Duality principles and *a posteriori* error estimation for DPG methods

Brendan Keith

TUM Postdoctoral Fellowship Presentation
January 26, 2018

Outline

Overview of PhD Work

Prelude: Background

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Numerical Methods

- Discrete least-squares methods
- DPG methods
- DPG* methods
- Goal-oriented methods

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Duality in DPG Methods

Goal-Oriented AMR

- The influence function
- Optimal test functions
- Duality in the errors
- Crude upper bounds & marking strategies

A Posteriori Error Estimation

- DPG methods
- DPG* methods
- Reliability & efficiency proof

Overview of PhD Work

About My Work

PhD candidate at ICES

Supervisor: Dr Leszek Demkowicz

- Study discontinuous Petrov–Galerkin (DPG) methods

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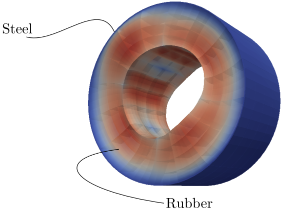
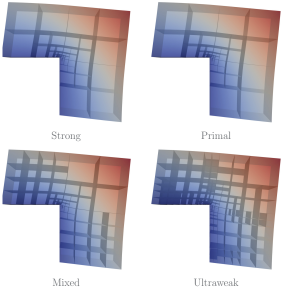
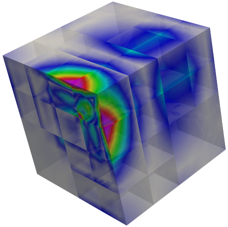
PhD candidate at ICES

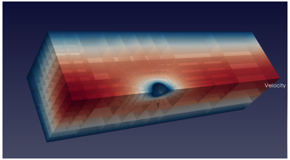
Supervisor: Dr Leszek Demkowicz

- Study discontinuous Petrov–Galerkin (DPG) methods
- Discovered DPG* methods
- Defined discrete least-squares (DLS) methods
- 3D simulations
- Adaptive mesh refinement
- Goal-oriented methods
- Applications!

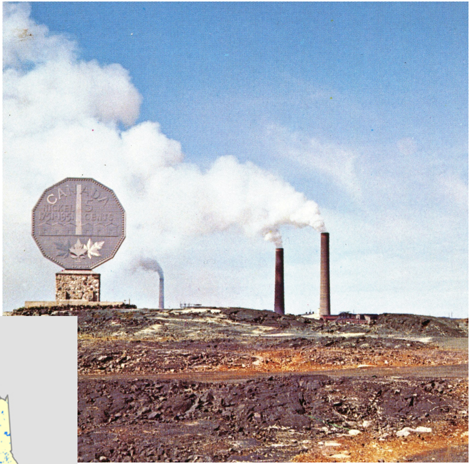




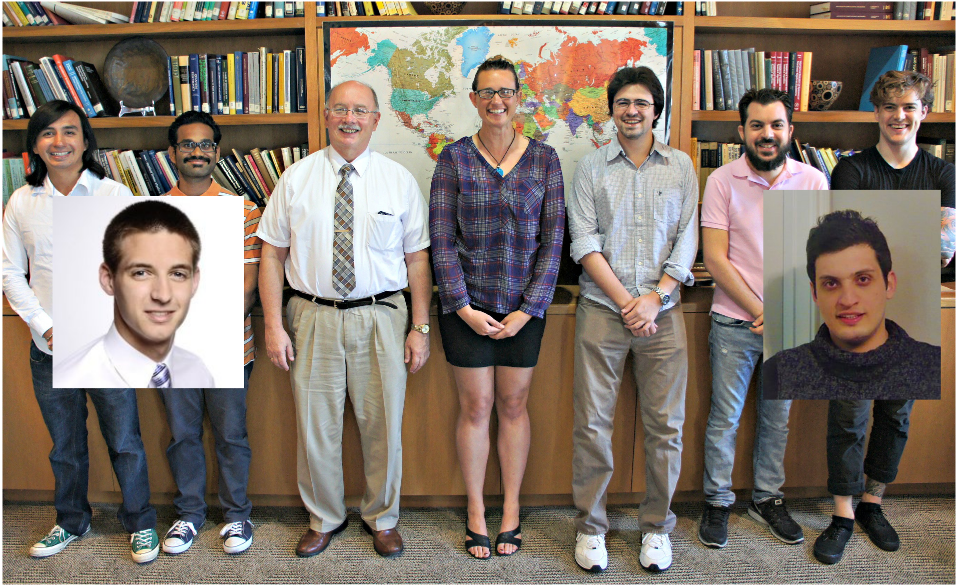




About Me



The DPG Group



Life at UT

The Texas Applied Mathematics and Engineering Symposium (http://tames.io)

Numerical Methods

Stability

Petrov–Galerkin methods

Let $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ be a continuous bilinear form.

