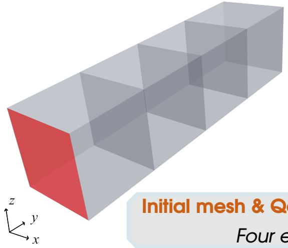
Goal-oriented error estimation and adaptivity

Quantity of interest (boundary)

$$g(\mathbf{u}) = \int_{\partial\Omega} \hat{u}(\boldsymbol{x})\, \hat{g}_1(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

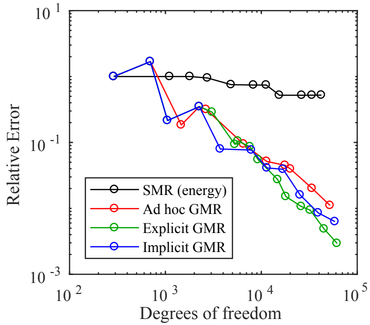
where

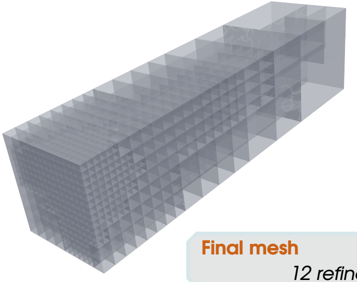
$$\hat{g}_1(\boldsymbol{x}) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



Initial mesh & QoI

Four elements





Final mesh

12 refinements

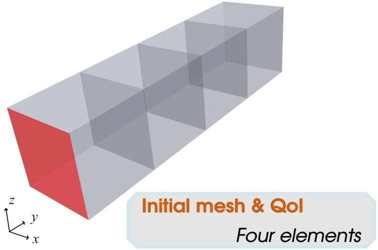
Goal-oriented error estimation and adaptivity

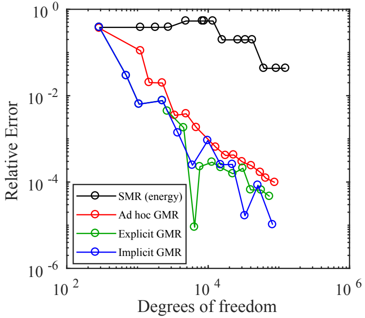
Quantity of interest (boundary flux)

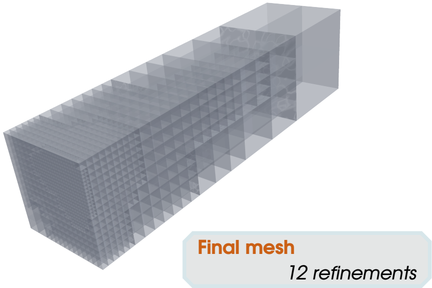
$$g(\mathbf{u}) = \int_{\partial\Omega} \hat{\sigma}_n(\boldsymbol{x})\, \hat{g}_2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

where

$$\hat{g}_2(\boldsymbol{x}) \approx \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

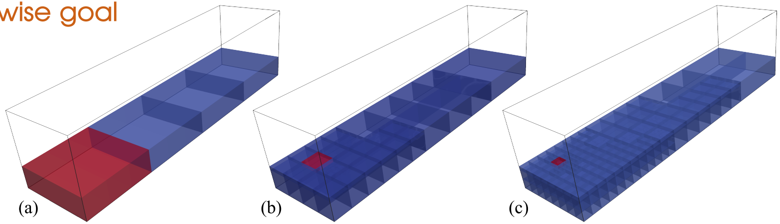


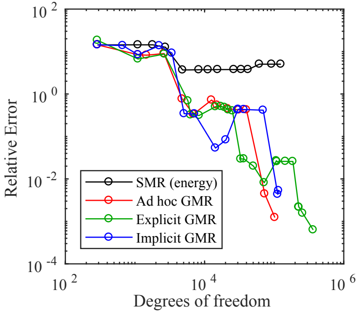


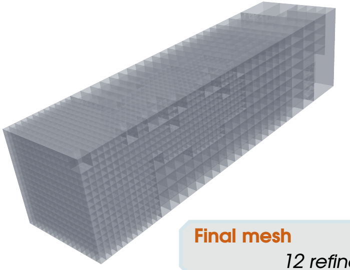


Goal-oriented error estimation and adaptivity

Pointwise goal







Final mesh

12 refinements