Zero-inflated Poisson model

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モデルの specification

- “Zero-inflated”
  - π: 頻度が無条件に 0 となる確率
  - $1-\pi$: 頻度がパラメータ λ の Poisson 分布に従う確率
- 同時分布

$$p(\boldsymbol{x}, \boldsymbol{z}; \pi, \lambda) = \prod_{i=1}^{N} \left(\pi \delta(x_i = 0) \right)^{1-z_i} \left((1-\pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)^{z_i}$$

  - $\delta(\cdot)$ は、丸括弧内の条件が成立するとき 1、そうでないとき 0。

潜在変数の周辺化

$$
\begin{aligned}
p(\boldsymbol{x}; \pi, \lambda) &= \sum_{\boldsymbol{z}} p(\boldsymbol{x}, \boldsymbol{z}; \pi, \lambda) \\
&= \prod_{i=1}^{N} \left(\pi \delta(x_i = 0) + (1 - \pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)
\end{aligned}
$$

$$
\therefore \log p(\boldsymbol{x}; \pi, \lambda) = \sum_{i=1}^{N} \log \left(\pi \delta(x_i = 0) + (1 - \pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)
$$

Jensenの不等式の適用

$$
\begin{aligned}
\log p(\boldsymbol{x}; \pi, \lambda) &= \sum_{i=1}^{N} \log \left(\pi \delta(x_i = 0) + (1 - \pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right) \\
&= \sum_{i=1}^{N} \log \left(q_i \frac{\pi \delta(x_i = 0)}{q_i} + (1 - q_i) \frac{(1 - \pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}}{1 - q_i} \right) \\
&\geq \sum_{i=1}^{N} \left(q_i \log \frac{\pi \delta(x_i = 0)}{q_i} + (1 - q_i) \log \frac{(1 - \pi) \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}}{1 - q_i} \right) \\
&= \sum_{i=1}^{N} \Big(q_i \log \pi + q_i \log \delta(x_i = 0) - q_i \log q_i + (1 - q_i) \log (1 - \pi) + (1 - q_i) x_i \log \lambda \\
&\qquad - (1 - q_i) \lambda - (1 - q_i) \log (x_i!) - (1 - q_i) \log (1 - q_i) \Big) \\
&\equiv L(\pi, \lambda)
\end{aligned}
$$

M step

$$\frac{\partial L(\pi,\lambda)}{\partial \pi} = \frac{\sum_{i=1}^{N} q_i}{\pi} - \frac{\sum_{i=1}^{N}(1-q_i)}{1-\pi}$$

$\frac{\partial L(\pi,\lambda)}{\partial \pi} = 0$ とおくと、$\pi = \frac{\sum_{i=1}^{N} q_i}{N}$ を得る。

$$\frac{\partial L(\pi,\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{N}(1-q_i)x_i}{\lambda} - \sum_{i=1}^{N}(1-q_i)$$

$\frac{\partial L(\pi,\lambda)}{\partial \lambda} = 0$ とおくと、$\lambda = \frac{\sum_{i=1}^{N}(1-q_i)x_i}{\sum_{i=1}^{N}(1-q_i)}$ を得る。

E step

$$
\begin{aligned}
\frac{\partial L(\pi, \lambda)}{\partial q_i} = \log \pi + \log \delta(x_i = 0) - \log q_i - 1 - \log(1 - \pi) \\
- x_i \log \lambda + \lambda + \log(x_i!) + \log(1 - q_i) + 1
\end{aligned}
$$

$\frac{\partial L(\pi,\lambda)}{\partial q_i} = 0$ とおくと、以下を得る。

$$
q_i = \frac{\pi \delta(x_i = 0)}{\pi \delta(x_i = 0) + (1 - \pi)\frac{\lambda^{x_i} e^{-\lambda}}{x_i!}}
$$