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Continuous stability **Discrete stability**

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A: Fix $\mathcal{U}_h$ and increase the dimension of $\mathcal{V}_h$ until satisfied.

Discrete Least-Squares Methods

Q: How to define a discrete solution when $\dim(\mathcal{V}_h) > \dim(\mathcal{U}_h)$?

B. Keith, S. Petrides, F. Fuentes, and L. Demkowicz.
Discrete least-squares finite element methods.
Comput. Methods Appl. Mech. Engrg., 327:226--255, 2017.

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Stiffness matrix:

$$\mathsf{B}_{ij} = b(\mathfrak{u}_j, \mathfrak{v}_i)$$

Load vector:

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DPG: G is block-diagonal

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L. Demkowicz, J. Gopalakrishnan, and B. Keith.
Analysis of a DPG* method for the Poisson equation.
In preparation, 2018.

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With DPG, we solve

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With DPG*, we solve

$$\mathsf{B}\mathsf{G}^{-1}\mathsf{B}^\mathsf{T}\mathsf{w} = \mathsf{l}$$

and then post-process **w**:

$$\mathsf{u} = \mathsf{G}^{-1}\mathsf{B}^\mathsf{T}\mathsf{w}$$

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Highlights

- ⋆ DPG/DPG* always deliver SPD (HPD) stiffness matrices

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- ⋆ DPG/DPG* always deliver SPD (HPD) stiffness matrices
- ⋆ DPG/DPG* have built-in stability
- ⋆ DPG has a built-in error estimator
- ⋆ DPG* works for some problems without uniqueness

Goal-Oriented Methods

Consider the following abstract model:

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathcal{F}(\mathbf{x}, \mathbf{u}; \mu)\,, \\ \mathbf{y} = \mathcal{G}(\mathbf{x})\,. \end{cases}$$

Solution variable: $\mathbf{x}$
Input: $\mathbf{u}$
Model parameter(s): μ
QoI (or output): $\mathbf{y}$.

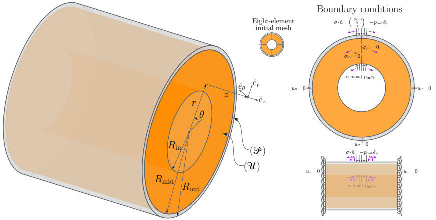
Research interest:

- By designing numerical methods with only the given output $\mathbf{y}$ in mind, efficiency can sometimes be greatly improved.
- Example: goal-oriented adaptive mesh refinement.

Applications

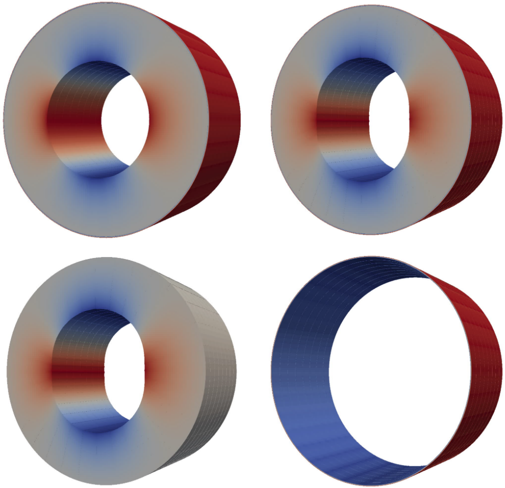
Structural Mechanics

Sheathed hose



$(E_{\text{steel}} = 200\text{GPA}, E_{\text{rubber}} = 0.01\text{GPA})$

Circumferential stress: $\boldsymbol{\sigma}_{\theta\theta}$



Coupled formulations

- All coupled formulations are mutually compatible throughout the domain.

F. Fuentes, B. Keith, L. Demkowicz, and P. L. Tallec.
Coupled variational formulations of linear elasticity and the DPG methodology.
J. Comput. Phys., 348:715--731, 2017.

