

# A Saddle Point Paradigm for Finite Element Analysis and Its Role in the DPG Methodology

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Continuing Fellowship

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The Institute for Computational Engineering and Sciences  
The University of Texas at Austin  
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# Outline

## Introduction

- Babuška to Brezzi
- DPG methods
- DPG\* methods

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## Applications

- Poisson
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# Prelude: Other Research

# Shape Function Software Library

- Energy Spaces

$$H^1 \xrightarrow{\nabla} H(\text{curl}) \xrightarrow{\nabla \times} H(\text{div}) \xrightarrow{\nabla \cdot} L^2$$

- Hybrid Meshes

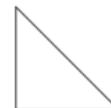
1D:



2D:

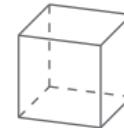


Segment



Quadrilateral

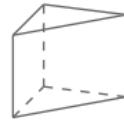
3D:



Hexahedron



Tetrahedron



Prism



Pyramid

- High order, *anisotropic*, hierarchical shape functions
- Affine invariance to master element geometries
- Sparse stiffness matrices
- Orientation embeddings

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# Shape Function Software Library

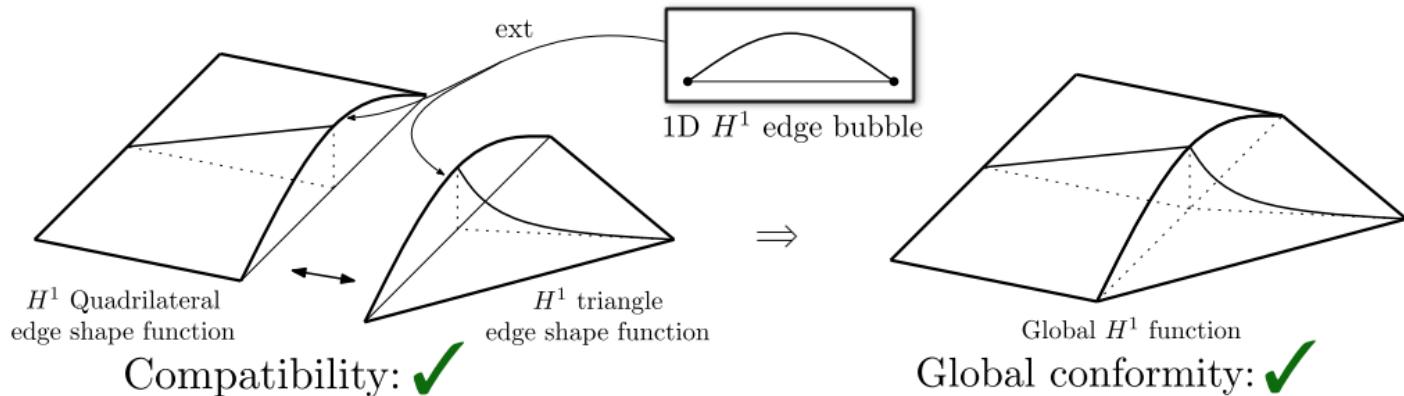
$$\begin{array}{ccccccc} \text{3D: } & H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\nabla \times} & H(\text{div}) & \xrightarrow{\nabla \cdot} L^2 \\ & \text{ext} \uparrow & & \text{ext} \uparrow & & \text{ext} \uparrow & \\ \text{2D: } & H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\nabla \times} & L^2 & \\ & \text{ext} \uparrow & & \text{ext} \uparrow & & & \\ \text{1D: } & H^1 & \xrightarrow{\nabla} & L^2, & & & \end{array}$$

Classification into *subtopologies*

1D: vertex and edge shape functions,

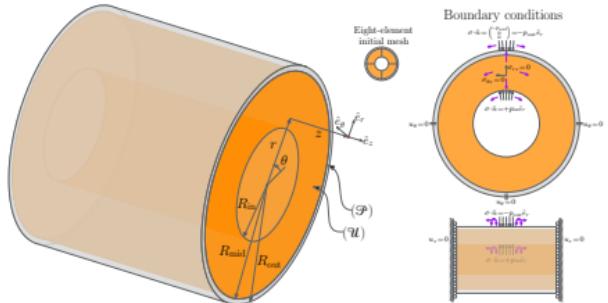
2D: vertex, edge, and face shape functions,

3D: vertex, edge, face and interior shape functions.



# Structural Mechanics

## Sheathed hose

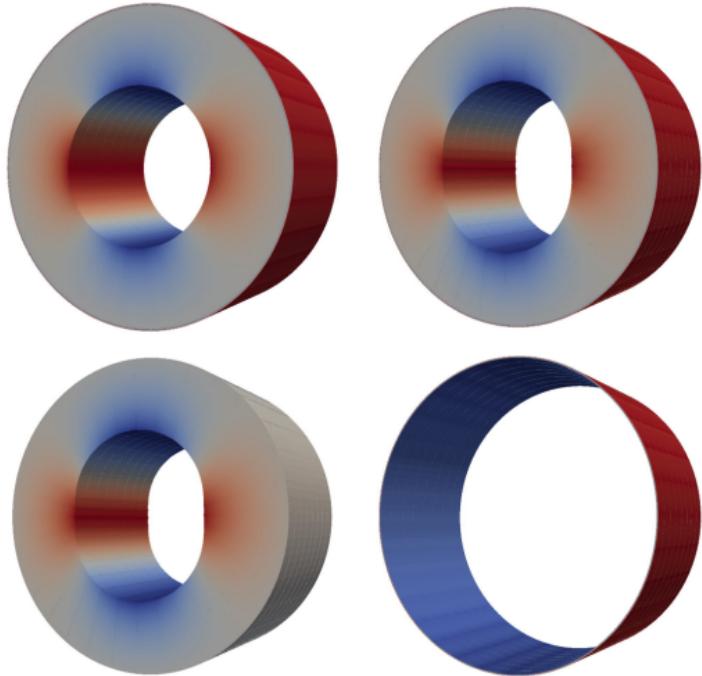


$$(E_{\text{steel}} = 200 \text{GPa}, E_{\text{Rubber}} = 0.01 \text{GPa})$$

## Coupled formulations

- All coupled formulations are **mutually compatible** throughout the domain.

Circumferential stress:  $\sigma_{\theta\theta}$



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# Fluid Mechanics

## Oldroyd-B fluid

Conservation of mass and momentum:

$$\begin{aligned} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma} &= \rho \mathbf{f} \quad \text{on } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{on } \Omega \times (0, T). \end{aligned}$$

Constitutive law:

$$\boldsymbol{\sigma} = -p \mathbf{I} + 2\mu_S \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{T},$$

where

$$\mathbf{T} + \lambda \mathcal{L}_u \mathbf{T} = 2\mu_P \boldsymbol{\varepsilon}(\mathbf{u}).$$

Lie derivative:

$$\mathcal{L}_u \mathbf{T} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u} \mathbf{T} - \mathbf{T} \nabla^T \mathbf{u}).$$

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Quantity of interest

Drag coefficient

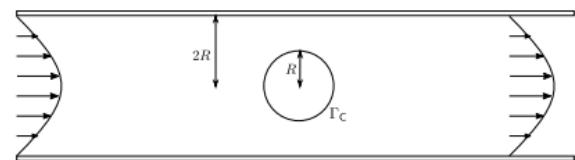
$$c_D(\boldsymbol{\sigma}) = \frac{1}{\mu \bar{u}} \int_{\Gamma_C} (\boldsymbol{\sigma} \vec{n}) \cdot \mathbf{e}_x \, ds.$$

$\Gamma_C$ : boundary of cylinder

$\vec{n}$ : traction

$\mu = \mu_S + \mu_P$ : viscosity

$\bar{u}$ : average inflow velocity.



Confined cylinder domain.

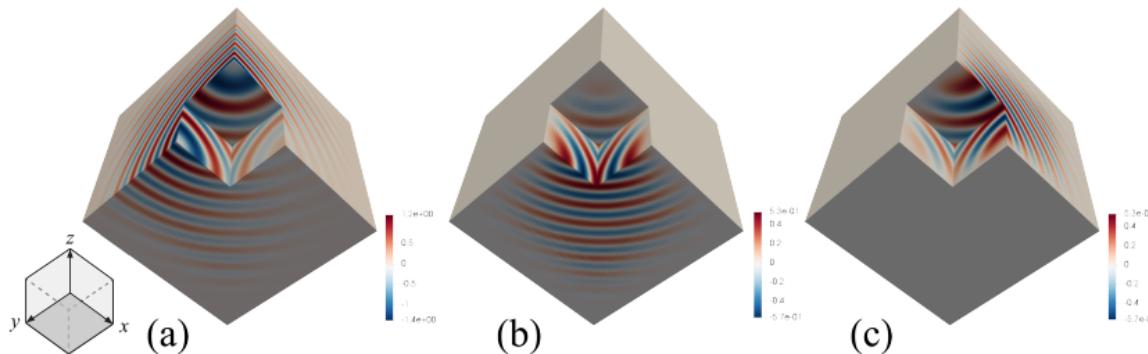
# Wave Mechanics

- Acoustics
- Electromagnetics
- Elastodynamics

$$-\Delta p - \omega^2 p = f$$

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{E} - \omega^2 \epsilon \mathbf{E} = i\omega \mathbf{J}^{\text{imp}}$$

$$-\nabla \cdot (\mathbf{C} : \nabla \mathbf{u}) - \rho \omega^2 \mathbf{u} = \mathbf{f}$$



Electromagnetic wave scattering of the discrete electric field  $\mathbf{E}$ .

(a) The  $x$ -component; (b) the  $y$ -component; (c) the  $z$ -component. Only the real part of the solution is visualized.

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# Discrete Least Squares

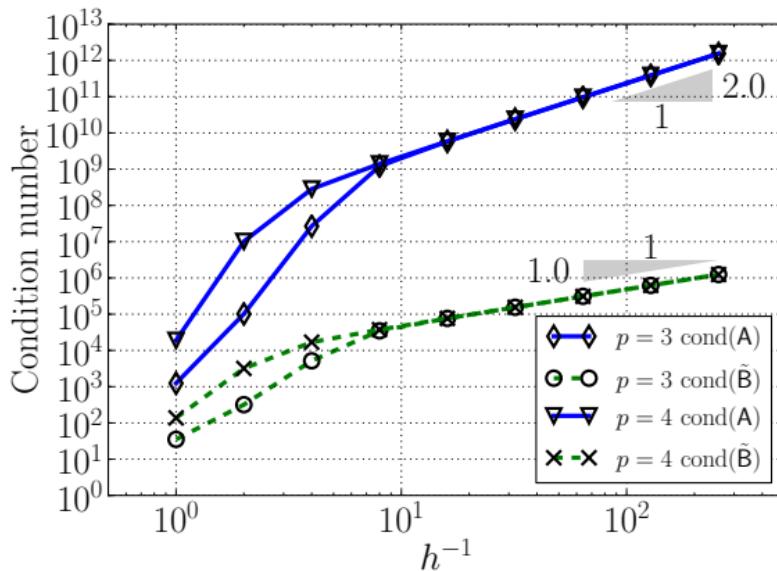
Two solution paradigms

Normal equations

$$\underbrace{B^T G^{-1} B}_A u = B^T G^{-1} f$$

Overdetermined system

$$\underbrace{L^{-1} B}_\sim u = L^{-1} f$$



Acoustics with DPG near resonance

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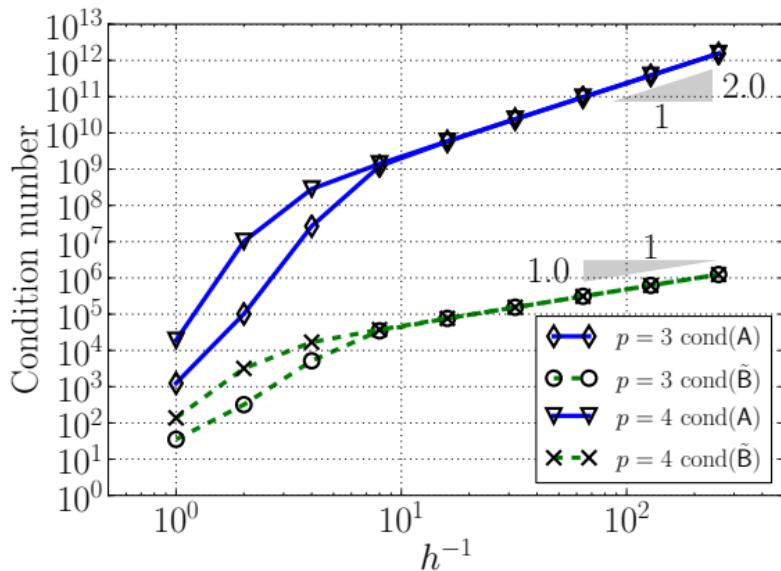
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- ★  $L^{-1}B$  has  $O(h^{-1})$  condition number!
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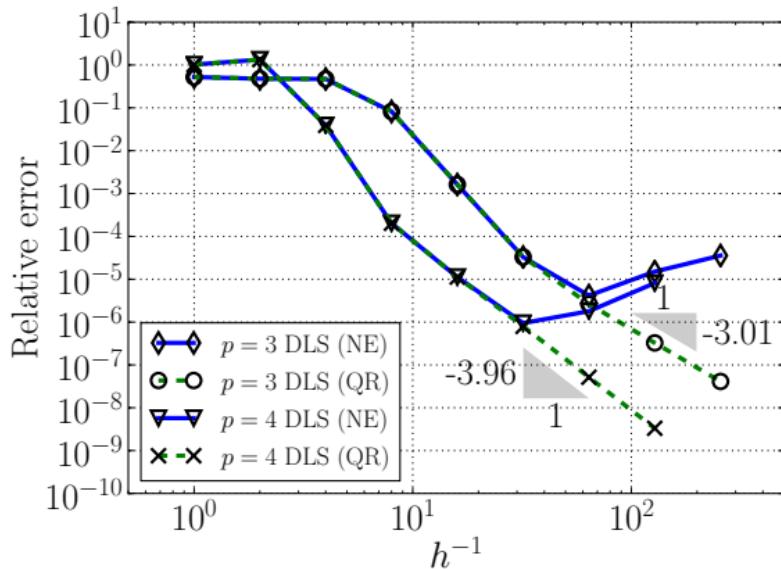
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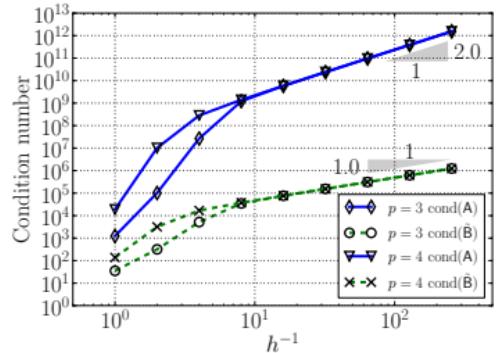
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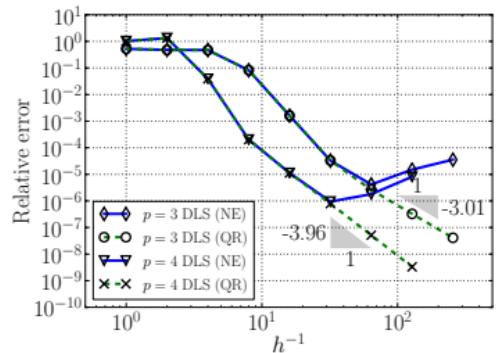


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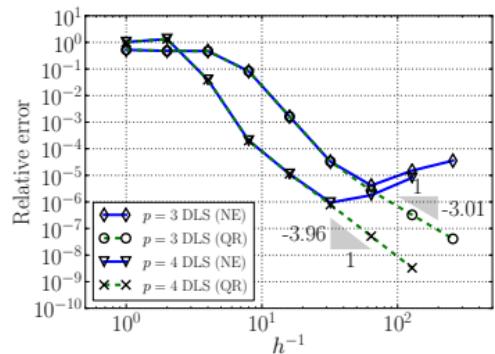
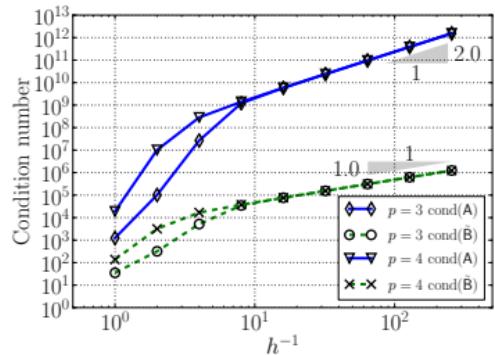
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Let  $b : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  be a continuous bilinear form.

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Problem:

Find  $\textcolor{brown}{u}_h \in \mathcal{U}_h$  satisfying  $b(\textcolor{brown}{u}_h, v_h) = F(v_h) \quad \forall v_h \in \mathcal{V}_h$

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The following **continuous problem** has a unique solution:

Find  $u \in \mathcal{U}$  satisfying  $b(u, v) = F(v) \quad \forall v \in \mathcal{V},$

$\|u\|_{\mathcal{U}} \leq \gamma^{-1} \|F\|_{\mathcal{V}'},$  if and only if

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Babuška to Brezzi

## Continuous stability

$$\inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{b(u, v)}{\|u\|_u \|v\|_v} = \gamma > 0 \quad \not\Rightarrow$$

## Discrete stability

$$\inf_{u_h \in \mathcal{U}_h} \sup_{v_h \in \mathcal{V}_h} \frac{b(u_h, v_h)}{\|u_h\|_u \|v_h\|_v} = \gamma_h > 0$$

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Q: How to satisfy the discrete inf-sup condition

$$\sup_{v_h \in \mathcal{V}_h} \frac{b(u_h, v_h)}{\|v_h\|_\mathcal{V}} \geq \gamma_h \|u_h\|_U, \quad \forall u_h \in \mathcal{U}_h \quad ?$$

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**A:** Fix  $\mathcal{U}_h$  and increase the dimension of  $\mathcal{V}_h$  until satisfied.

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- An inf-sup condition *in the kernel* of the dual operator 

# Stability

Babuška to Brezzi

$$\dim \mathcal{V}_h > \dim \mathcal{U}_h$$

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**Q:** The dual problem is now *underdetermined*. What to do?

**A:** Consider a different embedding in the saddle point system.

# Stability

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## Primal:

Find  $\mathbf{u}_h \in \mathcal{U}_h$  and  $\boldsymbol{\varepsilon}_h \in \mathcal{V}_h$  :

$$\begin{aligned} (\boldsymbol{\varepsilon}_h, v_h)_V + b(\mathbf{u}_h, v_h) &= F(v_h) \quad \forall v_h \in \mathcal{V}_h \\ b(w_h, \boldsymbol{\varepsilon}_h) &= 0 \quad \forall w_h \in \mathcal{U}_h \end{aligned}$$

## Dual:

Find  $\mathbf{v}_h \in \mathcal{V}_h$  and  $\boldsymbol{\lambda}_h \in \mathcal{U}_h$  :

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# Implementation

General problem:

Find  $v_h \in \mathcal{V}_h$  and  $w_h \in \mathcal{U}_h$  :

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Discretization:

$$\begin{bmatrix} \mathbf{G} & \mathbf{B} \\ \mathbf{B}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

Schur complement:

$$\begin{aligned} \mathbf{B}^T \mathbf{G}^{-1} \mathbf{B} \mathbf{w} &= \mathbf{B}^T \mathbf{G}^{-1} \mathbf{f} - \mathbf{g} \\ \mathbf{v} &= \mathbf{G}^{-1} (\mathbf{f} - \mathbf{B} \mathbf{w}) \end{aligned}$$

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DPG:  $\mathbf{G}$  is block-diagonal

# DPG and DPG\* comparison

Minimum norm derivation

Stiffness matrix: Gram matrix: Load vector: Goal vector:

$$B_{ij} = b(u_j, v_i)$$

$$G_{ij} = (v_j, v_i)_V$$

$$f_i = F(v_i)$$

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In DPG, we solve:

$$B^T G^{-1} Bu = B^T G^{-1} f$$

In DPG\*, we solve:

$$B^T G^{-1} Bw = g$$

and then post-process  $w$ :

$$v = G^{-1} Bw$$

# Discrete least squares finite element methods

Cholesky factorization

$$G = LL^T.$$

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Cholesky factorization

$$\mathbf{G} = \mathbf{L}\mathbf{L}^T.$$

Residual minimization problem

$$\mathbf{u} = \arg \min_{\mathbf{w} \in \mathbb{R}^N} (\mathbf{B}\mathbf{w} - \mathbf{f})^T \mathbf{G}^{-1} (\mathbf{B}\mathbf{w} - \mathbf{f})$$

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# Discrete least squares finite element methods

## Two solution paradigms

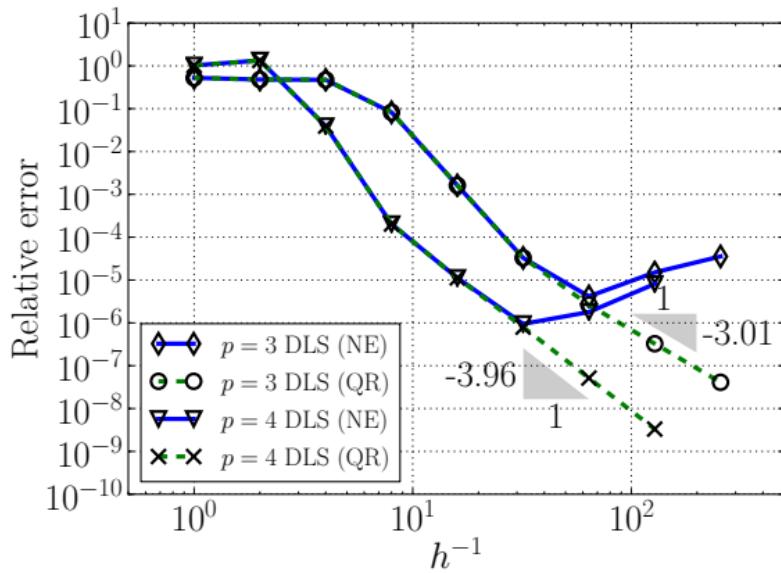
Normal equations

$$\mathbf{B}^T \mathbf{G}^{-1} \mathbf{B} \mathbf{u} = \mathbf{B}^T \mathbf{G}^{-1} \mathbf{f}$$

Overdetermined system

$$\mathbf{L}^{-1} \mathbf{B} \mathbf{u} = \mathbf{L}^{-1} \mathbf{f}$$

- ★ Static condensation for  $\mathbf{L}^{-1} \mathbf{B}$  is explained in the article
- ★  $\mathbf{L}^{-1} \mathbf{B}$  has  $O(h^{-1})$  condition number
- ★ Less round off error when handling  $\mathbf{L}^{-1} \mathbf{B}$  directly



Acoustics with DPG near resonance

# Highlights

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# *A priori* error estimation

Continuous problem:

Find  $\mathbf{v} \in \mathcal{V}$  and  $\mathbf{w} \in \mathcal{U}$  :

$$(\mathbf{v}, \nu)_{\mathcal{V}} + b(\mathbf{w}, \nu) = F(\nu), \quad \nu \in \mathcal{V}$$

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Find  $\mathbf{v}_h \in \mathcal{V}_h$  and  $\mathbf{w}_h \in \mathcal{U}_h$  :

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Fortin operator:

Continuous operator  $\Pi_h : \mathcal{V} \rightarrow \mathcal{V}_h$  such that

$$b(\mu, \nu - \Pi_h \nu) = 0 \quad \forall \mu \in \mathcal{U}_h, \nu \in \mathcal{V}.$$

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## Theorem (Brezzi)

Let  $F \in \mathcal{V}'$  and  $G \in \mathcal{U}'$ . There is a constant  $C$  such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{V}} + \|\mathbf{w} - \mathbf{w}_h\|_{\mathcal{U}} \leq C \left[ \inf_{\nu \in \mathcal{V}_h} \|\mathbf{v} - \nu\|_{\mathcal{V}} + \inf_{\mu \in \mathcal{U}_h} \|\mathbf{w} - \mu\|_{\mathcal{U}} \right].$$

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## Theorem

Let  $F \in (\text{Null } \mathcal{B}')^\perp$ . There is a constant  $C$  such that

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Recall:

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Find  $\boldsymbol{v} \in \mathcal{V}$  and  $\lambda \in \mathcal{U}$  :

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DPG\*: The discretization error is affected by  $\boldsymbol{\lambda}$ !