B. Keith, F. Fuentes, and L. Demkowicz.
The DPG methodology applied to different variational formulations of linear elasticity.
Comput. Methods Appl. Mech. Engrg., 309:579--609, 2016.

Fluid Mechanics

Oldroyd-B fluid

Conservation of mass and momentum:

$$\rho\left(\frac{\partial\,\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \rho\,\mathbf{f} \quad \text{on } \Omega\times(0,T)\,.$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad \text{on } \Omega\times(0,T)\,.$$

Constitutive law:

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2\mu_S\,\varepsilon(\mathbf{u}) + \mathbf{T}\,,$$

where

$$\mathbf{T} + \lambda\,\mathfrak{L}_{\mathbf{u}}\mathbf{T} = 2\mu_P\,\varepsilon(\mathbf{u})\,.$$

Lie derivative:

$$\mathfrak{L}_{\mathbf{u}}\mathbf{T} = \frac{\partial\,\mathbf{T}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{T} - \left(\boldsymbol{\nabla}\mathbf{u}\,\mathbf{T} - \mathbf{T}\,\boldsymbol{\nabla}^{\mathsf{T}}\mathbf{u}\right)\,.$$

B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr, and L. Demkowicz.
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Quantity of interest

Drag coefficient

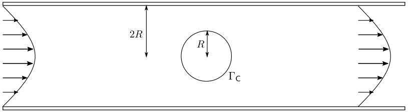
$$\mathfrak{K}(\hat{t}) = \frac{1}{\mu\bar{u}}\int_{\Gamma_C} \hat{t}\cdot\hat{\mathbf{e}}_x\,\mathrm{d}\,s\,.$$

Γ_C: boundary of cylinder

$\hat{t}$: traction

$\mu = \mu_S + \mu_P$: viscosity

$\bar{u}$: average inflow velocity.



Confined cylinder domain.

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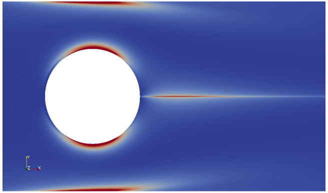
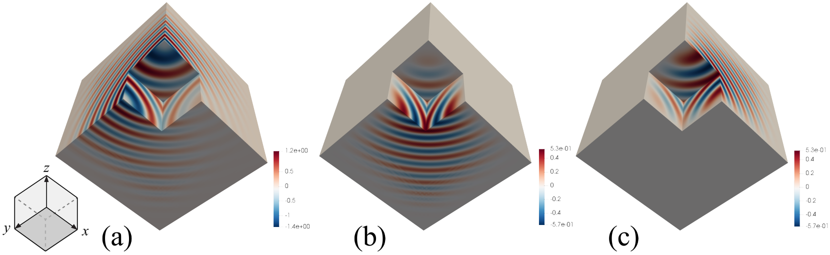


Close-up of rescaled T_{11}-component from $\lambda = 0.9$.

Wave Mechanics

- Acoustics $-\Delta p - \omega^2 p = f$
- Electromagnetics $\frac{1}{\mu}\nabla \times \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{J}^{\mathrm{imp}}$
- Elastodynamics $-\boldsymbol{\nabla} \cdot (\mathsf{C} : \boldsymbol{\nabla} \boldsymbol{u}) - \rho\omega^2 \boldsymbol{u} = \boldsymbol{f}$



Electromagnetic wave scattering of the discrete electric field $\mathbf{E}$.

(a) The x-component; (b) the y-component; (c) the z-component. Only the real part of the solution is visualized.

A. Vaziri Astaneh, B. Keith, and L. Demkowicz.
On perfectly matched layers and non-symmetric variational formulations.
Submitted, 2017.

Duality in DPG Methods

The Influence Function

Problem:

$$\begin{cases} \text{Find } \mathfrak{u} \in \mathcal{U} \ : \\ b(\mathfrak{u}, \mathfrak{v}) = \ell(\mathfrak{v}) \,, \quad \forall\, \mathfrak{v} \in \mathcal{V} \end{cases}$$

J. T. Oden and S. Prudhomme.
Goal-oriented error estimation and adaptivity for the finite element method.
Comput. Math. Appl., 41(5-6):735--756, 2001.

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The Influence Function

Problem:

$$\begin{cases} \text{Find } \mathfrak{u} \in \mathcal{U} : \\ \langle \mathcal{B}\mathfrak{u}, \mathfrak{v} \rangle = \ell(\mathfrak{v}), \quad \forall\, \mathfrak{v} \in \mathcal{V} \end{cases}$$

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Quantity of interest:

$$g(\mathfrak{u}), \quad \text{where} \quad g \in \mathcal{U}'$$

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where $\mathfrak{v} = (\mathcal{B}')^{-1} g$ is the *influence function*.

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Optimal Test Functions

Problem:

$$\begin{cases} \text{Find } \mathfrak{u}_h \in \mathcal{U}_h \; : \\ b(\mathfrak{u}_r, \mathfrak{v}_r) = \ell(\mathfrak{v}_r)\,, \quad \forall\, \mathfrak{v}_r \in \mathcal{V}_r \end{cases}$$

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Where $\Theta_r : \mathcal{U}_h \to \mathcal{V}_r$ is defined by the inner product $(\cdot,\cdot)_{\mathcal{V}}$:

$$(\Theta_r \mathfrak{w}_h, \mathfrak{v}_r)_{\mathcal{V}} = b(\mathfrak{w}_h, \mathfrak{v}_r) \quad \forall\, \mathfrak{w}_h \in \mathcal{U}_h\,,\ \mathfrak{v}_r \in \mathcal{V}_r$$

Any function in the range of Θ_r is called an *optimal test function*.

Duality in the Errors

Theorem

Let $\mathfrak{u}_h$ be the discrete primal solution and $\mathfrak{v}_r$ be the discrete dual solution. Define $e_h = \mathfrak{u} - \mathfrak{u}_h$ and $e_r = \mathfrak{v} - \mathfrak{v}_r$. Then the following identity holds:

$$g(e_h) = b(e_h, e_r) = \ell(e_r)\,.$$

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Proof: Due to Galerkin orthogonality in the primal and dual problems, respectively, observe that $b(e_h, \Theta_r \mathfrak{w}_h) = b(\mathfrak{w}_h, e_r) = 0$, for any $\mathfrak{w}_h \in \mathcal{U}$. Therefore,

$$g(e_h) = b(e_h, \mathfrak{v}) = b(e_h, e_r) = b(\mathfrak{u}, e_r) = \ell(e_r)\,.$$

□

A Crude Upper Bound

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$$\begin{cases} \text{Find } \mathfrak{u}_h \in \mathcal{U}_h : \\ b(\mathfrak{u}_h, \mathfrak{v}_r) = \ell(\mathfrak{v}_r)\,, \quad \forall \mathfrak{v}_r \in \Theta_r(\mathcal{U}_h) \end{cases}$$

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$$g(\mathfrak{u} - \mathfrak{u}_h) = b(\mathfrak{u} - \mathfrak{u}_h, \mathfrak{v})$$

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$$\begin{aligned} g(\mathfrak{u} - \mathfrak{u}_h) = b(\mathfrak{u} - \mathfrak{u}_h, \mathfrak{v}) &= b(\mathfrak{u} - \mathfrak{u}_h, \mathfrak{v} - \mathfrak{v}_r) \\ &\leq \underbrace{\|\mathcal{B}\mathfrak{u}_h - \ell\|_{\mathcal{V}'}}_{\lesssim \eta(\mathfrak{u}_h)} \underbrace{\|\mathfrak{v} - \mathfrak{v}_r\|_{\mathcal{V}}}_{\lesssim \eta^*(\mathfrak{v}_r)} \end{aligned}$$

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$$\leq \underbrace{\|\mathcal{B}\mathfrak{u}_h - \ell\|_{\mathcal{V}'}}_{\lesssim \eta(\mathfrak{u}_h)} \underbrace{\|\mathfrak{v} - \mathfrak{v}_r\|_{\mathcal{V}}}_{\lesssim \eta^*(\mathfrak{v}_r)}$$

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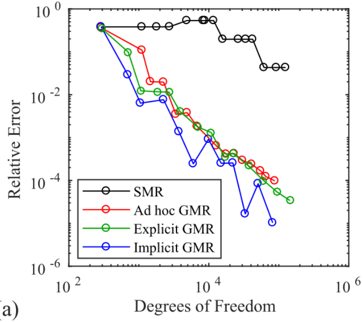
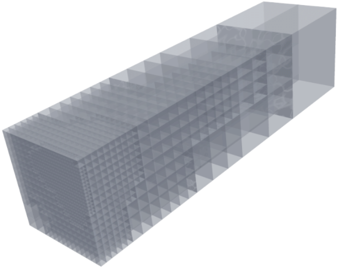
Goal-Oriented Adaptive Mesh Refinement in 3D



(a)



(b)

(a) The error in the average flux; (b) final mesh after goal-oriented adaptive mesh refinement.