# *A priori* error estimation: DPG\*

Poisson's equation (ultraweak)

Consider

Broken ultraweak formulation

$$\begin{aligned} -\Delta v = f \quad \implies \quad & b((u, \vec{q}, \hat{u}, \hat{q}_n), (v, \vec{p})) \\ &= (\vec{q}, \vec{p} - \text{grad}_h v)_\Omega - (u, \text{div}_h \vec{p})_\Omega \\ &+ \langle \hat{u}, \vec{p} \cdot \vec{n} \rangle_h + \langle \hat{q}_n, v \rangle_h \end{aligned}$$

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where  $\mathcal{L}_h(v, \vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h v)$ .

# *A priori* error estimation: DPG\*

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DPG\* formulation:

$$\begin{aligned} ((\boldsymbol{v}, \vec{\boldsymbol{p}}), (\boldsymbol{\nu}, \vec{\boldsymbol{\tau}}))_{\boldsymbol{\nu}} - b((\lambda, \vec{\zeta}, \hat{\lambda}, \hat{\zeta}), (\boldsymbol{\nu}, \vec{\boldsymbol{\tau}})) &= 0 \quad \forall \boldsymbol{\nu}, \vec{\boldsymbol{\tau}} \\ b((\mu, \vec{\sigma}, \hat{\mu}, \hat{\sigma}_n), (\boldsymbol{v}, \vec{\boldsymbol{p}})) &= (f, \mu)_{\Omega} \quad \forall \mu, \vec{\sigma}, \hat{\mu}, \hat{\sigma}_n \end{aligned}$$

# *A priori* error estimation: DPG\*

Poisson's equation (ultraweak)

## Proposition

Let  $\|(\mathbf{v}, \vec{p})\|_{\mathcal{V}}^2 = \|\operatorname{grad}_h \mathbf{v}\|_{\Omega}^2 + \|\mathbf{v}\|_{\Omega}^2 + \|\operatorname{div}_h \vec{p}\|_{\Omega}^2 + \|\vec{p}\|_{\Omega}^2$ . The solution components  $\lambda, \vec{\zeta}, \hat{\lambda}, \hat{\zeta}$  of the system above can be characterized using the remaining solution components  $\mathbf{v}, \vec{p}$  and  $f$  as

$$\begin{aligned}\lambda &= f + \mathbf{e}, & \hat{\lambda} &= \mathbf{e} \\ \vec{\zeta} &= \vec{p} + \vec{r}, & \hat{\zeta} &= 2\vec{p} \cdot \vec{n} + \vec{r} \cdot \vec{n}\end{aligned}$$

where  $\mathbf{e} \in H_0^1(\Omega)$  satisfies the Dirichlet problem  $-\Delta \mathbf{e} = \mathbf{v} + 2f$  and  $\vec{r} = -\operatorname{grad} \mathbf{e}$ .

---

L. Demkowicz, J. Gopalakrishnan, and B. Keith.

The DPG-star method.

*In preparation*, 2018.

T. Führer.

Superconvergence in a DPG method for an ultra-weak formulation.

*Comput. Math. Appl.*, 75(5):1705 – 1718, 2018.

## Sketch of proof

- Use the second line to characterize the solution  $\textcolor{brown}{v}, \vec{p}$ .

Note that  $b((\mu, \vec{\sigma}, \hat{\mu}, \hat{\sigma}_n), (\textcolor{brown}{v}, \vec{p})) = (f, \mu)_\Omega$ .

$$\begin{aligned} \implies ((\mu, \vec{\sigma}), \mathcal{L}_h(\textcolor{brown}{v}, \vec{p}))_\Omega &= (f, \mu)_\Omega & \forall \mu, \vec{\sigma} \\ \langle \hat{\mu}, \vec{p} \cdot \vec{n} \rangle_h &= 0 & \forall \hat{\mu} \\ \langle \hat{\sigma}_n, \textcolor{brown}{v} \rangle_h &= 0 & \forall \hat{\sigma}_n \end{aligned}$$

where  $\mathcal{L}_h(v, \vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h v)$ .

# Sketch of proof

- Use the second line to characterize the solution  $\mathbf{v}, \vec{p}$ .

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where  $\mathcal{L}_h(\mathbf{v}, \vec{p}) = (-\operatorname{div}_h \vec{p}, \vec{p} - \operatorname{grad}_h \mathbf{v})$ .

- Show that this implies

$$\vec{p} - \operatorname{grad} \mathbf{v} = \vec{0}, \quad -\operatorname{div} \vec{p} = f.$$

## Sketch of proof

- Manipulate first term in first line:

$$\begin{aligned} ((\mathbf{v}, \vec{\mathbf{p}}))_{\mathcal{V}} &= (\vec{\mathbf{p}}, \vec{\tau})_{\Omega} + (\operatorname{div} \vec{\mathbf{p}}, \operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} + (\operatorname{grad} \mathbf{v}, \operatorname{grad} \nu)_{\Omega} \\ &= (\vec{\mathbf{p}}, \vec{\tau} - \operatorname{grad} \nu)_{\Omega} + (\operatorname{div} \vec{\mathbf{p}}, \operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} + 2(\operatorname{grad} \mathbf{v}, \operatorname{grad} \nu)_{\Omega} \\ &= (\vec{\mathbf{p}}, \vec{\tau} - \operatorname{grad} \nu)_{\Omega} + (f, -\operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} \\ &\quad + 2 \sum_{K \in \Omega} \left[ \langle \vec{n} \cdot \operatorname{grad} \mathbf{v}, \nu \rangle_{H^{1/2}(\partial K)} - (\Delta \mathbf{v}, \nu)_K \right] \\ &= b((f, \vec{\mathbf{p}}, 0, 2\vec{\mathbf{p}} \cdot \vec{n}), (\nu, \vec{\tau})) + (\mathbf{v} + 2f, \nu)_{\Omega} \end{aligned}$$

## Sketch of proof

- Manipulate first term in first line:

$$\begin{aligned} ((\mathbf{v}, \vec{p}))_{\mathcal{V}} &= (\vec{p}, \vec{\tau})_{\Omega} + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} + (\operatorname{grad} \mathbf{v}, \operatorname{grad} \nu)_{\Omega} \\ &= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_{\Omega} + (\operatorname{div} \vec{p}, \operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} + 2(\operatorname{grad} \mathbf{v}, \operatorname{grad} \nu)_{\Omega} \\ &= (\vec{p}, \vec{\tau} - \operatorname{grad} \nu)_{\Omega} + (f, -\operatorname{div} \vec{\tau})_{\Omega} + (\mathbf{v}, \nu)_{\Omega} \\ &\quad + 2 \sum_{K \in \Omega} \left[ \langle \vec{n} \cdot \operatorname{grad} \mathbf{v}, \nu \rangle_{H^{1/2}(\partial K)} - (\Delta \mathbf{v}, \nu)_K \right] \\ &= b((f, \vec{p}, 0, 2\vec{p} \cdot \vec{n}), (\nu, \vec{\tau})) + (\mathbf{v} + 2f, \nu)_{\Omega} \end{aligned}$$

- Define  $\mathbf{e}$ ,  $\vec{r}$ ,  $\hat{\mathbf{e}}$ , and  $\hat{\mathbf{r}}$  such that

$$b((\mathbf{e}, \vec{r}, \hat{\mathbf{e}}, \hat{\mathbf{r}}), (\nu, \vec{\tau})) = (\mathbf{v} + 2f, \nu)_{\Omega} \implies -\Delta \mathbf{e} = \mathbf{v} + 2f$$

# *A priori* error estimation: DPG\*

Poisson's equation

Bramble-Hilbert argument:

## Corollary

Let  $\vec{v}_h = (\mathbf{v}_h, \vec{p}_h) \in \mathcal{V}_h$  and  $\vec{\lambda}_h = (\lambda_h, \vec{\zeta}_h, \hat{\lambda}_h, \hat{\zeta}_h) \in \mathcal{U}_h$  be the DPG\* solutions.

Let  $\mathbf{e} \in H_0^1(\Omega)$  satisfy  $-\Delta \mathbf{e} = \mathbf{v} + 2f$ . Then

$$\|\vec{v} - \vec{v}_h\|_{\mathcal{V}} + \|\vec{\lambda} - \vec{\lambda}_h\|_{\mathcal{U}} \leq Ch^s (\|\mathbf{v}\|_{H^{s+2}(\Omega)} + \|\mathbf{e}\|_{H^{s+2}(\Omega)})$$

for all  $1/2 < s < p + 1$ .

# *A priori* error estimation: DPG\*

Poisson's equation

Manufactured solution 1:  $v = \sin(\pi x) \sin(\pi y)$ ,  $\Omega = (0, 1)^2$

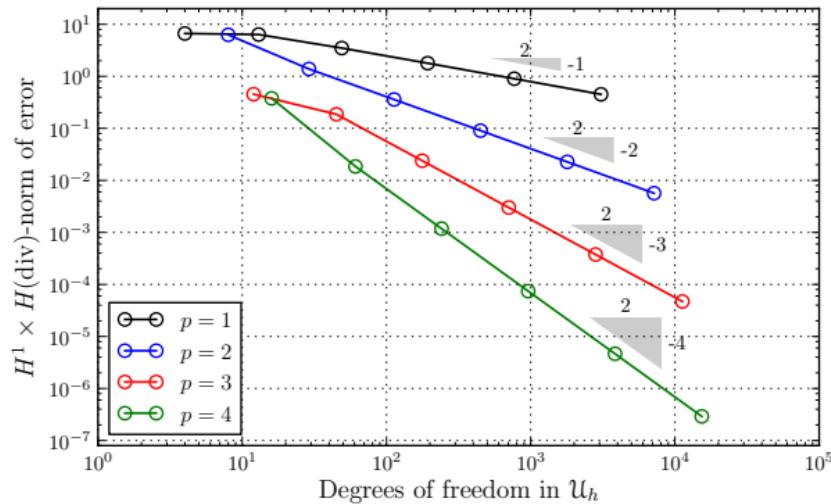


Figure: Uniform  $h$ -refinements

# *A priori* error estimation: DPG\*

Poisson's equation

Manufactured solution 2:  $v = 1$

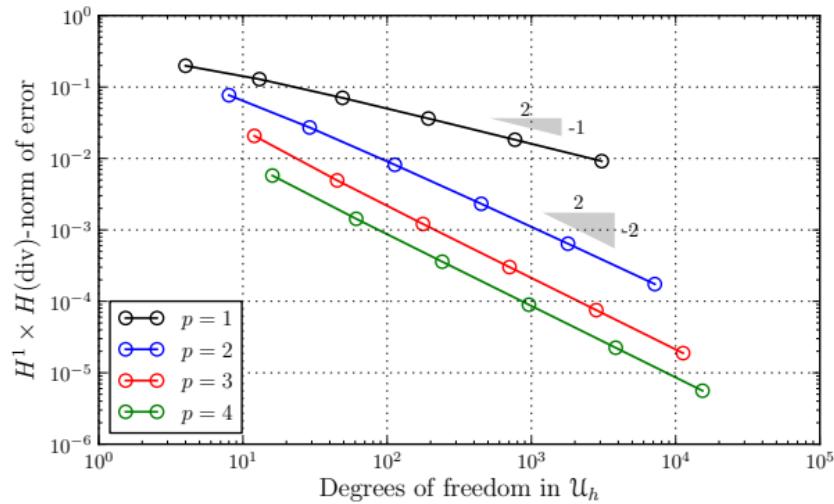


Figure: Uniform  $h$ -refinements

# *A priori* error estimation: DPG\*

Poisson's equation

Manufactured solution 2:  $v = 1$

$$-\Delta e = 1 \text{ in } \Omega, \quad e|_{\partial\Omega} = 0$$

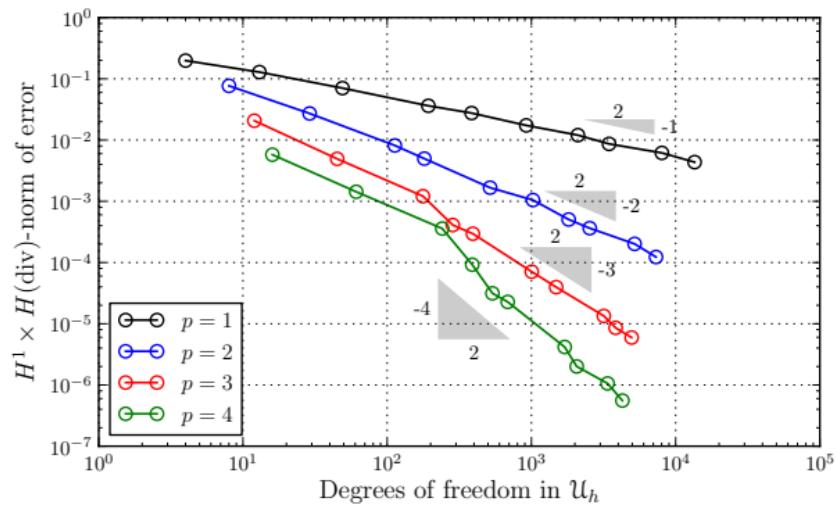


Figure: Adaptive  $h$ -refinements

## *A priori* error estimation: DPG\*

Poisson's equation

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# *A priori* error estimation: DPG\*