Algorithm and Marking Strategies

Algorithm 1 Adaptive mesh refinement

Input: initial mesh $\mathcal{T}$, marking strategy, tolerance $\mathtt{TOL}$.
while $\eta > \mathtt{TOL}$ **do**
(1) Solve for $\mathfrak{u}_h$ and $\mathfrak{v}_r$ on $\mathcal{T}$.
(2) Compute refinement indicators $\{\eta_K\}_{K\in\mathcal{T}}$ or $\{\eta_K^*\}_{K\in\mathcal{T}}$.
(3) Mark elements for refinement $\mathcal{M} \subset \mathcal{T}$, as dictated by *the marking strategy*.
(4) Refine all marked elements $K \in \mathcal{M}$ and construct new mesh $\mathcal{T}$.
return solution $\mathfrak{u}_h$ and $g(\mathfrak{u}_h)$.

- Mark all elements $K \in \mathcal{T}$ such that $\theta \cdot \widetilde{\eta}_{\max} \leq \widetilde{\eta}_K$
- Choices for $\widetilde{\eta}$: $\quad \widetilde{\eta}_K = \eta_K\,, \quad \widetilde{\eta}_K = \eta_K^*\,, \quad \widetilde{\eta}_K = \eta_K \cdot \eta_K^*\,, \quad \ldots$

A Posteriori Error Estimation

DPG Methods

Define a Fortin operator $\Pi_r : \mathcal{V} \to \mathcal{V}_r$:

$$b(\mathfrak{u}_h, \mathfrak{v} - \Pi_r \mathfrak{v}) = 0\,, \quad \forall \mathfrak{v} \in \mathcal{V}\,.$$

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$$b(\mathfrak{u}_h, \mathfrak{v} - \Pi_r \mathfrak{v}) = 0\,, \quad \forall \mathfrak{v} \in \mathcal{V}\,.$$

Theorem

Assume that $\ell \in \mathrm{Range}(\mathcal{B})$ and that Π_r exists and is a projection, $\Pi_r \circ \Pi_r = \Pi_r$. Then the computable residual $\eta(\mathfrak{u}) = \|\ell - \mathcal{B}\mathfrak{u}\|_{\mathcal{V}_r'}$ and the data approximation error $\mathrm{osc}(\ell) = \|\ell \circ (1 - \Pi_r)\|_{\mathcal{V}'}$ satisfy

$$\gamma^2 \|\mathfrak{u} - \mathfrak{u}_h\|_{\mathcal{U}}^2 \leq \eta(\mathfrak{u}_h)^2 + \Big(\eta(\mathfrak{u}_h)\sqrt{\|\Pi_r\|^2 - 1} + \mathrm{osc}(\ell)\Big)^2\,,$$

$$\eta(\mathfrak{u}_h) \leq M \|\mathfrak{u} - \mathfrak{u}_h\|_{\mathcal{U}}\,,$$

and $$\mathrm{osc}(\ell) \leq M \|\Pi_r\| \min_{\mathfrak{w}_h \in \mathcal{U}_h} \|\mathfrak{u} - \mathfrak{w}_h\|_{\mathcal{U}}\,.$$

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- $b(\mathfrak{u}, \mathfrak{v}) = (u, \mathcal{L}^*\mathfrak{v})_\Omega - \langle \hat{u}, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}} - \langle \hat{\sigma}, v \rangle_{\partial\mathcal{T}}$