Poisson's equation

Manufactured solution 2:  $v = 1$

$$-\Delta\lambda = 1 \text{ in } \Omega, \quad \lambda|_{\partial\Omega} = 0$$

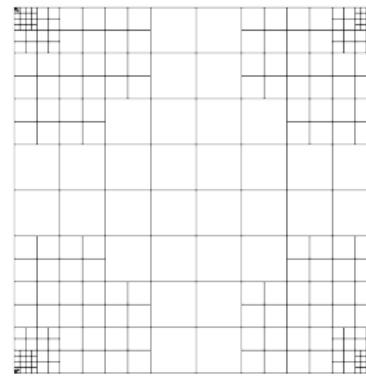
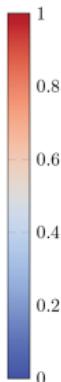
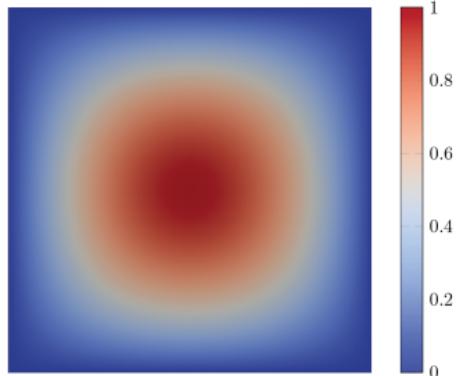


Figure: Left: The solution component  $\lambda$  when  $v = 1$ . Right: The corresponding mesh (ten refinements).

# Duality

DPG:

$$(1) \quad (\varepsilon, z)_V + b(\textcolor{brown}{u}, z) = F(z)$$

$$(2) \quad b(w, \varepsilon) = 0$$

DPG\*:

$$(3) \quad (\textcolor{brown}{v}, z)_V - b(\lambda, z) = 0$$

$$(4) \quad b(w, \textcolor{brown}{v}) = G(w)$$

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Observe:

$$G(\mathbf{u} - \mathbf{u}_h) = b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) \quad \text{by (4)}$$

DPG:

$$(1) \quad (\varepsilon, z)_V + b(\mathbf{u}, z) = F(z)$$

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$$(3) \quad (\mathbf{v}, z)_V - b(\lambda, z) = 0$$

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Observe:

$$\begin{aligned} G(\mathbf{u} - \mathbf{u}_h) &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) && \text{by (4)} \\ &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + (\mathbf{v}, \varepsilon - \varepsilon_h)_V - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \end{aligned}$$

**DPG:**

$$(5) \quad (\varepsilon_h, z_h)_V + b(\mathbf{u}_h, z_h) = F(z_h)$$

$$(6) \quad b(w_h, \varepsilon_h) = 0$$

**DPG\*:**

$$(7) \quad (\mathbf{v}_h, z_h)_V - b(\lambda_h, z_h) = 0$$

$$(8) \quad b(w_h, \mathbf{v}_h) = G(w_h)$$

Observe:

$$\begin{aligned} G(\mathbf{u} - \mathbf{u}_h) &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) && \text{by (4)} \\ &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + (\mathbf{v}, \varepsilon - \varepsilon_h)_V - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \end{aligned}$$

**DPG:**

$$(9) \quad (\varepsilon - \varepsilon_h, z_h)_\gamma + b(\mathbf{u} - \mathbf{u}_h, z_h) = 0$$

$$(10) \quad b(w_h, \varepsilon - \varepsilon_h) = 0$$

**DPG\*:**

$$(11) \quad (\mathbf{v} - \mathbf{v}_h, z_h)_\gamma - b(\lambda - \lambda_h, z_h) = 0$$

$$(12) \quad b(w_h, \mathbf{v} - \mathbf{v}_h) = 0$$

Observe:

$$\begin{aligned} G(\mathbf{u} - \mathbf{u}_h) &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) && \text{by (4)} \\ &= b(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + (\mathbf{v}, \varepsilon - \varepsilon_h)_\gamma - b(\lambda, \varepsilon - \varepsilon_h) && \text{by (3)} \end{aligned}$$

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DPG:

$$(9) \quad (\varepsilon - \varepsilon_h, z_h)_\gamma + b(\mathbf{u} - \mathbf{u}_h, z_h) = 0$$

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DPG:

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$$(1) \quad (\varepsilon, z)_V + b(\mathbf{u}, z) = F(z)$$

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# Aubin-Nitsche

## Corollary

The following crude upper bounds hold:

$$b(\mathbf{u} - \mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) \lesssim \begin{cases} \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{U}} \inf_{\mathbf{z}_h \in \mathcal{V}_h, \mathbf{w}_h \in \mathcal{U}_h} (\|\mathbf{v} - \mathbf{z}_h\|_{\mathcal{V}} + \|\boldsymbol{\lambda} - \mathbf{w}_h\|_{\mathcal{U}}) \\ \|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{V}} \inf_{\mathbf{w}_h \in \mathcal{U}_h} \|\mathbf{u} - \mathbf{w}_h\|_{\mathcal{U}} \end{cases}$$

# Aubin-Nitsche

## Theorem

Suppose that there is a positive function  $c_1(h)$  that goes to 0 as  $h \rightarrow 0$  satisfying

$$\inf_{w_h \in \mathcal{U}_h} \|\mathbf{u} - w_h\|_{\mathcal{U}} \leq c_1(h) \|F\|_{\mathcal{V}'}$$

Then the error in the DPG\* solution component  $\mathbf{u}_h$  satisfies

$$G(\mathbf{u} - \mathbf{u}_h) \leq c_0(h) \|G\|_{\mathcal{U}'} \|\mathcal{B}\| \inf_{w_h \in \mathcal{U}_h} \|\mathbf{u} - w_h\|_{\mathcal{U}}$$

- Can be used to demonstrate accelerated convergence of DPG\* solution components)

# *A priori* error estimation: DPG\*

Poisson's equation

Manufactured solution 2:  $v = 1$

$$-\Delta\lambda = 1 \text{ in } \Omega, \quad \lambda|_{\partial\Omega} = 0$$

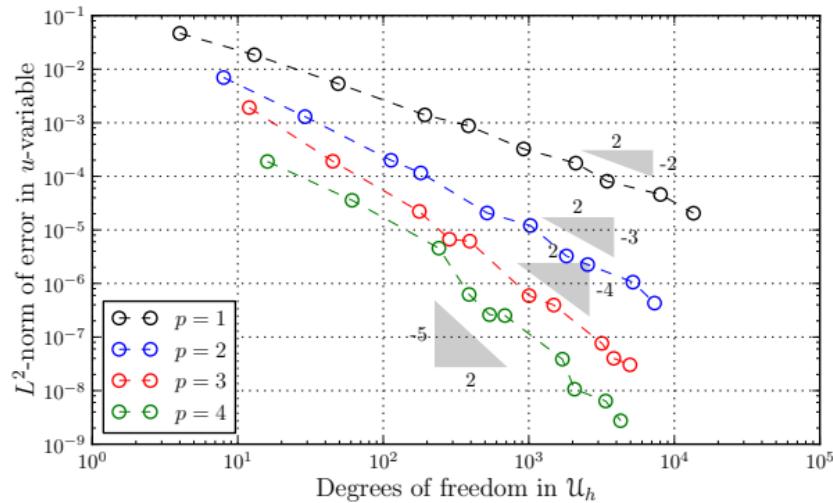


Figure: Adaptive  $h$ -refinements.

# *A posteriori* error estimation

# *A posteriori* error estimation: DPG

DPG minimizes the error in the **energy norm**

$$\|w\|_{\mathcal{U}} := \|\mathcal{B}w\|_{\mathcal{V}'} \quad \forall w \in \mathcal{U}$$

## *A posteriori* error estimation: DPG

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The energy norm  $\|\cdot\|_{\mathcal{U}}$  is equivalent to the norm  $\|\cdot\|_{\mathcal{U}}$

$$\|\mathcal{B}^{-1}\|^{-1}\|w\|_{\mathcal{U}} \leq \|w\|_{\mathcal{U}} \leq \|\mathcal{B}\| \|w\|_{\mathcal{U}}$$

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A posteriori error estimator for **broken** test spaces

$$\|\mathbf{u} - w\|_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2$$

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A posteriori error estimator for **broken test spaces**

$$\|\mathbf{u} - w\|_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2 = \sum_{K \in \mathcal{T}} \|\mathcal{B}w - F\|_{\mathcal{V}(K)'}^2$$

# *A posteriori* error estimation: DPG

DPG minimizes the error in the **energy norm**

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The energy norm  $\|\cdot\|_{\mathcal{U}}$  is equivalent to the norm  $\|\cdot\|_{\mathcal{U}}$

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A posteriori error estimator for **broken test spaces**

$$\|\mathbf{u} - w\|_{\mathcal{U}}^2 = \|\mathcal{B}w - F\|_{\mathcal{V}'}^2 = \sum_{K \in \mathcal{T}} \|\mathcal{B}w - F\|_{\mathcal{V}(K)'}^2 \simeq \sum_{K \in \mathcal{T}} \left( \underbrace{\|\mathcal{B}w - F\|_{\mathcal{V}_{\textcolor{brown}{h}(K)'}}} _{:= \eta_K(w)} \right)^2$$

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$$\eta_K(\mathbf{u}_h) = \sup_{z_h \in \mathcal{V}_h(K)} \frac{b(\mathbf{u}_h, z_h) - F(z_h)}{\|z_h\|_{\mathcal{V}}}$$

C. Carstensen, L. Demkowicz, and J. Gopalakrishnan.

A posteriori error control for DPG methods.

SIAM J. Numer. Anal., 52(3):1335–1353, 2014.

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# Residual bounds

## Theorem

Let  $w \in \mathcal{U}$  and  $z \in \mathcal{V}$  be arbitrary. Then the following upper bound holds:

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## Proof:

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$$\begin{aligned}|b(\mathbf{u} - w, \mathbf{v} - z)| &= |b(\mathbf{u} - w, \mathbf{v} - z - z_0)| \\&\leq \|\mathcal{B}w - F\|_{\mathcal{V}'} \|v - z - z_0\| \\&= \|\mathcal{B}w - F\|_{\mathcal{V}'} (\|\mathcal{P}(v - z - z_0)\|_{\mathcal{V}}^2 + \|\mathcal{B}'z - G\|_{\mathcal{U}'}^2)^{1/2}.\end{aligned}$$

- Set  $z_0 = \mathcal{P}(v - z) \in \text{Null } \mathcal{B}'$ .

# Algorithm and Marking Strategies

. . . Solve–Estimate–Estimate–Mark–Refine. . .

## Adaptive mesh refinement

1. Solve for  $u_h$  and/or  $v_h$

- Mark all elements  $K \in \mathcal{T}$  such that  $\theta \cdot \tilde{\eta}_{\max} \leq \tilde{\eta}_K$ ,  $\theta \in (0, 1)$
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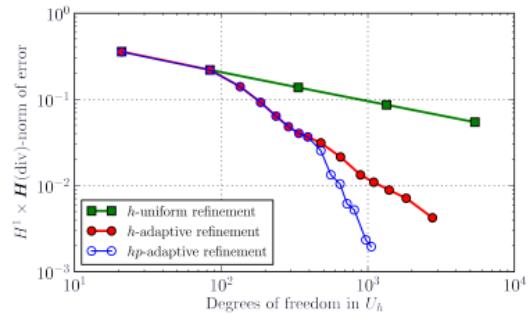
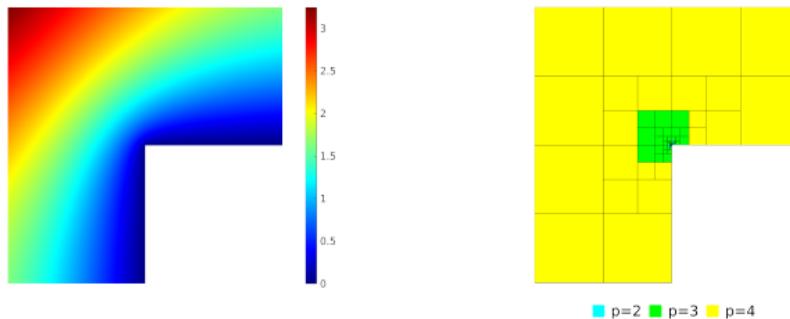
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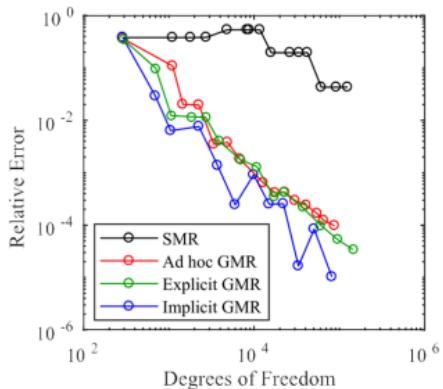
# Applications

# *hp*-adaptive mesh refinement in DPG\* methods

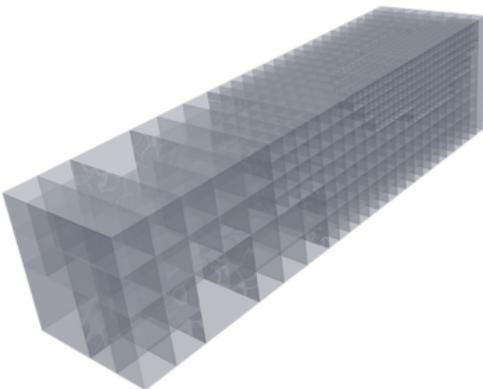


Singular solution and *hp*-adaptive mesh refinements with a DPG\* method for Poisson's equation.

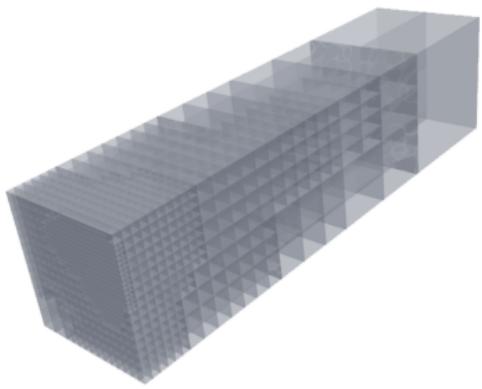
# Goal-Oriented Adaptive Mesh Refinement (AMR) in 3D



The error in the average flux.



Solution-oriented AMR.



Goal-oriented AMR

# Fluid Mechanics

## Oldroyd-B fluid

Conservation of mass and momentum:

$$\begin{aligned} \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma} &= \rho \mathbf{f} \quad \text{on } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{on } \Omega \times (0, T). \end{aligned}$$

Constitutive law:

$$\boldsymbol{\sigma} = -p \mathbf{I} + 2\mu_S \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{T},$$

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Lie derivative:

$$\mathcal{L}_u \mathbf{T} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{T} - (\nabla \mathbf{u} \mathbf{T} - \mathbf{T} \nabla^T \mathbf{u}).$$

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Quantity of interest

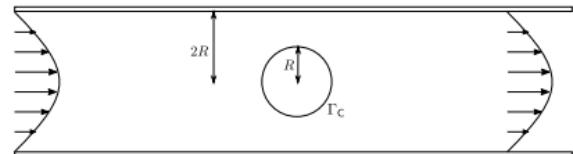
Drag coefficient

$$c_D(\boldsymbol{\sigma}) = \frac{1}{\mu \bar{u}} \int_{\Gamma_c} (\boldsymbol{\sigma} \vec{n}) \cdot \mathbf{e}_x \, ds.$$

$\Gamma_c$ : boundary of cylinder

$\mu = \mu_S + \mu_P$ : viscosity

$\bar{u}$ : average inflow velocity.



Confined cylinder domain.

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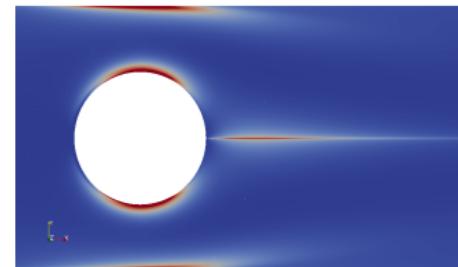
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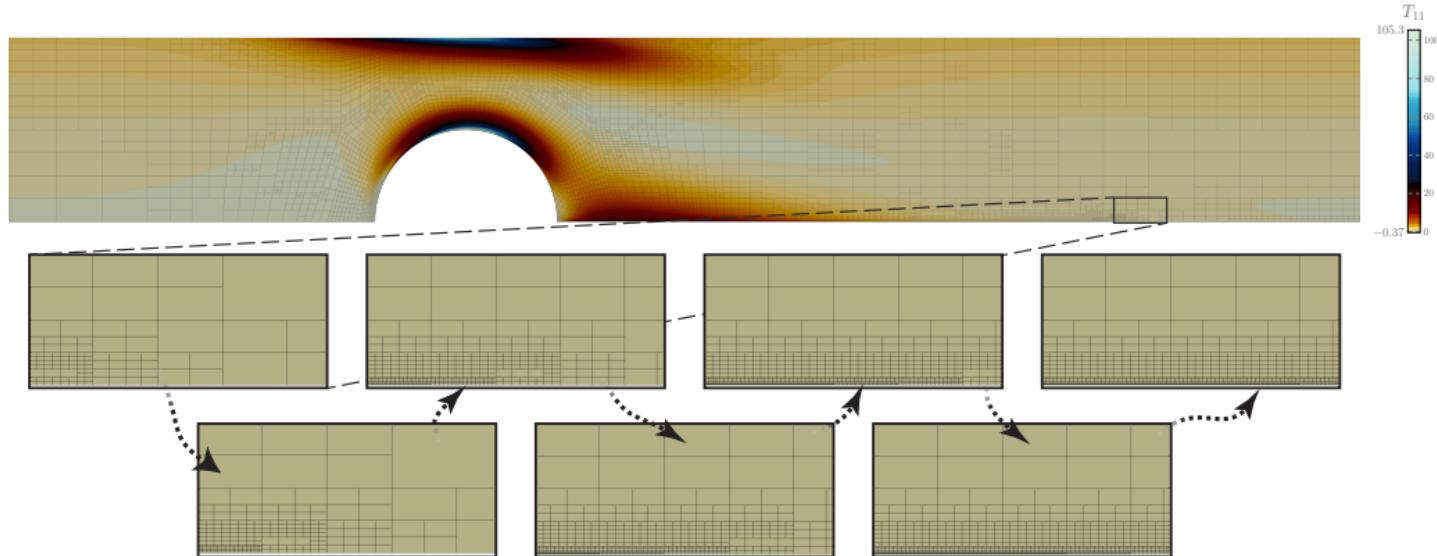
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Close-up of rescaled  $T_{11}$ -component from  $\lambda = 0.9$ .

# Fluid Mechanics

## Solution-oriented adaptive mesh refinement



Close-up of rescaled  $T_{11}$ -component from  $\lambda = 0.7$ .

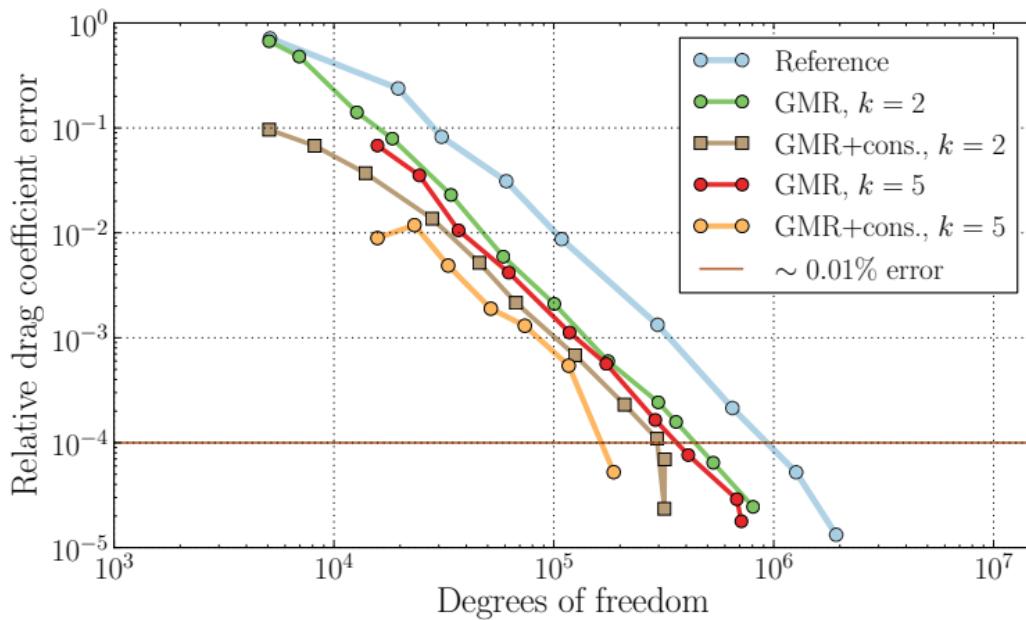
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B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr, and L. Demkowicz.

An ultraweak DPG method for viscoelastic fluids.

*J. Non-Newton. Fluid Mech.*, 247:107–122, 2017.

# Goal-oriented adaptivity in nonlinear problems



The error in the drag coefficient of the confined cylinder ( $\lambda = 0.4$ ).

# A Posteriori Error Estimation

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- $G(\vec{u}) = (f, u)_\Omega$

Theorem (Demkowicz, Gopalakrishnan, & K.)