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Theorem

Assume $\Omega \subset \mathbb{R}^2$ with sufficiently regular boundary. If $\mathfrak{v}_r = (v_r, \boldsymbol{\tau}_r) \in \mathcal{V}_r$ satisfies

$$b(\mathfrak{u}_h, \mathfrak{v}_r) = g(\mathfrak{u}_h)\,, \quad \forall\, \mathfrak{u}_h = (u_h, \hat{u}_h, \hat{\sigma}_h) \in \mathcal{U}_h\,,$$

then $\exists\; C_2 > C_1 > 0$, independent of the element sizes h_K, such that

$$C_1 \eta^*(\mathfrak{v}_r) \leq \|\mathfrak{v} - \mathfrak{v}_r\|_{\mathcal{V}} \leq C_2 \eta^*(\mathfrak{v}_r)\,,$$

where

$$\eta^*(\mathfrak{v}_r)^2 = \big\|\mathcal{L}^*\mathfrak{v}_r - g_\Omega\big\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K \left(\sum_{E\in\partial K\cap\mathcal{E}_{\mathrm{int}}} \big\|[\![\boldsymbol{\tau}_r \cdot \boldsymbol{n}_K]\!]\big\|_{L^2(E)}^2 + \sum_{E\in\partial K} \big\|[\![v_r]\!]\big\|_{H^1(E)}^2 \right)$$

Proof

Recall $(\mathcal{B}\mathfrak{u})(\cdot) = b(\mathfrak{u}, \cdot)$, for all $\mathfrak{u} \in \mathcal{U}$.

- $\|\mathfrak{v}\|_{\mathcal{V}}^2 = \sup_{\mathfrak{u}\in\mathcal{U}} \frac{b(\mathfrak{u},\mathfrak{v})}{\|\mathcal{B}\mathfrak{u}\|_{\mathcal{V}'}}$

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- Decompose supremum into terms:

$$\left(\sup_{\mathfrak{u}\in\mathcal{U}} \frac{b(\mathfrak{u},\mathfrak{v}_r) - g(\mathfrak{u})}{\|\mathfrak{u}\|_{\mathcal{U}}}\right)^2 = \left(\sup_{u\in L^2(\Omega)} \frac{(u, \mathcal{L}^*\mathfrak{v}_r - g_\Omega)_\Omega}{\|u\|_\Omega}\right)^2 + \left(\sup_{\hat{u}\in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \boldsymbol{\tau}_r\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}}}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}}\right)^2 + \left(\sup_{\hat{\sigma}\in H^{-1/2}(\mathcal{S})} \frac{\langle \hat{\sigma}, v_r\rangle_{\partial\mathcal{T}}}{\|\hat{\sigma}\|_{H^{-1/2}(\partial\mathcal{T})}}\right)^2$$

Proof

- $$\sup_{u \in L^2(\Omega)} \frac{(u, \mathcal{L}^* \mathfrak{v}_r - g_\Omega)_\Omega}{\|u\|_\Omega} = \left\| \mathcal{L}^* \mathfrak{v}_r - g_\Omega \right\|_\Omega$$

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Lemma 1:

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Lemma 2:

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Proof

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- Let $\boldsymbol{H} = \mathbf{curl}(H^1(\Omega))$

$$\left(\sup_{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), v_r \rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}\right)^2 = \left(\sup_{\boldsymbol{\sigma} \in \boldsymbol{H}} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), v_r \rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}\right)^2 + \left(\sup_{\boldsymbol{\sigma} \in \boldsymbol{H}^\perp} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), v_r \rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}\right)^2$$

Proof

$$\sup_{u \in H_0^1(\Omega)} \frac{\langle \mathrm{tr}(u), \boldsymbol{\tau}_r \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}}}{\|u\|_{H^1(\Omega)}}$$

Proof

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- (Galerkin orthogonality) Let $\hat{u}_h$ be in the trial space:

$$\langle \operatorname{tr}(u) - \hat{u}_h, \boldsymbol{\tau}_r \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}} = \sum_{K \in \mathcal{T}} (\operatorname{tr}(u) - \hat{u}_h, \boldsymbol{\tau}_r \cdot \boldsymbol{n}_K)_{\partial K} = \sum_{E \in \mathcal{E}_{\text{int}}} (\operatorname{tr}(u) - \hat{u}_h, [\![\boldsymbol{\tau}_r \cdot \boldsymbol{n}_K]\!])_E$$

Proof

$$\sup_{u\in H_0^1(\Omega)} \frac{\langle \mathrm{tr}(u), \boldsymbol{\tau}_r \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}}}{\|u\|_{H^1(\Omega)}}$$

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- Set $\hat{u}_h = (\mathcal{I}u)|_E$, where $\mathcal{I}$ is the corresponding Clément interpolation operator.