Assume  $\Omega \subset \mathbb{R}^2$  with sufficiently regular boundary. If  $\vec{\mathbf{v}}_h = (\mathbf{v}_h, \vec{\mathbf{p}}_h) \in \mathcal{V}_h$  satisfies

$$b(\vec{w}_h, \vec{\mathbf{v}}_h) = G(\vec{w}_h) \quad \forall \vec{w}_h \in \mathcal{U}_h,$$

then  $\exists C_2 > C_1 > 0$ , independent of the element sizes  $h_K$ , such that

$$C_1 \eta^*(\vec{\mathbf{v}}_h) \leq \|\vec{\mathbf{v}} - \vec{\mathbf{v}}_h\|_{\mathcal{V}} \leq C_2 \eta^*(\vec{\mathbf{v}}_h),$$

where

$$\eta^*(\vec{\mathbf{v}}_h)^2 = \left\| \mathcal{L}_h \vec{\mathbf{v}}_h - f \right\|_\Omega^2 + \sum_{E \in \mathcal{E}_{\text{int}}} h_E \|[\![\vec{\mathbf{p}}_h]\!] \|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}} h_E \|[\![\mathbf{v}_h]\!] \|_{H^1(E)}^2$$

## Sketch of the Proof

- $\mathcal{B}$  is a bijection so

$$\|\textcolor{brown}{v} - \textcolor{brown}{v}_h\|_{\mathcal{V}}^2 \asymp \|\mathcal{B}' \textcolor{brown}{v}_h - G\|_{\mathcal{U}'}^2$$

# Sketch of the Proof

- $\mathcal{B}$  is a bijection so

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{V}}^2 \asymp \|\mathcal{B}' \mathbf{v}_h - G\|_{\mathcal{U}'}^2$$

- Decompose supremum into three terms:

$$\begin{aligned} \left( \sup_{\vec{u} \in \mathcal{U}} \frac{b(\vec{u}, \vec{v}_h) - G(\vec{u})}{\|\vec{u}\|_{\mathcal{U}}} \right)^2 &= \left( \sup_{(u, \vec{q}) \in L^2(\Omega) \times L^2(\Omega)^d} \frac{((u, \vec{q}), \mathcal{L}_h \vec{v}_h - f)_{\Omega}}{\|(u, \vec{q})\|_{\Omega}} \right)^2 \\ &\quad + \left( \sup_{\hat{u} \in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \vec{p}_h \cdot \vec{n} \rangle_h}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}} \right)^2 \\ &\quad + \left( \sup_{\hat{q}_n \in H^{-1/2}(\mathcal{S})} \frac{\langle \hat{q}_n, v_h \rangle_h}{\|\hat{q}_n\|_{H^{-1/2}(\partial\mathcal{T})}} \right)^2 \end{aligned}$$

# Sketch of the Proof

Term 1:

- $$\sup_{(u, \vec{q}) \in L^2(\Omega) \times L^2(\Omega)^d} \frac{((u, \vec{q}), \mathcal{L}_h \vec{v}_h - f)_\Omega}{\|(u, \vec{q})\|_\Omega} = \|\mathcal{L}_h v_h - f\|_\Omega$$

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Term 2\*:

- $$\sup_{\hat{u} \in H_0^{1/2}(\mathcal{S})} \frac{\langle \hat{u}, \vec{p}_h \cdot \vec{n} \rangle_h}{\|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}} = \sup_{u \in H_0^1(\Omega)} \frac{\langle \text{tr}(u), \vec{p}_h \cdot \vec{n} \rangle_h}{\|u\|_{H^1(\Omega)}}$$

# Sketch of the Proof

Term 1:

- $$\sup_{(u, \vec{q}) \in L^2(\Omega) \times L^2(\Omega)^d} \frac{((u, \vec{q}), \mathcal{L}_h \vec{v}_h - f)_\Omega}{\|(u, \vec{q})\|_\Omega} = \|\mathcal{L}_h v_h - f\|_\Omega$$

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Term 3\*:

- $$\sup_{\hat{q}_n \in H^{-1/2}(\mathcal{S})} \frac{\langle \hat{q}_n, v_h \rangle_h}{\|\hat{q}_n\|_{H^{-1/2}(\partial\mathcal{T})}} = \sup_{\vec{q} \in H(\text{div}, \Omega)} \frac{\langle \text{tr}_n(\vec{q}), v_h \rangle_h}{\|\vec{q}\|_{H(\text{div}, \Omega)}}$$

\*requires a lemma

## Sketch of the Proof

Theorem (Demkowicz, Gopalakrishnan, K.)

If  $\int_{\partial K} \hat{p}_K [\vec{p}_h] = 0$  for all  $\hat{p}_K \in P_C^1(\partial K)$  then

$$\sup_{\mu \in H_0^1(\Omega)} \frac{\langle \operatorname{tr} \mu, \vec{p}_h \cdot \vec{n} \rangle_h^2}{\|\mu\|_{H^1(\Omega)}^2} \asymp \sum_{E \in \mathcal{E}} h_E \|[\vec{p}_h]\|_{L^2(E)}^2.$$

Theorem (Demkowicz, Gopalakrishnan, K.)

If  $\int_E [v_h] = 0$  for all  $E \in \mathcal{E}$  then

$$\sup_{\vec{q} \in H(\operatorname{div}, \Omega)} \frac{\langle \operatorname{tr}_n(\vec{q}), v_h \rangle_h^2}{\|\vec{q}\|_{H(\operatorname{div}, \Omega)}^2} \asymp \sum_{E \in \mathcal{E}} h_E \| [v_h] \|_{H^1(E)}^2 \asymp \sum_{E \in \mathcal{E}} h_E^{-1} \| [v_h] \|_{L^2(E)}^2.$$

# Sketch of the Proof

## Theorem (Regular decompositions)

Given any  $\vec{q} \in H(\text{div}, \Omega)$ , there exist  $\varphi \in H^1(\Omega)$  and  $\vec{\psi} \in H^1(\Omega)$  such that  $\vec{q} = \text{curl}(\varphi) + \vec{\psi}$  and

$$\|\varphi\|_{H^1(\Omega)} + \|\vec{\psi}\|_{H^1(\Omega)} \lesssim \|\vec{q}\|_{H(\text{div}, \Omega)}.$$

In this situation,  $\vec{q} = \text{curl}(\varphi) + \vec{\psi}$  is called a *regular decomposition* of  $\vec{q}$ .

# Sketch of the Proof

## Theorem (Demlow, Hirani, Schöberl)

Let  $E \in \mathcal{E}$ ,  $u \in H^1(\Omega)$ , and  $\vec{q} \in H(\text{div}, \Omega)$  be arbitrary. Let  $h_E$  denote the length of the edge  $E$  and let  $\varphi \in H^1(\Omega)$  and  $\vec{\psi} \in H^1(\Omega)$  belong to a *regular decomposition*  $\vec{q} = \text{curl}(\varphi) + \vec{\psi}$ . There exist *commuting quasi-interpolation operators*  $\mathcal{I} : H^1(\Omega) \rightarrow P_C^1(\mathcal{T})$  and  $\vec{\mathcal{I}} : H(\text{div}, \Omega) \rightarrow \mathcal{RT}^0(\mathcal{T})$ , such that  $\text{curl} \circ \mathcal{I} = \vec{\mathcal{I}} \circ \text{curl}$  and the following inequalities hold:

$$(5) \quad \|u - \mathcal{I}u\|_{L^2(E)}^2 \lesssim h_E \|u\|_{H^1(\Omega_E)}^2,$$

$$(6) \quad \|\varphi - \mathcal{I}\varphi\|_{L^2(E)}^2 + \|(\vec{\psi} - \vec{\mathcal{I}}\vec{\psi}) \cdot \vec{n}\|_{L^2(E)}^2 \lesssim h_E \|\vec{q}\|_{H(\text{div}, \Omega_E)}^2.$$

Each of the statements above hold with  $H^1(\Omega)$  replaced by  $H_0^1(\Omega)$ ,  $H^1(\Omega)$  replaced by  $H_0^1(\Omega)$ , and  $H(\text{div}, \Omega)$  replaced by  $H_0(\text{div}, \Omega)$ . In this case,  $\mathcal{I} : H_0^1(\Omega) \rightarrow P_{C,0}^1(\mathcal{T})$  and  $\vec{\mathcal{I}} : H_0(\text{div}, \Omega) \rightarrow \mathcal{RT}_0^0(\mathcal{T})$ .

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A. Demlow and A. N. Hirani.

A posteriori error estimates for finite element exterior calculus: The de Rham complex.  
*Found. Computat. Math.*, 14(6):1337–1371, 2014.

J. Schöberl.

A posteriori error estimates for Maxwell equations.  
*Math. Comput.*, 77(262):633–649, 2008.

# Sketch of the Proof

$$\sup_{u \in H^1(\Omega)} \frac{\langle \text{tr}(u), \vec{p}_h \cdot \vec{n} \rangle_h}{\|u\|_{H^1(\Omega)}}$$

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- Let  $u_h = \mathcal{I}u$ . Then

$$\langle u - u_h, \vec{p}_h \cdot \vec{n} \rangle_h = \sum_{K \in \mathcal{T}} (u - u_h, \vec{p}_h \cdot \vec{n}_K)_{\partial K} = \sum_{E \in \mathcal{E}} (u - u_h, [\![\vec{p}_h]\!])_E$$

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- For each edge  $E$ ,

$$(u - u_h, [\![\vec{p}_h]\!])_E \leq \|u - \mathcal{I}u\|_E \|[\![\vec{p}_h]\!]\|_E \lesssim h_K^{1/2} \|u\|_{H^1(\Omega_E)} \|[\![\vec{p}_h]\!]\|_E$$

where  $\Omega_E$  denotes the patch of elements associated with the edge  $E$ .

## Sketch of the Proof

$$\sup_{u \in H^1(\Omega)} \frac{\langle \text{tr}(u), \vec{p}_h \cdot \vec{n} \rangle_h}{\|u\|_{H^1(\Omega)}}$$

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where  $\Omega_E$  denotes the patch of elements associated with the edge  $E$ .

- Upper bound follows from manipulating the supremum
- Lower bound follows from Verfürth's bubble function technique

# Sketch of the Proof

$$\sup_{\vec{q} \in H(\text{div}, \Omega)} \frac{\langle \text{tr}_n(\vec{q}), v_h \rangle_h}{\|\vec{q}\|_{H(\text{div}, \Omega)}}$$

- Gauss & Stokes:

$$\langle (\text{curl } \phi) \cdot \vec{n}_K, v \rangle_{\partial K} = (\text{curl } \phi, \text{grad } v)_K + (\text{div curl } \phi, v)_K = \langle \phi, (\text{grad } v) \cdot \vec{t}_K \rangle_{\partial K}$$

## Sketch of the Proof

$$\sup_{\vec{q} \in H(\text{div}, \Omega)} \frac{\langle \text{tr}_n(\vec{q}), v_h \rangle_h}{\|\vec{q}\|_{H(\text{div}, \Omega)}}$$

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- Let  $\sigma_h = \vec{\mathcal{I}} \text{curl } \phi = \text{curl } \mathcal{I}\phi$ . Then

$$\begin{aligned} \langle (\phi - \sigma_h) \cdot \vec{n}_K, v \rangle_{\partial K} &= \langle \phi - \mathcal{I}\phi, (\text{grad } v) \cdot \vec{t}_K \rangle_{\partial K} \\ &= \sum_{K \in \mathcal{T}} (\phi - \mathcal{I}\phi, (\text{grad } v) \cdot \vec{t}_K)_{\partial K} \\ &= \sum_{E \in \mathcal{E}} (\phi - \mathcal{I}\phi, [(\text{grad } v) \cdot \vec{t}_K])_E \end{aligned}$$

## Sketch of the Proof

$$\sup_{\vec{q} \in H(\text{div}, \Omega)} \frac{\langle \text{tr}_n(\vec{q}), v_h \rangle_h}{\|\vec{q}\|_{H(\text{div}, \Omega)}}$$

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- Manipulate as before to get upper bound

# Sketch of the Proof

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- Manipulate as before to get upper bound
- Lower bound from Verfürth

# Summary

- Early work and applications
- DPG/DPG\* paradigm
- A *priori* error analysis
  - DPG error controlled by regularity of solution
  - DPG\* error controlled by regularity of solution **and** Lagrange multiplier
- Duality
- A *posteriori* error control
- Goal-oriented adaptivity

# CSEM objectives

## Area A:

- Analysis of the saddle point paradigm
- *A priori* and *a posteriori* error analysis of DPG\* methods
- Goal-oriented adaptive mesh refinement

## Area B:

- Implement finite element methods and compute *a posteriori* error indicators

## Area C:

- Analysis of viscoelastic fluid flow

# CSEM objectives

## Area A:

- Analysis of the saddle point paradigm
- *A priori* and *a posteriori* error analysis of DPG\* methods
- Goal-oriented adaptive mesh refinement
- Well-posedness of multiple/coupled variational formulations (linear elasticity)

## Area B:

- Implement finite element methods and compute *a posteriori* error indicators
- ESEAS shape functions library
- New assembly algorithm for DPG methods

## Area C:

- Analysis of viscoelastic fluid flow
- Structural mechanics and wave propagation problems

## References

- L. Demkowicz, J. Gopalakrishnan, and B. Keith.  
The DPG-star method.  
*In preparation*, 2018.
- F. Fuentes, B. Keith, L. Demkowicz, and S. Nagaraj.  
Orientation embedded high order shape functions for  
the exact sequence elements of all shapes.  
*Comput. Math. Appl.*, 70(4):353–458, 2015.
- F. Fuentes, B. Keith, L. Demkowicz, and P. L. Tallec.  
Coupled variational formulations of linear elasticity and  
the DPG methodology.  
*J. Comput. Phys.*, 348:715–731, 2017.
- B. Keith, L. Demkowicz, and J. Gopalakrishnan.  
DPG\* method.  
ICES Report 17-25, The University of Texas at Austin, 2017.
- B. Keith, F. Fuentes, and L. Demkowicz.  
The DPG methodology applied to different variational  
formulations of linear elasticity.  
*Comput. Methods Appl. Mech. Engrg.*, 309:579–609,  
2016.
- B. Keith, P. Knechtges, N. V. Roberts, S. Elgeti, M. Behr,  
and L. Demkowicz.  
An ultraweak DPG method for viscoelastic fluids.  
*J. Non-Newton. Fluid Mech.*, 247:107–122, 2017.
- B. Keith, S. Petrides, F. Fuentes, and L. Demkowicz.  
Discrete least-squares finite element methods.  
*Comput. Methods Appl. Mech. Engrg.*, 327:226–255,  
2017.
- B. Keith, N. V. Roberts, and L. Demkowicz.  
Goal-oriented adaptive mesh refinement with a DPG  
method for viscoelastic fluids.  
*In preparation*, 2018.
- B. Keith, A. Vaziri Astaneh, and L. Demkowicz.  
Goal-oriented adaptive mesh refinement for  
non-symmetric functional settings.  
*arXiv:1711.01996 (math.NA)*, 2017.
- A. Vaziri Astaneh, B. Keith, and L. Demkowicz.  
On perfectly matched layers for discontinuous  
Petrov–Galerkin methods.  
*arXiv:1804.04496 (math.NA)*, 2018.

Thank you!

# Extra slides

# The ultraweak form of Poisson's problem

Strong form:

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(Ultra)weak form:

$$\begin{aligned}(\boldsymbol{\sigma}, \operatorname{grad} v)_\Omega &= f, \quad \forall v \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (u, \operatorname{div} \boldsymbol{\tau})_\Omega &= 0, \quad \forall \boldsymbol{\tau}\end{aligned}$$

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Broken ultraweak form:

$$\begin{aligned}(\boldsymbol{\sigma}, \operatorname{grad} v)_\tau - \langle \hat{\sigma}_n, v \rangle_{\partial\tau} &= f, \quad \forall v \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_\tau + (u, \operatorname{div} \boldsymbol{\tau})_\tau - \langle \hat{u}, \boldsymbol{\tau} \cdot \hat{n} \rangle_{\partial\tau} &= 0, \quad \forall \boldsymbol{\tau}\end{aligned}$$

# The ultraweak form of Poisson's problem

Strong form:

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(Ultra)weak form:

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Broken ultraweak form:

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N.B.

- The functions  $u$  and  $\boldsymbol{\sigma}$  are discontinuous

$$u \in L^2(\Omega) \quad \boldsymbol{\sigma} \in (L^2(\Omega))^3 \quad u = (u, \boldsymbol{\sigma}, \hat{u}, \hat{\sigma}_n)$$

- All QoI can be written

$$g(\mathbf{u}) = \int_{\Omega} g_1 \cdot u + \int_{\Omega} \mathbf{g}_2 \cdot \boldsymbol{\sigma} + \int_{\partial\Omega} \hat{g}_1 \cdot \hat{u} + \int_{\partial\Omega} \hat{g}_2 \cdot \hat{\sigma}_n$$

# Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\mathbf{u}) = \int_{\Omega} u(\mathbf{x}) g_1(\mathbf{x}) d\mathbf{x}$$

where

$$g_1(\mathbf{x}) = \begin{cases} 1 & x \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Manufactured solution

$$u^{\text{man}}(x, y, z) = f(x/4)f(y)f(z)$$

where

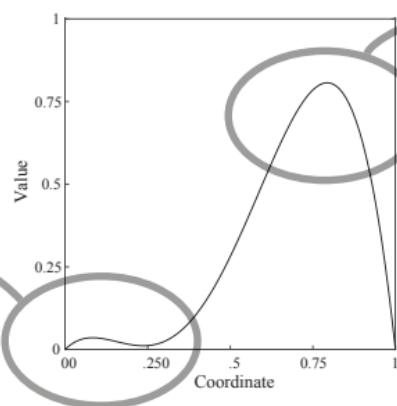
$$f(x) = x(1-x)((x/4) + (1-4x)^2)$$

and

$$\Omega = [0, 4] \times [0, 1] \times [0, 1]$$

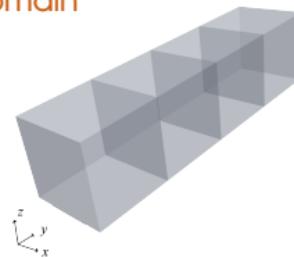
1D perspective

Shallow gradient



Steep gradient

Domain



# Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\mathbf{u}) = \int_{\Omega} u(\mathbf{x}) g_1(\mathbf{x}) \, d\mathbf{x}$$

where

$$g_1(\mathbf{x}) = \begin{cases} 1 & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Manufactured solution

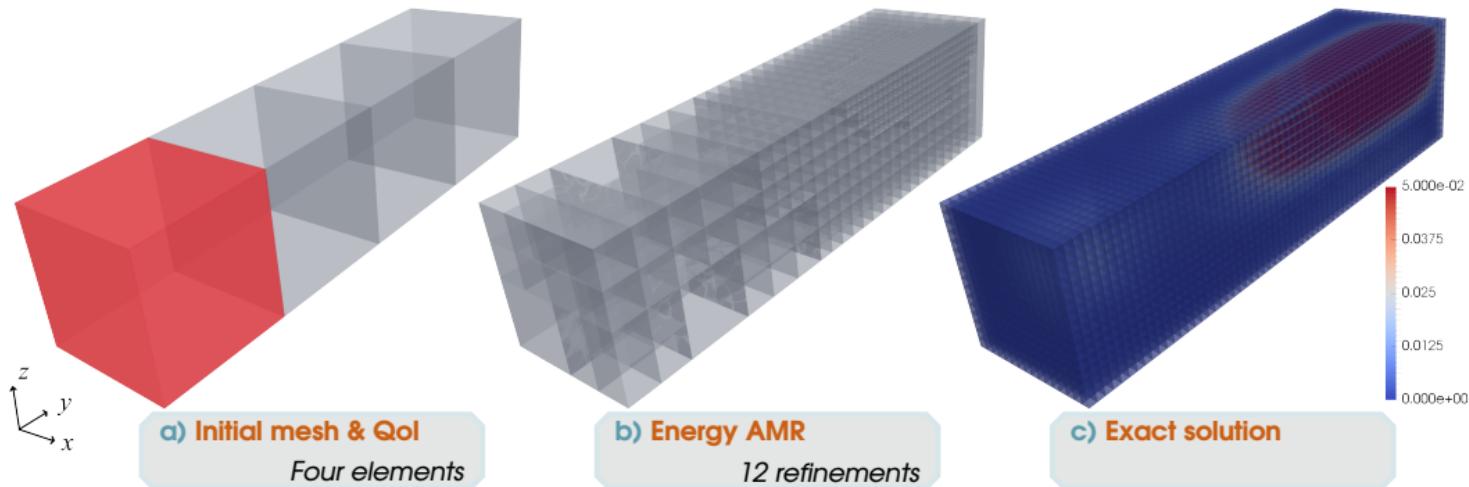
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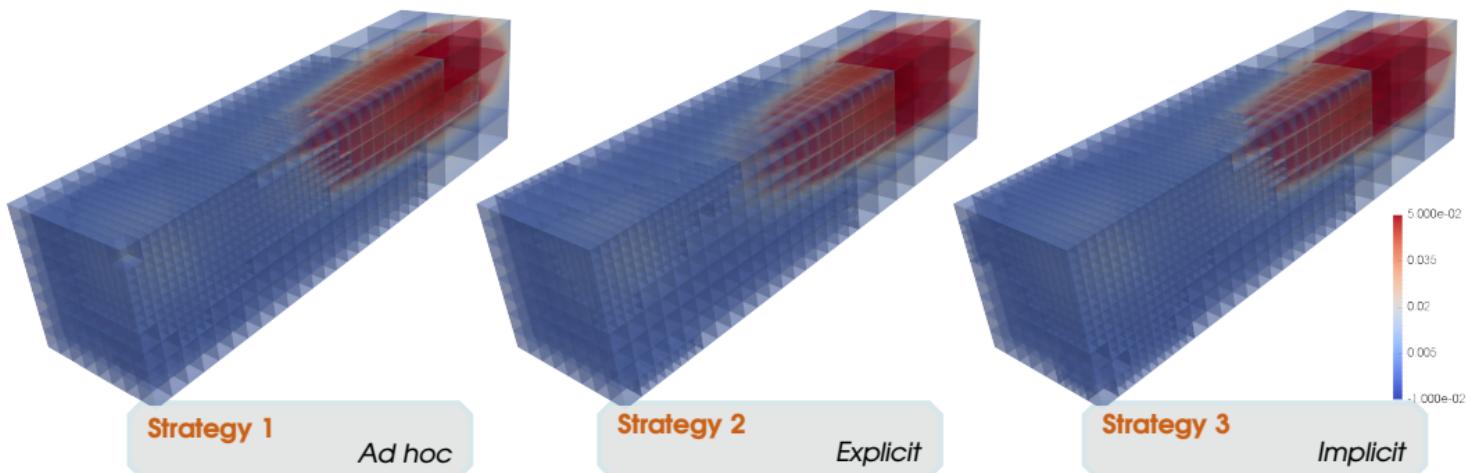
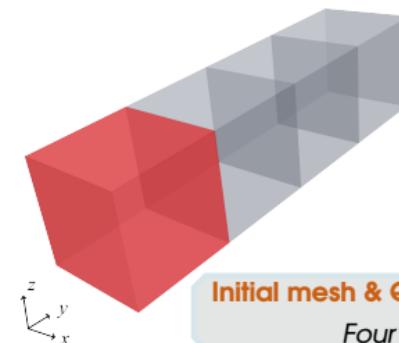
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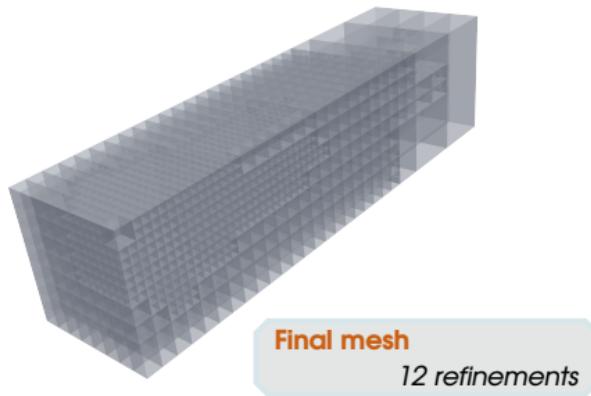
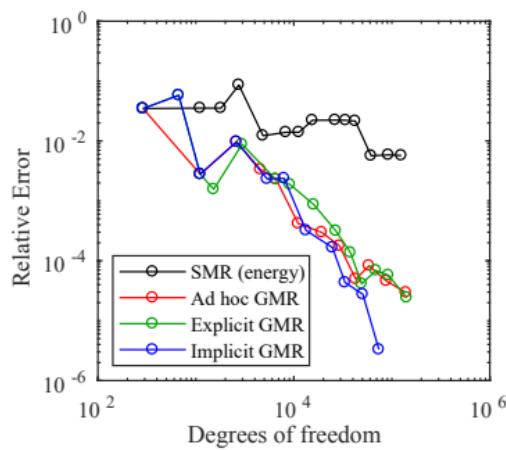
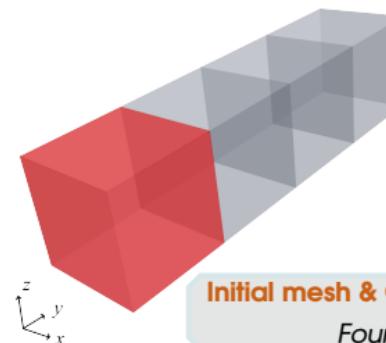
# Goal-oriented error estimation and adaptivity