For each edge E,

$$(\mathrm{tr}(u) - \hat{u}_h, [\![\boldsymbol{\tau}_r \cdot \boldsymbol{n}_K]\!])_E \leq \|\mathrm{tr}(u - \mathcal{I}u)\|_E \|[\![\boldsymbol{\tau}_r \cdot \boldsymbol{n}_K]\!]\|_E \lesssim h_K^{1/2} \|u\|_{H^1(\widetilde{K})} \|[\![\boldsymbol{\tau}_r \cdot \boldsymbol{n}_K]\!]\|_E$$

where $\widetilde{K}$ denotes the patch of elements associated with the edge E.

Proof

$$\sup_{u\in H_0^1(\Omega)} \frac{\langle \mathrm{tr}(u), \boldsymbol{\tau}_r\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}}}{\|u\|_{H^1(\Omega)}}$$

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- Upper bound follows from manipulating the supremum

Proof

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- Upper bound follows from manipulating the supremum
- Lower bound follows from Verfürth's bubble function technique

Proof

$$\sup_{\boldsymbol{\sigma}\in\boldsymbol{H}} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), \upsilon_r\rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}$$

- Gauss & Stokes:

$$\langle (\mathbf{curl}\,\phi)\cdot\boldsymbol{n}_K, \upsilon\rangle_{\partial K} = (\mathbf{curl}\,\phi, \mathbf{grad}\,\upsilon)_K + (\mathrm{div}\,\mathbf{curl}\,\phi, \upsilon)_K = \langle \phi, (\mathbf{grad}\,\upsilon)\cdot\boldsymbol{t}_K\rangle_{\partial K}$$

Proof

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- Use Clément interpolation as before to get upper bound

Proof

$$\sup_{\boldsymbol{\sigma}\in\boldsymbol{H}} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), v_r\rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}$$

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- Use Clément interpolation as before to get upper bound
- Lower bound from Verfürth

Proof

$$\sup_{\boldsymbol{\sigma}\in\boldsymbol{H}^{\perp}} \frac{\langle \mathrm{tr}_n(\boldsymbol{\sigma}), v_r\rangle_{\partial\mathcal{T}}}{\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div},\Omega)}}$$

- Helmholtz embedding:

$$\boldsymbol{H}^{\perp} \hookrightarrow (\boldsymbol{H}^1(\Omega))^2$$

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- Use Clément interpolation again to get upper bound

Proof

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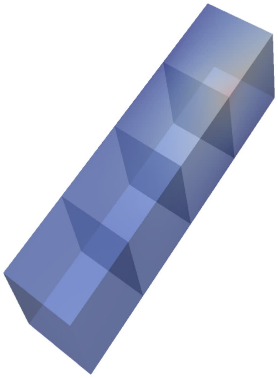
A. Vaziri Astaneh, B. Keith, and L. Demkowicz.
On perfectly matched layers and non-symmetric variational formulations.
Submitted, 2017.

Thank you!

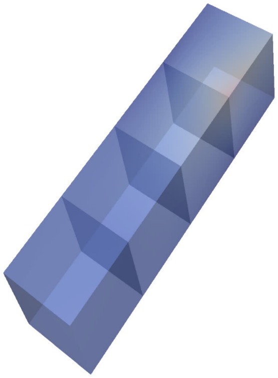
Extra slides

Goal-Oriented Adaptivity

Solution-oriented strategy



Goal-oriented strategy (1 of 3)



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Discrete Least-Squares Methods

Factorization, etc.

We are solving the discrete weighted least squares problem

$$\mathbf{u}^{\text{opt}} = \arg\min_{\mathbf{u}\in\mathbb{R}^N} \left\| \mathsf{L}^{-1}(\mathsf{B}\mathbf{u} - \mathsf{l}) \right\|_2 .$$

or

$$(\star\star) \qquad (\mathsf{L}^{-1}\mathsf{B})^{\mathsf{T}}(\mathsf{L}^{-1}\mathsf{B})\,\mathbf{u} \;=\; (\mathsf{L}^{-1}\mathsf{B})^{\mathsf{T}}(\mathsf{L}^{-1}\mathsf{l}) .$$

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- Note that $(\star\star)$ suggests an efficient way to locally construct $\mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{B}$.
- It hints at solution strategies based on least squares solvers.

  Trefethen and Bau (SIAM, 1997) belabour least squares problems and there is a untapped literature on sparse least squares solvers.
- There are many stable solution techniques for SPD linear systems, however the matrix $\mathsf{B}^{\mathsf{T}}\mathsf{G}^{-1}\mathsf{B}$ has a squared condition number.
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