Quantity of interest

$$g(\mathbf{u}) = \int_{\Omega} u(\mathbf{x}) g_1(\mathbf{x}) d\mathbf{x}$$

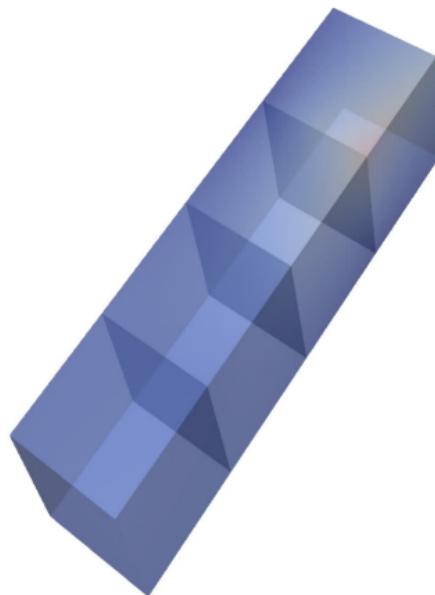
where

$$g_1(\mathbf{x}) = \begin{cases} 1 & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

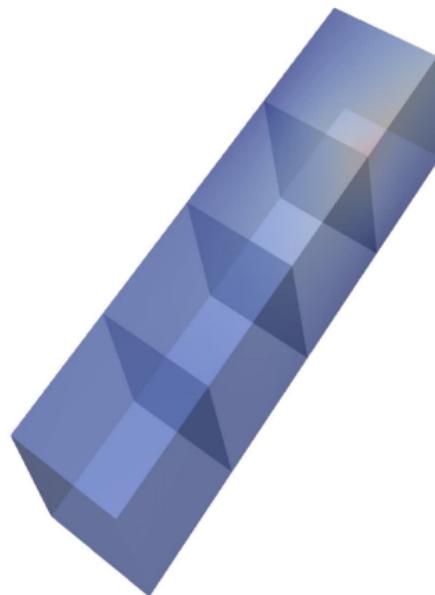


# Goal-oriented error estimation and adaptivity

Energy strategy



Goal-oriented strategy



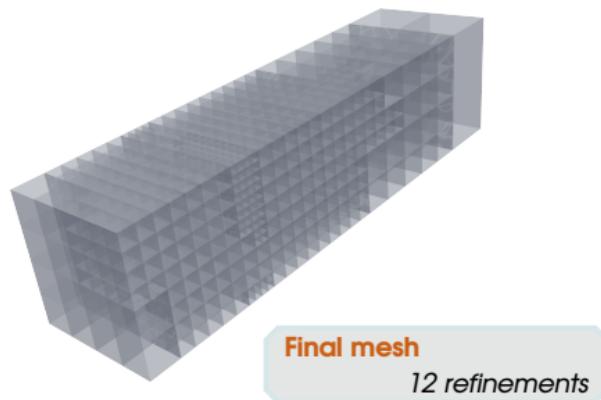
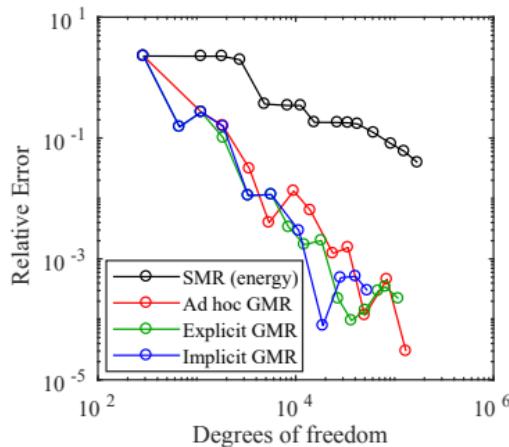
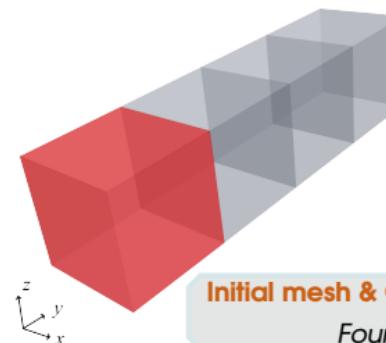
# Goal-oriented error estimation and adaptivity

Quantity of interest (flux)

$$g(\mathbf{u}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{g}_2(\mathbf{x}) \, d\mathbf{x}$$

where

$$\mathbf{g}_2(\mathbf{x}) = \begin{cases} [1, 0, 0]^T & x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



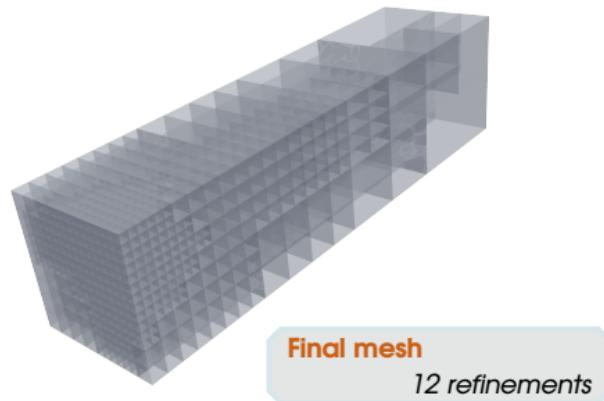
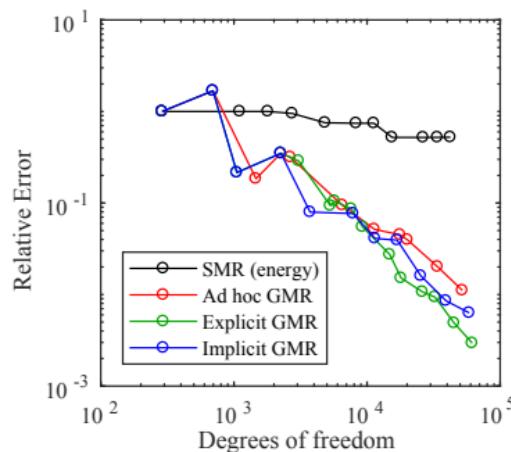
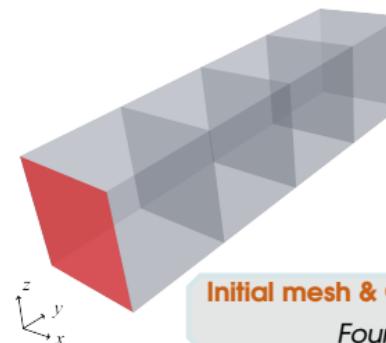
# Goal-oriented error estimation and adaptivity

Quantity of interest (boundary)

$$g(\mathbf{u}) = \int_{\partial\Omega} \hat{u}(\mathbf{x}) \hat{g}_1(\mathbf{x}) d\mathbf{x}$$

where

$$\hat{g}_1(\mathbf{x}) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



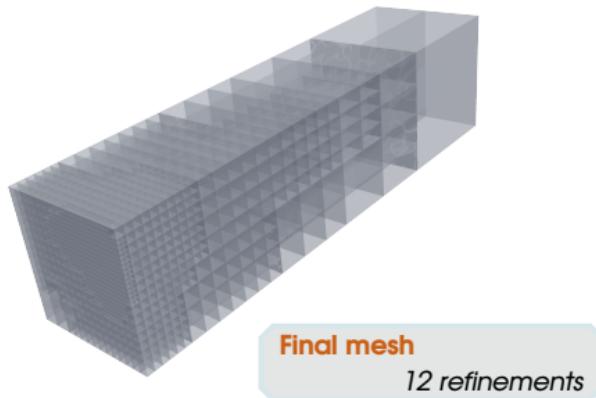
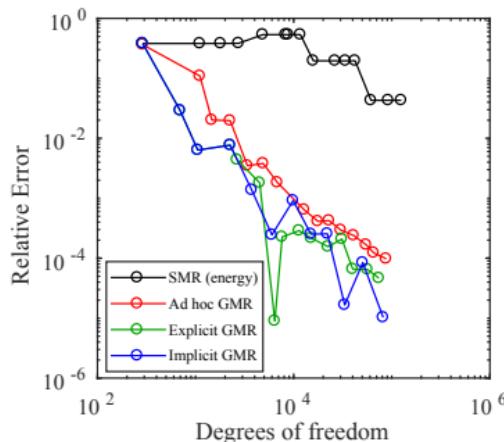
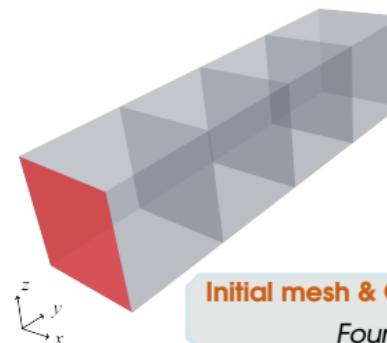
# Goal-oriented error estimation and adaptivity

Quantity of interest (boundary flux)

$$g(\mathbf{u}) = \int_{\partial\Omega} \hat{\sigma}_n(\mathbf{x}) \hat{g}_2(\mathbf{x}) d\mathbf{x}$$

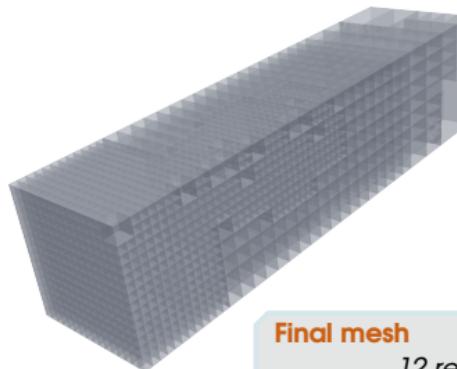
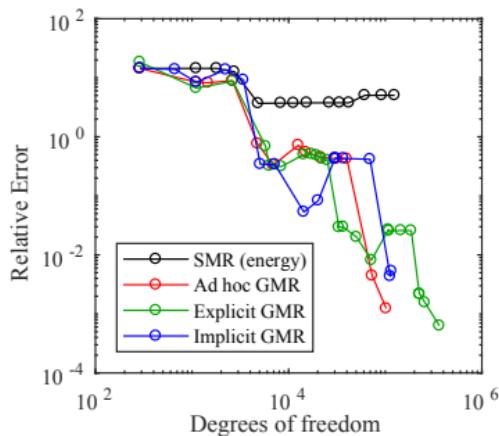
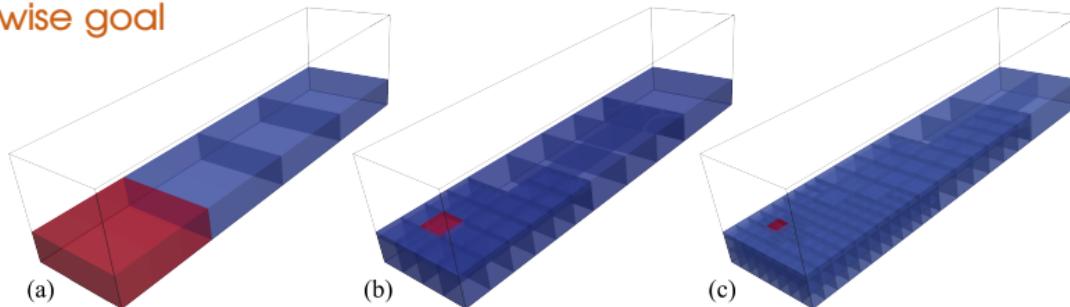
where

$$\hat{g}_2(\mathbf{x}) \approx \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



# Goal-oriented error estimation and adaptivity

Pointwise goal



Final mesh

12 refinements